JOURNAL OF
PURE AND
APPLIED ALGEBRA

ELSEVIER Journal of Pure and Applied Algebra 98 (1995) 119–134

# Ladder determinantal rings

Aldo Conca [1]

*FB 6 Mathematik, Universität-Gesamthochschule Essen, Universitätstr. 3, 45117 Essen, Germany*

Communicated by T. Hibi; received 17 July 1993

**Abstract**

In this paper we show that ladder determinantal rings are normal. In the case of a ladder determinantal ring associated with a one-sided ladder, we compute the divisor class group, the canonical class, and we obtain a characterization of the Gorensteinness in terms of the shape of the ladder.

## 1. Introduction

Let $K$ be a field, and $X = (X_{ij})$ an $m \times n$ matrix of indeterminates. A subset $Y$ of $X$ with shape as in Fig. 1 is called a *ladder*. One defines $I_t(Y)$ to be the ideal generated by all the $t$-minors of $X$ which involve only indeterminates of $Y$. The ring $R_t(Y) = K[Y]/I_t(Y)$ is called a *ladder determinantal ring*. Narasimhan proved that $R_t(Y)$ is a domain using the fact that the set of the $t$-minors of $X$ is a Gröbner basis of the ideal $I_t(X)$, [11]. In [8], Herzog and Trung proved the ideal of the leading terms of $I_t(Y)$, with respect to a suitable term order, is the square-free monomial ideal associated with a shellable simplicial complex. This result implies the Cohen–Macaulayness of $R_t(Y)$. As we shall see, Gröbner bases play a role also in the proof of normality of $R_t(Y)$.

In Section 1 we introduce definitions and notation. In Section 2 we deal with the ring $R_2(Y)$. We use localization tricks in order to show that $R_2(Y)$ is a normal domain and to compute its divisor class group and its canonical class. It should be noted that part of the results of this section are covered by results of Hibi [9] and Hashimoto et al. [6].

In Section 3 we show that ladder determinantal rings are normal domains. First we show that every ladder determinantal ring is intersection of ladder determinantal rings

[1] Supported by a grant of Consiglio Nazionale delle Ricerche, Italy. This paper is part of the author's Ph.D. thesis.


*SSDI* 0022-4049(94)00039-L

associated with ladders with only one inside corner, see Fig. 3. Then for these ladders we deduce the desired result from the normality of classical determinantal rings.

In the last section we restrict our attention to one-sided ladders, see Fig. 2. For these ladders we get nice localizations. It turns out that, after inversion of a certain $(t-1)$-minor $f$ of $Y$, the ring $R_t(Y)$ becomes regular. Therefore the divisor class group of $R_t(Y)$ is generated by the minimal prime ideals of $f$. Our approach to compute the minimal prime ideals of $f$ is based again on the knowledge of the Gröbner bases of certain ideals of minors. In order to compute the divisor class group and the canonical class of $R_t(Y)$ we argue as in the case of the classical determinantal rings [2, Section 8]. We use suitable localizations which allow the reduction to the case of the 2-minors. We get the following result.

**Theorem.** *Let $Y$ be a one-sided ladder with $k$ inside corners and $t \in \mathbb{N}$, with $t > 1$. Assume that all the indeterminates of $Y$ are involved in some $t$-minors of $Y$ and that the size of the smallest matrix which contains $Y$ is $m \times n$. Then*

(a) *The divisor class group of $R_t(Y)$ is free of rank $k+1$.*

(b) *$R_t(Y)$ is Gorenstein if and only if $m = n$ and all the inside corners of $Y$ lie on the line of equation $x + y = m + t - 1$.*

## 1. Notations

Let $K$ be a field, and let $X = (X_{ij})$ be an $m \times n$ matrix of indeterminates over $K$. Denote by $K[X]$ the polynomial ring $K[X_{ij}\colon 1 \le i \le m,\ 1 \le j \le n]$ and denote by $[a_1, \ldots, a_t \mid b_1, \ldots, b_t]$ the $t$-minor $\det(X_{a_i b_j})$ of $X$. Here we always assume that $1 \le a_1 < \cdots < a_t \le m$ and $1 \le b_1 < \cdots < b_t \le n$. We define the main diagonal of $[a_1, \ldots, a_t \mid b_1, \ldots, b_t]$ to be the set $\{X_{a_1 b_2}, \ldots, X_{a_t b_t}\}$. The main diagonal of a $t$-minor of $X$ or the product of the elements of the main diagonal of a $t$-minor is called a $t$-diagonal of $X$.

In accordance with [11], one calls as subset $Y$ of $X$ a *ladder* (saturated subset in [1]) if whenever $X_{ij}, X_{hk} \in Y$ and $i \le h$, $j \le k$, then $X_{ik}, X_{hj} \in Y$. A ladder $Y$ has the following property: if the main diagonal of a minor is a subset of $Y$, then all the indeterminates of the minor are in $Y$.

Let $Y$ be a ladder, we denote by $K[Y]$ the polynomial ring $K[X_{ij}\colon X_{ij} \in Y]$ and by $R_t(Y)$ the quotient ring $K[Y]/I_t(Y)$, where $I_t(Y)$ is the ideal generated by all the $t$-minors of $X$ which involve only indeterminates of $Y$. The ideal $I_t(Y)$ is called a *ladder determinantal ideal* and the ring $R_t(Y)$ a *ladder determinantal ring.* Narasimhan [11, Theorem 4.1] proved that the ladder determinantal rings are domains, and Herzog and Trung [8, Corollary 4.10] proved that they are Cohen–Macaulay rings. Gröbner bases plays a fundamental role in the proofs of the domain property and Cohen–Macaulayness of the ladder determinantal rings. Let $\tau$ denote the lexicographic term order induced by the variable order $X_{11} > X_{12} > \cdots > X_{1n} > X_{21} > \cdots > X_{2n} > \cdots > X_{m-1n} > X_{m1} > \cdots > X_{mn}$.

Note that the leading term of a minor with respect to $\tau$ is the corresponding $t$-diagonal. The set of all the $t$-minors of $X$ is a Gröbner basis of $I_t(X)$ with respect to $\tau$, see [11, Corollary 3.1] or [13, Theorem 1]. Moreover the set of the $t$-minors of $Y$ is a Gröbner basis of $I_t(Y)$ with respect to $\tau$ [11, Corollary 3.4]. Herzog and Trung gave also a characterization of the dimension and multiplicity of $R_t(Y)$ in terms of the "shape" of $Y$ [8, Corollaries 4.7 and 4.8].

Note that in general $R_t(Y)$ is not an ASL (in a natural way as subring of $R_t(X)$), but $R_2(Y)$ is an ASL on the poset $Y$ [9].

We identify the indeterminates of $X$ with the points of the set $\{(i,j) \in \mathbb{N}^2: 1 \le i \le m, 1 \le j \le n\}$ in the plane. Similarly we identify ladders with subsets of points. If $X$ we introduce two partial orders $\le$ and $\preceq$. We define

$$(i,j) \le (h,k) \Leftrightarrow i \le h \text{ and } j \le k, \qquad (i,j) \preceq (h,k) \Leftrightarrow i \ge h \text{ and } j \le k.$$

It is clear that $X$ is a distributive lattice with respect to both partial orders. Further a subset $Y$ of $X$ is a ladder if and only if it is a sublattice of $X$ with respect to $\preceq$.

Since we are interested in the study of the ring $R_t(Y)$ we may assume that

(a) $\max_{\preceq} Y = (1,n)$ and $\min_{\preceq} Y = (m,1)$. Otherwise we replace $X$ by its smallest submatrix which contains $Y$.

(b) For all $a$, $1 \le a \le m$, there exists $b$, $1 \le b \le n$, such that $(a,b) \in Y$ and for all $b$, $1 \le b \le n$, there exists $a$, $1 \le a \le m$, such that $(a,b) \in Y$. Otherwise we delete from $X$ the row or the column which has an empty intersection with $Y$.

