PAOLO BOERO

TRANSFORMATION AND ANTICIPATION AS KEY PROCESSES IN ALGEBRAIC PROBLEM SOLVING

This chapter aims to deepen the idea that one of the crucial aspects of algebraic problem solving is the transformation of the mathematical structure of the problem in order to be able to manage it better, and that anticipation allows the process of transformation to be directed towards simplifying and resolving the task. Different cases of the transformation of the problem (without, before and/or after algebraic formalization) are analysed. Some educational implications are discussed.

1. INTRODUCTION

This chapter aims to deepen the idea that one of the crucial aspects of algebraic problem solving (which might be used to characterize it) is the transformation of the mathematical structure of the problem in order to be able to manage it better, and that anticipation allows the process of transformation to be directed towards simplifying and resolving the task.

The process of transformation may happen without, before and/or after algebraic formalization .When it happens without or before algebraic formalization, it is frequently based on the transformation of the problem situation through arithmetic or geometric or physical manipulation of variables (adding, subtracting, translating, equilibrating...). These problem solving strategies can be called "pre-algebraic" (see section 3). When the transformation happens after algebraic formalization, it is frequently based upon the "transformation function" of the algebraic code. In this case, the manipulation of the algebraic expression extends enormously the range of possibilities of transformation. At least a partial "suspension of the original meaning" of the transformed expression may happen during the transformation process (see section 2; cf Bednarz & al., 1992). The process of transformation needs specific prerequisites and skills. In the case of transformation after formalization, a crucial prerequisite is the mastery of standard patterns of transformation (see section 4).

A common ingredient of all the processes of transformation (without, before and/or after formalization) is anticipation. In order to direct the transformation in an efficient way, the subject needs to foresee some aspects of the final shape of the object to be transformed related to the goal to be reached, and some possibilities of transformation. This "anticipation" allows planning and continuous feed-back. In the case of transformations performed after formalization, anticipation is based on some peculiar properties of the external algebraic representation (see section 5).

One focus of this chapter is to consider the educational strategies which could enhance the development of the "anticipation process". In section 6 an analysis of some traditional and innovative practices will be carried out.

Examples related to different school levels will be integrated into the presentation, in order to show different aspects of the same topics.

2. AN HEURISTIC MODEL FOR THE TRANSFORMATION PROCESS

For heuristic purposes, I will use the following kinds of diagrams:

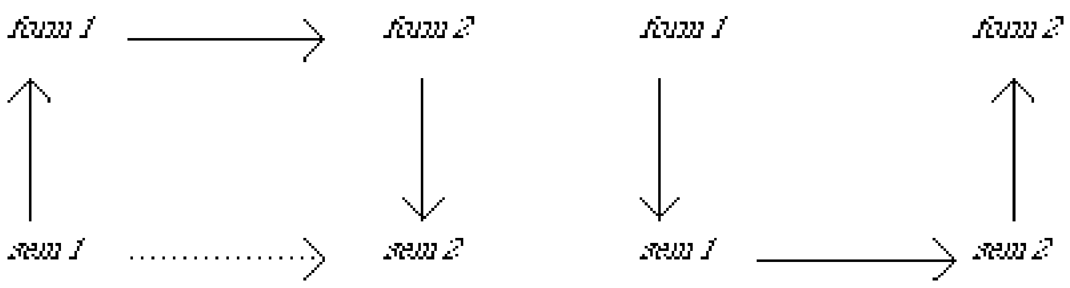


Fig. 1 - *sem/form* diagrams

In these diagrams, *form* means any written expression based on the use of the algebraic language; this wide definition covers a great deal of mathematical expressions (eventually integrating special symbols used in different mathematical fields: mathematical analysis, linear algebra, probability....): from arithmetic expressions (such as $3 * [2+ 5 * (7+2 * 3)]$) to algebraic equations, from trigonometric equations (such as $sin(2x+\pi/2)+2cos(x-\pi/2)=1$) to differential equations (such as $y'(x)=ay(x)-by^2(x)$), from functional expressions (such as $T(ax+by)=aT(x)+bT(y)$) to matrix expressions.

I will consider *sem* as a mathematical or non-mathematical cultural object (a mathematical statement, a relationship between physical or economical variables, and so on). *sem* consists of a mental representation *and* an external non-algebraic representation. These two expressions are suggested by the classification proposed by C.Janvier (1987, pp. 148-149): *"the word "representation" has roughly three different acceptations in the psychology literature: at first, ... material organization of symbols,...,which refers to other entities or 'modelizes' various mental processes....*(see "external representation" in the text above) *; the second meaning... refer to a certain organization of knowledge in the human mental "system" or in the long-term memory...* (see "mental representation" in the text above); *the third meaning refers to mental images. In fact, it is a special case of the second one".* See also Duval (1995, pp. 25-26) for similar definitions.

I would like to point out the fact that the *form* - *sem* distinction, as proposed in this article, does not follow the traditional *syntax* - *semantics* distinction. Indeed, *sem* brings its own external representation (for instance, a geometric figure - and/or a sentence of natural language). This choice can be justified by the need of analysing some algebraic problem solving processes, especially the activities performed at the *form* level and their relationships with the problem situation "represented" at the *sem* level (see subsections 2.1., 2.2 and 2.3.). For a discussion about possible "meanings" attributed by students to *form* , see Demby, 1996; for an in-depth study of some, delicate questions related to the *form* - *sem* distinction, see Arzarello, Bazzini & Chiappini, 1994 **(!!! and this volume !!!)**

In *form/sem* diagrams, upward arrows mean "formalization", downward arrows mean "interpretation". Formalization consists of a translation from *sem* into an expression of the algebraic language. Interpretation means generating a mental representation and an external non-algebraic representation coherent with *form* .

