JOURNAL OF ALGEBRA 166, 406–421 (1994)

Gröbner Bases of Ideals of Minors of a Symmetric Matrix

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Communicated by Melvin Hochster

Received September 3, 1993

INTRODUCTION

The aim of this paper is to determine Gröbner bases of the determinantal ideals of a symmetric matrix of indeterminates and to compute the multiplicity of the corresponding quotient rings.

This has already been done for determinantal ideals of an ordinary matrix of indeterminates X, (ordinary means that the entries of X are distinct indeterminates). Gröbner bases for these ideals are computed by Narasimhan [10], Caniglia–Guccione–Guccione [3], Sturmfels [15], and Herzog–Trung [7]. In particular the method of Sturmfels is based on two combinatorial facts:

(a) The polynomial ring $K[X]$ over a field K is an algebra with straightening law on the poset of the minors of X [2].

(b) The Knuth–Robinson–Schensted correspondence maps standard monomials of $K[X]$ to ordinary monomials of $K[X]$ [8, 15].

Let I_t be the ideal generated by all the t-minors of X. If we consider a suitable term order then the ideal in (I_t) of the initial term of I_t is generated by square-free monomials. Therefore the quotient ring $K[X]/\text{in}(I_t)$ is the Stanley–Reisner ring associated with a simplicial complex Δ. Using this fact Herzog–Trung [7] compute the multiplicity of the ring $K[X]/I_t$.

Now let $X=(X_{ij})$, with $X_{ij}=X_{ji}$, be an $n\times n$ symmetric matrix of indeterminates, and let $R=K[X_{ij}: 1\leqslant i\leqslant j\leqslant n]$ be the polynomial ring over a field K. We denote by $I_t(X)$ the ideal generated by all the t-minors

* Supported by a grant of Consiglio Nazionale delle Ricerche, Italy.

0021-8693/94 $6.00

of X and by $R_t(X)$ the corresponding quotient ring. We extend to this case the method of Sturmfels to compute a Gröbner basis for $I_t(X)$, and the method of Herzog–Trung to compute the multiplicity of $R_t(X)$.

The combinatorial structure of R, with respect to the minors of X, is described in [4]. It turns out that R is not an algebra with straightening law on a poset but on a doset. The standard monomials are products of certain minors. They form a "nice" K-basis of R. We may represent the standard monomials as standard tableaux of double shape, see Definition 1.5. Therefore we have to modify the Knuth–Robinson–Schensted correspondence suitably in order to obtain a correspondence between the standard tableaux of double shape and the ordinary monomials of R. We deal with this problem in the first section.

In the second section we define a class of ideals of minors compatible with the combinatorial structure of R. Let $\alpha = \{\alpha_1, ..., \alpha_t\}$ be a subset of the set $\{1, ..., n\}$ and suppose that $\alpha_1 < \cdots < \alpha_t$, and define $I_\alpha(X)$ to be the ideal generated by all the i-minors of the first $\alpha_i - 1$ rows of X with $i = 1, ..., t$, and by all the $(t+1)$-minors. We call $I_\alpha(X)$ the ideal cogenerated by α and denote by $R_\alpha(X)$ the corresponding quotient ring. This class of ideals contains the ideals $I_t(X)$ and it is essentially the class of ideals studied by Kutz [9]. Using the new version of the Knuth–Robinson–Schensted correspondence we compute Gröbner bases for this class of ideals. A minor $[a_1, ..., a_t | b_1, ..., b_t]$ of X is called a doset minor if $a_i \leqslant b_i$ for $i = 1, ..., t$. Our result for the ideal $I_t(X)$ is the following:

The set of the doset t-minors of X is a Gröbner basis for the ideal $I_t(X)$ with respect to the lexicographic term order τ induced by the variable order $X_{11} > \cdots > X_{1n} > X_{22} > \cdots > X_{2n} > \cdots > X_{n-1n} > X_{nn}$.

As in the case of the determinantal ideals of an ordinary matrix of indeterminates, the ideal $\mathrm{in}(I_\alpha(X))$ of the initial terms of the elements of $I_\alpha(X)$ is generated by square-free monomials. Therefore we may interpret the ring $R/\mathrm{in}(I_\alpha(X))$ as a Stanley–Reisner ring $K[\Delta_\alpha]$ associated with a simplicial complex Δ_α. The Hilbert function of $K[\Delta_\alpha]$ can be expressed completely in terms of the f-vector of the numbers of faces of different dimension of Δ_α, see [13]. In particular, $e(K[\Delta_\alpha]) = f_d$, where f_d is the number of the faces of maximal dimension of Δ_α. In the third section we characterize the faces of maximal dimension of Δ_α as families of disjoint paths. Then, using a method of Gessel–Viennot [6], we compute f_d. Since the Hilbert functions of $R/\mathrm{in}(I_\alpha(X))$ and $R_\alpha(X)$ coincide, we have $e(R_\alpha(X)) = f_d$. Our result for the ring $R_t(X)$ is

$$e(R_t(X)) = \sum_{1 \leqslant j_1 < \cdots < j_{t-1} \leqslant n} \det \left[\binom{n-h}{n-j_k} \right]_{1 \leqslant h, k \leqslant t-1}.$$

This paper wil be part of the author's Ph. D. thesis.

1. Standard Tableaux of Double Shape

The original Knuth–Robinson–Schensted correspondence, KRS for short, is a one-to-one correspondence between standard Young bitableaux and matrices of non-negative integers. It is constructed by Knuth [8] and has its origin in a combinatorial algorithm of Robinson and Schensted [12]. Sturmfels in [15] and Herzog–Trung in [7] use a version of the KRS correspondence in the computation of Gröbner bases of determinantal ideals of an ordinary matrix of indeterminates. Throughout this paper we call a matrix whose entries are distinct indeterminates an ordinary matrix of indeterminates. The purpose of this section is define a suitable modification of the KRS correspondence in order to use it for the computation of Gröbner bases of determinantal ideals of a symmetric matrix. The combinatorial structure of this class of ideals [5, 4] is quite different from that of the determinantal ideals of an ordinary matrix of indeterminates. This fact leads us to consider a special class of tableaux [1] the standard tableaux of double shape.