Of course these assumptions do not affect the ring $R_t(Y)$. We say that a ladder $Y$ is disconnected if there exist two ladders $\emptyset \ne Y_1, Y_2 \subset Y$ such that $Y_1 \cap Y_2 = \emptyset$, $Y_1 \cup Y_2 = Y$ and every minor of $Y$ is contained in $Y_1$ or in $Y_2$. If $Y$ is disconnected we get $I_t(Y) = I_t(Y_1) + I_t(Y_2)$ and then $R_t(Y) = R_t(Y_1) \otimes_K R_t(Y_2)$. In this case we can consider the ring $R_t(Y)$ as the ladder determinantal ring associated with the ladder $Y_1$ and with base ring $R_t(Y_2)$. We need the following lemma.

**Lemma 1.1.** *Let $Y$ be a ladder and $S = \mathbb{Z}[Y]/I_t(Y)$ the ladder determinantal ring defined over the based ring $\mathbb{Z}$. Then $S$ is a free $\mathbb{Z}$-module.*

**Proof.** First we show that $I_t(Y) = I_t(X) \cap \mathbb{Z}[Y]$. Let $g$ be an homogeneous polynomial in $I_t(X) \cap \mathbb{Z}[Y]$. We consider $g$ as a polynomial in $\mathbb{Q}[X]$ and Let $\mathrm{Lt}(g) = as$, with $a \in \mathbb{Z}\setminus\{0\}$ and $s = \prod_{ij} X_{ij}^{n_{ij}}$, be the leading term of $g$ with respect to $\tau$. Of course $s \in \mathbb{Z}[Y]$, and, by [11, Corollary 3.4], $s$ is divisible by the leading term of a $t$-minor $M$ of $X$, $s = \mathrm{Lt}(M)r$. But $\mathrm{Lt}(M) \in \mathbb{Z}[Y]$ implies $M \in \mathbb{Z}[Y]$. Therefore we get $g_1 = aMr - g \in I_t(X) \cap \mathbb{Z}[Y]$, and $\mathrm{Lt}(g_1) < \mathrm{Lt}(g)$ or $g_1 = 0$. Since $g_1$ and $g$ are homogeneous of the same degree, by induction, we may suppose that $g_1 \in I_t(Y)$ and we are done. From the previous assertion we deduce that $S$ is a subring of $\mathbb{Z}[X]/I_t(X)$. The ring $\mathbb{Z}[X]/I_t(X)$ is a free $\mathbb{Z}$-module since it is an ASL [2]. Therefore $S$ is also $\mathbb{Z}$-free. $\square$

The ring $\mathbb{Z}[Y]/I_t(Y)$ is a free $\mathbb{Z}$-module and hence faithfully flat. By generic flatness arguments, it is not a restriction, for our purpose, to consider only ladder determinantal rings associated with connected ladders, see for instance [2, Section 3].

It is easy to see that connected ladders with satisfy the assumptions (a) and (b) have the shape shown in Fig. 1.

We call the points $S_1', \dots, S_h'$ *inside lower corners*, $T_1', \dots, T_k'$ *inside upper corners*, $S_1, \dots, S_{h+1}$ *outside lower corners and* $T_1, \dots, T_{k+1}$ *outside upper corners of* $Y$. For the next applications we fix the coordinates of these points:

$$S_i' = (a_i, b_i) \quad \text{for } i = 1, \dots, h, \qquad T_i' = (c_i, d_i) \quad \text{for } i = 1, \dots, k,$$

$$S_i = (a_{i-1}, b_i) \quad \text{for } i = 1, \dots, h+1, \quad T_i' = (c_i, d_{i-1}) \quad \text{for } i = 1, \dots, k+1.$$

where $1 = a_0 < a_1 < \cdots < a_h < a_{h+1} = m$, $1 = c_0 < c_1 < \cdots < c_k < c_{k+1} = m$, $n = b_0 > b_1 > \cdots > b_h > b_{h+1} = 1$ and $n = d_0 > d_1 > \cdots > d_k > d_{k+1} = 1$. For

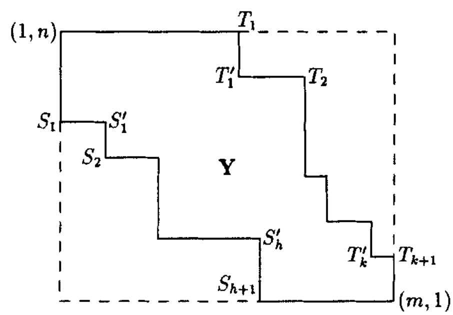


Fig. 1.

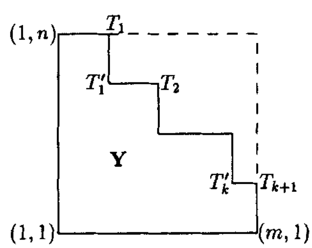


Fig. 2.

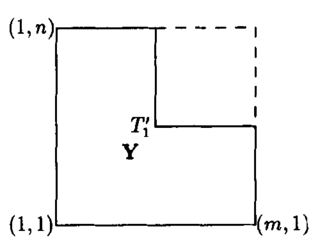


Fig. 3.

systematic reasons we let $S'_0 = T'_0 = (1, n)$, and $S'_{h+1} = T'_{k+1} = (m, 1)$. With these notations we have $Y = \{(a,b) \in X$: $a \le c_i$ or $b \le d_i$ for all $i = 1, \dots, k$, and $a \ge a_i$ or $b \ge b_i$ for all $i = 1, \dots, h\}$.

We say that a ladder is a *one-sided ladder* if it has no inside lower corners (or no inside upper corners), see Fig. 2. We say that a ladder is a *one-sided one-corner ladder* if it has no inside lower corners and one inside upper corner (or no inside upper corners and one inside lower corner), see Fig. 3.

## 2. The case of the 2-minors

In this section we study the ring $R_2(Y) = K[Y]/I_2(Y)$ where $K$ is a field and $Y$ a connected ladder. We show that $R_2(Y)$ is a normal domain and that its divisor class group is free of rank $c + 1$, where $c$ is the number of the inside corners of $Y$. Then we compute the canonical class, the class of the canoncial module in the divisor class group, and we give a necessary and sufficient condition for the Gorensteinness of $R_2(Y)$ in terms of the shape of $Y$. The results of Hibi [9] and Hashimoto et al. [6] concerning the ASL domain associated with a distributive lattice cover part of our results. However, we include the proofs because they are different from the ones in [6, 9], and because we need two steps of the proofs in later sections.

Throughout this section we fix a connected ladder $Y$ and we keep the notation of Fig. 1. Let us denote by $x_{ij}$ the residue class of $X_{ij}$ in $R_2(Y)$. First we define some ideals of $K[Y]$ which play a role in our investigation of $R_2(Y)$. For all $i = 1, \dots, h+1$, let

$$Q_i = (X_{a_{i-1},j}\text{: for all } j \text{ such that } X_{a_{i-1},j} \in Y) + I_2(Y),$$

$$Q'_i = (X_{j,b_i}\text{: for all } j \text{ such that } X_{j,b_i} \in Y) + I_2(Y)$$

and for all $i = 1, \dots, k$, let

$$P_i = (X_{pq}\text{: } X_{pq} Y, (p,q) \le T'_i) + I_2(Y).$$

Let us denote by $\mathfrak{q}_i, \mathfrak{q}'_i, \mathfrak{p}_i$, respectively, the residue classes of $Q_i, Q'_i, P_i$ in $R_2(Y)$. Let $Q$ be one of the ideals $Q_i, Q'_i$ or $P_i$. It is clear that $K[Y]/Q = R_2(Y_1)$, where $Y_1$ is the ladder obtained from $Y$ by deleting the $a_{i-1}$th row, the $b_i$th column or all the points $\le T'_i$. Using the dimension formula [8, Corollary 4.7] one obtains $\dim R_2(Y_1) = \dim R_2(Y) - 1$ in all the cases. Therefore we have shown that $\mathfrak{q}_i, \mathfrak{q}'_i$ and $\mathfrak{p}_i$ are prime ideals of height one in $R_2(Y)$.

For all $i = 1, \dots, h+1$, we denote by $I_i$ the set $\{j\text{: } 1 \le j \le k, S_i \le T'_j\}$. Of course $I_i$ can be empty, but if it is not empty is an interval, that is $I_i = \{j\text{: } j_i \le j \le j'_i\}$. Note also that for all $j = 1, \dots, k$, there exists $i$, $1 \le i \le h+1$, such that $j \in I_i$.

