Regularity jumps for powers of ideals

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1 Abstract

The Castelnuovo-Mumford regularity $\operatorname{reg}(I)$ is one of the most important invariants of a homogeneous ideal I in a polynomial ring. A basic question is how the regularity behaves with respect to taking powers of ideals. It is known that in the long-run $\operatorname{reg}(I^k)$ is a linear function of k, i.e. there exist integers $a(I), b(I), c(I)$ such that $\operatorname{reg}(I^k) = a(I)k + b(I)$ for all $k \geq c(I)$. For any given integer $d > 1$ we construct an ideal J generated by $d+5$ monomials of degree $d+1$ in 4 variables such that $\operatorname{reg}(J^k) = k(d+1)$ for every $k < d$ and $\operatorname{reg}(J^d) \geq d(d+1) + d - 1$. In particular, this shows that the invariant $c(I)$ cannot be bounded above in terms of the number of variables only, not even for monomial ideals.

2 Introduction

Let K be a field. Let $R = K[x_1, \dots, x_n]$ be the polynomial ring over K. Let $I = \oplus_{i\in\mathbb{N}} I_i$ be a homogeneous ideal. For every $i, j \in \mathbb{N}$ one defines the ijth graded Betti number of I as

$$\beta_{ij}(I) = \dim_K \operatorname{Tor}_i^R(I,K)_j$$

and set

$$t_i(I) = \max\{j \mid \beta_{ij}(I) \neq 0\}$$

with $t_i(I) = -\infty$ if it happens that $\operatorname{Tor}_i^R(I,K) = 0$. The Castelnuovo-Mumford regularity $\operatorname{reg}(I)$ of I is defined as

$$\operatorname{reg}(I) = \sup\{t_i(I) - i \, : i \in \mathbb{N}\}.$$

By construction $t_0(I)$ is the largest degree of a minimal generator of I. The initial degree of I is the smallest degree of a minimal generator of I, i.e. it is the least index i such that $I_i \neq 0$. The ideal I has a linear resolution if its regularity is equal to its initial degree. In other words, I has a linear resolution if its minimal generators all have the same degree and the non-zero entries of the matrices of the minimal free resolution of I all have degree 1.

Bertram, Ein and Lazarsfeld proved in [BEL] that if $X \subset \mathbb{P}^n$ is a smooth complex variety of codimension s which is cut out scheme-theoretically by hypersurfaces of degree $d_1 \geq \cdots \geq d_m$, then $H^i(\mathbb{P}^n, \mathcal{I}_X^k(a)) = 0$ for $i \geq 1$ and $a \leq kd_1 + d_2 + \cdots + d_s - n$. Their result has initiated the study of the Castelnuovo-Mumford regularity of the powers of a homogeneous ideal. Note that the highest degree of a generator the k-th power I^k of I is bounded above by k times the highest degree of a generator of I, i.e. $t_0(I^k) \leq kt_0(I)$. One may wonder whether the same relation holds also for the Castelnuovo-Mumford regularity, that is, whether the inequality

$$\operatorname{reg}(I^k) \leq k \operatorname{reg}(I) \tag{1}$$

holds for every k. If $\dim R/I \leq 1$ then equation (1) holds, see e.g. [C, GGP] and [Ca, CH, Si, EHU] for further generalizations. But in general (1) does not hold. We will present later some examples of ideals with linear resolution whose square does not have a linear resolution. On the other hand, Cutkosky, Herzog and Trung [CHT], and Kodiyalam [K] proved (independently) that for every homogeneous ideal I one has

$$\operatorname{reg}(I^k) = a(I)k + b(I) \quad \text{ for all } k \geq c(I) \tag{2}$$

where $a(I), b(I)$ and $c(I)$ are integers depending on I. They also shown that $a(I)$ is bounded above by the largest degree of a generator of I. Bounds for $b(I)$ and $c(I)$ are given in [R] in terms of invariants related to the Rees algebra of I.

Remark/Definition 2.1 *Let I be an ideal generated in a single degree s. Then it is easy to see that $a(I) = s$. Hence it follows that $\operatorname{reg}(I^{k+1}) - \operatorname{reg}(I^k) = s$ for all $k \geq c(I)$. We say that the (function) regularity of the powers of I jumps at place k if $\operatorname{reg}(I^k) - \operatorname{reg}(I^{k-1}) > s$.*

One of the most powerful tools in proving that a (monomial) ideal has a linear resolution is the following notion:

DEFINITION 2.2 *An ideal I generated in a single degree is said to have linear quotients if there exists a system of minimal generators $f_1, \dots, f_s$ of I such that for every $k \leq s$ the colon ideal $(f_1, \dots, f_{k-1}) : f_k$ is generated by linear forms.*

One has (see [CH]):

LEMMA 2.3 *(a) If I has linear quotients then I has a linear resolution.*

(b) If I is a monomial ideal, then the property of having linear quotients with respect to its monomial generators is independent of the characteristic of the base filed.