Continuous horizontal arrows mean "transformation" ; or more precisely:

- horizontal arrows between *form 1* and *form 2* mean "transformation according to the rules of the algebraic language", including not only standard algebraic transformations of a literal expression, but also resolution of differential equations, of systems of linear equations, etc.; also included are substitutions of numerical values to letters. In general, "transformation" will mean any process, based on direct algebraic transformations or substitutions or general theorems proved through algebraic transformations, and expressed through formulas, which allow to get some new algebraic expressions from the original one. The following are some examples illustrating the above ideas:

  i) transformation from : $(a^4-b^4)/(a+b)$ *to:* $a^3 - a^2b + ab^2 - b^3$ *: it can be performed through decomposition:* $a^4 - b^4 = (a-b)(a^3 - a^2b + ab^2 - b^3)$ *and simplification;*

  ii) transformation from : (sinx exp2x)' *to:* (cosx + 2sinx)exp2x : *it can be performed according to the theorem concerning the derivative of a product, and application of the distributive property;*

  iii) transformation from: y"(x)+4y(x)=1 *to:* y(x)=A sin2x + B cos2x + 1/4 : *it can be performed through standard methods of resolution of linear differential equations.*

- horizontal continuous arrows between *sem 1* and *sem 2* mean "transformation of mental and corresponding external representations". The example concerning the evaluation of the area of a rectangular trapezium, illustrated in subsection 2.2., shows how this transformation can be performed in that case (through a change of the decomposition of the trapezium).

I observe that *form 1* may be equivalent (through reversible algebraic transformations) to *form 2* , and *sem1* may be equivalent to *sem 2* (same example quoted above).

- horizontal dotted arrows indicate a "guess" (conjecture to be proved, etc.).

I will consider now some examples of usage of the heuristic model I have introduced. These examples will show how some algebraic problem solving activities can be schematized through the model and prepare the analyses performed in the following sections. The complexity of mental operations involved in algebraic problem solving and revealed through the *sem - form* diagrams suggests some, possible reasons of the difficulties met by students.

2.1. *Applying a formula to solve a mathematical or non mathematical standard problem.*

In this case, we start from *sem 1*, we put the problem we must solve into a formula (*form 1*), we operate a standard algebraic transformation (for instance: solving a standard algebraic equation), and we produce a "result" *form 2;* the interpretation of *form 2* produces a new "meaning" , *sem 2*. In many cases, this process is a multi-step process (with a chain of fundamental diagrams of the type considered before).

Example: *it is well known that the "stopping distance"* s *of a car, from the point where the driver sees the danger, can be determined by adding a distance proportional to the square of the speed* v *to a distance proportional to the speed v (depending on the quickness of reflex). Let us consider the problem of determining the range of the speed which is compatible with the "stopping distance" of 100 m; we may put the law stated before* (sem 1) *into a formula; we get, as* form 1 : $s= Av^2 + Bv \leq 100$; *then we may give values to* A *and* B *depending on the conditions of the road, on the condition of the braking system of the car, and on quickness of reflex (particularization of the situation, bringing to* sem 2 *and, correspondingly,* form 2).

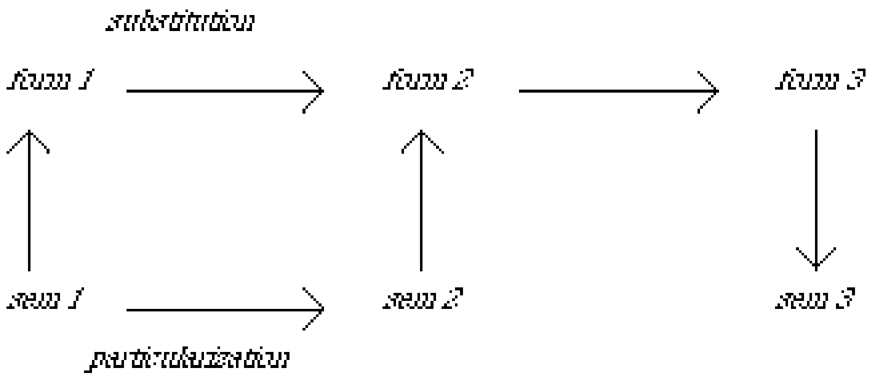
For a common situation (modern cars, normal conditions of the road, mean reflex speed) we may pose: A= 0.006 ; B= 0.08 *(if* v *is expressed in km/h, and* s *in metres).*

So, we may solve the inequality : $s = 0.006 v^2 + 0.08 v \leq 100$ (form 2).

We get (through standard formulas): $-136 \leq v \leq 123$ (form 3).

Interpreting this result, we may say that the speed must not exceed 123 km/h (sem 3).

The following diagram synthetizes the whole process:



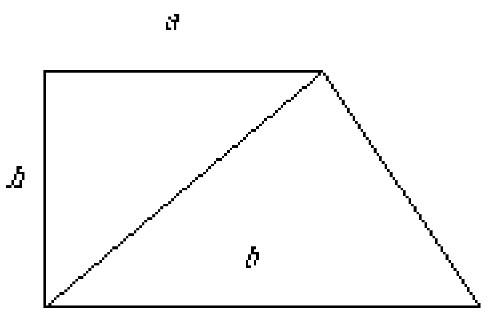
We may observe that the root -136 *(obtained through the resolution of the equation) is not relevant to our problem; this shows the importance of the "interpretation" phase of the "algebraic result"* form 3 .

