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Divisor class group and canonical class of determinantal rings defined by ideals of minors of a symmetric matrix

By

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In this paper we study the rings defined by ideals of minors, not necessarily of fixed size, of a symmetric matrix of indeterminates. We prove that they are normal, compute the divisor class group and the canonical class. We determine the Gorenstein rings among the rings under consideration.

Introduction. Let K be a field and let $X = (X_{ij})$ be an $n \times n$ symmetric matrix of indeterminates over K. Denote by $K[X]$ the polynomial ring $K[X_{ij}: 1 \leqq i \leqq j \leqq n]$.

Let $\alpha = \{\alpha_1, \ldots, \alpha_t\}$ be a non-empty subset of $\{1, \ldots, n\}$, with $\alpha_i < \alpha_{i+1}$. We define $I_\alpha(X)$ to be the ideal generated by all the i-minors of the first $\alpha_i - 1$ rows of X, for $i = 1, \ldots, t$, and by all the $(t+1)$-minors of X.

Denote by $R_\alpha(X)$ the ring $K[X]/I_\alpha(X)$. In particular, if $\alpha = \{1, \ldots, t-1\}$, then $I_\alpha(X)$ is the ideal generated by all the t-minors of X; in this case we denote by $R_t(X)$ the quotient ring.

This class of ideals and rings was investigated by Kutz in his paper [12]. In Kutz's notation the ideal $I_{H,0}$, $H = (s_0, \ldots, s_m)$, defined in [12, 1], corresponds to the ideal $I_\alpha(X)$ with $\alpha = \{1, s_1 + 1, \ldots, s_{m-1} + 1\}$. Kutz showed that $R_\alpha(X)$ is a Cohen-Macaulay domain of dimension $(n+1)t - \sum_{i=1}^{t} \alpha_i$.

By results of Goto [10], [11], the ring $R_t(X)$ is known to be normal with divisor class group $\mathbb{Z}_2$, and is Gorenstein if and only if $n \equiv t \bmod (2)$.

The aim of this paper is to prove that the ring $R_\alpha(X)$ is a normal domain, to compute its divisor class group and its canonical class, and to decide whether it is Gorenstein or not.

Our approach is based on the knowlewdge of the combinatorial structure of $R_\alpha(X)$, [7] and [6], and on the methods developed by Bruns and Vetter to study generic determinantal rings, [1], [2] and [4].

1. Normality and divisor class group. Let us recall the combinatorial structure of $R_\alpha(X)$. Let H be the set of the non-empty subsets of $\{1, \ldots, n\}$. Given $a \in H$, we will always write its elements

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in ascending order $1 \leq a_1 < \ldots < a_t \leqq n$. On H we define the following partial order:

$$a = \{a_1, \ldots, a_t\} \leqq b = \{b_1, \ldots, b_r\} \Leftrightarrow r \leqq t \quad \text{and} \quad \alpha_i \leqq b_i \quad \text{for } i = 1, \ldots, r.$$

As usual, we denote by $[a_1, \ldots, a_t \mid b_1, \ldots, b_t]$ the minor $\det(X_{a_i b_j})$ of X. A minor $[a_1, \ldots, a_t \mid b_1, \ldots, b_t]$ of X is called a *doset minor* if $\{a_1, \ldots, a_t\} \leqq \{b_1, \ldots, b_t\}$ in H. Let us denote by D the set of all the doset minors. Let $a_1 = \{a_{11}, \ldots, a_{1t_1}\}$, $b_1 = \{b_{11}, \ldots, b_{1t_1}\}, \ldots, a_s = \{a_{s1}, \ldots, a_{st_s}\}$, $b_s = \{b_{s1}, \ldots, b_{st_s}\}$ be elements of H and suppose that $[a_1 \mid b_1], \ldots, [a_s \mid b_s] \in D$. We say that the product $[a_1 \mid b_1] \cdots [a_s \mid b_s]$ is a *standard monomial* if $b_i \leqq a_{i+1}$ in H for all $i = 1, \ldots, s-1$. Given a standard monomial $M = [a_1 \mid b_1] \cdots [a_s \mid b_s]$, $\min(M)$ denotes the "minimum" $[a_1 \mid b_1]$ of its factors. The combinatorial structure of $K[X]$ with respect to products of minors is clarified by the following results due to De Concini and Procesi, [6, pp. 82], [7, 5.1] and [7, 5.2].

Theorem 1.1. (a) *The standard monomials form a K-basis for* $K[X]$.