We present now some (known and some new) examples of ideals with a linear resolution such that the square does have non-linear syzygies.

The first example of such an ideal was discovered by Terai. It is an ideal well-known for having another pathology: it is a square-free monomial ideal whose Betti numbers, regularity and projective dimension depend on the characteristic of the base field.

EXAMPLE 2.4 *Consider the ideal*

$$I = (abc, abd, ace, adf, aef, bcf, bde, bef, cde, cdf)$$

of $K[a, b, c, d, e, f]$. In characteristic 0 one has $\operatorname{reg}(I) = 3$ and $\operatorname{reg}(I^2) = 7$. The only non-linear syzygy for I^2 comes at the very end of the resolution. In characteristic 2 the ideal I does not have a linear resolution.

The second example is taken from [CH]. It is monomial and characteristic free. The ideal is defined by 5 monomials. To the best of our knowledge, no ideal is known with the studied pathology and with less than 5 generators.

EXAMPLE 2.5 *Consider the ideal*

$$I = (a^2b, a^2c, ac^2, bc^2, acd)$$

of $K[a, b, c, d]$. It is easy to check that I has linear quotients (with respect to the monomial generators in the given order). It follows that I has a linear resolution independently of $\operatorname{char} K$. Furthermore I^2 has a quadratic first-syzygy in characteristic 0. But the first syzygies of a monomial ideals are independent of $\operatorname{char} K$. So we may conclude that $\operatorname{reg}(I) = 3$ and $\operatorname{reg}(I^2) > 6$ for every base field K.

The third example is due to Sturmfels [St]. It is monomial, square-free and characteristic free. It is defined by 8 square-free monomials and, according to Sturmfels [St], there are no such examples with less than 8 generators.

EXAMPLE 2.6 *Consider the ideal*

$$I = (def, cef, cdf, cde, bef, bcd, acf, ade)$$

of $K[a, b, c, d, e, f]$. *One checks that* I *has linear quotients (with respect to the monomial generators in the given order) and so it has a linear resolution independently of* $\operatorname{char} K$. *Furthermore* I^2 *has a quadratic first-syzygy. One concludes that* $\operatorname{reg}(I) = 3$ *and* $\operatorname{reg}(I^2) > 6$ *for every base field* K.

So far all the examples were monomial ideals generated in degree 3. Can we find examples generated in degree 2? In view of the main result of [HHZ], we have to allow also non-monomial generators. One binomial generator is enough:

EXAMPLE 2.7 *Consider the ideal*

$$I = (a^2, ab, ac, ad, b^2, ae + bd, d^2)$$

of $K[a, b, c, d, e]$. *One checks that* $\operatorname{reg}(I) = 2$ *and* $\operatorname{reg}(I^2) = 5$ *in characteristic* 0. *The ideal has linear quotients with respect to the generators in the given order. We have not checked whether the example is characteristic free.*

One may wonder whether there exists a prime ideal with this behavior. Surprisingly, one can find such an example already among the most beautiful and studied prime ideals, the generic determinantal ideals.

EXAMPLE 2.8 *Let* I *be the ideal of* $K[x_{ij} : 1 \leq i \leq j \leq 4]$ *generated by the* 3*-minors of the generic symmetric matrix* (x_{ij}). *It is well-known that* I *is a prime ideal defining a Cohen-Macaulay ring and that* I *has a linear resolution. One checks that* I^2 *does not have a linear resolution in characteristic* 0.

