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ON THE HILBERT FUNCTION OF DETERMINANTAL RINGS AND THEIR CANONICAL MODULE

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ABSTRACT. We determine the Hilbert function of a determinantal ring and of its canonical module using a combinatorial result of Krattenthaler. This gives a new proof of Abhyankar's formula.

Let K be a field and $X = (X_{ij})$ be an $m \times n$-matrix of indeterminates, with $m \leq n$. We denote by $K[X]$ the polynomial ring over K in the indeterminates X_{ij} and by $I_{r+1}(X)$ the ideal in $K[X]$ generated by the $r+1$-minors of X and set

$$R_{r+1} = K[X]/I_{r+1}(X).$$

The purpose of this note is to derive a compact formula for the Hilbert series of R_{r+1} and its canonical module. The result is actually a simple rewriting of Abhyankar's formula [1]. However, we want to point out that our formula can as well be obtained from a combinatorial result of Krattenthaler [7] and thus gives a new proof of Abhyankar's formula. Of course, the burden of the proof is hidden in the combinatorial part.

Krattenthaler counts the nonintersecting paths with a given number of corners. To be precise consider the set of points $V = \{(i, j): 1 \leq i \leq m, 1 \leq j \leq n\}$. We define a partial order on V by setting $(i, j) \leq (i', j')$ if $i \geq i'$ and $j \leq j'$. Let $P, Q \in V$; a maximal chain C in V with end points P and Q will be called a *path* from P to Q. A *corner* of C is an element $(i, j) \in C$ for which $(i-1, j)$ and $(i, j-1)$ belong to C as well. The path in Figure 1 on the next page has two corners.

Let P_i, Q_i, $i = 1, \ldots, r$, be points of V. A subset $W \subset V$ is called an *r-tuple of nonintersecting paths from* P_i *to* Q_i $(i = 1, \ldots, r)$ if $W = C_1 \cup C_2 \cup \cdots \cup C_r$ where each C_i is a path from P_i to Q_i and where $C_i \cap C_j = \varnothing$ if $i \neq j$. The number of corners $c(W)$ of W is the sum of the number of corners of the C_i, $i = 1, \ldots, r$.

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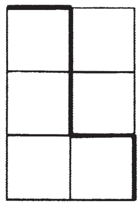

FIGURE 1

Theorem (Kulkarni, Krattenthaler). *Let* $P_i = (a_i, n)$, $1 \le a_1 < \cdots < a_r \le m$, *and* $Q_i = (m, b_i)$, $1 \le b_1 < \cdots < b_r \le n$. *Then the number of nonintersecting paths from* P_i *to* Q_i, $i = 1, \dots, r$, *with exactly* k *corners is given by*

$$\sum \det\left(\binom{m - a_j - i + j}{k_i}\binom{n - b_i + i - j}{k_i + i - j}\right)_{i,j=1,\dots,r},$$

where the sum is taken over all sequences $(k_1, \dots, k_r)$ *such that* $\sum_{i=1}^r k_i = k$.

Kulkarni [8] deduces this theorem from Abhyankar's formula for the Hilbert function of a determinantal ring, while Krattenthaler [7] gives a purely combinatorial proof of it. Actually his result is more general, that is, he allows the end points to be in a more general position.

Let us indicate, as described in [5] and [4], or in Kulkarni's paper [8], how the Hilbert series of determinantal rings is related to the nonintersecting paths. Given integers $1 \le a_1 < \cdots < a_r \le m$ and $1 \le b_1 < \cdots < b_r \le n$, we denote by $[a_1, \dots, a_r | b_1, \dots, b_r]$ the minor with the rows $a_1, \dots, a_r$ and the columns $b_1, \dots, b_r$. The set of all minors $P(X)$ is a poset with the following partial order:

$$[a_1, \dots, a_r | b_1, \dots, b_r] \le [c_1, \dots, c_s | d_1, \dots, d_s]$$

if $r \ge s$ and $a_1 \le c_1, \dots, a_s \le c_s$, $b_1 \le d_1, \dots, b_s \le d_s$. Let $\sigma \in P(X)$; then we denote by $I_\sigma(X)$ the ideal generated by the minors in the set $\{\eta \in P(X) : \eta \not\ge \sigma\}$. In case $\sigma = [1, \dots, r | 1, \dots, r]$ one obtains $I_\sigma(X) = I_{r+1}(X)$.