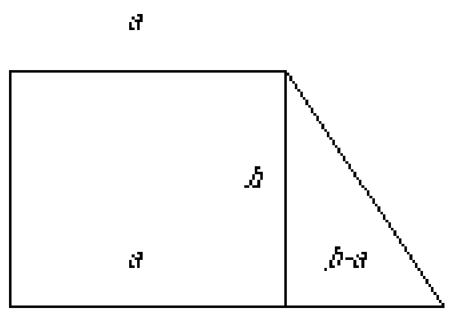
2.2. *Producing new knowledge about an open problem situation*

Suitable transformations at the *sem* level and/or at the *form* level can produce new knowledge.The new knowledge may concern:

- a conjecture about the existence of a transformation between *form 1* and *form 2*, suggested by relationships existing between *sem 1* and *sem 2*

This is illustrated by a simple **example** concerning the evaluation of the area of a rectangular trapezium:

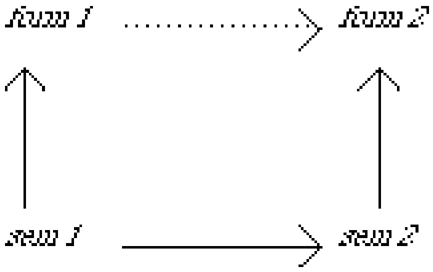




form 1 = ah/2 + bh/2

form 2 = ah + (b-a)h/2

In this case, two different decompositions of the original figure into simpler figures generate two different formulas; but sem 1 *is equivalent to* sem 2 *, and this suggests that a transformation may exist between* form 1 *and* form 2 .



- the existence of an "object" related to *sem 1*, whose existence is a consequence of the interpretation of *form 2*, derived from *form 1* according to more or less standard transformations

This is illustrated by an **example** suggested by Paolo Guidoni:

it is not difficult to verify through measure that the equilibrium temperature T_f *reached by the mixture of two quantities of water ,* m_1 *and* m_2 *, is related to their respective temperatures* T_1 *and* T_2 *at the moment of the mixture according to the following formula* (form 1):

$T_f = (m_1T_1 + m_2T_2)/(m_1+m_2)$

This formula may be interpreted as:

" T_f is the weighted mean of the temperatures T_1 and T_2 " (sem 1)

By a very easy algebraic transformation we may write :

$T_f(m_1+m_2) = m_1T_1 + m_2T_2$

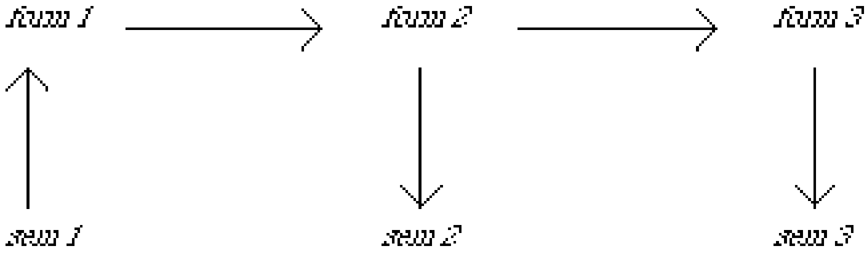
This formula (form 2) *may be interpreted as* "conservation of the quantity of heat" (sem 2)

By a suitable algebraic transformation of this formula we may write the following formula:

$(T_f - T_1)/(T_2 - T_f) = m_2/m_1$ (form 3)

This formula may be interpreted as "inverse proportionality between the quantities of water and the absolute variations of temperatures" (sem 3).

The following diagram synthetizes the whole process:

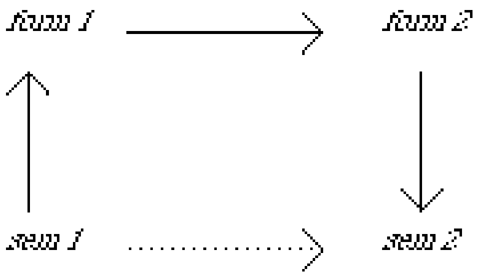


Some of the more spectacular applications of mathematics to physics concern this kind of usage of mathematics; physicists "put a physical situation (*sem 1*) into a formula" (*form1*) (an algebraic formula, a differential equation, etc.); suitable (more or less standard) transformations of the formula may generate a new "formula" *(form 2*), the interpretation of which *(sem 2)* may increase our knowledge of the physical world.

Another remark: a kind of principle of "neutrality"(in relation with the real world) of algebraic transformations (already realised by Galilei) allows us to operate in such a way, that if an hypothesis *sem 1* is appropriately put into a formula *form 1* , the interpretation of a transformed *form 2* formula (obtained from *form 1*) may be used as a tool to validate *sem1*.

2.3. *Proving a conjecture (in mathematics, physics, and so on)*

Algebraic formalism is a current tool in proving conjectures. In this case, frequently, *sem 2* is known (the "content" of the conjecture), *sem 1* is known (information about data: physical situation, relationships between variables), *form 1* and *form 2* must be expressed in a convenient way in order to get *form 2* starting from *form 1* with suitable transformations.



In many situations, the passage from *form 1* to *form 2* (and, consequently, from *sem 1* to *sem 2*) needs intermediate steps, according to a chain which may be more or less complex.

As a very simple **example**, we may consider the following theorem :

"The sum of two consecutive odd numbers is a multiple of 4"

Through a suitable formalization, we may write the sum of two consecutive odd numbers as: (2K + 1) + (2K + 3); *performing standard algebraic transformations we get:*

(2K + 1) + (2K + 3) = 2K + 1 + 2K + 3 = 4K + 4 = 4 (K +1) ;

the interpretation of this formula allows the validation of the thesis.