(b) *Let* $M_1, \ldots, M_s \in D$, $M_i = [a_{i1}, \ldots, a_{ir_i} \mid b_{i1}, \ldots, b_{ir_i}]$. *Let* $N = [c_{11}, \ldots, c_{1s_1} \mid d_{11}, \ldots, d_{1s_1}] \cdots [c_{r1}, \ldots, c_{rs_r} \mid d_{r1}, \ldots, d_{rs_r}]$ *be one of the standard monomials which appear in the (unique) representation of the product* $M_1 \cdots M_s$ *as linear combination of standard monomials. Set* $c_i = \{c_{i1}, \ldots, c_{is_i}\}$, $d_i = \{d_{i1}, \ldots, d_{is_i}\}$, $a_i = \{a_{i1}, \ldots, a_{ir_i}\}$ *and* $b_i = \{b_{i1}, \ldots, b_{ir_i}\}$. *Then, in lexicographic order of the sequences of elements of H, the sequence* $c_1, d_1, \ldots, c_r, d_r$ *is less than or equal to every sequence which is obtained permuting the elements* $a_1, b_1, \ldots, a_s, b_s$.

(c) *Every t-minor* $M = [a_1, \ldots, a_t \mid b_1, \ldots, b_t]$ *is a linear combination of doset t-minors. Moreover if* $[c_1, \ldots, c_t \mid d_1, \ldots, d_t]$ *is a doset t-minor which appears in the representation of M, then* $c_i \leqq a_i$ *for all* $i = 1, \ldots, t$.

Given $\alpha = \{\alpha_1, \ldots, \alpha_t\} \in H$, we define $I_\alpha(X)$ to be the ideal generated by all the i-minors of the first $\alpha_i - 1$ rows of X, for $i = 1, \ldots, t$, and by all the $(t+1)$-minors of X. Denote by $R_\alpha(X)$ the ring $K[X]/I_\alpha(X)$.

From 1.1(c), one deduces that $I_\alpha(X)$ is generated by the set $\Omega_\alpha = \{[a \mid b] \in D : a \ngeq \alpha)$. Using the straightening laws one shows immediately:

Corollary 1.2. (a) *The set of all the standard monomials M such that* $\min(M) \in \Omega_\alpha$ *forms a K-basis of* $I_\alpha(X)$.

(b) *Set* $D_\alpha = \{[a \mid b] \in D : a \geqq \alpha\}$. *Then set* S_α *of (the residue classes of) all the standard monomials M such that* $\min(M) \in D_\alpha$ *forms a K-basis of the ring* $R_\alpha(X)$.

Let us denote by f the residue class in $R_\alpha(X)$ of the minor $[\alpha \mid \alpha]$. If $M \in S_\alpha$, then $[\alpha \mid \alpha] M \in S_\alpha$. Therefore f is a non zero divisor of $R_\alpha(X)$. We want to describe the localization of $R_\alpha(X)$ with respect to f. Let Ψ be the set of the residue classes in $R_\alpha(X)$ of the elements of the set $\{[\alpha_i \mid \alpha_j] : 1 \leqq i \leqq j \leqq t\} \cup \{[\alpha \mid \beta] \in D_\alpha : \beta$ differs from α in exactly one index$\}$. Denote by $K[\Psi]$ the K-subalgebra of $R_\alpha(X)$ generated by the elements in the set Ψ. In order to prove the following lemma one has just to imitate the argument of [4, 6.4].

Lemma 1.3. *The set* Ψ *is algebraically independent over K and* $R_\alpha(X)[f^{-1}] = K[\Psi][f^{-1}]$.

Now we describe the minimal prime ideals of f. Let J be the set of the minimal elements of the set $\{\gamma \in H : \gamma > \alpha\}$, that is, J is the set of the upper neighbours of α in H. For systematic reason it is convenient to set $J = \{\emptyset\}$ if $\alpha = \{n\}$, and $I_\emptyset(X) = (X_{ij}, 1 \leqq i \leqq j \leqq n)$. Given $\beta \in J$, one has $I_\beta(X) \supset I_\alpha(X)$. Let us denote by $I_\beta(x)$ the prime ideal $I_\beta(X)/I_\alpha(X)$ of $R_\alpha(X)$. Then:

Lemma 1.4. *The ideals* $I_\beta(x)$, $\beta \in J$, *are the minimal prime ideals of* (f).

Proof. First we show that $\bigcap_{\beta \in J} I_\beta(X) = ([\alpha \mid \delta] \in D_\alpha) + I_\alpha(X)$. The inclusion $\supset$ is clear. Let $F \in \bigcap_{\beta \in J} I_\beta(X)$, we have to show that $F \in ([\alpha \mid \delta] \in D_\alpha) + I_\alpha(X)$. We may assume that F is a standard monomial, and let $\min(F) = [\gamma \mid \delta]$. Since $\gamma \ngeq \beta$ for all $\beta \in J$, we deduce that $\gamma = \alpha$ or $\gamma \ngeq \alpha$, therefore $F \in ([\alpha \mid \delta] \in D_\alpha) + I_\alpha(X)$.