It is shown in [5] that for a suitable term order (order of the monomials) the ideal of initial forms $I_\sigma(X)^*$ of $I_\sigma(X)$ is generated by squarefree monomials. Thus $K[X]/I_\sigma(X)^*$ may be viewed as a Stanley-Reisner ring of a certain simplicial complex Δ_σ. Further it is shown in [5] that Δ_σ is shellable and that its facets may be identified with the nonintersecting paths connecting $P_i = (a_i, n)$ with $Q_i = (m, b_i)$ for $i = 1, \dots, r$. Using the fact that $R_\sigma = K[X]/I_\sigma(X)$ and $K[X]/I_\sigma(X)^*$ have the same Hilbert series and that Δ_σ is shellable, one deduces as in [4] from the McMullen-Walkup formula that the Hilbert series $H_{R_\sigma}(t)$ of R_σ is of the form

$$H_{R_\sigma}(t) = \frac{\sum_k h_k t^k}{(1-t)^d}, \tag{1}$$

where $d = \dim R_\sigma(X)$ and where h_k is the number of the nonintersecting paths from P_i to Q_i $(i = 1, \dots, r)$ with exactly k corners. Thus applying the theorem we get

$$H_{R_\sigma}(t) = \frac{\det(\sum_k \binom{m-a_j-i+j}{k}\binom{n-b_i+i-j}{k+i-j} t^k)_{i,j=1,\dots,r}}{(1-t)^d}, \tag{2}$$

where $d = (m+n+1)r - \sum_{i=1}^r (a_i + b_i)$ is the dimension of R_σ.

From now on we concentrate our attention to the Hilbert function of the ring R_{r+1} which is defined by the ideal of minors $I_{r+1}(X)$ of size $r+1$. In this case we have to consider the nonintersecting paths from $P_i = (i, n)$ to $Q_i = (m, i)$. Let us denote by $\mathscr{C}(P_i, Q_j)_k$ the number of paths from P_i to Q_j with exactly k corners and by $\mathscr{C}(P_i, Q_j)$ the polynomial $\sum_k \mathscr{C}(P_i, Q_j)_k t^k$. Then we have

Corollary 1.

$$H_{R_{r+1}}(t) = \frac{\det(\mathscr{C}(P_i, Q_j))_{i,j=1,\dots,r}}{t^{\binom{r}{2}}(1-t)^d} = \frac{\det(\sum_k \binom{m-i}{k}\binom{n-j}{k})_{i,j=1,\dots,r}}{t^{\binom{r}{2}}(1-t)^d}.$$

Proof. For $P = (a, b)$ we set $|P| = a^2 + b^2$. One proves easily by induction on $|P_i - Q_j|$ that $\mathscr{C}(P_i, Q_j)_k = \binom{m-i}{k}\binom{n-j}{k}$. Thus it remains to show that

$$\det\left(\sum_k \binom{m-i}{k}\binom{n-j}{k+i-j} t^k\right)_{i,j=1,\dots,r} = \frac{1}{t^{\binom{r}{2}}} \det\left(\sum_k \binom{m-i}{k}\binom{n-j}{k} t^k\right)_{i,j=1,\dots,r}.$$

This identity is a special case $(u = 0)$ of the next

Lemma. *Let $u \geq 0$ an integer, and consider the following $r \times r$ matrices of polynomials*

$$H_u = \left(\sum_k \binom{m-i}{k}\binom{n-j}{k+u+i-j} t^k\right),$$

$$H'_u = \left(\frac{1}{t^{i-1}} \sum_k \binom{m-i}{k}\binom{n-j}{k+u} t^k\right),$$

$$A = \left((-1)^{i-j}\binom{i-1}{j-1}\frac{1}{t^{i-j}}\right), \quad \text{and} \quad B = \left((-1)^{j-i}\binom{j-1}{i-1}\right).$$

Then $H'_u = AH_uB$. In particular, since A and B are triangular matrices whose diagonal elements are all 1, it follows that $\det H'_u = \det H_u$.