3. TRANSFORMATION BEFORE ALGEBRAIC FORMALIZATION

Some transformations of the problem situation, having a counterpart at the *form* level, can be performed without any algebraic formalism. This subject was investigated in collaboration with Lora Shapiro. First of all we recall some essential contents of our research report (Boero&Shapiro, 1992).

The purpose of this study was to understand better the mental processes (i.e. planning activities, management of memory, etc.), underlying students' problem solving strategies in a "complex" situation. To this end the following problem was administered to students from grade IV to grade VIII: *"With* T *liras for stamps one may mail a letter weighing no more than* M *grams. Maria has an envelop weighing* E *grams; how many drawing sheets , weighing* S *grams each, may she put in the envelop in order not to surmount (with the envelop) the weight of* M *grams ?"*

Various numerical versions have been proposed to different groups of pupils:

type of data	money needed (T)	maximum admissible weight(M)	weight of the envelop(E)	weight of each sheet of paper (S)
(50,7,8)	1500	50	7	8
(100,14,16)	2000	100	14	16
(100,7,8)	2000	100	7	8

(250,14,16)	3800	250	14	16

The students' resolution strategies have been analysed according to a classification scheme suggested by the data from a pilot study, and corresponding to the aim of exploring the mental processes underlying these strategies.

Strategies were coded in the following manner:

- "Pre-algebraic " strategies (PRE-ALG.). In this category the strategies involved taking the maximum admissible weight and subtracting the weight of the envelop from it.The number of sheets is then found by multiplying the weight of one sheet and comparing the product with the remaining weight, or dividing the remaining weight by the weight of a sheet of paper , or through mental estimates.If the problem would be represented in algebraic form, these strategies would correspond to transformations of the form :

$Sx + E \leq M$ to : $Sx \leq M-E$, up to : $x = (M - E)/ S$

For the purposes of this research, we have adopted the denomination "pre-algebraic" in order to emphasise two important , strictly connected aspects of algebraic reasoning, namely the transformation of the mathematical structure of the problem ("reducing" it to a problem of division by performing a prior subtraction) ; and the discharge of information from memory in order to simplify mental work .

- "Envelop and sheets" strategies (ENV&SH).This "situational" denomination was chosen by us because it best represented students' production of a solution where the weight of the envelop and the weight of the sheet are managed together . These strategies include "mental calculation strategies", in which the result is reached by immediate, simultaneous intuition of the maximum admissible number of sheets with respect to the added weight of the envelop; "trial and error" strategies in which the solution is reached by a succession of numerical trials, keeping into account the results of the preceding trials (for instance, one works on the weight of some number of sheets and adds the weight of the envelope, checking for the compatibility with the maximum allowable weight) ; "hypothetical strategies", in which one keeps into account the fact that the weight of one sheet is near to the weight of the envelop, and thus hypothesizes that the maximum allowable weight is filled by sheets, and then decreases the number of sheets by one, etc.

A preliminary review of the results (see Boero & Shapiro,1992) showed that there is a clear evolution with respect to age and instruction from ENV&SH. strategies towards PRE-ALG.strategies within and between numerical versions (this is found in homogeneous groups of pupils: transition from IV grade to V grade; and from VI grade to VIII grade). We see that the motivations and access to pre-algebraic strategies may be different; but in all of them there is a form of reasoning that may derive from a wide experience involving production of "anticipatory thinking". That is to say , with the aim of economizing efforts, pupils plan operations which reduce the complexity of mental work . This interpretation provides a coherence amongst different results, concerning the evolution towards PRE-ALG. strategies with respect to age, as shown in the solutions produced in grade IV to grade V and in grade VI to grade VIII, as well as with respect to the results involving more difficult numerical data (in the case (250, 14, 16), results show an higher percentage of PRE - ALG. strategies at every age level).

Concerning research findings in the domain of pre-algebraic thinking, we may observe that there is some coherence between:

- our results, concerning the influence of numerical data on strategies in an applied mathematical word problem, proposed to students prior to any experience of representation of a word problem by an equation and prior to any instruction in the domain of equations; and

- Herscovics & Linchewski's results (1991) , concerning numerical equations proposed to seventh graders prior to any instruction in the domain of equations. For instance, they find that the equation $4n + 17 = 65$ is solved by 41% of seventh graders by performing $4n= 65-17=48$ and then $n= 48/4$, while the equation $13n + 196 = 391$ is solved in a similar way by 77% of seventh graders.

Taking into account the Herscovics&Linchewski's (1991) and Filloy&Rojano's (1989) findings, we have performed a further analysis of our data which gives evidence of two extreme opposite patterns, and many intermediate behaviours of pupils engaging in a PRE-

ALG. strategy. Some students seem to transform the problem situation by thinking about the number of sheets and the weight of the envelope as physical variables; indeed they subtract the weight of the envelope and work with the remaining weight. Other students "put into a numerical equation" the problem situation (even if they do not formally write the equation!) and transform the equation (they perform a subtraction, and then a division on pure numbers). The presence of these extreme patterns in the same problem situation in the same classes may explain a deeper relationship between our findings and other findings concerning purely numerical equations. It also allows us to understand better the degree to which different approaches to the "transforming function" of the algebraic languageare complementary.