**Proposition 2.1.** *For all $i = 1, \dots, h+1$, $(x_{S_i})$ is a radical ideal and its minimal prime ideals are $\mathfrak{q}_i, \mathfrak{q}'_i$ and $\mathfrak{p}_j$ with $j \in I_i$, that is,*

$$(x_{S_i}) = \mathfrak{q}_i \cap \mathfrak{q}'_i \cap \left( \bigcap_{j \in I_i} \mathfrak{p}_j \right).$$

**Proof.** In this proof we use several times the following fact. Let $S$ be a ring and $M$ an $S$-module; if $N_1, N_2, N_3$ are submodules of $M$ and $N_2 \subset N_1$, then $N_1 \cap (N_2 + N_3) = N_2 + (N_1 \cap N_3)$.

First we discuss the case $I_i = \emptyset$. We have to show that $Q_i \cap Q_i' = (X_{S_i}) + I_2(Y)$. We have

$$\begin{aligned}
&[I_2(Y) + (X_{a_{i-1},j} \in Y) \cap [I_2(Y) + (X_{j,b_i} \in Y)] \\
&\quad = I_2(Y) + [(X_{a_{i-1},j} \in Y) \cap [I_2(Y) + (X_{j,b_i} \in Y)]] \\
&\quad = I_2(Y) + (X_{S_i}) + [(X_{a_{i-1},j} \in Y) \cap [I_2(Y_1) + (X_{j,b_i} \in Y \setminus \{X_{S_i}\})]],
\end{aligned}$$

where $Y_1$ is the ladder obtained from $Y$ by deleting the $a_{i-1}$th row. Since $\{X_{a_{i-1},j} \in Y\} \cap Y_1 = \emptyset$, we may replace the intersection in the last expression by the product. But $(X_{a_{i-1},j} \in Y)[I_2(Y_1) + (X_{j,b_i} \in Y \setminus \{X_{S_i}\})] \subset I_2(Y) + (X_{S_i})$ and therefore we obtain the desired result.

If $I_i \neq \emptyset$, we define for $j_i \leq j \leq j_i'$ the ideals: $J_j = (X_{pq} \in Y\colon (p,q) \leq T_j'$ and $q < b_i)$, and $L_j = (X_{a_{i-1},d_j+1}, X_{a_{i-1},d_j+2}, \ldots, X_{a_{i-1},d_{j+1}})$. By induction, using the same method as before in order to replace intersections by products, we get for all $j = j_i, \ldots, j_i'$

$$Q_i \cap P_{j_i} \cap \cdots \cap P_j = (X_{a_{i-1},b_i}, X_{a_{i-1},b_i+1}, \ldots, X_{a_{i-1},d_j}) + I_2(Y) + \sum_{z=j_i}^{j} J_z L_z.$$

Finally, when we intersect $Q_i \cap P_{j_i} \cap \cdots \cap P_{j_i'}$ with $Q_i'$, the desired conclusion follows if we note that $J_z L_z Q_i' \subset I_2(Y) + (X_{S_i})$. $\square$

In order to show that the ring $R_2(Y)$ is normal and to get information about its divisor class group we study the localization of $R_2(Y)$ with respect to the set of the powers of $x_{S_i}$. We show that we may interpret $R_2(Y)[x_{S_1}^{-1}]$ as a polynomial extension of a ladder determinantal ring.

**Proposition 2.2.** *Let $B$ be the set of points of the lower border of $Y$, $B_1 = \{P \in Y : S_1 \preceq P$ or $S_1' \prec P \preceq S_1\}$ and $Y_1 = \{P \in Y\colon P \preceq S_1'\}$ (note that $Y_1$ is a subladder of $Y$). We set $x = x_{S_1}$ and $y = \prod_{i=1}^{h+1} x_{S_i}$. Then we have*

(a) $R_2(Y)[x^{-1}] \simeq R_2(Y_1)[B_1][X_{S_1}^{-1}]$ *and*

(b) $R_2(Y)[y^{-1}] \simeq K[B][X_{S_1}^{-1}, \ldots, X_{S_{h+1}}^{-1}]$,

*where $R_2(Y_1)[B_1]$ is the polynomial extension of $R_2(Y_1)$ with the set of indeterminates $B_1$ and $K[B]$ is the polynomial ring over the set of indeterminates $B$.*

**Proof.** Let $K[Z]$ be the $K$-subalgebra of $R_2(Y)$ generated by the residue classes of the indeterminates in the set $Z = B_1 \cup Y_1$. Let $S \in Y \setminus Z$, it is clear that there exists a 2-minor of $Y$ of the form $X_{S_1}X_S - X_E X_F$ with $E, F \in Z$. Therefore, in $R_2(Y)[x^{-1}]$, we may write $x_S = x_E x_F x^{-1}$, and from this we deduce $R_2(Y)[x^{-1}] = K[Z][x^{-1}]$. Since the affine $K$-algebra $R_2(Y)$ is a domain, we have $\dim R_2(Y)[x^{-1}] = \dim R_2(Y)$.

By the dimension formula [8, Corollary 4.7], we know that $\dim R_2(Y) = \dim R_2(Y_1) + |B_1|$. We conclude that $B_1$ is a set of algebraically independent elements and that $K[S][x^{-1}] = R_2(Y_1)[B_1][X_{S_1}^{-1}]$.

The localization of $R_2(Y)$ with respect to the powers of $y$ is the localization of $R_2(Y)[x^{-1}]$ with respect to the powers of $\prod_{i=2}^{h+1} x_{S_i}$. Therefore, by induction, we get $R_2(Y)[y^{-1}] = K[B][X_{S_1}^{-1}, \dots, X_{S_{h+1}}^{-1}]$. □

Of course the results of the previous proposition hold also if we localize with respect to the powers of the residue classes of indeterminates in the upper outside corners, we have only to choose the right $B$, $B_1$ and $Y_1$.

We are ready to show that $R_2(Y)$ is normal and to compute its divisor class group. For the theory of divisorial ideals and divisor class group we refer the reader to [4]. We denote by $\mathrm{Cl}(S)$ the divisor class group of a normal domain $S$ and by $\mathrm{cl}(I)$ the class in $\mathrm{Cl}(S)$ of a divisorial ideal $I$ of $S$.

**Corollary 2.3.** (a) *The ring $R_2(Y)$ is a normal domain.*

(b) *The group $\mathrm{Cl}(R_2(Y))$ is free of rank $h + k + 1$.*

(c) *The elements $\mathrm{cl}(\mathfrak{q}_1), \dots, \mathrm{cl}(\mathfrak{q}_{h+1}), \mathrm{cl}(\mathfrak{p}_1), \dots, \mathrm{cl}(\mathfrak{p}_k)$ form a basis of $\mathrm{Cl}(R_2(Y))$.*

**Proof.** (a) We use Serre's normality criterion [4, Theorem 4.1]. From Proposition 2.2, we obtain by induction on $h$, that $R_2(Y)[x^{-1}]$ is a normal domain, where $x = x_{S_1}$. But $(x)$ is a radical ideal (see Proposition 2.1), and therefore $R_2(Y)_{\mathfrak{p}}$ is a regular ring for all the height one prime ideals $\mathfrak{p}$ of $R_2(Y)$. Since $R_2(Y)$ is a Cohen–Macaulay ring, we conclude that $R_2(Y)$ is a normal domain. (b), (c) from Nagata's theorem [4, Corollary 7.2] and Proposition 2.2, we deduce that $\mathrm{Cl}(R_2(Y))$ is generated by the classes of the minimal prime ideals of $y$. By Proposition 2.1, we know that $\min(y) = \{\mathfrak{q}_1, \dots, \mathfrak{q}_{h+1}, \mathfrak{q}'_1, \dots, \mathfrak{q}'_{h+1}, \mathfrak{p}_1, \dots, \mathfrak{p}_k\}$, and that for all $i = 1, \dots, h+1$

$$(i) \qquad \mathrm{cl}(\mathfrak{q}_i) + \mathrm{cl}(\mathfrak{q}'_i) + \sum_{j \in I_i} \mathrm{cl}(\mathfrak{p}_j) = 0.$$