REMARK 2.9 Denote by I the ideal of 2.8 and by J the ideal of 2.4. It is interesting to note that the graded Betti numbers of I and J as well as those of I^2 and J^2 coincide. Is this just an accident? There might be some hidden relationship between the two ideals. In order to be able to reproduce the pathology it would be important to understand whether the non-linear syzygies of I^2 and J^2 have a common "explanation". We can ask whether J is an initial ideal or a specialization (or an initial ideal of a specialization) of the ideal I. Concretely, we may ask whether J can be represented as the (initial) ideal of (the ideal of) 3-minors of a 4×4 symmetric matrix of linear forms in 6 variables. We have not been able to answer this question. Note that the most natural way of filling a 4×4 symmetric matrix with 6 variables would be to put 0's on the main diagonal and to fill the remaining positions with the 6 variables. Taking 3-minors one gets an ideal, say J_1, which shares many invariants with J. For instance, we have checked that J_1 and J as well as their squares have the same graded Betti numbers (respectively of course). The ideals J and J_1 are both reduced but J has 10 components of degree 1 while J_1 has 4 components of degree 1 and 3 of degree 2. We have also checked that, in the given coordinates, J cannot be an initial ideal of J_1.

Finally, the last example shows that even among the Cohen-Macaulay ideals with minimal multiplicity one can find ideals whose square have a non-linear resolution.

EXAMPLE 2.10 *Consider the ideal*

$$I = (x^2, xy_1, xy_2, xy_3, y_1^2, y_2^2, y_3^2, y_1y_2 - xz_{12}, y_1y_3 - xz_{13}, y_2y_3 - xz_{23})$$

of $R = K[x, y_1, y_2, y_3, z_{12}, z_{13}, z_{23}]$. *Then* R/I *is Cohen-Macaulay with minimal multiplicity (and hence* I *has a* 2*-linear resolution) but* I^2 *does not have a linear resolution in characterisitc* 0*. Indeed,* I^2 *has a non-linear second syzygy.*

More generally one can consider variables, $x, y_1, \dots, y_n$ and z_{ij} with $1 \leq i < j \leq n$. Then the ideal I_n generated by

$$x^2, xy_1, \dots, xy_n, y_1^2, \dots, y_n^2$$

and

$$y_iy_j - xz_{ij} \text{ with } 1 \leq i < j \leq n$$

defines a Cohen-Macaualy ring with minimal multiplicity. For $n = 3$ one gets the ideal of 2.10. For $n > 3$ there is some computational evidence that I_n^2 has a non-linear first syzygy. It seems to be difficult to compute or bound the invariants $b(I_n)$ and $c(I_n)$ for this family.

3 Regularity Jumps

The goal of this section is to show that the regularity of the powers of an ideal can jump for the first time at any place. This happens already in 4 variables. Let R be the polynomial ring $K[x_1, x_2, z_1, z_2]$ and for $d > 1$ define the ideal

$$J = (x_1z_1^d, x_1z_2^d, x_2z_1^{d-1}z_2) + (z_1, z_2)^{d+1}$$

We will prove the following:

Theorem 3.1 *The ideal J^k has linear quotients for all $k < d$ and J^d has a first-syzygy of degree d. In particular* $\operatorname{reg}(J^k) = k(d+1)$ *for all $k < d$ and* $\operatorname{reg}(J^d) \geq d(d+1) + d - 1$ *and this holds independently of K.*

We may deduce:

Corollary 3.2 *The invariant $c(I)$ cannot be bounded above in terms of the number of variables only, not even for monomial ideals.*

In order to prove that J^k has linear quotients we need the following technical construction.

Given three sets of variables $x = x_1, \ldots, x_m$, $z = z_1, \ldots, z_n$ and $t = t_1, \ldots, t_k$ we consider monomials $m_1, \ldots, m_k$ of degree d is the variables z. Let ϕ be a map

$$\phi : \{t_1, \ldots, t_k\} \to \{x_1, \ldots, x_m\}.$$

Extend its action to arbitrary monomials by setting

$$\phi(\prod t_i^{a_i}) = \prod \phi(t_i)^{a_i}.$$

Set $W = (m_1, \ldots, m_k)$. Consider the bigraded presentation

$$\Phi : K[z_1, \ldots, z_n, t_1, \ldots, t_k] \to R(W) = K[z_1, \ldots, z_n, m_1 s, \ldots, m_k s]$$

of the Rees algebra of the ideal W obtained by setting $\Phi(z_i) = z_i$ and $\Phi(t_i) = m_i s$ (s a new variable) and giving degree $(1, 0)$ to the z's and degree $(0, 1)$ to the t's. We set

$$H = \operatorname{Ker} \Phi.$$

and note that H is a binomial ideal.