Our study gives some information about the cognitive roots of algebraic transformations. As we saw in the preceding paragraph, algebraic transformations (especially the more open and complex ones) require the student to integrate two or more of the following activities:

- transforming the nature of the problem (through horizontal and vertical arrows), in order to be able to manage the transformed problem in an easier way;
- anticipating (i.e. imagining the consequences of some choices operated on algebraic espressions and/or on the variables, and/or through the formalization process)
- making choices in order to obtain the solution in an economic way;
- suspending the original reference meaning (at the *sem* level) of algebraic expressions, and working at the level of algebraic transformations;
- using the reference to the meaning (at the *sem* level) to plan further steps of transformation of *form* (cfr. Radford, 1994: *"semantic deduction"*), or to interpret the consequences of performed transformations.

If we consider the "sheets and envelop" problem and the resolutions achieved by students, we realize that (depending on age and instruction) many of them, while producing and managing PRE-ALG. strategies, were able to integrate some of these activities in an effective way.

4. DEEPENING THE TRANSFORMATION FUNCTION OF THE ALGEBRAIC LANGUAGE

We observe that any algebraic expression may be transformed into different expressions, and any transformation may be achieved through different patterns, according to different aims and criteria. I will try to explore some aspects of this "transformation" process related to its aims and components. To do this, I will start by analysing an example in some details:

Example: *this is the case of trigonometric equations deriving from mechanics or geometry problems; the trasformation process is performed in order to bring them to a well known, easy to process expression:*

$\sin^2 x + \cos^2 2x = 3/2$ *becomes:* $\sin^2 x + (\cos^2 x - \sin^2 x)^2 = 3/2$, *and then:*

$\sin^2 x + (1 - 2\sin^2 x)^2 = 3/2$,.............., *and, finally,* $4 \sin^4 x - 3 \sin^2 x - 1/2 = 0$,

which is easy to solve through the substitution: $\sin^2 x = y$.

As regards the next points i), ii), I observe that the (standard) transformation of $\cos^2 2x$ in terms of $\sin^2 x$ is suggested by a guess concerning the possibility of writing down an equation in the "unknown" $\sin^2 x$: in the case of a high school student familiar with trigonometric equations, the experience gained in similar situations and the initial shape of the equation allow a transformation suitable to facilitate the task of solving the (transformed) equation. The transformation process, guided by this intuition, is performed according to standard patterns.

Taking into account the analysis performed in the preceding example, we may consider the following working hypotheses:

i) The transformation function is performed through a **dialectic relationship** between **standard patterns of transformation**, deriving from instruction and practice, which produce the transformations, (for instance, considering the equality:
$b^2 - a^2 = (b-a)(b+a)$; or: $a/b + c/d = (ad+bc)/bd$; or: $(fg)' = f'g + fg'$)) and **anticipations** which suggest a suitable "shape" for the formula to be processed and the direction of transformations.

Concerning the words utilised to express this working hypothesis, I observe that the word "anticipation" means the mental process through which the subject foresees the final (and/or some intermediate) shape of an algebraic expression useful for solving the problem, and the general direction of the trasformations needed to get it. Different elements may be concurrent in this process: the memory of past, successful trasformations performed in similar situations (i.e., experience); the intuition of possible, final or intermediate shapes of the algebraic expression, suggested by its present shape; the capacity of relating the shape of a possible transformed expression to the aim of solving the problem.

With the word "dialectic" I want to emphasise the fact that if the subject has the necessary prerequisites and experience to attack a problem needing algebraic transformations, his success depends on a functional dynamic relationship between the two "poles" (standard pattern of transformation and anticipation) whose characteristics are different and, in some senses, opposite. The continuous tension between "foreseeing" and "applying", "guessing" and "testing the effectiveness" allows the productive development of the process of algebraic transformation.

In general, standard patterns of transformation without anticipation offer blind perspectives - with the exception, for expert people, of some easy school exercises on "simplification" of algebraic expressions or standard resolutions of equations. Concerning the expression "blind perspectives", here are two very simple **examples**:

- (**example,** grade VIII): *the student is requested to generalize, in the case of the sum of four subsequent odd numbers, the property according to which the sum of two subsequent odd numbers is a multiple of* 4.

He immediately writes down:

p +1 + p+ 3 + p + 5 + p + 7= 4p + 16 (*the choice of the letter* p *probably depends on the first letter of* "pari", *which means* "even" *in Italian*), *then he stops: he does not anticipate the divisibility by 8; the probable, original meaning of* p ("even") *and the divisibility of* p *by* 2 *remain hidden; at first glance the student only finds the divisibility by* 4; *later on he writes :*

4p +16 = 4p + 8 + 8 = (4p + 8) + 8, *then he stops again;*

- (**example**, first university year): *the student already knows the proof according to which if* f *and* g *are derivable functions, then* fg *is derivable and* (fg)'= f 'g+fg'; *the student must find out what happens with* 1/f *(if* f *is a derivable positive function). The student writes down the "incremental ratio" of* 1/f *at point* x:

(1/f(x+h)-1/f(x))/h=((f(x)-f(x+h))/f(x+h)f(x))/h;

then he stops: no relationship is recognized with known derivatives, no connection is made with a formula to be proved.

On the contrary, anticipation may suggest the final expression of the transformed formula or intermediate steps, but - apart from very easy problems- such results cannot be obtained without a sufficient mastery of the standard patterns of transformation.