We claim that all the relations between the classes of the minimal prime ideals of $y$ are linear combinations of the relations $(i)$. Suppose that

$$\sum_i u_i \,\mathrm{cl}(\mathfrak{q}_i) + \sum_i v_i \,\mathrm{cl}(\mathfrak{q}'_i) + \sum_i z_i \,\mathrm{cl}(\mathfrak{p}_i) = 0. \qquad (*)$$

Then $\sum_i u_i \,\mathrm{div}(\mathfrak{q}_i) + \sum_i v_i \,\mathrm{div}(\mathfrak{q}'_i) + \sum_i z_i \,\mathrm{div}(\mathfrak{p}_i)$ is a principal divisor, say $\mathrm{div}(g)$. The divisors $\mathrm{div}(\mathfrak{q}_i)$, $\mathrm{div}(\mathfrak{q}'_i)$, $\mathrm{div}(\mathfrak{p}_i)$ are in the kernel of the homomorphism $\mathrm{Div}(R_2(Y)) \to \mathrm{Div}(R_2(Y)[y^{-1}])$, therefore $g$ is a unit of $R_2(Y)[y^{-1}]$. By Proposition 2.2, we know that a unit of $R_2(Y)[y^{-1}]$ is of the form $g = \lambda x_{S_1}^{n_1} \cdots x_{S_{h+1}}^{n_{h+1}}$, with $\lambda \in K \setminus \{0\}$ and $n_1, \dots, n_{h+1} \in \mathbb{Z}$. Since the divisors of prime ideals of height 1 are linearly independent in $\mathrm{Div}(R_2(Y))$, the relation $(*)$ is the sum of the relations of type $(i)$ with coefficients $n_i$. Then, using the relation $(i)$, we cancel $\mathrm{cl}(\mathfrak{q}'_i)$ and we are done. □

A canonical module of a Cohen–Macaulay ring $S$ is a finitely generated module $S$-module $\omega$ such that $\omega_{\mathfrak{m}}$ is the canonical module of the local ring $S_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of $S$. Concerning general facts about the canonical module we refer the reader to [7]. If $S$ is a normal Cohen–Macaulay domain and $\omega$ a canonical module, then $\omega$ is isomorphic to a divisorial ideal of $S$. A canonical class of $S$ is the class in the divisor class group of a divisorial ideal which is a canonical module. Suppose that $S$ is a positively graded $K$-algebra, $S = \bigoplus_{i \geq 0} S_i$, with $S_0 = K$ and denote by $\mathfrak{m} = \bigoplus_{i \geq 1} S_i$ its unique maximal homogeneous ideal. Then, by [4, Corollary 10.3], $\mathrm{Cl}(S) = \mathrm{Cl}(S_{\mathfrak{m}})$. It follows that the canonical module of $S$ is unique up to isomorphisms and it is determinated by the canonical class.

**Proposition 2.4.** *Let* $\mathrm{cl}(\omega)$ *be the canonical class of* $R_2(Y)$ *and let* $\sum_{i=1}^{h+1} \lambda_i \mathrm{cl}(\mathfrak{q}_i) + \sum_{j=1}^{k} \delta_j \mathrm{cl}(\mathfrak{p}_j)$ *be the representation of* $\mathrm{cl}(\omega)$ *with respect to the basis of* $\mathrm{Cl}(R_2(Y))$ *given in Corollary* 2.3. *Then*

$$\lambda_i = a_i + b_i - a_{i-1} - b_{i-1} \quad \text{for all } i = 1, \ldots, h+1$$

$$\delta_j = a_{i_j} + b_{i_j} - c_j - d_j \quad \text{for all } j = 1, \ldots, k,$$

*where* $i_j = \min\{i\colon a_i > c_j\}$.

**Proof.** We begin with an observation. Let $Y_1, B_1$ and $x$ as in Proposition 2.2. The isomorphism 2.2(a) induces an isomorphism $\mathrm{Cl}(R_2(Y)[x^{-1}]) \simeq \mathrm{Cl}(R_2(Y_1)[B_1][X_{S_1}^{-1}])$. Since $X_{S_1}$ is prime in $R_2(Y_1)[B_1]$, and a polynomial extension does not affect the divisor class group, one has $\mathrm{Cl}(R_2(Y_1)[B_1]\,[X_{S_1}^{-1}]) \simeq \mathrm{Cl}(R_2(Y_1))$. The ring $R_2(Y_1)[B_1][X_{S_1}^{-1}]$ is free as $R_2(Y_1)$ module. It follows that in isomorphism $\mathrm{Cl}(R_2(Y_1)[B_1]\,[X_{S_1}^{-1}]) \simeq \mathrm{Cl}(R_2(Y_1))$, a canonical class is mapped to a canonical class. If we consider the composition of the canonical epimorphism $\mathrm{Cl}(R_2(Y)) \to \mathrm{Cl}(R_2(Y)[x^{-1}])$ with the previous isomorphisms we get an epimorphism $\mathrm{Cl}(R_2(Y)) \to \mathrm{Cl}(R_2(Y_1))$ which maps the canonical class to the canonical class.

This observation allows inductive arguments, one only needs to control the images under the map $\mathrm{Cl}(R_2(Y)) \to \mathrm{Cl}(R_2(Y_1))$ of the elements of the given basis of $\mathrm{Cl}(R_2(Y))$.

The proof is by induction on $h + k$. If $h = k = 0$, then $Y$ is an $m \times n$ matrix, $\mathfrak{q}_1$ is the ideal generated by the first row of $Y$. In this case $a_1 + b_1 - a_0 - b_0 = m - n$ and, by [2, Theorem 8.8], $\lambda_1 = m - n$.

Now we suppose $h + k > 0$. First we discuss the case $h = 0$. We localize with respect to the powers of $x = x_{T_1}$. By Proposition 2.2 we have $R_2(Y)[x^{-1}] \simeq R_2(Y_1)[C_1][X_{T_1}^{-1}]$, where $Y_1 = \{X_P \in Y\colon P \preceq T_1'\}$ and $C_1 = \{X_P \in Y\colon T_1' \prec P \preceq T_1$ or $T_1 \preceq P\}$. The ideal $\mathfrak{q}_1 R_2(Y)[x^{-1}]$ is the principal ideal generated by $x_{1n}$. The ideal $\mathfrak{p}_1 R_2(Y)[x^{-1}]$ is mapped, by the previous isomorphism, to the extension of the ideal $\mathfrak{q}_1(Y_1)$ generated by the first row of $Y_1$ in $R_2(Y_1)$. The ideals $\mathfrak{p}_2 R_2(Y)[x^{-1}], \ldots, \mathfrak{p}_k R_2(Y)[x^{-1}]$ are mapped to the extensions of the ideals $\mathfrak{p}_1(Y_1), \ldots, \mathfrak{p}_{k-1}(Y_1)$ of $R_2(Y_1)$ which correspond to the upper inside corners of $Y_1$. Therefore we may describe exactly the epimorphism $\mathrm{Cl}(R_2(Y)) \to \mathrm{Cl}(R_2(Y_1))$ in terms of the given bases. Since the canonical class of $R_2(Y)$ is mapped to the canonical class $R_2(Y_1)$, be induction, we get $\delta_j = m + 1 - c_j - d_j$ for all $j = 1, \dots, k$. It remains to determine $\lambda_1$. We localize $R_2(Y)$ with respect to $x_{T_{k+1}}$ and obtain a surjection $\mathrm{Cl}(R_2(Y)) \to \mathrm{Cl}(R_2(Y_1))$, where now $Y_1 = \{X_P \in Y : T_{k+1} \preceq P\}$. The classes of $\mathfrak{p}_1, \dots, \mathfrak{p}_{k-1}$ are mapped by to the classes of the ideals $\mathfrak{p}_1(Y_1), \dots, \mathfrak{p}_{k-1}(Y_1)$ which correspond to the upper inside corners of $Y_1$. The class of $\mathfrak{q}_1$ is mapped to the class of $\mathfrak{q}_1(Y_1)$ the ideal generated by the first row of $Y_1$. The class of $\mathfrak{p}_k$ is mapped to the class of the ideal $\mathfrak{q}'_1(Y_1)$ generated by the first column of $Y_1$. Since in $\mathrm{Cl}(R_2(Y_1))$ we have $\mathrm{cl}(\mathfrak{q}_1(Y_1)) + \mathrm{cl}(\mathfrak{q}'_1(Y_1)) + \sum_{j=1}^{k-1} \mathrm{cl}(\mathfrak{p}_i(Y_1)) = 0$, by induction, we deduce that $\lambda_1 - \delta_k = c_k + d_k - 1 - n$. But we know already that $\delta_k = m + 1 - c_k - d_k$ and we conclude $\lambda_1 = m - n$.