Definition 3.3 *We say that $m_1 \ldots, m_k$ are pseudo-linear of order p with respect to ϕ if for every $1 \leq b \leq a \leq p$ and for every binomial $MA - NB$ in H with M, N monomials in the z of degree $(a-b)(d+1)$ and A, B monomials in the t of degree b such that $\phi(A) > \phi(B)$ in the lex-order there exists an element of the form $M_1 t_i - N_1 t_j$ in H where M_1, N_1 are monomials in the z such that the following conditions are satisfied:*

(1) $N_1 | N$,

(2) $t_i | A$, $t_j | B$,

(3) $\phi(t_i) | \phi(A) / \operatorname{GCD}(\phi(A), \phi(B))$,

(4) $\phi(t_i) > \phi(t_j)$ *in the lex-order.*

The important consequence is the following:

Lemma 3.4 *Assume that* $m_1 \ldots, m_k$ *are monomials of degree* d *in the* z *which are pseudo-linear of order* p *with respect to* ϕ. *Set*

$$J = (z_1, \ldots, z_n)^{d+1} + (\phi(t_1)m_1, \phi(t_2)m_2, \ldots, \phi(t_k)m_k).$$

Then J^a *has linear quotients for all* $a = 1, \ldots, p$.

Proof: Set $Z = (z_1, \ldots, z_n)^{d+1}$ and $I = (\phi(t_1)m_1, \phi(t_2)m_2, \ldots, \phi(t_k)m_k)$. Take a with $1 \leq a \leq p$ and order the generators of J^a according to the following decomposition: $J^a = Z^a + Z^{a-1}I + \cdots + Z^b I^{a-b} + \cdots + I^a$. In the block Z^a we order the generators so that they have linear quotients; this is easy since Z^a is just a power of the $(z_1, \ldots, z_n)$. In the the block $Z^b I^{a-b}$ with $b < a$ we order the generators extending (in anyway) the lex-order in the x. We claim that, with this order, the ideal J^a has linear quotients. Let us check this. As long as we deal with elements of the block Z^a there is nothing to check. So let us take some monomial, say u from the block $Z^b I^{a-b}$ with $b < a$ and denote by V the ideal of generated by the monomials which are earlier in the list. We have to show that the colon ideal $V : (u)$ is generated by variables. Note that $V : (u)$ contains $(z_1, \ldots, z_n)$ since $(z_1, \ldots, z_n)u \subset Z^{b+1} I^{a-b-1} \subset V$. Let v be a generator of V. If v comes from a block $Z^c I^{a-c}$ with $c > b$ then we are done since $(v) : (u)$ is contained in $(z_1, \ldots, z_n)$ by degree reason. So we can assume that also v comes from the block $Z^b I^{a-b}$. Again, if the generator $v/\operatorname{GCD}(v, u)$ of $(v) : (u)$ involves the variables z we are done. So we are left with the case in which $v/\operatorname{GCD}(v, u)$ does not involves the variables z. It is now the time to use the assumption that the m_i's are pseudo-linear. Say $u = N m_{s_1}\phi(t_{s_1}) \cdots m_{s_a}\phi(t_{s_a})$ and $v = M m_{r_1}\phi(t_{r_1}) \cdots m_{r_a}\phi(t_{r_a})$ with M, N monomials of degree $(b-a)(d+1)$ in the z. Set $A = t_{r_1} \cdots t_{r_a}$ and $B = t_{s_1} \cdots t_{s_a}$. Since v is earlier than u in the generators of J^a we have $\phi(A) > \phi(B)$ in the lex-order. Note also that $(v) : (u)$ is generated by $\phi(A)/\operatorname{GCD}(\phi(A), \phi(B))$. Now the fact that $v/\operatorname{GCD}(v, u)$ does not involves the variables z is equivalent to say that $MA - NB$ belongs to H. By assumption there exists $L = M_1 t_i - N_1 t_j$ in H such that the conditions (1)–(4) of Definition 3.3 hold. Multiplying L with $(N/N_1)(B/t_j)$, we have that $M_1(N/N_1)t_i(B/t_j) - NB$ is in H and by construction

$$v_1 = M_1(N/N_1)m_i\phi(t_i)m_{s_1}\phi(t_{s_1}) \cdots m_{s_a}\phi(t_{s_a})/m_j\phi(t_j)$$

is a monomial of the block $Z^b I^{a-b}$ which is in V by construction and such that $(v) : (u) \subseteq (v_1) : (u) = (\phi(t_i))$. This concludes the proof. □

Now we can prove:

Lemma 3.5 *For every integer* $d > 1$ *the monomials*

$$m_1 = z_1^d, \quad m_2 = z_2^d, \quad m_3 = z_1^{d-1} z_2$$

are pseudo-linear of order $(d-1)$ *with respect to the map*

$$\phi : \{t_1, t_2, t_3\} \to \{x_1, x_2\}$$

defined by $\phi(t_1) = x_1$, $\phi(t_2) = x_1$, $\phi(t_3) = x_2$.