Note that if $\beta, \beta_1 \in J, \beta \neq \beta_1$, then $[\beta \mid \beta] \in I_{\beta_1}(x) \backslash I_\beta(x)$. Thus the ideals $I_\beta(x), \beta \in J$, are distinct. Applying the straightening relations 1.1(b), one has $([\alpha \mid \gamma] \in D_\alpha)^2 \subset I_\alpha(X) + (f)$, and the desired conclusion follows. □

We want to describe more precisely the set J. We decompose α into a sequence $B_1, \ldots, B_r$, where $B_i = \{\alpha_{k_{i-1}+1}, \ldots, \alpha_{k_i}\}$ is a block of consecutive integers, ($k_0 = 0$, $k_r = t$). Let us denote by $\chi_1, \ldots, \chi_r$ the "gaps sequence" associated with α, that is, for $i = 1, \ldots, r-1$, $\chi_i = \{\alpha_{k_i} + 1, \ldots, \alpha_{k_i+1} - 1\}$ and $\chi_r = \{\alpha_t + 1, \ldots, n\}$, $\chi_r = \emptyset$ if $\alpha_t = n$. For every $i = 1, \ldots, r$, we get β_i an upper neighbour of α in the following way. If $i < r$, or $i = r$ and $\alpha_t < n$, then β_i is obtained from α by replacing α_{k_i} with $\alpha_{k_i} + 1$. If $\alpha_t = n$, then β_r is obtained from α by deleting α_t. It is not difficult to show that $J = \{\beta_1, \ldots, \beta_r\}$.

In order to carry further our investigation we adapt the usual inversion trick [4, 2.4] to our situation. Let $\bar{X}$ be a symmetric matrix of indeterminates of size $(n-1) \times (n-1)$. Consider the isomorphism of K-algebras

$$\psi : K[X][X_{11}^{-1}] \to K[\bar{X}][X_{11}, \ldots, X_{1n}][X_{11}^{-1}]$$

defined by the assignment $\psi(X_{1j}) = X_{1j}$ for all $j = 1, \ldots, n$, and $\psi(X_{ij}) = \bar{X}_{i-1j-1} + X_{1i} X_{1j} X_{11}^{-1}$ for all $1 < i \leqq j \leqq n$.

Suppose $\alpha_1 = 1$. Then $X_{11} \notin I_\alpha(X)$. As in the generic case one shows that $\psi(I_\alpha(X))$ is the extension of the ideal $I_{\bar{\alpha}}(\bar{X})$ to the ring $K[\bar{X}][X_{11}, \ldots, X_{1n}][X_{11}^{-1}]$, where $\bar{\alpha} = \{\alpha_2 - 1, \ldots, \alpha_t - 1\}$. Thus we get an isomorphism

$$\bar{\psi} : R_\alpha(X)[x_{11}^{-1}] \to R_{\bar{\alpha}}(\bar{X})[X_{11}, \ldots, X_{1n}][X_{11}^{-1}] = S \tag{1}$$

where x_{11} denotes the residue class of X_{11} in $R_\alpha(X)$. Let us denote by $\bar{f}$ the residue class in $R_{\bar{\alpha}}(\bar{X})$ of the minor $[\bar{\alpha} \mid \bar{\alpha}]$, by $\bar{\beta}_i$ the upper neighbours of $\bar{\alpha}$, and by $I_{\bar{\beta}_i}(\bar{x})$ the corresponding minimal prime ideals of $\bar{f}$. One has $\bar{\psi}(f) = X_{11} \bar{f}$. Furthermore,

$$\bar{\psi}(I_{\beta_i}(x)) = \begin{cases} I_{\bar{\beta}_i}(\bar{x}) S & \text{if } \alpha_2 = 2 \text{ and for all } i = 1, \ldots, r. \\ I_{\bar{\beta}_{i-1}}(\bar{x}) S & \text{if } \alpha_2 > 2 \text{ and for all } i = 2, \ldots, r. \\ S & \text{if } \alpha_2 > 2 \text{ and } i = 1. \end{cases} \tag{2}$$

Lemma 1.5. *Set $R = R_\alpha(X)$, and $P_i = I_{\beta_i}(x)$. Then:*
(a) *For all $i = 1, \ldots, r$, R_{P_i} is a regular ring. Let v_i denote the valuation one the* DVR R_{P_i}.
(b) *If $i < r$, or if $i = r$ and $\alpha_t < n$, then $v_i(f) = 2$. If $\alpha_t = n$, then $v_r(f) = 1$.*

Proof. By induction on n. In the case $n = 2$ the assertions are trivial. Now suppose $n > 2$. If $\alpha_1 > 1$, then $R_\alpha(X) \cong R_\gamma(Y)$, where Y is an $(n + 1 - \alpha_1) \times (n + 1 - \alpha_1)$ symmetric matrix of indeterminates and $\gamma = \{1, \alpha_2 + 1 - \alpha_1, \ldots, \alpha_t + 1 - \alpha_1\}$. By induction, we may assume $\alpha_1 = 1$. Suppose $\alpha_2 = 2$, then R_{P_i} is a localization of $R[x_{11}^{-1}]$ for all $i = 1, \ldots, r$. Denote $\bar{R}$ the ring $R_{\bar{\alpha}}(\bar{X})$, and by $\bar{P}_i$ the ideal $I_{\bar{\beta}_i}(\bar{x})$. By (1) and (2), we get that

R_{P_i} is isomorphic to a localization of $\bar{R}_{\bar{P}_i}[X_{11}, \ldots, X_{1n}][X_{11}^{-1}]$. The last is regular by induction, hence R_{P_i} is regular. Since X_{11} is a unit, we have $v_i(f) = \bar{v}_i(\bar{f})$, where $\bar{v}_i$ denotes the valuation on $\bar{R}_{\bar{P}_i}$. Again by induction $\bar{v}_r(\bar{f}) = 1$ if $\alpha_t - 1 = n - 1$, and $\bar{v}_i(\bar{f}) = 2$ in the other cases.