Here are two simple **examples**:

- (**example**, grade VIII): *through examples, students have discovered that the sum of two subsequent odd numbers is divisible by* 4; *the teacher helped them to write down the initial and final step of the proof of this conjecture:*

2k + 1+ 2k + 3 = C * 4 *(where* C *is a suitable number depending on* k*)*

A student writes: 2k + 1 + 2k + 3 = 2k + 2k + 4, *then he stops and says:* "I see, 4 seems to be there... But I cannot figure it out in the formula". *The standard transformations* 2k+2k = 4k *and* 4k+4=(K+1) ∗ 4 *seem to be out of reach of this student.*

Another student writes: 2k + 1 + 2k + 3 = 2k + 2k + 4 = 4k +4; *then he says: "* how can it be proved that 4k + 4 is a multiple of 4? ". *The standard transformation* 4k+4=(K+1) ∗ 4 *seems to be out of reach of this student, although he knew the distributive property for numbers, as we realized by a previous interview.*

- (**example**, first university year): *students must prove that* "if f is derivable and positive, then 1/f is derivable and (1/f)' =- f'/f^2 "; *a student writes down the same expression already considered in the previous example concerning derivation, then he says:* "Oh, yes, I see: *f(x+h)f(x)* approaches $f^2(x)$ when *h* approaches *0*.... oh, yes, *f(x)-f(x+h)* is like - *(f(x+h)-f(x))*....but how can I bring *h* under the difference *-(f(x+h)-f(x))* ? The place of *h* is occupied by *f(x+h)f(x)* ! "

ii) - such **dialectic relationship** may have different characteristics and develop in different ways in different problems, as shown in the examples under i) and in the following two extreme cases:

- proving a conjecture (see 2.3.):

frequently, the "shape" of the final formula may be easily determined - or it is given; a convenient algebraic representation of the relationship between data must be constructed in order to facilitate the process of transformation towards the final formula, anticipating some aspects of this process, and standard patterns of transformation must be applied to achieve the transformations;

- constructing a conjecture (see 2.2., second example):

the final formula is unknown; exploring, anticipating, transforming algebraic expressions must take place, based on generalizing and synthetizing numerical experiments and/or extablishing algebraic relationships between the variables involved;

iii) - such **dialectic relationship** needs support by algebraic, external representations (see Janvier, 1987 and section 2) with suitable characteristics in order to manage "patterns of trasformation" and "anticipations" (symmetry, references of literal signs to their meanings,.....). In particular, as we have seen in the preceding examples, and we will see later in more details, sometimes the shape of the written algebraic expression may provide hints for the process of transformation (thus supporting anticipation), sometimes the shape of the written algebraic expression is suggested by the guess of a possible transformation suitable for solving the problem.

According to our experiences, the shape of the algebraic expression, autonomously written by students (or suggested to them by the teacher) in order to solve a problem, has a very strong influence on their performances.

As an **example**, *few eight graders are able to prove that the sum of two consecutive odd numbers is a multiple of* 4 (see 2.3.) *if the teacher suggests writing two consecutive odd numbers as* d, d+2 *or* p+1, p+3. *On the contrary if the teacher suggests taking into account that an odd number may be written as* 2k +1, *then proof becomes accessible to many students.*

Another **example***: in order to prove that* (p-1)(q^2 - 1) *is divisible by* 16 *if* p *and* q *are odd numbers, frequently high school or university students write* p=2m+1 *and* q=2n+1 *and finally get (by standard transformations) the following espression:* (2m-2)(2n+2)2n= 8(m-1)(n+1)n *(cf Arzarello, Bazzini & Chiappini, 1994). At this point, if the teacher does not intervene, many students abandon this track because they do not "see" that* (n+1)n *is the product of an even number and an odd number ! The presence of m-1 acts as a distractor, the shape* (n+1)n *hides the existence of an even number in this product!*

These working hypotheses, which offer a "way of viewing" the process of transformation of algebraic expressions, have been used to plan some experiments with students. Collected data will allow some cognitive aspects of the transformation function of the algebraic language (see 5.), and some educational problems concerning it (see 6.) to be analyzed. Thus, it will be possible to understand if the "way of viewing" realized through the previous hypotheses provides some insight into the process of transformation; it will also be possible to deepen the meaning of these hypotheses.

5. COGNITIVE ASPECTS OF ALGEBRAIC TRANSFORMATIONS: THE PROCESS OF ANTICIPATION AND THE ROLE OF EXTERNAL REPRESENTATION

From the cognitive point of view, I will try to deepen the role of suitable written algebraic representations in enhancing the previously mentioned dialectic relationship or preventing it from taking place.

Concerning this issue, two experiments were realized in 1994/95.

The **first experiment** concerned university students with a wide, common university background in algebraic transformations (fourth year mathematics students).

The aim of this experiment was to explore the dependence of the two poles of that dialectic relationship (standard patterns of transformation; anticipation) on the possibility of writing algebraic expressions. Do written algebraic expressions, when they are allowed, enhance standard patterns of transformation and/or anticipation? How do limitations in using written algebraic expression affect standard patterns of transformation and/or anticipation?

This experiment consisted of proposing different tasks (proving a conjecture; constructing a conjecture) without any restrictions for one group of students (group A), and with the following restrictions for the other parallel group (group B):

- when "proving a conjecture", only the "final" and the "initial" algebraic expressions may be written; the proof must be written verbally without using algebraic signs.
- when "constructing a conjecture", all the explorations had to be expressed verbally; no algebraic sign was allowed neither in the explorations nor in the expression of the conjecture.