We recall the definition of standard tableaux and the definition of the algorithms Delete and Insert.

Definition 1.1. A *standard tableau* is an array $A=(a_{ij})$ of positive integers with $1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant r_i$, where $r_1 \geqslant \cdots \geqslant r_t$, such that the rows are strictly increasing and the columns are not decreasing (that is $a_{ij} < a_{ij+1}$ for all $i=1, ..., t$, $j=1, ..., r_i-1$ and $a_{ij} \leqslant a_{i+1j}$ for all $i=1, ..., t-1$, $j=1, ..., r_{i+1}$). We call the sequence $r_1, ..., r_t$ the shape of A. For systematic reason let $r_{t+1}=0$.

Definition 1.2. The inputs of *Delete* are a standard tableau $A=(a_{ij})$ of shape $r_1, ..., r_t$ and an integer i, $1 \leqslant i \leqslant t$, such that $r_i > r_{i+1}$. The outputs are a standard tableau B of shape $r_1, ..., r_{i-1}, r_i - 1, r_{i+1}, ..., r_t$ and an element $a \in A$.

We define recursively a sequence of integers $a_i, ..., a_1$. Let $a_i = a_{ir_i}$ and for $j=i-1, ..., 1$ let a_j be the largest element of the jth row of A less than or equal to a_{j+1}. Then we define B as the standard tableau obtained from A by replacing a_j with a_{j+1} for $j=i, ..., 2$, and set $a=a_1$. We write $\mathrm{Del}(A, i)=(B, a)$.

Definition 1.3. The inputs of *Insert* are a standard tableau $B=(b_{ij})$ of shape $s_1, ..., s_t$ and an integer a. The outputs are a standard tableau A and an integer i such that the shape of A is $s_1, ..., s_{i-1}, s_i+1, s_{i+1}, ..., s_t$.

We define recursively a sequence of integers. Let $a_1=a$; suppose $a_1, ..., a_j$ has already been defined. If $a_j > b_{js_j}$ or $j=t+1$, then the sequence terminates. If $a_j \leqslant b_{js_j}$, we let a_{j+1} be the minimum of the elements of the jth row of B greater than or equal to a_j. Therefore we have a sequence $a_1, ..., a_i$ and $a_i > b_{is_i}$ or $i = t+1$. Then we define A to be the standard tableau obtained from B by replacing a_{j+1} with a_j for $j = 1, ..., i-1$, and by adding a_i in the place $(i, s_i + 1)$. The outputs are A and i, and we write $\mathrm{Ins}(B, a) = (A, i)$.

The definition of standard tableaux and the algorithms Delete and Insert correspond to the definition of dual tableau and to the algorithms Delete* and Insert* in the Knuth's paper [8]. The proof of the following theorem can be found in [8, Theorem 1*, p. 720].

THEOREM 1.4. (a) *Let A be a standard tableau and i an integer such that we can apply* $\mathrm{Del}(A, i)$, *then*

$$\mathrm{Del}(A, i) = (B, a) \Leftrightarrow \mathrm{Ins}(B, a) = (A, i).$$

(b) *If* $\mathrm{Del}(A, i) = (A_1, a)$ *and* $\mathrm{Del}(A_1, j) = (A_2, b)$, *then* $i > j \Leftrightarrow a \leqslant b$.

(c) *If* $\mathrm{Ins}(A_2, b) = (A_1, j)$ *and* $\mathrm{Ins}(A_1, a) = (A, i)$, *then* $i > j \Leftrightarrow a \leqslant b$.

DEFINITION 1.5. A standard tableau T of shape $r_1, r_1, ..., r_i, r_i, ..., r_t, r_t$ is called *standard tableau of double shape*, d-tableau for short. We define $\deg(T) = \sum_{i=1}^{t} r_i$, the degree of T.

Now we are ready to describe a correspondence between d-tableaux and monomials of the set of indeterminates $X = \{X_{ij} : i, j \in \mathbb{N},\ 1 \leqslant i \leqslant j\}$.

Remark 1.6. We note that every monomial $\prod_{i \leqslant j} X_{ij}^{a_{ij}}$ can be uniquely rewritten in the form $X_{v_1 u_1} \cdots X_{v_c u_c}$ with $v_i \leqslant u_i$, $u_1 \leqslant \cdots \leqslant u_c$ and $v_i \geqslant v_{i+1}$ if $u_i = u_{i+1}$. Therefore we have an identification between the set of monomials of X and the set of two-line arrays of positive integers

$$\begin{pmatrix} u_1 & \cdots & u_c \\ v_1 & \cdots & v_c \end{pmatrix}$$

which satisfy the conditions $v_i \leqslant u_i$, $u_1 \leqslant \cdots \leqslant u_c$ and $v_i \geqslant v_{i+1}$ if $u_i = u_{i+1}$. We denote by (S) this set of conditions.

The KRS Correspondence for d-*Tableaux*

We define a map Φ between the set of the d-tableaux and the set of the two-line arrays of positive integers which satisfy the conditions (S). It is defined by the following algorithm:

Let T be a d-tableau of degree c. We define recursively a sequence of d-tableaux $T_c, ..., T_1, T_0$, with $T_0 = \varnothing$, and two sequences of integers $u_c, ..., u_1$ and $v_c, ..., v_1$. First put $T_c = T$. Then, for $i = c, ..., 1$, let u_i be the maximum of the elements of T_i, and p_i the largest index such that u_i can be found in the p_ith row of T_i. Since T_i is a d-tableau p_i is even, say $p_i = 2s_i$. Then:

(1) Apply $\text{Del}(T_i, p_i)$ and let (T_i', v_i) be the output.

Comment. If the shape of T_i is $r_1, r_1, ..., r_q, r_q$, then the shape of T_i' is $r_1, r_1, ..., r_{s_i}, r_{s_i} - 1, ..., r_q, r_q$ and u_i is in the $(p_i - 1)$th row of T_i'.