Proof: It is easy to see that the defining ideal H of the Rees algebra of $W = (m_1, m_2, m_3)$ contains

$$(3)\ z_2t_1 - z_1t_3 \qquad (4)\ z_1^{d-1}t_2 - z_2^{d-1}t_3 \qquad (5)\ t_1^{d-1}t_2 - t_3^d$$

and the algebra $K[m_1, m_2, m_3]$ is defined by the equation (5).

NOTE: In the printed version of the paper I claimed that the polynomials (3),(4), (5) generate H. That is not true. However the only properties that are used in the following are those stated above.

Let $1 \leq b \leq a \leq d-1$ and $F = MA - NB$ a binomial of bidegree $((a-b)(d+1), b)$ in H such that $\phi(A) > \phi(B)$ in the lex-order. Denote by $v = (v_1, v_2), u = (u_1, u_2)$ the exponents of M and N and by $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$ the exponents of A and B. We collect all the relations that hold by assumption:

$$\begin{array}{ll} (i) & 1 \leq b \leq a \leq d-1 \\ (ii) & \alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3 = b \\ (iii) & v_1 + v_2 = u_1 + u_2 = (a-b)(d+1) \\ (iv) & v_1 + d\alpha_1 + (d-1)\alpha_3 = u_1 + d\beta_1 + (d-1)\beta_3 \\ (v) & v_2 + d\alpha_2 + \alpha_3 = u_2 + d\beta_2 + \beta_3 \\ (vi) & \alpha_1 + \alpha_2 > \beta_1 + \beta_2 \end{array}$$

Note that (vi) holds since $\phi(A) > \phi(B)$ in the lex-order. If

$$\alpha_1 > 0 \text{ and } \beta_3 > 0 \text{ and } u_1 > 0 \tag{6}$$

then equation (3) does the job. If instead

$$\alpha_2 > 0 \text{ and } \beta_3 > 0 \text{ and } u_2 \geq d-1 \tag{7}$$

then equation (4) does the job. So it is enough to show that either (6) or (7) hold. By contradiction, assume that both (6) and (7) do not hold. Note that (ii) and (vi) imply that $\beta_3 > 0$. Also note that if $a = b$ then the equation F has bidegree $(0, a)$ and hence must be divisible by (3) which is impossible since $a < d$. So we may assume that $b < a$. But then $u_1 + u_2 = (a-b)(d+1) \geq d+1$ and hence either $u_1 > 0$ or $u_2 \geq d-1$. Summing up, if both (6) and (7) do not hold and taking into consideration that $\beta_3 > 0$, that either $u_1 > 0$ or $u_2 \geq d-1$ and that $\alpha_1 + \alpha_2 > 0$, then one of the following conditions hold:

(8) $\alpha_1 = 0$ and $u_2 < d-1$
(9) $\alpha_2 = 0$ and $u_1 = 0$

If (8) holds then $\alpha_2 > \beta_1 + \beta_2$. Using (v) we may write

$$u_2 = v_2 + d(\alpha_2 - \beta_1 - \beta_2) + d\beta_1 + \alpha_3 - \beta_3. \tag{10}$$

If $\beta_1 > 0$ we conclude that $u_2 \geq d + d - \beta_3$ and hence $u_2 \geq d+1$ since $\beta_3 \leq b \leq d-1$, and this is a contradiction.

If instead $\beta_1 = 0$ then $\alpha_2 + \alpha_3 = \beta_2 + \beta_3$ and hence (10) yields $u_2 = (d-1)(\alpha_2 - \beta_2) + v_2 \geq (d-1)$, a contradiction.

If (9) holds then $\alpha_1 > \beta_1 + \beta_2$. By (iv) we have

$$v_1 + d(\alpha_1 - \beta_1) + (d-1)(\alpha_3 - \beta_3) = 0$$

Hence

$$v_1 + d(\alpha_1 - \beta_1 - \beta_2) + (d-1)(\alpha_3 - \beta_3) + d\beta_2 = 0$$

But $\alpha_3 - \beta_3 = -\alpha_1 + \beta_1 + \beta_2$, so that

$$v_1 + (\alpha_1 - \beta_1 - \beta_2) + d\beta_2 = 0$$

which is impossible since $\alpha_1 - \beta_1 - \beta_2 > 0$ by assumption. □

Now we are ready to complete the proof of the theorem.