Suppose $\alpha_2 > 2$. As above, for all $i > 1$, by (1) and (2), and by induction, we get the desired results. It remains the case $i = 1$ and $\alpha_2 > 2$. By definition $P_1 = (x_{11}, \ldots, x_{1n})$. The 2-minors of the first two row of X vanish in R. Hence $x_{1i} = x_{12} x_{i2} x_{22}^{-1}$ in R_{P_1}, and $P_1 R_{P_1} = (x_{12}) R_{P_1}$. Therefore R_{P_1} is regular. In R_{P_1}, we have:

$$f = \begin{vmatrix} x_{11} & \cdots & x_{1\alpha_t} \\ \cdots & \cdots & \cdots \\ x_{1\alpha_t} & \cdots & x_{\alpha_t\alpha_t} \end{vmatrix} = \begin{vmatrix} x_{22}^{-1} x_{12}^2 & \cdots & x_{22}^{-1} x_{12} x_{2\alpha_t} \\ \cdots & \cdots & \cdots \\ x_{22}^{-1} x_{12} x_{2\alpha_t} & \cdots & x_{\alpha_t\alpha_t} \end{vmatrix} = [\beta_1 \mid \beta_1] x_{22}^{-2} x_{12}^2 .$$

But $[\beta_1 \mid \beta_1] x_{22}^{-2}$ is a unit in R_{P_1}, and we are done. □

We are ready to prove that $R_\alpha(X)$ is normal and to compute its divisor class group. For the theory of the divisor class group we refer the reader to [8]. Let us denote by $\mathrm{Cl}(A)$ the divisor class group of a normal domain A, and by $\mathrm{cl}(I)$ the class in $\mathrm{Cl}(A)$ of a divisorial ideal I of A.

Theorem 1.6. *The ring $R_\alpha(X)$ is a normal domain, and its divisor class group is generated by $\mathrm{cl}(I_{\beta_1}(x)), \ldots, \mathrm{cl}(I_{\beta_r}(x))$. Furthermore, the only relation between the given generators is $\sum_{i=1}^{r} v_i(f)\, \mathrm{cl}(I_{\beta_i}(x)) = 0$, and we have $\mathrm{Cl}(R_\alpha(X)) = \mathbb{Z}^{r-1} \oplus \mathbb{Z}_2$ if $\alpha_t < n$, and $\mathrm{Cl}(R_\alpha(X)) = \mathbb{Z}^{r-1}$ if $\alpha_t = n$.*

Proof. By 1.3, the localization of $R_\alpha(X)$ to a prime which does not contain f is a localization of a polynomial ring, and therefore it is regular. By 1.5 (a), every localization of $R_\alpha(X)$ to a minimal prime ideal of f is regular. Therefore $R_\alpha(X)$ satisfies Serre's condition (R_1). Since $R_\alpha(X)$ is Cohen-Macaulay, by Serre's normality criterion [8, 4.1], it is normal. From Nagata's theorem [8, 7.2], and 1.3, we deduce that $\mathrm{Cl}(R_\alpha(X))$ is generated by the classes of the minimal prime ideals of f, $\mathrm{cl}(I_{\beta_1}(x)), \ldots, \mathrm{cl}(I_{\beta_r}(x))$. By 1.3, $R_\alpha(X)[f^{-1}]$ is (isomorphic to) a polynomial ring after inversion of the determinant of a symmetric matrix of indeterminates. The last is a prime element, hence the units of $R_\alpha(X)[f^{-1}]$ are elements of the form kf^s, with $k \in K \setminus \{0\}$ and $s \in \mathbb{Z}$. Then we may argue as in the generic case, see [4, pp. 94], to show that the only relation between $\mathrm{cl}(I_{\beta_1}(x)), \ldots, \mathrm{cl}(I_{\beta_r}(x))$ is $\sum_{i=1}^{r} v_i(f)\ \mathrm{cl}((I_{\beta_i}(x)) = 0$. The rest follows immediately from 1.5 (b). □