Now suppose $h > 0$. Again we localize with respect to residue class of outside corner indeterminates and use induction. First we localize with respect to $x_{S_1}$ and obtain $\lambda_i = a_i + b_i - a_{i-1} - b_{i-1}$, for all $i = 2, \dots, h + 1$. Then we localize with respect to $x_{S_{h+1}}$ and obtain $\lambda_1 = a_1 + b_1 - a_0 - b_0$. Finally, if $k > 0$, we localize with respect to $x_{T_1}$ and obtain $\delta_j = a_{i_j} + b_{i_j} - c_j - d_j$, for all $j = 1, \dots, k$. □

**Proposition 2.5.** *The ring $R_2(Y)$ is Gorenstein if and only if $m = n$ and all the inside corners of $Y$ lie on the main anti-diagonal of $X$, that is $\{(i,j) \in X : i + j = m + 1\}$.*

**Proof.** The ring $R_2(Y)$ is Gorenstein if and only if $\mathrm{cl}(\omega) = 0$. Therefore the ring $R_2(Y)$ is Gorenstein if and only if $m = n$ and all the differences listed in Proposition 2.4 vanish. This is the case if and only if $m = n$ and all the inside corners of $Y$ lie on the main anti-diagonal of $X$. □

## 3. Normality

In this section we prove that ladder determinantal rings are normal domains. We use the knowledge of a Gröbner basis of the ideal $I_t(Y)$ to reduce the problem to the case of a ladder determinantal ring associated with a one-sided ladder with only one inside corner.

We fix a ladder $Y$ and we keep the notation of Fig. 1. By [11, Corollary 3.4] the set of the $t$-minors of $Y$ is a Gröbner basis of $I_t(Y)$ with respect to $\tau$. We denote by $B_t(Y)$ the set of all the monomials of $R_t(Y)$ which are not divisible by the leading term of a $t$-minor of $Y$. It is well-known that $B_t(Y)$ is a basis of $R_t(Y)$ as $K$-vector space. Since $I_t(Y) = I_t(X) \cap K[Y]$, see [11, Theorem 4.1], we know that $R_t(Y)$ is a subring of $R_t(X)$ and $B_t(Y) \subset B_t(X)$.

**Lemma 3.1.** *Let $Y_1, \dots, Y_c$ be ladders contained in $X$ and $Y = \bigcap_{i=1}^c Y_i$. Then $R_t(Y) = \bigcap_{i=1}^c R_t(Y_i)$.*

**Proof.** Since $B_t(Y_1), \dots, B_t(Y_c)$ are subsets of $B_t(X)$ the $K$-basis of $R_t(X)$, then a $K$-basis of $\bigcap_{i=1}^c R_t(Y_i)$ is $\bigcap_{i=1}^c B_t(Y_i)$. But it is easy to see that $B_t(Y) = \bigcap_{i=1}^c B_t(Y_i)$ and therefore $R_t(Y) = \bigcap_{i=1}^c R_t(Y_i)$. □

**Lemma 3.2.** *The ladder $Y$ is the intersection of one-sided one-corner ladders.*

**Proof.** For all $i = 1, \ldots, k$, let $Y_i = \{(a, b) \in X: a \le c_i \text{ or } b \le d_i\}$ and for all $i = 1, \ldots, h$, let $Z_i = = \{(a, b) \in X: a \ge a_i \text{ or } b \ge b_i\}$. It is clear that $Y = Y_1 \cap \cdots \cap Y_k \cap Z_1 \cap \cdots \cap Z_h$. □

**Proposition 3.3.** *The ring $R_t(Y)$ is a normal domain.*

**Proof.** By Lemmas 3.1 and 3.2 we may write $R_t(Y) = \bigcap_{i=1}^{c} R_t(Y_i)$, where the $Y_i$ are one-sided one-corner ladders. The intersection of normal domains is a normal domain. Therefore we may assume that $Y$ is a one-sided one-corner ladder. Let $(c, d)$ be the unique inside corner of $Y$, of course we may suppose that it is an inside upper corner. If $c < t$ (or $d < t$), then $R_t(Y)$ is the polynomial extension of the determinantal ring $R_t(X_1)$, where $X_1$ is the matrix formed by the first $d$ columns (or by the first $c$ rows) of $X$. Therefore we conclude that $R_t(Y)$ is normal since $R_t(X_1)$ is, see [2, Theorem 6.3]. In the case $c \ge t$ and $d \ge t$, we define $\delta$ to be $[1, \ldots, t-1, c+1, \ldots, c+r \mid 1, \ldots, t-1, d+1, \ldots, d+r]$, with $r = \min\{m-c, n-d\}$. Then we consider $I_\delta(X)$ is the ideal generated by all the minors of $X$ which are not greater than or equal to $\delta$, see [2]. It is easy to see that the ideal $I_\delta(X)$ is generated by the $t$-minors of the first $c$ rows and by the $t$-minors of the first $d$ columns of $X$, in other words $I_\delta(X) = I_t(Y)K[X]$. By [2, Theorem 6.3] we know that $K[X]/I_\delta(X) = R_t(Y)[X \setminus Y]$ is a normal domain, and therefore $R_t(Y)$ is a normal domain. □

**Remark 3.4.** Let $Y$ be an one-sided one-corner ladder with the unique upper inside corner in $(c, d)$, and suppose $c \ge t$ and $d \ge t$. We have seen in the proof of the previous proposition that $R_t(Y)[X \setminus Y] = K[X]/I_\delta(X)$. From [2, Theorems 8.3 and 8.14] one has that $\mathrm{Cl}(R_t(Y)) = \mathbb{Z}^2$ and that $R_t(Y)$ is Gorenstein if and only if $m = n$ and $c + d = m + t - 1$. In the next section we generalize this results to one-sided ladder determinantal rings.

## 4. Divisor class group and Gorensteinness of one-sided ladder determinantal rings

In order to compute the divisor class group and to characterize the Gorensteinness we restrict our attention to the one-sided ladders since for these ladders suitable localizations allow the reduction to the case of the 2-minors. From now on let $Y$ be an one-sided ladder, and we keep the notation of Fig. 2. We want to study the ring $R_t(Y)$, and we always assume that $Y$ contains $t$-minors and $t > 1$. Moreover we may assume that $c_1 \ge t$ and $d_k \ge t$, since otherwise the indeterminates of the $n$th column or of the $m$th row of $Y$ are not involved in a $t$-minor of $Y$ and we may delete them. Under these hypotheses the dimensions of $R_t(Y)$ and $R_t(X)$ coincide, in particular $\dim R_t(Y) = \dim R_t(X) = (m + n - t + 1)(t - 1)$.

Let $X_{ij} \in Y$; we denote by $x_{ij}$ the residue class of $X_{ij}$ in $R_t(Y)$. Note that the minor $\delta = [1, \ldots, t-1 \mid 1, \ldots, t-1]$ and $X_{11}$ are in $Y$.

**Proposition 4.1.** *Let $f$ be the residue class of $\delta$ in $R_t(Y)$ and $x = x_{11}$. Let $Z$ be the ladder which is obtained from $Y$ by deleting the first row and the first column. Then we have*
(1) $R_t(Y)[f^{-1}] \simeq K[X_{ij} \in X: i \leq t-1 \text{ or } j \leq t-1][\delta^{-1}]$,
(2) $R_t(Y)[x^{-1}] \simeq R_{t-1}(Z)[X_{11}, X_{12}, \ldots, X_{1n}, X_{21}, \ldots, X_{m1}][X_{11}^{-1}]$.