(2) Cancel u_i from the tableau T_i' in the $(p_i - 1)$th row to obtain T_{i-1}.

Comment. T_{i-1} is a d-tableau of shape $r_1, r_1, ..., r_{s_i} - 1, r_{s_i} - 1, ..., r_q, r_q$. Then we define

$$\Phi(T) = \begin{pmatrix} u_1 & \cdots & u_c \\ v_1 & \cdots & v_c \end{pmatrix}.$$

We note that the maximum of the elements of the $(p_i - 1)$th row of T_i which are less than or equal to u_i is the element at the end of this row. Therefore we can replace the steps (1) and (2) by the steps:

(1*) Cancel u_i from the tableau T_i in the p_ith row to obtain the standard tableau T_i^*.

Comment. If the shape of T_i is $r_1, r_1, ..., r_q, r_q$, then the shape of T_i^* is $r_1, r_1, ..., r_{s_i}, r_{s_i} - 1, ..., r_q, r_q$.

(2*) Apply $\text{Del}(T_i^*, p_i - 1)$; then the output is (T_{i-1}, v_i).

It is clear that $v_i \leqslant u_i$ and that $u_1 \leqslant \cdots \leqslant u_c$. In order to show that the array $\Phi(T)$ satisfies the condition $v_i \geqslant v_{i+1}$ if $u_i = u_{i+1}$, we observe that $u_i = u_{i+1}$ implies $p_i < p_{i+1}$. Now using first (2*) and then (1), we have $\text{Del}(T_{i+1}^*, p_{i+1} - 1) = (T_i, v_{i+1})$ and $\text{Del}(T_i, p_i) = (T_i', v_i)$. We note that $p_i < p_{i+1} - 1$ since p_i and p_{i+1} are even. Therefore, as a consequence of Theorem 1.4, we obtain $v_i \geqslant v_{i+1}$.

EXAMPLE 1.7. In this example we apply the algorithm to a d-tableau of degree 5. In the following description $\xrightarrow{(i)}$ means "apply step (i) of the algorithm."

$$T = T_5 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \hline 3 & 5 & \\ \cline{1-2} 4 & 5 & \\ \cline{1-2} \end{array} \quad u_5 = 5, \qquad p_5 = 4 \xrightarrow{(1)} v_5 = 3, \qquad T_5' = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 & 5 \\ \hline 3 & 5 & \\ \cline{1-2} 4 & & \\ \cline{1-1} \end{array}$$

$$\xrightarrow{(2)} T_4 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 & 5 \\ \hline 3 & & \\ \cline{1-1} 4 & & \\ \cline{1-1} \end{array} \quad u_4 = 5, \qquad p_4 = 2 \xrightarrow{(1)} v_4 = 4, \qquad T_4' = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 2 & 3 & \\ \cline{1-2} 3 & & \\ \cline{1-1} 4 & & \\ \cline{1-1} \end{array}$$

$$\xrightarrow{(2)} T_3 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 & \\ \cline{1-1} 4 & \\ \cline{1-1} \end{array} \quad u_3 = 4, \qquad p_3 = 4 \xrightarrow{(1)} v_3 = 2, \qquad T_3' = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 3 \\ \hline 4 & \\ \cline{1-1} \end{array}$$

$$\xrightarrow{(2)} T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 3 \\ \hline \end{array} \quad u_2 = 3, \qquad p_2 = 2 \xrightarrow{(1)} v_2 = 3, \qquad T_2' = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}$$

$$\xrightarrow{(2)} T_1 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \quad u_1 = 2, \qquad p_1 = 2 \xrightarrow{(1)} v_1 = 1, \qquad T_1' = \begin{array}{|c|}\hline 2 \\ \hline \end{array} \xrightarrow{(2)} \varnothing$$

$$\Phi(T) = \begin{pmatrix} 2 & 3 & 4 & 5 & 5 \\ 1 & 3 & 2 & 4 & 3 \end{pmatrix}.$$

In order to prove that Φ is a one-to-one correspondence we define a map Γ between the set of the two-line arrays of positive integers which satisfy (S) and the set of the d-tableaux. Then we show that Φ and Γ are inverse to each other.

Let $U = \binom{u_1 \cdots u_c}{v_1 \cdots v_c}$ be a two-line array which satisfies (S). We define recursively a sequence of d-tableau $T_1, ..., T_c$ and a sequence of even integers $p_1, ..., p_c$ such that:

(a) The entries of T_i are the elements $u_1, ..., u_i, v_1, ..., v_i$.

(b) The last element of the p_ith row of T_i is u_i.

(c) All the elements of T_i which are equal to u_i are in the first p_i rows.

Put $T_1 = \begin{array}{|c|}\hline v_1 \\ \hline u_1 \\ \hline \end{array}$, and for all $i = 1, ..., c-1$, proceed as follows:

(1) Apply $\text{Ins}(T_i, v_{i+1})$ and let $(T^*_{i+1}, p_{i+1} - 1)$ be the output.

Comment. The row index $p_{i+1}-1$ is odd since T_i is a *d*-tableau, say $p_{i+1}=2s_{i+1}=2s_{i+1}$. If the shape of T_i is $r_1, r_1, ..., r_q, r_q, r_q$, then the shape of T^*_{i+1} is $r_1, r_1, .., r_{s+1}+1, r_{s_{i+1}}, ..., r_q, r_q$.

(2) T_{i+1} is obtained from T^*_{i+1} is obtained from T^*_{i+1} by adding the element u_{i+1} to the end of the (p_{i+1})-row.

Then we define $\Gamma(U)=T_c$.