Here are the conjectures to be proved:

(C1) (following an idea by Arzarello): *prove that the number* $(p-1)(q^2 - 1)/8$ *is an even number, when* p *and* q *are odd numbers*

(C2) *if* K *is a natural number*, *prove that the sum of* 2K *consecutive odd numbers is a multiple of* 4K

The conjecture to be constructed concerned possible generalizations, different from (C2), of the property: *"The sum of two consecutive odd numbers is a multiple of* 4*"*

The conclusion of the analysis of collected data may be summarized as follows:

- for a very simple initial expression (conjecture C1), anticipation and standard patterns seem to be more easily developed by the students of group B in comparison with the students of group A (having no restriction in writing the intermediate steps): the possibility of intermediate steps seem to reduce the engagement in planning activities;
- in the case of a more complex task (conjecture C2), or more general conjectures constructed, both anticipation and recourse to standard patterns of transformation are substantially enhanced by the possibility of managing and transforming written expressions in a written form, even if the two processes seem of very different nature (see the later issues: "externalization" and"internalization"). It is also interesting to see that students of group B produce only simple conjectures, compared to those produced by the students of group A;

- again, in the case of the conjecture C2, the choices of the initial expressions (when produced) are more carefully made by the students of group B with suitable letters and "shapes" : the prevention from transforming (in written form) the algebraic expressions seems to enhance anticipation; this puts into evidence the nature of the planning process, which is inherent in writing down the starting expression.

All this seems to be very strictly related to our observations concerning the role of external representation in problem solving (see Ferrari, 1992): good problem solvers orient the external representation of the problem situation towards its resolution; this means that, from its very beginning, external representation is "solving process oriented", with a balanced relationship between "externalization" (external realization of shapes and steps anticipated in the mind) and "internalization" (taking and exploiting products of one's own external actions, or products of other people).

The following excerpt, concerning a group A student who tries to prove C2, shows what "externalization" and "internalization" mean in the case of algebraic expressions:

1+3+5+7 = 16 *divisible by* 4x2=8 , *OK;* 5+7+9+11+13+15= 60 *divisible by* 4x3=12, *OK*

(after 2 minutes):

(*) (2m + 1) + (2m + 3) + ... + (2m + 2K-1) = 4KS

(NOTE: I am not sure about the last term of the sum . However, we will see).

(after 3 minutes):

(**) *Perhaps, I can balance!*

(2m + 1) + (2m + 3) + (2m+5) ...+ (2m +2K - 5) + (2m+2K -3) + (2m + 2K-1) = 4KS

(2m + 1 +2m + 2K-1) + (2m+3 + 2m + 2K-3) +(2m + 5+ 2m+2K - 5)+ ....=

(4m +2K) K *times* .... = 2 (2m +K)K

It is not yet divisible by 4K. *Let us try with some values of* K

K = 1 *means: two odd, consecutive numbers :* 2m + 1 + 2m+1

It does not work! They are not consecutive!

Perhaps, the first is 2m +1 *and the last is* 2m + 2K + 1 *for* K=1 *is O.K.*

For K= 2 *I must consider 4 consecutive, odd numbers:* 2K+1 =5

(2m + 1) +(2m + 3) + (2m + 5) + (2m + 7) = (2m + 1) +.... +(2m+5) . *It does not work.*

I consider only the second terms of the sums : K=1 : 2K=2 ------> 1, 3

K=2 : 2K=4 ------> 1, 3, 5, 7

K=3 : 2K=6 ------> 1, 3, 5, 7, 9,11

K=4 : 2K=8 ------> 1, 3, 5, 7, 9, 11, 13,15

3, 7, 11, 15 *: it means* 4K-1 *!*

OK, let us come back to 2 (2m +K)K:

(***) *now it works very well, I see* 4K:

(2m + 1) + (2m + 3) + (2m+5) .....+ (2m +4K - 5) + (2m+4K -3) + (2m + 4K-1)=

(2m + 1 +2m + 4K-1) + (2m+3 + 2m + 4K-3) +(2m + 5+ 2m+4K - 5)+ .... =

(4m+4K)K= 4K(m+K)

We can see that:

(*) is a mental product (from "inside" to "outside"), probably realized in order to fix (and gain access to further analysis) the intuition of a "shape" suggested by numerical experiments. The "heuristic" intention of this step is confirmed by the "NOTE".

(**) the "shape" produced suggests (from "outside" to "inside") the process of balancing and, probably, the possibility of putting into evidence multiples of *2m* and *2K.* In order to better realize the process of balancing, another expression is produced (from "inside" to "outside").

(***) the student foresees (probably with the help of the expression written after (**) , with a process from "inside" to "outside" and subsequently from "outside" to "inside") that the change decided in the formalization, and imagined in the expression, fits in with the need to show the divisibility by 4 . Then the student produces the new expression, following the old one carefully.

The **second experiment** was made with students of different ages and school experience in the domain of algebraic trasformations (grade VIII, comparing groups of students having followed different curricula; first year university students coming from different kinds of high schools). The aim of the experiment was to explore the dependence of mental and written transformations on previous experience in this domain.

Two kinds of tasks were proposed:

- transforming a given expression into a given expression without/with permission to write down intermediate steps of the algebraic transformation. Here is an example of a test used in this experiment for first year university students:

Consider the expression: $(b^3-a^3)(b^2+a^2)/(b^4-a^4)$;

Can you transform it into : $b + a$ *?*

Or into: $b - a$ *?*

Or into: $(a^2 + ab + b^2)/(a+b)$ *?*

- proving a conjecture, expressed in verbal terms, with permission to write down the initial and final steps, and without/with permission to write down intermediate steps of the algebraic transformation.

The conjecture to be proved, for grade VIII students, concerned the fact that "*the sum of four consecutive odd numbers is a multiple of eight"*.

For first year university students, the conjecture to be proved concerned the fact that

"if K *is a natural number, the sum of* 2K *consecutive odd numbers is a multiple of* 4K*"*

In both situations, in the case of students who are allowed to perform a completely written transformation process the previous school experience (quality and quantity) seems very relevant; in the case of students not allowed to write down the intermediate steps, only the quality of the previous school experience seems to be influential (especially as regards the "anticipation" aspects of the transformation process).