Proof: [of Theorem 3.1]: Combining 3.4 and 3.5 we have that J^k has linear quotients, and hence a linear resolution, for all $k < d$. It remains to show that J^d has a first-syzygy of degree d. Denote by V the ideal generated by all the monomial generators of J^d but $u = (z_1^{d-1} z_2 x_2)^d$. We claim that x_1^d is a minimal generator of $V : u$ and this is clearly enough to conclude that J^d has a first-syzygy of degree d. First note that $x_1^d u = x_2^d (z_1^d x_1)^{d-1} (z_2^d x_1) \in V$, hence $x_1^d \in V : u$. Suppose, by contradiction, x_1^d is not a minimal generator of $V : u$. Then there exists an integer $s < d$ such that $x_1^s u \in V$. In other words we may write $x_1^s u$ as the product of d generators of J, say $f_1, \ldots, f_d$, not all equal to $z_1^{d-1} z_2 x_2$, times a monomial m of degree s. Since the total degree in the x in $x_1^s u$ is $s + d$ at each generator of J has degree at most 1 in the x, it follows that the f_i are all of the of type $z_1^d x_1, z_2^d x_2, z_1^{d-1} z_2 x_2$ and m involves only x. Since x_2 has degree d in u, $s < d$ and $z_1^{d-1} z_2 x_2$ is the only generator of J containing x_2 it follows that at least one of the f_i is equal to $z_1^{d-1} z_2 x_2$. Getting rid of those common factors we obtain a relation of type $x_1^s (z_1^{d-1} z_2 x_2)^r = m (z_1^d x_1)^{r_1} (z_2^d x_1)^{r_2}$ with $r = r_1 + r_2 < d$. In the z-variables it gives $(z_1^{d-1} z_2)^r = (z_1^d)^{r_1} (z_2^d)^{r_2}$ with $r = r_1 + r_2 < d$ which is clearly impossible. □

What is the regularity of J^k for $k \geq d$? There is some computational evidence that the first guess, i.e. $\operatorname{reg}(J^k) = k(d+1)+d-1$ for $k \geq d$, might be correct.

The ideas and the strategy of the previous construction can be used, in principle, to create other kinds of "bad" behaviors. We give some hints and examples but no detailed proofs.

HINT 3.6 Given $d > 1$ consider the ideal

$$H = (x_1z_1^d, x_1z_2^d, x_2z_1^{d-1}z_2) + z_1z_2(z_1, z_2)^{d-1}$$

We conjecture that $\operatorname{reg}(H^k) = k(d+1)$ for all $k < d$ and $\operatorname{reg}(H^d) \geq d(d+1)+d-1$. Note that H has two generators less than the ideal J of 3.1. In the case $d = 2$, H is exactly the ideal of 2.5.

One can ask whether there are radical ideals with a behavior as the ideal in 3.1. One would need a square-free version of the construction above. This suggests the following:

HINT 3.7 For every d consider variables $z_1, z_2, \ldots, z_{2d}$ and x_1, x_2 and the ideal

$$J = \left(\begin{array}{c} x_1z_1z_2, x_1z_3z_4, \ldots, x_1z_{2d-1}z_{2d}, \\ x_2z_2z_3, x_2z_4z_5, \ldots, x_2z_{2d}z_1 \end{array} \right) + \operatorname{Sq}^3(z)$$

where $\operatorname{Sq}^3(z)$ denote the square-free cube of $(z_1, \ldots, z_{2d})$, i.e. the ideal generated by the square-free monomials of degree 3 in the z's. We conjecture that $\operatorname{reg}(J^k) = 3k$ for $k < d$ and $\operatorname{reg}(J^d) > 3d$. Note that for $d = 2$ one obtains Sturmfels' Example 2.6.

We do not know how to construct prime ideals with a behavior as the ideal in 3.1. If one wants two (or more) jumps one can try with:

HINT 3.8 Let $1 < a < b$ be integers. Define the ideal

$$I = (y_2z_1^b, y_2z_2^b, xz_1^{b-1}z_2) + z_1^{b-a}(y_1z_1^a, y_1z_2^a, xz_1^{a-1}z_2) + z_1z_2(z_1, z_2)^{b-1}$$

of the polynomial ring $K[x, y_1, y_2, z_1, z_2]$. We expect that $\operatorname{reg}(I) = b+1$ and $\operatorname{reg}(I^k) - \operatorname{reg}(I^{k-1}) > (b+1)$ if $k = a$ or $k = b$.

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