In order to show that the algorithm is well-defined we have to verify that step (2) gives a d-tableau and that the elements of T_{i+1} which are equal to u_{i+1} are in the first p_{i+1} rows. It is enough to show that all the elements in the tableau T^*_{i+1} which are equal to u_{i+1} are in the first $p_{i+1}-1$ rows. If $u_i<u_{i+1}$, then this is clear since the entries of T_i are less than or equal to u_i and v_{i+1} is in the first row of T^*_{i+1}. If $u_i=u_{i+1}$, then $v_i \geqslant v_{i+1}$ since U satisfies (S). It is clear that $\mathrm{Del}(T_i, p_i)=(T_i', v_i)$, where the tableau T_i' is obtained from T_{i-1} by adding u_i at the end of the (p_i-1)th row. Then $\mathrm{Ins}(T_i', v_i)=(T_i, p_i)$ and $\mathrm{Ins}(T_i, v_{i+1})=(T^*_{i+1}, p_{i+1}-1)$. Hence, by Theorem 1.4, we obtain $p_{i+1}-1>p_i$. By induction we know that all the elements in the tableau T_i which are equal to u_i are in the first p_i rows. Then all the elements in the tableau T^*_{i+1} which are equal to u_i are in the first p_{i+1} rows. Since p_i and p_{i+1} are even we have $p_{i+1}-1 \geqslant p_i+1$. Therefore all the elements in the tableau T^*_{i+1} which are equal to u_{i+1} are in the first $p_{i+1}-1$ rows.

THEOREM 1.8. *The maps Φ and Γ are the inverse to each other.*

Proof. Let T be a d-tableau of degree c and $\Phi(T)=U=X_{v_1u_1}\cdots X_{v_cu_c}$ (resp. $U=X_{v_1u_1}\cdots X_{v_cu_c}$ a monomial of degree c and $\Gamma(U)=T$). By induction on c we prove that $\Gamma(U)=T$ (resp. $\Phi(T)=U$). The case $c=1$ is trivial. Suppose $c>1$. Let T_{c-1} be the d-tableau which is defined in the definition of Φ, then $\Phi(T_{c-1})=U_{c-1}=X_{v_1u_1}\cdots X_{v_{c-1}u_{c-1}}$ (resp. $U_{c-1}=X_{v_1u_1}\cdots X_{v_{c-1}u_{c-1}}$ and $\Gamma(U_{c-1})=T_{c-1}$). By induction $\Gamma(U_{c-1})=T_{c-1}$ (resp. $\Phi(T_{c-1})=U_{c-1}$). Now we observe that the steps (1) and (2) in the definition of Γ and the steps (2*) and (1*) in the definition of Φ are, respectively, inverse to each other. Therefore $\Gamma(U)=T$ (resp. $\Phi(T)=U$). ∎

We have shown that the map Φ is a one-to-one correspondence between the set of the d-tableaux and the set of the monomials of the set of indeterminates $X=\{X_{ij}: i, j \in \mathbb{N},\ 1 \leqslant i \leqslant j\}$. We call it the Knuth–Robinson–Schensted correspondence for d-tableau. If we consider only d-tableaux with entries less than or equal to n, then Φ is a one-to-one correspondence between the set of the d-tableaux and the monomials of the set of indeterminates $\{X_{ij}: 1 \leqslant i \leqslant j \leqslant n\}$. We close this section with an important property of the KRS correspondence.

LEMMA 1.9. *Let T be a d-tableau of degree c and $a_1, ..., a_{r_1}$ its first row. If $\Phi(T) = X_{v_1u_1} \cdots X_{v_cu_c}$, then for every $i = 1, ..., r_1$ there exists a sequence $1 \leqslant \alpha_1 < \cdots < \alpha_i \leqslant c$ such that $v_{\alpha_1} < \cdots < v_{\alpha_i}$ and $u_{\alpha_1} < \cdots < u_{\alpha_i}$.*

Proof. By induction on c. The case $c = 1$ is trivial. Suppose $c > 1$. The element v_c is of course an element of the first row of T, suppose $v_c = a_k$ with $1 \leqslant k \leqslant r_1$. Now consider the d-tableau T_{c-1} which is the one defined in the definition of Φ. We have $\Phi(T_{c-1}) = X_{c_1u_1} \cdots X_{v_{c-1}u_{c-1}}$ and the first row of T_{c-1} differs from that of T only in the position k. Let $i \in \{1, ..., r_1\}$, we distinguish two cases. If $i \neq k$, then by induction there exists a sequence $\alpha_1 < \cdots < \alpha_i$ such that $v_{\alpha_1} < \cdots < v_{\alpha_i} = a_i$. If $i = k$, then we may take $\alpha_1 = c$ in case $k = 1$, and $\alpha_1 < \cdots < \alpha_{k-1} < \alpha_k = c$ with $v_{\alpha_1} < \cdots < v_{\alpha_{k-1}} = a_{k-1}$ in case $k \neq 1$. Note that $u_{\alpha_1} < \cdots < u_{\alpha_i}$ since $\Phi(T)$ satisfies the conditions (S). ∎

2. GRÖBNER BASES OF DETERMINANTAL IDEALS OF A SYMMETRIC MATRIX

Let $X = (X_{ij})$, with $X_{ij} = X_{ji}$, be an $n \times n$ symmetric matrix of indeterminates and $R = K[X_{ij} : 1 \leqslant i \leqslant j \leqslant n]$ the polynomial ring over a field K. In order to study the ideals generated by minors of X we first discuss the combinatorial structure of R. Let H be the set of the non-empty subsets of $\{1, ..., n\}$. Let $a \in H$; we will always write the elements of a in ascending order $1 \leqslant a_1 < \cdots < a_t \leqslant n$ and define on H the partial order

$$a = \{a_1, ..., a_t\} \leqslant b = \{b_1, ..., b_r\} \Leftrightarrow r \leqslant t$$

and

$$a_i \leqslant b_i \qquad \text{for} \quad i = 1, ..., r$$

We denote by $[a_1, ..., a_t \mid b_1, .., b_t]$ the minor $\det(X_{a_ib_j})$, $1 \leqslant i, j \leqslant t$, of X. Since X is symmetric it is clear that $[a \mid b] = [b \mid a]$. A minor $[a_1, ..., a_t \mid b_1, ..., b_t]$ of X is called a *doset minor* if $a \leqslant b$ in H. We denote by D the set of all the doset minors. Let $[a \mid b]$, $[c \mid d] \in D$, we define on D the following transitive relation: $[a \mid b] \preccurlyeq [c \mid d] \Leftrightarrow b \leqslant c$ in H. A product $M = M_1 \cdots M_s$ of elements of D is called a *standard monomial* if $M_1 \preccurlyeq \cdots \preccurlyeq M_s$, and $\min(M) = M_1$ denotes the minimum of its factors.