2. The canonical class of $R_\alpha(X)$. Given a normal Cohen-Macaulay domain A with a canonical module ω_A, it is known that ω_A is a rank 1 reflexive module. Therefore ω_A is isomorphic to a divisorial ideal, and its class in $\mathrm{Cl}(A)$ is called a *canonical class*. Furthermore, if A is a positively graded K-algebra, then its canonical module is unique up to isomorphism, so that there exists a unique canonical class, the canonical class of A. The ring A is Gorenstein if and only if its canonical module is principal, that is, if its canonical class vanishes in $\mathrm{Cl}(A)$. We want to determine the canonical class ω of $R_\alpha(X)$, and decide whether $R_\alpha(X)$ is Gorenstein or not. We compute the canonical class of $R_\alpha(X)$, by induction, using the isomorphism (1) and the following result due to Bruns [3, 4.2]:

Lemma 2.1. *Let A be a normal Cohen-Macaulay domain, and I a prime ideal of height 1 in A such that A/I is again a normal Cohen-Macaulay domain. Let $Q_1, \ldots, Q_u$ be prime ideals of height 1 in A and suppose that the class of I and the class of a canonical module ω_A have representations $\mathrm{cl}(I) = \sum_{i=1}^{u} s_i \,\mathrm{cl}(Q_i)$ and $\mathrm{cl}(\omega_A) = \sum_{i=1}^{u} r_i \,\mathrm{cl}(Q_i)$. Assume further that:*

(i) $r_i - s_i \geqq 0$ *for* $i = 1, \ldots, u$.
(ii) $\mathrm{Ann}(Q_i^{(r_i - s_i)}/Q_i^{r_i - s_i}) \not\subset Q_i + I$ *for* $i = 1, \ldots, u$.
(iii) *The ideals $\bar{Q}_i = (Q_i + I)/I$ are distinct prime ideals of height 1 in A/I.*

Then A/I has a canonical module with class $\sum_{i=1}^{u} (r_i - s_i)\,\mathrm{cl}(\bar{Q}_i)$.

With the notation introduced in the first section, we get:

Theorem 2.2. *Let $\mathrm{cl}(\omega)$ be the canonical class of $R_\alpha(X)$, and let $\mathrm{cl}(\omega) = \sum_{i=1}^{r} \lambda_i \,\mathrm{cl}(I_{\beta_i}(x))$ be a representation of $\mathrm{cl}(\omega)$ with respect to the system of generators of $\mathrm{Cl}(R_\alpha(X))$ given in 1.6. Then, if $\alpha_t < n$:*

$$\begin{cases} \lambda_{i-1} - \lambda_i = |\chi_{i-1}| - |B_i| & \text{for all} \quad i = 2, \ldots, r, \\ \lambda_r \equiv |\chi_r| + 1 \bmod (2) \end{cases}$$

and if $\alpha_t = n$:

$$\begin{cases} \lambda_{i-1} - \lambda_i = |\chi_{i-1}| - |B_i| & \text{for all} \quad i = 2, \ldots, r-1 \\ \lambda_{r-1} - 2\lambda_r = |\chi_{r-1}| - |B_r| - 1 \end{cases}.$$

P r o o f. The proof is by induction on n. In the case $n = 2$ everything is trivial. If $\alpha_t = n$ and $r = 1$, then $R_\alpha(X)$ is a polynomial ring. Hence we may assume that $r > 1$ if $\alpha_t = n$. As in the proof of 1.5, we may also assume $\alpha_1 = 1$. The isomorphism (1) induces an isomorphism of divisor class groups $\psi^*: \mathrm{Cl}(R_\alpha(X)[x_{11}^{-1}]) \to \mathrm{Cl}(R_{\bar{\alpha}}(\bar{X}))$. Since the extension $R_{\bar{\alpha}}(\bar{X}) \to R_{\bar{\alpha}}(\bar{X})[X_{11}, \ldots, X_{1n}][X_{11}^{-1}]$ is faithfully flat, ψ^* maps the canonical class to the canonical class. The compositon of the canonical epimorphism $\mathrm{Cl}(R_\alpha(X)) \to \mathrm{Cl}(R_\alpha(X)[x_{11}^{-1}])$ with ψ^* gives an epimorphism

$$h: \mathrm{Cl}(R_\alpha(X)) \to \mathrm{Cl}(R_{\bar{\alpha}}(\bar{X})).$$

The localization of a canonical module is a canonical module. Hence h maps the canonical class to the canonical class.

If $\alpha_2 = 2$, then by (2) we get $h(\mathrm{cl}(I_{\beta_i}(x))) = \mathrm{cl}(I_{\bar{\beta}_i}(\bar{x}))$ for all $i = 1, \ldots, r$. By induction we get the desired result. If $\alpha_2 > 2$, then by (2) we get $h(\mathrm{cl}(I_{\beta_i}(x))) = \mathrm{cl}(I_{\bar{\beta}_{i-1}}(\bar{x}))$ for all $i = 2, \ldots, r$, and $h(\mathrm{cl}(I_{\beta_1}(x))) = 0$. Again by induction, it is enough to prove only the relation which involves λ_1. We have to distinguish 3 cases:

Case 1. $\alpha_t < n$ and $r = 1$. Then $\alpha = \{1\}$, and $R_\alpha(X) = R_2(X)$ is the second Veronese subring of $K[X_1, \ldots, X_n]$. It is known that $R_2(X)$ is Gorenstein if and only if $n \equiv 0 \bmod (2)$, see [9, pp. 54]. Therefore $\lambda_1 \equiv n = |\chi_1| + 1 \bmod (2)$.