The proof follows by straightforward calculation using the identity

$$\sum_{q \geq 1} (-1)^{p-q}\binom{p-1}{q-1}\binom{a-q}{b-q} = (-1)^{p-1}\binom{a-p}{b-1}.$$

In order to compute the Hilbert series of the graded canonical module ω_{r+1} of R_{r+1} we use the equation

$$H_{\omega_{r+1}}(t) = (-1)^d H_{R_{r+1}}(t^{-1}), \tag{3}$$

where $d = (n+m-r)r$ is the dimension of R_{r+1}, and obtain

Corollary 2.

$$H_{\omega_{r+1}}(t) = \frac{t^{nr}\det(\sum_k \binom{m-i}{k}\binom{n-j}{n-m+k+i-j}t^k)}{(1-t)^d} = \frac{t^{nr}\det(\sum_k \binom{m-i}{k}\binom{n-j}{n-m+k}t^k)}{t^{\binom{r}{2}}(1-t)^d}.$$

Proof. The first equation follows directly from the substitution of t by t^{-1}, while for the second equation we use our lemma with $u = n - m$.

Recall that, by a result of Stanley, a Cohen-Macaulay domain homogeneous K-algebra R is Gorenstein if and only if

$$H_R(t) = (-1)^d t^a H_R(t^{-1}) \tag{4}$$

for some a. If this is the case, then a is the degree of the rational function $H_R(t)$, the so-called a-invariant of R.

Comparing the formulas in Corollaries 1 and 2 we deduce the well-known fact that R_{r+1} is Gorenstein if and only if $m = n$.

As a last application we compute the Cohen-Macaulay type $r(R_{r+1})$ of R_{r+1}, that is, the minimal number of generators of ω_{r+1}. For this we use the fact, proved by Bruns [2] (see also [3]), that R_{r+1} is a level ring, which means that all generators of ω_{r+1} have the same degree. Therefore, by (3) and Corollary 1, the type of R_{r+1} is the leading coefficient of the polynomial $\det(\sum_k \binom{m-i}{k}\binom{n-j}{k}t^k)$. The (i, j)th polynomial in the matrix has degree $m-i$ for $j \leq n-m+i$ and degree $n-j$ for $j > n-m+i$. Hence we see that

$$r(R_{r+1}) = \det\left(\binom{n-j}{m-i}\right)_{i,j=1,\dots,r}.$$

Using the Vandermonde determinant we get

$$r(R_{r+1}) = \prod_{i=1}^{r} \binom{n-i}{m-r} \Big/ \binom{m-i}{m-r}.$$

In [3] Bruns and Vetter quote the formula $r(R_{r+1}) = \prod_{i=1}^{m-r}\binom{n-i}{r}/\binom{m-i}{r}$. This formula was obtained by J. Brennan from the explicit computation of the Hilbert series of Schubert varieties due to Hodge and Pedoe [6, Theorem 3, p. 387]. It can be shown directly, by induction on r, that these formulas agree. In particular, one sees that

$$r(R_{r+1}) = r(R_{m-r+1}).$$

We conclude with one observation. The formula in Corollary 1 has the following combinatorial interpretation. Let σ be an element of S_r, the group of permutations of r elements. Let us denote by $\mathscr{C}(P_\sigma, Q)_k$ the number of the families of paths from $P_{\sigma(i)} = (\sigma(i), n)$ to $Q_i = (m, i)$, $i = 1, \dots, r$, with k corners, and by $\mathscr{C}(P, Q)_k^+$ the number of the nonintersecting paths from (i, n) to (m, i), $i = 1, \dots, r$, with k corners. Expanding the determinant

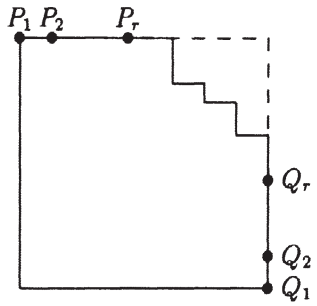

FIGURE 2

in the formula in Corollary 1, one gets

$$\mathscr{C}(P, Q)^{+}_{k-\binom{r}{2}} = \sum_{\sigma \in S_r} (-1)^{\sigma} \mathscr{C}(P_\sigma, Q)_k$$

for all $k \geq 0$. It would be nice to have a direct proof of this identity. We believe that it is true even when we consider paths in a restricted region having the shape of a one-sided ladder, as indicated in Figure 2. We can give a proof for the case when $r = 2$.

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