Remark 2.1. The standard monomial $M_1 \cdots M_s$, $M_i = [a_{i1}, ..., a_{ir_i} \mid b_{i1}, ..., b_{ir_i}]$, can be identified with the d-tableau T, where $a_{i1}, ..., a_{ir_i}$ is the $(2i-1)$th row and $b_{i1}, ..., b_{ir_i}$ is the $2i$th row of T.

The combinatorial structure of R is described in the following theorem of De Concini and Procesi [4, p. 82; 5, Theorem 5.1, p. 344].

THEOREM 2.2. *The ring R is a doset algebra on D over K, that is*:

(a) *The standard monomials form a K-basis for R.*

(b) (*The straightening law*) *Let $M_1, ..., M_s \in D$, $M_i = [a_{i1}, ..., a_{ir_i} \mid b_{i1}, ..., b_{ir_i}]$, and let $\sum k_i N_i$ be the representation as linear combination of standard monomials of the product $M_1 \cdots M_s$. Let $\min(N_i) = [c_{i1}, ..., c_{is_i} \mid d_{i1}, ..., d_{is_i}]$; we set $c_i = \{c_{i1}, ..., c_{is_i}\}$, $d_i = \{d_{i1}, ..., d_{is_i}\}$, $a_i = \{a_{i1}, ..., a_{ir_i}\}$, and $b_i = \{b_{i1}, ..., b_{ir_i}\}$. Then for all i and j we have $c_i, d_i \leqslant a_j, b_i$ in the lexicographic order, that is $c_i < a_j$ or $c_i = a_j$ and $d_i \leqslant b_j$.*

The following lemma is a consequence of the identity [5, Lemma 5.2, p. 345].

LEMMA 2.3. *Every t-minor $M = [a_1, ..., a_t \mid b_1, ..., a_t]$ is a linear combination of doset t-minors. Moreover if $[c_1, ..., c_t \mid d_1, ..., d_t]$ is a doset t-minor which appears in the representation of M, then $c_i \leqslant a_i$ for all $i = 1 ..., t$.*

Now we define an important class of ideals which is compatible with the doset structure of the algebra. From now on let $\alpha = \{\alpha_1, ..., \alpha_t\} \in H$. For systematic reasons it is convenient to set $\alpha_{t+1} = n + 1$.

DEFINITION 2.4. We set

$$\Omega_\alpha = \{[a \mid b] \in D : a \ngeqslant \alpha\}, \quad I_\alpha(X) = \Omega_\alpha R, \quad \text{and} \quad R_\alpha(X) = R/I_\alpha(X).$$

Note that Ω_α is the set of all the doset minors whose sequence of row indices is not greater than or equal to α, $I_\alpha(X)$ is the ideal generated by this set in R and $R_\alpha(X)$ is the corresponding quotient ring. We call $I_\alpha(X)$ the ideal cogenerated by α.

Remark 2.5. (a) We note that, by Lemma 2.3, the ideal $I_\alpha(X)$ coincides with the ideal generated by all the minors whose sequence of row indices is not greater than or equal to α. This class of ideals was investigated by Kutz in his paper [9]. More precisely the ideal $I_{H,0}$, $H = (s_0, ..., s_m)$, defined in [9, Theorem 1, p. 116], corresponds in our notation to the ideal cogenerated by $\alpha = \{1, s_1 + 1, ..., s_{m-1} + 1\}$. Kutz showed that $I_\alpha(X)$ is a perfect prime ideal of height $n(n+1)/2 - (n+1)t + \sum_{i=1}^{t} \alpha_i$.

(b) If $\alpha = \{1, ..., t-1\}$, then $I_\alpha(X)$ is the ideal generated by the t-minors of X. We denote this ideal by $I_t(X)$ and by $R_t(X)$ the quotient ring.

(c) Let X^* be an $m \times n$ partially symmetric matrix of indeterminates, with $m < n$. Partially symmetric means that $X_{ij} = X_{ji}$ for all $1 \leqslant i, j \leqslant m$. If we are interested in the study of the ideal $I_t(X^*)$ generated by the t-minors of X^* we complete X^* to an $n \times n$ smmetric matrix of indeterminates X by adding a set $Y = \{X_{ij} : m+1 \leqslant i \leqslant j \leqslant n\}$ of new indeterminates. Now we consider $\alpha = \{1, 2, ..., t-1, \ m+1, \ m+2, ..., n\}$. We will see in Lemma 2.7(b) that the ideal $I_\alpha(X)$ is generated by the doset t-minors of the first m rows of X. Therefore, by Lemma 2.3, $I_\alpha(X)$ coincide with the extension of $I_t(X^*)$ to the ring $K[X^*, Y] = K[X]$ and $R_\alpha(X) = R_t(X^*)[Y]$.

LEMMA 2.6. *The set of all the standard monomials M such that* $\min(M) \in \Omega_\alpha$ *forms a K-basis of $I_\alpha(X)$.*

Proof. Let M be a standard monomial and $N \in \Omega_\alpha$. By the straightening law we know that the minimum of the factors of every standard monomial which appears in the representation of NM is in Ω_α. ∎

Let J_α be the set of all the doset minors $[a_1, ..., a_r \mid b_1, ..., b_r]$ with $1 \leqslant r \leqslant t+1$, such that $a_i \geqslant \alpha_i$ for $i = 1, ..., r-1$, and $a_r < \alpha_r$.

LEMMA 2.7. *The set J_α is a minimal system of generators of $I_\alpha(X)$.*

Proof. Let $M = [a_1, ..., a_s \mid b_1, ..., b_s]$ be a doset minor of X. First we observe that if we expand M with respect to the last row we may write it as a linear combination, with polynomial coefficients, of the doset minors $[a_1, ..., a_{s-1} \mid b_1, ..., \hat{b}_j, ..., b_s]$, $j = 1, ..., s$.