Case 2. $\alpha_t = n$ and $r = 2$. Set $\chi_1 = m - 1$, with $n > m > 1$, then $\alpha = \{1, m+1, \ldots, n\}$. We have to show that $\lambda_1 - 2\lambda_2 = m - 1 - (n - m) - 1 = 2m - n - 2$, that is, $\mathrm{cl}(\omega) = (2m - n - 2)\,\mathrm{cl}(I_{\beta_1}(x))$.

Let X^* be the submatrix of X of the first m rows. Denote by S the ring $K[X^*]/I_2(X^*)$, where $I_2(X^*)$ is the ideal generated by all the 2-minors of X^*. One has $I_2(X^*)K[X] = I_\alpha(X)$, and then $S[X \backslash X^*] = R_\alpha(X)$, see [5, 2.5(c)]. Hence S is a normal Cohen-Macaulay domain, and we have an isomorphism of divisor class groups $\mathrm{Cl}(S) \cong \mathrm{Cl}(R_\alpha(X))$, which maps the canonical class to the canonical class. The extension of the ideal $P = (x_{11}, \ldots, x_{1n})$ of S to $R_\alpha(X)$ is $I_{\beta_1}(x)$. Therefore it is enough to show that the canonical class of S is $\mathrm{cl}(\omega_S) = (2m - n - 2)\,\mathrm{cl}(P)$.

In order to apply 2.1, we need a more flexible system of generators of $\mathrm{Cl}(S)$. Let $Q = (x_{ij}, 1 \leqq i \leqq j \leqq m)$; it is not difficult to see that Q is a prime ideal of height 1 in S, that $\mathrm{cl}(P)$, $\mathrm{cl}(Q)$ are generators of $\mathrm{Cl}(S)$, and the only relation is $2\,\mathrm{cl}(P) + \mathrm{cl}(Q) = 0$.

First let $m = 2$, and we argue by induction on n. Suppose $n = 3$; since $(2m - n - 2)\,\mathrm{cl}(P) = -\mathrm{cl}(P) = \mathrm{cl}(P) + \mathrm{cl}(Q)$, we have to show that $J = P \cap Q = (x_{11}, x_{12})$ is the canonical module of S. In this case $\dim(S) = 3$, and we note that $S/J = K[X_{13}, X_{22}, X_{23}]/(X_{22}X_{13})$. Thus J is a maximal Cohen-Macaulay S-module. Since S is a domain and it has a canonical module, J is the canonical module of S if its Cohen-Macaulay type $r(J)$ is 1. The sequence $\bar{x} = x_{11}, x_{23}, x_{13} - x_{22}$, is a system of parameters of S. Hence $\bar{x}$ is a maximal regular S and J sequence. Comparing the Hilbert series of S and S/J, one shows that the Hilbert series of J is $2t + t^2/(1-t)^3$. Then the Hilbert series of $J/\bar{x}J$ is $2t + t^2$. The homogeneous component of degree 2 of $J/\bar{x}J$ is generated by $x_{11}x_{13} = x_{11}x_{22} = x_{12}^2$. Clearly no 1-forms of $J/\bar{x}J$ annihilate the irrelevant maximal ideal of S. Therefore $r(J) = 1$.

Now suppose $n > 3$. We apply 2.1 with respect to the ideal $I = (x_{1n}, x_{2n})$. It is clear that $I \cong (x_{11}, x_{12}) = P \cap Q$, so that $\mathrm{cl}(I) = \mathrm{cl}(P) + \mathrm{cl}(Q)$. We may write $\mathrm{cl}(\omega_S) = a\,\mathrm{cl}(P) + b\,\mathrm{cl}(Q)$, and we may assume $a, b > 0$. The ideals P, Q are principal after inversion of x_{23}. Then a power of x_{23} annihilates $P^{(a-1)}/P^{a-1}$ and $Q^{(b-1)}/Q^{b-1}$. Since $P + I$, and $Q + I$ are prime ideals and do not contain x_{23}, the assumption (ii) of 2.1 is satisfied. By induction we get $a - 1 - 2(b-1) = 4 - (n-1) - 2$, that is, $a - 2b = 4 - n - 2$.

When $m > 2$, the desired result is obtained by induction using 2.1 with respect to the ideal $I = (x_{1m}, x_{2m}, \ldots, x_{mm}, x_{mm+1}, \ldots, x_{mn})$. One has to note that $\mathrm{cl}(I) = \mathrm{cl}(P)$, and that P, Q are principal after inversion of x_{2n}. This concludes the Case 2.