**Proof.** Let $B = \{x_{ij}: i \leq t-1 \text{ or } j \leq t-1\}$ and denote by $K[B]$ the $K$-subalgebra of $R_t(Y)$ generated by the elements of $B$. Let $X_{pq} \in Y$ with $p, q \geq t$. Then we consider the minor $\gamma = [1, \ldots, t-1, p \,|\, 1, \ldots, t-1, q]$ which is in $Y$. In $R_t(Y)$ we have $0 = \gamma = x_{pq} f + g$, where $g \in K[B]$. Therefore we have shown that $x_{pq} \in K[B][f^{-1}]$, that is $R_t(Y)[f^{-1}] = K[B][f^{-1}]$. Since $R_t(Y)$ is an affine $K$-algebra domain we have $\dim R_t(Y)[f^{-1}] = \dim R_t(Y)$, and, by [8, Corollary 4.7], $\dim R_t(Y) = |B|$. We conclude that $B$ is a set of algebraically independent elements. Since $f$, as an element of $K[B]$, is the determinant of a matrix of indeterminates, the first isomorphism is proved.

We denote by $Z_{ij}$ the entries of the ladder $Z$ and consider the homomorphism of rings

$$\psi: K[Y][X_{11}^{-1}] \to K[Z][X_{11}, X_{12}, \ldots, X_{1n}, X_{21}, \ldots, X_{m1}][X_{11}^{-1}]$$

defined by the assignment $\psi(X_{ij}) = X_{ij}$ if $i = 1$ or $j = 1$, and $\psi(X_{ij}) = Z_{i-1\,j-1} + X_{i1} X_{1j} X_{11}^{-1}$ otherwise. It is clear that $\psi$ is an isomorphism. We denote by $[\ldots | \ldots]_Y$ and $[\ldots | \ldots]_Z$, respectively, the minors of $Y$ and $Z$. Let $M = [i_1, \ldots, i_s \,|\, j_1, \ldots, j_s]_Y$, then we have

$$\psi(M) = [i_1 - 1, \ldots, i_s - 1 \,|\, j_1 - 1, \ldots, j_s - 1]_Z + \sum_{p,q=1}^{s} (-1)^{p+q} X_{1 j_q} X_{i_p 1} X_{11}^{-1}$$
$$\times [i_1 - 1, \ldots, \widehat{i_p - 1}, \ldots, i_s - 1 \,|\, j_1 - 1, \ldots, \widehat{j_q - 1}, \ldots, j_s - 1]_Z,$$

where $\hat{v}$ means that the index $v$ is omitted and, by convention, a minor is 0 if one of its row or column index is 0. From the previous formula one deduces that $\psi(I_t(Y)) \subset I_{t-1}(Z)$. Since

$$\psi([1, i_2, \ldots, i_s \,|\, 1, j_2, \ldots, j_s]_Y) = X_{11} [i_2 - 1, \ldots, i_s - 1 \,|\, j_2 - 1, \ldots, j_s - 1]_Z$$

and $X_{11}$ is a unit, we get $\psi(I_t(Y)) = I_{t-1}(Z)$. Therefore $\psi$ induces the second isomorphism. □

We have seen that, after inversion of $f$, $R_t(Y)$ becomes a factorial ring. Hence, by Nagata's theorem [4, Corollary 7.2], the divisor class group of $R_t(Y)$ is generated by the classes of the minimal prime ideals of $f$. In the case of the classical determinantal rings it is possible to determine the minimal prime ideals of $f$ using the ASL structure, see [2, Corollary 6.5]. In general $R_t(Y)$ need not be an ASL, and therefore we have to argue in a different way. Note that if $t = 2$, then $f$ is the residue class of $X_{11}$ and we know already its minimal prime ideals, see Proposition 2.1. The natural extension to the general case is the following.

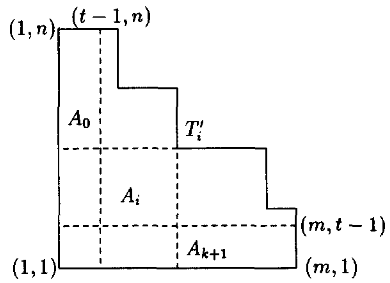


Fig. 4.

**Definition 4.2.** Let $A_0 = \{(p,q) \in Y: p \leq t-1\}$, $A_{k+1} = \{(p,q) \in Y: q \leq t-1\}$ and for $i = 1, \ldots, k$ let $A_i = \{(p,q) \in Y: (p,q) \leq T_i'\}$ (see Fig. 4).

For $i = 0, \ldots, k+1$ we define $P_i = I_{t-1}(A_i) + I_t(Y)$ and denote by $\mathfrak{p}_i$ the residue class of $P_i$ in $R_t(Y)$.

Note that, when $t = 2$, the ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ correspond to the ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ defined in the second section but the ideals $\mathfrak{p}_0, \mathfrak{p}_{k+1}$ correspond to $\mathfrak{q}_1$ and $\mathfrak{q}_1'$.

By construction, $f \in \mathfrak{p}_i$ for all $i = 0, \ldots, k+1$. The ideals $P_0$ and $P_{k+1}$ belong to the class of ideals cogenerated by one element in a ladder as defined in [8]. From [8, Theorem 4.1 and Corollary 4.7] we know that $\mathfrak{p}_0, \mathfrak{p}_{k+1}$ are prime ideals of height 1. In order to show that also $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ are prime ideals we need some preliminary results. For all $i = 1, \ldots, k$, let $J_i$ be the set of the $(t-1)$-minors of $A_i$ and $t$-minors of $Y$.

**Proposition 4.3.** *For all $i = 1, \ldots, k$, the set $J_i$ is a Gröbner basis of $P_i$ with respect to $\tau$.*

**Proof.** Let $Y_1 = \{(p,q) \in Y: p \leq c_i\}$, and $Y_2 = \{(p,q) \in Y: q \leq d_i\}$. The sets $Y_1, Y_2$ are ladders and $I_1 = I_{t-1}(A_i) + I_t(Y_1)$ and $I_2 = I_{t-1}(A_i) + I_t(Y_2)$ are ideals cogenerated by one element in the ladders $Y_1$ and $Y_2$, respectively. We denote by $C_j$ the set of the $(t-1)$-minors of $A_i$ and $t$-minors of $Y_j$, $j = 1, 2$. It is clear that $J_i = C_1 \cup C_2$. By [8, Theorem 4.2], we know that $C_j$ is a Gröbner basis of $I_j$ with respect to $\tau$, $j = 1, 2$. We prove that $J_i$ is Gröbner basis of $P_i$ by showing that in Buchberger's algorithm the $S$-resultant $S(v,s)$ of two elements $v, s \in J_i$ is always 0, see [12]. We distinguish 2 cases:

If $v$ and $s$ are $t$-minors, then $S(v,s) = 0$ since $J_i$ contains the set of the $t$-minors of $Y$, which is a Gröbner basis.

If $v$ (or $s$) is a $(t-1)$-minor, then it is clear that $v, s \in C_1$ or $v, s \in C_2$ and therefore $S(v,s) = 0$ since $C_1$ and $C_2$ are Gröbner bases. $\square$

The ideal $\mathrm{Lt}(P_i)$ of the leading terms of $P_i$ is generated by the $t$-diagonals of $Y$ and by the $(t-1)$-diagonals of $A_i$. Since $\mathrm{Lt}(P_i)$ is a square-free monomial ideal, the quotient ring $K[Y]/\mathrm{Lt}(P_i)$ is the Stanley-Reisner ring associated with the simplicial complex $\Delta_i = \{F \subset Y$: $F$ does not contain $t$-diagonals of $Y$ and $(t-1)$-diagonals of $A_i\}$.

**Proposition 4.4.** *For all $i = 1, \ldots, k$, the ring $R_t(Y)/\mathfrak{p}_i$ is Cohen–Macaulay of dimension* $\dim R_t(Y) - 1$.