Let $M \in \Omega_\alpha$, then there exists an integer $i \leqslant \min\{s, t+1\}$ such that $a_i < \alpha_i$ and $a_j \geqslant \alpha_j$ if $j < i$. Now we expand M with respect to the last $s-i$ rows. By the previous observation, we may write M as a linear combination of doset i-minors of J_α. Therefore J_α is a system of generators of $I_\alpha(X)$. If J_α is not a minimal system of generators, there exists $N \in J_\alpha$ such that $N = \sum V_i M_i$ with $N \neq M_i \in J_\alpha$ and $V_i \in R \setminus \{0\}$. We may suppose that V_i is a standard monomial. Since N is a doset minor it appears in the representation of some $M_i V_i$ as a linear combination of standard monomials. In this case, by Theorem 2.2, the sequence of row indices of N is less than or equal to that of M_i. By definition two elements of J_α of different order have incomparable sequences of row indices. Therefore, $\deg(N) = \deg(M_i)$. But this implies $\deg(V_i) = 0$ and $N = M_i$, a contradiction. ∎

Let τ be a term order on R such that the initial term of every doset minor $[a_1, ..., a_s \mid b_1, ..., b_s]$ is $\prod_{i=1}^{s} X_{a_i b_i}$. For instance, we can take the lexicographic term order induced by the variable order,

$$X_{11} > X_{12} > \cdots > X_{1n} > X_{22} > \cdots > X_{2n} > \cdots > X_{n-1n} > X_{nn}.$$

If $f \in R$ and I is an ideal of R we denote by $\mathrm{in}(f)$ the initial term of f and by $\mathrm{in}(I)$ the ideal generated by the initial terms of the elements of I. A subset J of the ideal I is called a Gröbner basis for I if $\mathrm{in}(I)$ is generated by the initial terms of the elements of J. For further information on the theory of Gröbner basis see [11].

THEOREM 2.8. *The set J_α is a Gröbner basis for $I_\alpha(X)$ with respect to τ.*

Proof. Let I be the ideal generated by the initial terms of the elements of J_α. Let M be a standard monomial of the K-basis of $I_\alpha(X)$ described in Lemma 2.6. By Remark 2.1 we may identify M with a d-tableau T, and let $f = X_{v_1u_1} \cdots X_{v_cu_c}$ be the image of T under the KRS correspondence. Since $\min(M) = [a_1, ..., a_r \mid b_1, ..., b_r] \in \Omega_\alpha$ there exists i, $1 \leqslant i \leqslant t+1$, such that $a_i < \alpha_i$. We take i minimal with respect to this property. By Lemma 1.9, we find a sequence $j_1 < \cdots < j_i$ such that $v_{j_1} < \cdots < v_{j_i} = a_i$, and $u_{j_1} < \cdots < u_{j_i}$. Therefore f is divisible by the initial term of a doset i-minor and this minor is in J_α since i is minimal, that is $f \in I$.

We note that the monomials in I are a K-basis of I. Therefore the KRS correspondence maps elements of the K-basis of $I_\alpha(X)$ to elements of the K-basis of I and is degree-preserving. Then, if we denote by L_i the K-vector space of the forms of degree i of an homogeneous ideal L, we obtain $\dim_K I_\alpha(X)_i \leqslant \dim_K I_i$. But we know that $\text{in}(I_\alpha(X)) \supseteq I$, and it is well-known that $\dim_K I_\alpha(X)_i = \dim_K in(I_\alpha(X))_i$. Therefore $\text{in}(I_\alpha(X)) = I$. ∎

In particular we have:

THEOREM 2.9. *The set of the doset t-minors is a Gröbner basis for $I_t(X)$ with respect to τ.*

Remark 2.10. In general J_α is not the reduced Gröbner basis of $I_\alpha(X)$. For instance, the minors $M_1 = [1, 3, 4 \mid 2, 3, 5]$, $M_2 = [1, 2, 3 \mid 3, 4, 5]$ are part of the Gröbner basis of the ideal $I_3(X)$ and the initial term of the M_2, $\text{in}(M_2) = X_{13}X_{24}X_{35}$, appears in $M_1 = X_{12}X_{33}X_{45} - X_{12}X_{34}X_{35} - X_{13}X_{23}X_{45} + X_{13}X_{24}X_{35} + X_{15}X_{23}X_{34} - X_{15}X_{24}X_{34}$.

3. MULTIPLICITY OF DETERMINANTAL RINGS OF A SYMMETRIC MATRIX

In the previous section we have computed a Gröbner basis of the determinantal ideal $I_\alpha(X)$. As in the case of an ordinary matrix of indeterminates, the ideal $\text{in}(I_\alpha(X))$ of the initial terms of $I_\alpha(X)$ is generated by square-fee monomials. Therefore, in order to compute the multiplicity of $R_\alpha(X)$ following the approach of [7], we associate with $\text{in}(I_\alpha(X))$ a simplicial complex Δ_α. Then using the theory of the Stanley–Reisner ring we may compute the multiplicity of $K[X]/\text{in}(I_\alpha(X))$ which coincide with that of $R_\alpha(X)$. In this section we set $A = \{(i, j) \in \mathbb{N}^2 : 1 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant n\}$ and $B = \{(i, j) \in A : i \leqslant j\}$. In A we introduce the partial order: $(i, j) \leqslant (k, h) \Leftrightarrow$

$i \geqslant k$ and $j \leqslant h$. It is clear that A with this poset structure is a distributive lattice if we set

$$(i, j) \vee (k, h) = (\min\{i, k\}, \max\{j, h\}),$$
$$(i, j) \wedge (k, h) = (\max\{i, k\}, \min\{j, h\}).$$

Note that B, as subposet of A, is an upper semi-lattice, that is, if (i, j), $(k, h) \in B$ then $(i, j) \vee (k, h) \in B$. A set of incomparable elements is called an antichain. We note that an antichain of A is a set $\{(v_1, u_1), ..., (v_p, u_p)\}$ such that $v_1 < \cdots < v_p$ and $u_1 < \cdots < u_p$. Let P be a poset and $x \in P$. We define the rank of x in P to be the maximum of the integers i such that there exists a chain $x_1 < \cdots < x_i = x$ and the rank of P to be the maximum of the ranks of its elements.