Case 3. $\alpha_t < n$ and $r > 1$, or $\alpha_t = n$ and $r > 2$. Just to simplify the notation set $R = R_\alpha(X)$. We have to show that $\lambda_1 - \lambda_2 = |\chi_1| - |B_2|$. Set $h = |B_2|$, and $C = B_3, \ldots, B_r$. Then $\alpha = \{1, \alpha_2, \alpha_2 + 1, \ldots, \alpha_2 + h - 1, C\}$. Let σ be the sequence obtained from α by replacing 1 with $\alpha_2 + h$ (note that $\alpha_2 + h \leqq n$), that is, $\sigma = \{\alpha_2, \alpha_2 + 1, \ldots, \alpha_2 + h, C\}$, and denote by $\sigma_1, \ldots, \sigma_t$ the indices of σ. Denote by g the residue class of the minor $[\sigma \,|\, \sigma]$ in R. By construction, $g \notin I_{\beta_i}(x)$ if and only if $\sigma \geqq \beta_i$, that is, if and only if $i = 1$ or $i = 2$. We may invert g to isolate λ_1 and λ_2 from the expression of $\mathrm{cl}(\omega)$. The class $\lambda_1\,\mathrm{cl}(I_{\beta_1}(x)) + \lambda_2\,\mathrm{cl}(I_{\beta_2}(x))$ is a canonical class of $R[g^{-1}]$. Following [4, 8.11], we may interpret

$R[g^{-1}]$ as a polynomial extension of a determinantal ring associated with the ideal of the 2-minors of a generic matrix of indeterminantes. In order to do this we consider the set Ψ_1 of the residue classes in R of the elements of $\{[\sigma_i \mid \sigma_j]: 1 \leqq i \leqq j \leqq t\} \cup \{[\sigma \mid \delta] \in D_\alpha : \delta$ differs from σ in exactly one index$\}$, and the set Ψ_2 of the residue classes in R of the elements of $\{[\delta \mid \sigma] \in D_\alpha : \delta$ differs from σ in exactly one index$\}$.

Denote by δ_{jk} the sequence $\{j, \alpha_2, \ldots, \widehat{(\alpha_2 + h + 1 - k)}, \ldots, \alpha_2 + h, C\}$, and let M_{jk} be the residue class in R of $[\delta_{jk} \mid \sigma]$ in R. One has $\Psi_2 = \{M_{jk} : 1 \leqq j < \alpha_2, 1 \leqq k \leqq h+1\}$. Let $K[\Psi_1 \cup \Psi_2]$ be the K-subalgebra of R generated by the elements in the set $\Psi_1 \cup \Psi_2$. As in [4, 6.4], one shows that $R[g^{-1}] = K[\Psi_1 \cup \Psi_2][g^{-1}]$. We have to determine the relations between the elements of $\Psi_1 \cup \Psi_2$. We claim that for all $1 \leqq j, j_1 < \alpha_2$ and $1 \leqq k \leqq k_1 \leqq h+1$

$$M_{jk} M_{j_1 k_1} = [\delta_{j \wedge j_1 k} \mid \delta_{j \vee j_1 k_1}][\sigma \mid \sigma] = M_{jk_1} M_{j_1 k}$$

where $j \wedge j_1 = \min\{j, j_1\}$ and $j \vee j_1 = \max\{j, j_1\}$. To prove the claim note that $M_{jk} M_{j_1 k_1} = [\delta_{jk} \mid \sigma][\sigma \mid \delta_{j_1 k_1}]$, since the matrix is symmetric. Then we consider the "generic straightening relation" of $[\delta_{jk} \mid \sigma][\sigma \mid \delta_{j_1 k_1}]$. Each standard monomial in the generic standard representation of $[\delta_{jk} \mid \sigma][\sigma \mid \delta_{j_1 k_1}]$ contains at most two factors. There are only two such standard monomials obtained from the indices of the minors under consideration, namely $[\delta_{jk} \mid \delta_{j_1 k_1}][\sigma \mid \sigma]$ and $[j, \sigma \mid j_1, \sigma][\sigma \backslash \{\alpha_2 + h + 1 - k\} \mid \sigma \backslash \{\alpha_2 + h + 1 - k_1\}]$. The first appears in the representation with coefficient 1 (to see this one can specialize $j = k$ in the generic expression) while the second vanishes in R. Therefore $[\delta_{jk} \mid \sigma][\sigma \mid \delta_{j_1 k_1}] = [\delta_{jk} \mid \delta_{j_1 k_1}][\sigma \mid \sigma]$ and it remains to prove that $[\delta_{jk} \mid \delta_{j_1 k_1}] = [\delta_{j \wedge j_1 k} \mid \delta_{j \vee j_1 k_1}]$. But since the matrix is symmetric, the last equality is straightforward from the fact that the 2-minors of the first $(\alpha_2 - 1)$-rows vanish in R.