In the second situation, we also obtained confirmation of the fact (found in the other experiment) that prevention from writing intermediate steps forces students to find more suitable "shapes" for their initial formulas (enhancing "anticipation").

6. EDUCATIONAL IMPLICATIONS

From the educational point of view, I will consider some (more or less) current activities which may hinder or enhance the development of the **dialectic relationship** considered in section 4.:

- calculating standard arithmetic expressions;
- transforming algebraic expressions in order to simplify them;
- understanding and repeating algebraic proofs;
- producing and proving conjectures expressed with algebraic formulas;
- discussing the direction of transformations needed to obtain an algebraic expression with given characteristics.

The content of this section largely agree with some findings obtained with different methods and from a different perspective by Y. Chevallard (1989).

Calculating standard arithmetic expressions: usually students are taught to manage algebraic notations according to very strict algorithmic rules *("calculate multiplications and divisions, then additions and subtractions, from the inner parentheses outwards")* ; the mechanism is blind, some small, formal changes modifying the usual "shape" of the expression may cause serious problems, as happens in the calculation of :

$3 * (2 + 5 * [1+ 4 * \{1 - 6x3\}] \{2+5 * 6\})$

We observe that anticipation is not stimulated, nor is the application of algebraic properties of operations (distributive property...). For these types of problems, teaching might be considerably improved, but no effort is usually made in this direction.

Transforming algebraic expressions in order to simplify them: in some cases, the final result is given (and it may stimulate some anticipation process, such as in an algebraic proof of a given formula); in other cases, anticipation is needed to perform transformations which (for some phases of the process) do not tend to "reduce the number of parentheses".

For instance, the simplification of the expression:

$(b^3-a^3)(b^2+a^2)/ (b^4-a^4)$

needs some parentheses to be added temporarily, anticipating the fact that b^4-a^4 *may "liberate"* b-a *and* b^2+a^2*, and that* b^3-a^3 *may "liberate"* b-a.

We observe that executing the multiplication $(b^3-a^3)(b^2+a^2)$ *results in a stalemate.*

Examining nine textbooks for Italian high schools, I found many examples of this kind only in two textbooks. On the contrary, most expressions proposed in the textbooks suggest moving towards progressive simplification, step after step, reducing the number of parentheses, as in this example:

$$[(b^2 - a^2)(b^2 + a^2) - b^4 - a^4] / a^3$$

Examples of this kind are very frequent in almost all the texbooks for Italian high school. Usually, the exercise is followed by the final, "simple" result (- *2a*) which suggests this process of standard, progressive simplification.

Understanding and repeating algebraic proofs: this activity is very common from high school onwards, in algebra (algebraic equations, theory of groups, vector spaces, etc.) and in other domains when using the algebraic code. No anticipation is usually requested, while standard patterns of transformation are performed and may be better understood.

Producing and proving conjectures expressed with algebraic formulas; discussing the direction of transformations needed to obtain an algebraic expression with given characteristics: these are uncommon activities in pre-university mathematics education; they might be used to enhance anticipation and (under suitable guidance from the teacher) to stimulate awareness about the nature of processes of transformation (metacognitive aspect). It is not difficult to plan this kind of activities. For instance, the example illustrated at the beginning of 2.2. might be proposed to comprehensive school students as an introduction to consciuos transformation of algebraic formulas. Unfortunately, no room is usually left for this kind of topics.

Striking a balance between 'common' and 'uncommon' activities performed using the algebraic code, we find that the activities more suitable for ensuring development of "anticipation" and a conscious management of the process of transformation are 'uncommon' in schools. At present, students are mainly forced to develop the "standard patterns of transformation" component of the transformation process.

7. CONCLUSION

The aim of this chapter was to show how the transformation of the mathematical structure of a problem is a crucial aspect in algebraic problem solving, and the role of anticipation in allowing the process of transformation to be directed towards simplifying and resolving the task.

The point of view illustrated in this article brings some elements of novelty in the debate concerning the school approach to algebra and the relationships between arithmetic and algebra. Indeed, the transformation of the mathematical structure of a problem finds a natural tool in the algebraic language, but can be performed also before algebraic formalization (as shown in the PRE-ALG. strategies of students, described in the section 3.). Suitable word problems, needing pre-algebraic strategies (like the "envelop and sheet problem"), could be largely introduced in early grades in order to develop anticipation.

Other findings illustrated in this chapter concern the transformation of algebraic expressions in algebraic problem solving. Processes of anticipation integrated with the recourse to appropriate standard patterns of transformation play a major role in ensuring effective transformations. This point of view suggests that current classroom activities in the field of algebra are not equilibrated - most exercises are aimed at developing only standard patterns of transformation; other activities, which might be very useful in order to develop anticipation, are missing.

This chapter introduces the *sem - form* diagrams as an original, heuristic tool to schematize some crucial algebraic problem solving activities. In spite of the lack of "formal" definitions of the elements of the *sem - form* diagrams, their usage allowed to clarify the complexity of mental operations involved in algebraic problem solving and might suggest possible interpretations of some difficulties met by students and some ways to overcome them. For instance, the first example in subsection 2.2 and the example in subsection 2.3., sometimes used to introduce students to algebraic trasformations, might also be exploited to make them aware of the "sense" of algebraic transformations in algebraic problem solving.

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Paolo Boero

Dipartimento di Matematica, Università di Genova
Via Dodecaneso, 35 - GENOVA, 16146 - ITALIA

E-mail: *boero@dima.unige.it*