For $k = 1, ..., t+1$, let $S_k = \{(i, j) \in A : i < \alpha_k \text{ or } j < \alpha_k\}$, $G_k = B \cap S_k$, $S'_k = A \setminus S_k$ and $G'_k = B \setminus G_k$. In particular, $S_{t+1} = A$ and $G_{t+1} = B$. Finally we define Δ'_α to be the simplicial complex of all the subsets of A which, for $k = 1, ..., t+1$, do not contain k-antichains (antichains with k elements) of S_k, and let Δ_α be the restriction of Δ'_α to B. It is clear that Δ_α is the simplicial complex of all the subsets of B which, for $k = 1, ..., t+1$, do not contain k-antichains of G_k.

Proposition 3.1. *Let $K[\Delta_\alpha]$ be the Stanley–Reisner ring associated with the simplicial complex Δ_α. Then $K[\Delta_\alpha] = K[X]/\text{in}(I_\alpha(X))$.*

Proof. By definition, see [13], $K[\Delta_\alpha] = K[X_{ij} : (i, j) \in B]/I$, where I is the ideal generated by the monomials $\prod_{(i,j) \in Z} X_{ij}$ with $Z \notin \Delta_\alpha$. In Theorem 2.8 we have shown that $\text{in}(I_\alpha(X))$ is generated by the k-antichains of G_k for $k = 1, ..., t+1$. Therefore $I = \text{in}(I_\alpha(X))$. ∎

Let Δ be a simplicial complex; its elements are called faces and its maximal elements under inclusion are called facets. A face of dimension i is a face with $i+1$ elements, the dimension of Δ is the maximum of the dimensions of its faces and f_i is the number of the faces of dimension i. The sequence $f_0, ..., f_d$, $d = \dim(\Delta)$, is called the f-vector of Δ. The Hilbert function of the Stanley–Reisner ring $k[\Delta]$ is determined by its f-vector [13]. In particular, $\dim k[\Delta] = d+1$ and $e(k[\Delta]) = f_d$. It is well-known that the Hilbert functions of $R_\alpha(X)$ and $K[X]/\text{in}(I_\alpha(X))$ coincide. Therefore, by virtue of Proposition 3.1, $\dim R_\alpha(X) = d+1$ and $e(R_\alpha(X)) = f_d$, where $f_0, .., f_d$ is the f-vector of Δ_α. Using the result of [7] which describes the facets of Δ'_α we will be able to describe the facets of Δ_α. First we note that the simplicial complex Δ'_α is the simplicial complex Δ_M defined in [7], where $M = [\alpha_1, ..., \alpha_t \mid \alpha_1, ..., \alpha_t]$. The facets of Δ'_α are described in [7, Theorem 4.6]: a facet $\bar{Z}$ of Δ'_α is the disjoint union of $\bar{Z}_1, ..., \bar{Z}_t$, where, for $k = 1, ..., t$, $\bar{Z}_k$ is a maximal chain of the distributive lattice S'_k. We may interpret a maximal chain of S'_k as a path from (n, α_k) to (α_k, n).

LEMMA 3.2. *Let $\bar{Z}$ be a facet of Δ'_α, then $|\bar{Z} \cap B| = (n+1)t - \sum_{i=1}^{t} \alpha_i$.*

Proof. It is clear that the rank of the poset $S'_k = \{(i, j) \in A : i, j \geqslant \alpha_k\}$ is $2(n-\alpha_k)+1$. The elements of G'_k are the elements of S'_k with rank $\geqslant n+1-\alpha_k$. Therefore every maximal chain of S'_k contains exactly $n+1-\alpha_k$ elements of G'_k. Then the desired result follows from the characterization of the facets of Δ'_α. ∎

PROPOSITION 3.3. *Let Z be a face of Δ_α. Then Z is a facet of Δ_α if and only if there exists a facet $\bar{Z}$ of Δ'_α such that $Z = \bar{Z} \cap B$.*

Proof. $\Rightarrow$. Since Δ_α is the restriction of Δ'_α to B there exists a facet $\bar{Z}$ of Δ'_α such that $Z \subset \bar{Z}$. But $\bar{Z} \cap B$ is a face of Δ_α and contains Z. Therefore $Z = \bar{Z} \cap B$.

$\Leftarrow$. Of course Z is contained in a facet Z_1 of Δ_α. We already know $Z_1 = \bar{Z}_1 \cap B$, where $\bar{Z}_1$ is a facet of Δ'_α. From Lemma 3.2 it follows that $|Z| = |Z_1|$, and hence $Z = Z_1$. ∎

In accordance with the Kutz's result [9] (see Remark 2.5), Lemma 3.2 and Proposition 3.3 immediately imply:

COROLLARY 3.4. *The dimension of $R_\alpha(X)$ is $(n+1)t - \sum_{i=1}^{t} \alpha_i$.*

For $i = 1, ..., t$, let $P_i = (\alpha_i, n)$ and $Q_i = \{(\alpha_i, \alpha_i), (\alpha_i+1, \alpha_i+1), ..., (n, n)\}$. We have the following description of the facets of Δ_α:

PROPOSITION 3.5. *The set Z is a facet of $\Delta_\alpha \Leftrightarrow Z$ is the disjoint union of $Z_1, ..., Z_t$, where Z_i is a path from P_i to one of the points of Q_i.*

Proof. Since Q_i is the set of the minimal elements of G'_k the desired result follows from Proposition 3.3 and from the description of the facets of Δ'_α. ∎

Now we are ready to compute the multiplicity of $R_\alpha(X)$. As in the case of the determinantal ideals of an ordinary matrix of indeterminates, [7, Theorem 3.5], it is a paths-counting problem, the only difference being that now the ending points are not fixed.