**Proof.** It is known [5] that it is enough to show that $K[\Delta_i] = K[Y]/\mathrm{Lt}(P_i)$ is a Cohen–Macaulay ring of dimension $\dim R_t(Y) - 1$. Let $Y_1$ be the one-sided ladder which is obtained by adding the point $E = (c_i + 1, d_i + 1)$ to $Y$. The ideal $\mathrm{Lt}(I_t(Y_1))$ is generated by the $t$-diagonals of $Y_1$ and it is the ideal associated with the simplicial complex $\Delta = \{F \subset Y_1 : F$ does not contain $t$-diagonals of $Y_1\}$. It is clear that the link of $E$ in $\Delta$, $\mathrm{L}_\Delta\{E\} := \{F \subset Y_1 \colon F \cup \{E\} \in \Delta,\ F \cap \{E\} = \emptyset\}$, is $\Delta_i$. From [8, Sections 4.7 and 4.9] we know that $\Delta$ is a Cohen–Macaulay simplicial complex of dimension $\dim R_t(Y) - 1$. The link of a vertex of a Cohen–Macaulay simplicial complex of dimension $\dim R_t(Y) - 1$ is a Cohen–Macaulay simplicial complex of dimension $\dim R_t(Y) - 2$ [10, Proposition 5.6] and we are done. □

**Remark 4.5.** With respect to the notations of the previous proposition one has that the simplicial complex $\Delta_i$ is shellable since $\Delta$ is.

Now we are already to show that the ideal $\mathfrak{p}_i$ is prime for all $i = 1, \ldots, k$. First we do that for one-sided one-corner ladders and then we deduce the result for the one-sided ladders.

**Lemma 4.6.** *If $Y$ has only one inside corner, then $\mathfrak{p}_1$ is a prime ideal.*

**Proof.** Just to simplify the notation let $\mathfrak{p} = \mathfrak{p}_1$, $A = A_1$, $d = \dim R_t(Y)$, $S = R_t(Y)/\mathfrak{p}$, and let $x$ the residue class of $X_{11}$ in $S$. We argue by induction on $t$. The statement is clear if $t = 2$, see the discussion before Proposition 2.1. Let $t > 2$. Let $V = (v_1, v_2)$ be an element of $A$ and denote by $v$ the residue class of $X_V$ in $S$. Note that the ideals $I_t(Y)$ and $I_t(Y) + I_{t-1}(A)$ are invariant under permutations of the first $c_1$ rows and $d_1$ columns of $Y$, where $(c_1, d_1)$ is the inside corner of $Y$. Therefore the isomorphism 4.1(2) holds when we localize $R_t(Y)$ with respect to the powers of the residue class of $X_V$, too. Under the isomorphism 4.1(2) the ideal $\mathfrak{p}$ is mapped to extension of the corresponding ideal $\mathfrak{p}'$ of the ring $R_{t-1}(Z)$. Therefore $S[v^{-1}] \simeq R_{t-1}(Z)/\mathfrak{p}'[X_{ij} \in Y\colon i = v_1 \text{ or } j = v_2][X_V^{-1}]$. By induction, $S[v^{-1}]$ is a domain, and in particular $S[x^{-1}]$ is a domain. Hence it is enough to show that $x$ is a non-zero divisor of $S$. Suppose the contrary. Then there exists a prime ideal $Q$ of $S$ and $y \in S$, $y \neq 0$, with $x \in Q = 0 : y$. We take $v$ in the same row or column of $x$. Since $xy = 0$ in $S$ and $x$ is an indeterminate in $S[v^{-1}]$, which is a domain, we deduce that $y = 0$ in $S[v^{-1}]$. Therefore $v^s y = 0$ in $S$, and, since $Q$ is prime, $v \in Q$. Repeating the previous argument we get $I_1(A) \subset Q$. It is easy to see from the dimensional formula [7, Section 4.7] that $\dim S/I_1(A) \leq \dim R_t(Y) - (t-1)^2 = d - (t-1)^2$. We know from Proposition 4.4 that $S$ is a graded Cohen–Macaulay ring of dimension $d - 1$ and since $I_1(A)$ is a graded ideal we get: height $Q \geq$ height $I_1(A) = \dim S - \dim S/I_1(A) \geq d - 1 - d + (t-1)^2 = t(t-2) > 0$. This is a contradiction since $\operatorname{grade} Q = 0$ and $S$ is a Cohen–Macaulay ring. □

**Proposition 4.7.** *For all $i = 1, \dots, k$, the ideal $\mathfrak{p}_i$ is prime.*

**Proof.** Let $Y_1$ be the one-sided one corner ladder with inside corner in $(c_i, d_i)$. Note that $Y \subset Y_1$ and that $\{(p,q) \in Y_1 : (p,q) \le (c_i, d_i)\} = A_i$. Let us denote by $P$ the ideal $I_t(Y_1) + I_{t-1}(A_i)$. By Lemma 4.6 we know that $P$ is prime in $K[Y_1]$. Therefore it is enough to show that $P_i = P \cap K[Y]$. Since we know a Gröbner basis of $P$ we can argue as in Lemma 1.1. Let $g$ be an homogeneous element of $P \cap K[Y]$ and denote by $\mathrm{Lt}(g)$ it leading term with respect to $\tau$. By Proposition 4.3 $\mathrm{Lt}(g)$ is divisible by the leading term of $M$, where $M$ is a $t$-minor of $Y_1$ or a $(t-1)$-minor of $A_i$. We may write $g = vM + v_1$, where $v$ is a monomial of $K[Y]$, $v_1 \in P \cap K[Y]$ and $\mathrm{Lt}(v_1) < \mathrm{Lt}(g)$ or $v_1 = 0$. But $M \in P_i$ since $Y$ is a ladder. Since $g$ and $v_1$ are homogeneous of the same degree, by induction, we may assume $v_1 \in P_i$. Therefore $g \in P_i$. □

**Proposition 4.8.** *Let $x, f$ be the residue classes in $R_t(Y)$ of $X_{11}$ and $[1, \dots, t-1 \mid 1, \dots, t-1]$, respectively.*

(a) *The set of the minimal prime ideals of $f$ is $\{\mathfrak{p}_0, \dots, \mathfrak{p}_{k+1}\}$.*
(b) *The ideal $(f)$ is radical.*
(c) *The ideal $(x)$ is prime if $t > 2$.*

**Proof.** (a) We know already that the ideals $\mathfrak{p}_0, \dots, \mathfrak{p}_{k+1}$ are prime ideals of height 1 and $f \in \mathfrak{p}_i$. Of course all are distinct, so it is enough to show that $\mathfrak{p}_0 \cdots \mathfrak{p}_{k+1} \subset (f)$. In order to do that, we define $B_0 = A_0$, $B_i = \{(p,q) \in A_0 : q \le d_i\}$ for all $i = 1, \dots, k$, $B_{k+1} = \{(p,q) \in Y : p, q \le t-1\}$. We show, by induction on $j$, that $\mathfrak{p}_0 \cdots \mathfrak{p}_j \subset I_{t-1}(B_j)$.

The case $j = 0$ is trivial. Now let $j > 0$. We observe that the straightening relation [2] of the product of two minors $M, N$ of a matrix involves only minors of the smallest submatrix which contains $M$ and $N$. Therefore whenever the smallest submatrix which contains two minors of a ladder is contained in the ladder, then the straightening relation of the product of the minors is inside the ladder. We note that $A_j$ and $B_{j-1}$ are contained in the set $\{(p,q) \in Y : (p,q) \le T_j\}$ which is a rectangle. Then, by the previous observation, we may use straightening relations and we get $\mathfrak{p}_0 \cdots \mathfrak{p}_j \subset I_{t-1}(B_{j-1})\mathfrak{p}_j \subset I_{t-1}(B_j)$. Now, since the only $(t-1)$-minor of $B_{k+1}$ is $f$, we conclude that $\mathfrak{p}_0 \cdots \mathfrak{p}_{k+1} \subset (f)$.

(b) By induction on $t$. The case $t = 2$ is covered by Proposition 2.1. Let $t > 2$. The ring $R_t(Y)$ is a Cohen–Macaulay domain and $x \notin \mathfrak{p}_i$ for all $i = 0, \dots, k+1$. Therefore $f, x$ is a regular sequence in $R_t(Y)$. But $x, f$ is a regular sequence, too, since $x$ and $f$ are homogeneous elements. Under the isomorphism 4.1(2) the ideal $(f)$ is mapped to the extension of the ideal generated by the residue class of the minor $[1, \dots, t-2 \mid 1, \dots, t-2]$ in the ladder $Z$. Hence, by induction, $(f)$ is radical in $R_t(Y)[x^{-1}]$ and therefore is radical in $R_t(Y)$.