Now we take an $(\alpha_2 - 1) \times (h+1)$ matrix of indeterminates $T = (T_{ij})$, and a set of distinct indeterminates $A = \{A_\psi : \psi \in \Psi_1\}$. Denote by a the determinant of the $t \times t$ symmetric matrix $(A_{[\sigma_i \mid \sigma_j]})$. Consider the following surjective ring homomorphism $K[T, A][a^{-1}] \to R[g^{-1}]$ by the assignment $T_{ij} \to M_{ij}$, and $A_\psi \to \psi$. The kernel of this homomorphism contains the ideal $I_2(T)$ of the 2-minors of T. Therefore we get a surjective homomorphism $L: K[T]/I_2(T)[A][a^{-1}] \to R[g^{-1}]$. We claim that L is an isomorphism. Since $K[T]/I_2(T)[A]$ and R are domains, the claim follows if we show that they have same dimension. Note that $\dim K[T]/I_2(T)[A] = |\Psi_1| + (\alpha_2 + h - 1)$. An element $\delta \geqq \sigma$, which differs from σ exactly in one index and has t entries, is obtained from σ by replacing an index σ_i with an index $k > \sigma_i$ and $k \neq \sigma_j$ for $j > i$. We have $|\Psi_1| + (\alpha_2 + h - 1) = t(t+1)/2 + \sum_{i=1}^{t} (n - \sigma_i - t + i) + (\alpha_2 + h - 1) = t(n+1) - \sum_{i=1}^{t} \alpha_i = \dim R$. Therefore L is an isomorphism.

Let P be the prime ideal of $K[T]/I_2(T)$ generated by the elements of the first row of T, and Q be the prime ideal generated by the elements of the first column. By construction $L(T_{1k}) \in I_{\beta_1}(x) R[g^{-1}]$, and $L(T_{j1}) \in I_{\beta_2}(x) R[g^{-1}]$. Then the extension of P is contained in $I_{\beta_1}(x) R[g^{-1}]$, but since both are prime ideals of height 1, they coincide. The same argument works as well for Q and $I_{\beta_2}(x) R[g^{-1}]$. Since a is a prime element in $K[T]/I_2(T)[A]$, one has an isomorphism $\mathrm{Cl}(K[T]/I_2(T)) \to \mathrm{Cl}(R[g^{-1}])$. It is not difficult to see that the ring $R[g^{-1}] = K[T]/I_2(T)[A][a^{-1}]$ has a unique canonical class. We deduce that $\lambda_1 \operatorname{cl}(P) + \lambda_2 \operatorname{cl}(Q)$ is the canonical class of $\operatorname{Cl}(K[T]/I_2(T))$. By [4, 8.8], we conclude that $\lambda_1 - \lambda_2 = \alpha_2 - 1 - (h+1) = (\alpha_2 - 2) - h = |\chi_1| - |B_2|$. $\square$

Taking into account the relations between the given generators of $\operatorname{Cl}(R_\alpha(X))$, the canonical class is completely determined by the theorem. In particular:

Corollary 2.3. *If $\alpha_t < n$, then $R_\alpha(X)$ is Gorenstein if and only if $|\chi_{i-1}| - |B_i| = 0$ for all $i = 2, \ldots, r$, and $|\chi_r| + 1 \equiv 0 \bmod (2)$.*

If $\alpha_t = n$, then $R_\alpha(X)$ is Gorenstein if and only if $|\chi_{i-1}| - |B_i| = 0$ for all $i = 2, \ldots, r-1$, and $|\chi_{r-1}| - |B_r| - 1 = 0$.

We single out the most important cases:

Theorem 2.4. (a) (S. Goto). *Let $1 < t \leqq n$. The ring $R_t(X)$ is a normal domain and its divisor class group is $\mathbb{Z}_2$. It is Gorenstein if and only if $n \equiv t \bmod (2)$. Furthermore, if $n \not\equiv t \bmod (2)$, its canonical module is the ideal P generated by the $(t-1)$-minors of the first $(t-1)$ rows of X, and its Cohen-Macaulay type is $\binom{n}{t-1}$.*

(b) *Let X^* be an $m \times n$, $m < n$, partially symmetric matrix of indeterminates, that is, $X^*_{ij} = X^*_{ji}$ for all $1 \leqq i, j \leqq m$. Let $1 < t \leqq m$. The ring $R_t(X^*)$ is a normal domain and its divisor class group is $\mathbb{Z}$. Furthermore it is Gorenstein if and only if $2m = n + t$.*

Proof. (a) One observes that P is minimally generated by the set of all (doset) $(t-1)$-minors of the first $(t-1)$ rows. The rest is a particular case of 1.6 and 2.3.

(b) Consider $\alpha = \{1, \ldots, t-1, m+1, \ldots, n\}$, then a minimal system of generators of $I_\alpha(X)$ is the set of the doset t-minors of the first m rows. In other words, $R_\alpha(X)$ is a polynomial extension of $R_t(X^*)$. Therefore we obtain the desired results as an application of 1.6 and 2.3. $\square$

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