THEOREM 3.6. *The multiplicity of $R_\alpha(X)$ is given by formula*

$$e(R_\alpha(X)) = \sum_{\substack{1 \leqslant j_1 < \cdots < j_t \leqslant n \\ \alpha_i \leqslant j_i}} \det \left[\binom{n-\alpha_h}{n-j_k} \right]_{1 \leqslant h, k \leqslant t}.$$

Proof. In order to compute the multiplicity of $R_\alpha(X)$ we compute f_d of the simplicial complex Δ_α, that is, the number of disjoint union of paths from P_i to a point of Q_i, $i=1, ..., t$. For every choice $E_i=(j_i, j_i)\in Q_i$, $j_i \geqslant \alpha_i$, $i=1, ..., t$ let $W(P_1, ..., P_t, E_1, ..., E_t)$ be the number of the disjoint paths from P_i to E_i $i=1, ..., t$. We consider only the choices with $j_1 < \cdots < j_t$, since otherwise there are no disjoint paths. By [14, Section 2.7], $W(P_1, ..., P_t, E_1, ..., E_t) = \det[W(P_h, E_k)]_{h,k=1,...,t}$, where $W(P_h, E_k)$ is the number of the paths from P_h to E_k. But it is easy to see that $W(P_h, E_k) = \binom{n-\alpha_h}{n-j_k}$. ▮

EXAMPLE 3.7. The ideal $I_t(X)$ of the t-minors of X and the corresponding quotient ring $R_t(X)$ arise, as we have seen in Remark 2.5(b), from $\alpha = \{1, ..., t-1\}$. Therefore we have

$$\dim R_t(X) = (n+1-t/2)(t-1)$$

$$e(R_t(X)) = \sum_{1\leqslant j_1 < \cdots < j_{t-1} \leqslant n} \det\left[\binom{n-h}{n-j_k}\right]_{1\leqslant h,k\leqslant t-1}.$$

In the particular cases $t=2, 3$ and $t=n-2$, $n-1$ we have

$$(1)\quad e(R_2(X)) = 2^{n-1}, \qquad (2)\quad e(R_3(X)) = \binom{2n-3}{n-2},$$

$$(3)\quad e(R_{n-2}(X)) = \binom{n+2}{6} + \binom{n+3}{6}, \qquad (4)\quad e(R_{n-1}(X)) = \binom{n+1}{3}.$$

The formula (1) is clear. In order to show (2) we note that

$$e(R_3(X)) = \sum_{1\leqslant j_1<j_2\leqslant n} \det\left[\binom{n-h}{n-j_k}\right]_{1\leqslant h,k\leqslant 2}$$

$$= \sum_{1\leqslant j_1\leqslant j_2\leqslant n} \frac{j_2-j_1}{n-1}\binom{n-1}{j_1-1}\binom{n-1}{j_2-1},$$

and, if we set $k=j_2-j_1$, then

$$e(R_3(X)) = \sum_{k=0}^{n-1} \frac{k}{n-1} \sum_{j_1=1}^{n} \binom{n-1}{j_1-1}\binom{n-1}{j_1+k-1} = \sum_{k=0}^{n-1} \frac{k}{n-1}\binom{2(n-1)}{n-1+k}.$$

Then, using the binomial identity $\sum_{k=0}^{s} (k/s)\binom{2s}{s+k} = \binom{2s-1}{s-1}$, we obtain the desired result.

In order to prove (4) we set $e(n) = e(R_{n-1}(X))$. In this case $P_i=(i, n)$ $i=1, ..., n-2$. Consider the disjoint paths from P_i to a point of Q_i, $i=1, ..., n-2$, where we fix the path starting from P_{n-2} to be $(n-2, n)$, $(n-1, n)$, (n, n). The number of these paths is $e(n-1)$. If the path starting from P_{n-2} is $(n-2, n)$, $(n-1, n)$, $(n-1, n-1)$, then the number is

$e(n-1)-e(n-2)$. Finally the number of the paths such that the path from P_{-2} begins with $(n-2, n)$, $(n-2, n-1)$ is $n-1$. It follows that $e(n)=2e(n-1)-e(n-2)+n-1$. Since $e(3)=4$ and $e(4)=10$, the desired result follows.

For (3) we argue in the same way and obtain the recursive formula $e(n)=3e(n-1)-3e(n-2)+e(n-3)+\binom{n-1}{3}+\binom{n}{3}$, where $e(n)=e(R_{n-2}(X))$.

EXAMPLE 3.8. Let X^* be an $m \times n$ partially symmetric matrix of indeterminates. We have seen in Remark 2.5(c) that we can study the ideal $I_t(X^*)$ of the t-minors of X^* as an ideal of the class $I_\alpha(X)$ with a suitable choice of X and setting $\alpha = \{1, 2, ..., t-1, m+1, m+2, ..., n\}$. Using the notation of Remark 2.5(c) we have $R_\alpha(X) = R_t(X^*)[Y]$. Therefore the rings $R_\alpha(X)$ and $R_t(X^*)$ have the same multiplicity and the difference between their dimensions is $|Y|$.

Let $r = t-1+n-m$ be the number of elements of α. We note that, for all $j = t, ..., r$, $P_j = (m+1+j-t, n)$ and $Q_j = \{(m+1+j-t, m+1+j-t), ..., (n, n)\}$. It is clear that there exists only one choice of disjoint paths from $P_t, ..., P_r$ to points of $Q_t, ..., Q_r$. Hence the multiplicity of $R_\alpha(X)$ is the number of the disjoint paths from P_i to one of the points of $Q_i^* = \{(i, i), ..., (m, m)\}$, $i = 1, ..., t-1$. Thus we obtain

$$\dim R_t(X^*) = (n+1-t/2)(t-1)$$

$$e(R_t(X^*)) = \sum_{1 \leq j_1 < \cdots < j_{t-1} \leq m} \det \left[\binom{n-h}{n-j_k} \right]_{1 \leq h, k \leq t-1}.$$

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Printed in Belgium
Uitgever: Academic Press, Inc.
Verantwoordelijke uitgever voor België:
Hubert Van Maele
Altenastraat 20, B-8310 Sint-Kruis