(c) We have seen that $f, x$ is a regular sequence. From the isomorphism 4.1(1) we deduce that $(x)$ is prime in $R_t(Y)[f^{-1}]$ and therefore $(x)$ is prime in $R_t(Y)$. □

We are ready to prove the main result of the paper. For the reader's convenience we repeat all the assumptions on the shape of the ladder.

**Theorem 4.9.** *Let $Y$ be an one-sided ladder with $k$ inside corners and $t \in \mathbb{N}$, $t > 1$. We assume that all the indeterminates of $Y$ are involved in some $t$-minors of $Y$ and that $X$ is the smallest matrix which contains $Y$. Let $(c_i, d_i)$ be the coordinates of the ith the inside corner of $Y$, $i = 1, \ldots, k$. Then*

(a) *The divisor class group $\mathrm{Cl}(R_t(Y))$ of $R_t(Y)$ is free of rank $k+1$, and a basis of $\mathrm{Cl}(R_t(Y))$ is $\mathrm{cl}(\mathfrak{p}_0), \ldots, \mathrm{cl}(\mathfrak{p}_k)$.*

(b) *Let $\mathrm{cl}(\omega) = \sum_{i=0}^{k} \lambda_i \mathrm{cl}(\mathfrak{p}_i)$ be the unique representation of the canonical class of $R_t(Y)$ in $\mathrm{Cl}(R_t(Y))$ with respect to the basis of (a). Then $\lambda_0 = m - n$ and $\lambda_i = m + t - 1 - c_i - d_i$ for $i = 1, \ldots, k$.*

(c) *The ring $R_t(Y)$ is Gorenstein if and only if $m = n$ and $c_i + d_i = m + t - 1$, for $i = 1, \ldots, k$.*

**Proof.** (a) We know from Proposition 4.1 that $R_t(Y)[f^{-1}]$ is isomorphic to a polynomial ring after inversion of a prime element. Therefore, by Nagata's theorem [4, Corollary 7.2], $\mathrm{Cl}(R_t(Y))$ is generated by $\mathrm{cl}(\mathfrak{p}_0), \ldots, \mathrm{cl}(\mathfrak{p}_{k+1})$. Since $(f)$ is radical, we have $\sum_{i=0}^{k+1} \mathrm{cl}(\mathfrak{p}_i) = 0$ and, as in Corollary 2.3, one shows that this is the only relation between the $\mathrm{cl}(\mathfrak{p}_i)$. Therefore $\mathrm{cl}(\mathfrak{p}_0), \ldots, \mathrm{cl}(\mathfrak{p}_k)$ is a set of linearly independent generators of $\mathrm{Cl}(R_t(Y))$.

(b) By induction on $t$. The case $t = 2$ is covered by Proposition 2.4. Let $t > 2$. Since $(x)$ is prime, from the isomorphism 4.1(2), we get an isomorphism $h: \mathrm{Cl}(R_t(Y)) \to \mathrm{Cl}(R_{t-1}(Z))$. As in Proposition 2.4, one shows that $h$ maps the canonical class of $R_t(Y)$ to the canonical class of $R_{t-1}(Z)$. The ideal $\mathfrak{p}_i R[x^{-1}]$ is mapped, under the isomorphism 4.1(2), to the extension of the ideal $\mathfrak{p}_i(Z)$. Therefore, by induction, we obtain the desired result since the data of $R_{t-1}(Z)$ are $m-1$, $n-1$, $t-1$ and the coordinates of the inside corners of $Z$ are $(c_i - 1, d_i - 1)$ for $i = 1, \ldots, k$.

(c) The ring $R_t(Y)$ is Gorenstein if and only if $\mathrm{cl}(\omega) = 0$. $\square$

**Example 4.10.** Consider the ladder given in Fig. 5.

$$
\begin{array}{cccccc}
 & X_{15} & X_{25} & X_{35} & & \\
 & X_{14} & X_{24} & X_{34} & X_{44} & \\
Y = & X_{13} & X_{23} & X_{33} & X_{43} & X_{53} \\
 & X_{12} & X_{22} & X_{32} & X_{42} & X_{52} \\
 & X_{11} & X_{21} & X_{31} & X_{41} & X_{51}
\end{array}
$$

Fig. 5.

The data of $Y$ are $m = n = 5$, $k = 2$, $(c_1, d_1) = (3,4)$ and $(c_2, d_2) = (4,3)$. Therefore the ring $R_3(Y)$ is a normal Gorenstein domain of dimension 16, and $\mathrm{Cl}(R_3(Y)) = \mathbb{Z}^3$.

The ring $R_2(Y)$ is a normal Cohen–Macaulay domain and it has dimension 9. Further $\mathrm{Cl}(R_2(Y) = \mathbb{Z}^3$. Moreover $R_2(Y)$ is not Gorenstein. The canonical class of

$R_2(Y)$ is $\mathrm{cl}(\omega) = -\mathrm{cl}(\mathfrak{p}_1) - \mathrm{cl}(\mathfrak{p}_2) = \mathrm{cl}(\mathfrak{p}_0) + \mathrm{cl}(\mathfrak{p}_4)$. Therefore $\omega \simeq \mathfrak{p}_0 \cap \mathfrak{p}_4 = (x_{11}, x_{12}, x_{13}, x_{14}, x_{15}) \cap (x_{11}, x_{21}, x_{31}, x_{41}, x_{51})$. It is easy to see that $x_{11}, x_{41}x_{15}, x_{51}x_{14}, x_{51}x_{15}$ is a minimal system of generators of $\mathfrak{p}_0 \cap \mathfrak{p}_4$. Hence the Cohen–Macaulay type of $R_2(Y)$ is 4.

Note that, from Theorem 4.9, we get a representation of the canonical module of $R_t(Y)$ as intersection of symbolic powers of the ideals $\mathfrak{p}_i$. In general, from this representation of the canonical module it is not easy to compute the Cohen–Macaulay type of $R_t(Y)$.

## Acknowledgements

The author would like to thank Prof. J. Herzog for several useful discussions during the preparation of this paper.

## References

[1] S.S. Abhyankar, Enumerative Combinatorics of Young tableaux (Marcel Dekker, New York, 1988).
[2] W. Bruns and U. Vetter, Determinantal rings, Lecture Notes in Mathematics, Vol. 1327, (Springer, Heidelberg, 1988).
[3] L. Caniglia, J.A. Guccione and J.J. Guccione, Ideals of generic minors, Commun. Algebra 18 (1990) 2633–2640.
[4] R. Fossum, The Divisor Class of Group of a Krull Domain (Springer, Berlin, 1973).
[5] H.G. Gräbe, Streckungsringe, Dissertation B, Pädagogische Hochschule Erfurt, 1988.
[6] M. Hashimoto, T. Hibi and A. Noma, Divisor class groups of affine semigroup rings associated with a distributive lattice, J. Algebra 149 (1992) 352–357.
[7] J. Herzog and E. Kunz, Der kanonische Modul eines Cohen–Macaulay Rings, Lecture Notes in Mathematics, Vol. 238 (Springer, Berlin, 1971).
[8] J. Herzog and N.V. Trung, Gröbner bases and multiplicity of determinantal and pfaffian ideals, Adv., in Math. 96 (1992) 1–37.
[9] T. Hibi, Distributive lattice, affine semigroup rings and algebras with straightening laws, Commutative algebra and combinatorics, Advanced Studies in Pure Mathematics, Vol. 11 (North-Holland, Amsterdam, 1987) pp. 93–109.
[10] M. Hochster, Cohen–Macaulay rings, combinatorics, and simplicial complexes, in: Macdonald, Magid and Smith eds., Ring theory II (Marvel Dekker, New York 1977) 171–223.
[11] H. Narasimhan, The irreducibility of ladder determinantal varieties, J. Algebra 102 (1986) 162–185.
[12] L. Robbiano, Introduction to the theory of Gröbner bases, Queen's Paper in Pure and Appl. Math., Kingston, Canada, Vol. V, No. 80 (1988).
[13] B. Sturmfels, Gröbner bases and Stanley decompositions of determinantal rings, Math. Z. 205 (1990) 137–144.