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Hilbert-Kunz function of monomial ideals and binomial hypersurfaces

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The aim of this note is to determine the Hilbert-Kunz functions of rings defined by monomial ideals and of rings defined by a single binomial equation $X^a - X^b$ with $\gcd(X^a, X^b) = 1$.

Introduction

The Hilbert-Kunz function HK_R of a local ring R of prime characteristic p with maximal ideal m is defined by the assignment:

$$\mathrm{HK}_R(q) = \ell(R/m^{[q]})$$

where ℓ denotes the length, $q = p^e$ is a power of p and $m^{[q]}$ denotes the q-th Frobenius power of m, that is, $m^{[q]} = (\{x^q : x \in m\})$.

The function HK_R was introduced by Kunz [5]. He showed [5,3.2,3.3] that $\mathrm{HK}_R(q) \geq q^d$ for all q, $d = \dim R$, and that the following conditions are equivalent:

(i) R is regular

(ii) $\mathrm{HK}_R(q) = q^d$ for all q,

(iii) $\mathrm{HK}_R(q) = q^d$ for some $q \geq p$.

Hence the difference between $\mathrm{HK}_R(q)$ and q^d can be taken as a measure of the singularity of R. By a result of Monsky [7,1.8] it is known that there exist a real constant $c(R) \geq 1$ and a function $f(q)$ such that

$$\mathrm{HK}_R(q) = c(R)q^d + f(q)$$

and there exists a constant α such that $|f(q)| \leq \alpha q^{d-1}$ for $q >> 0$, (see [9] for a generalization of this result). Monsky [7,3.10] has also shown that if the dimension of R is one then $c(R)$ is equal to the multiplicity $e(R)$ of R and that $f(q) = f(p^e)$ is a periodic function in e for large e. It also known that $c(R) \leq e(R)$ for Cohen-Macaulay rings, see [5,3.2]. But in general it is not clear how the constant $c(R)$ is related to the other invariants of R, and it is not even known whether $c(R)$ is a rational number. The nature of the remainder term $f(q)$ is also quite mysterious, see [4] for example.

Hilbert-Kunz functions of some classes of hypersurface rings were determined by Kunz, Han and Monsky, Chang, and Pardue [2,3,4,5,6,8]. The first Hilbert-Kunz function of a ring which is not an hypersurface is determined by Buchweitz and Pardue [1]. They succeeded in computing the Hilbert-Kunz function of the coordinate ring of a rational normal curve.

We introduce a more general Hilbert-Kunz function. Given a Noetherian ring R of arbitrary characteristic which is local or a standard graded K-algebra and a system of generators x of its maximal ideal, we define the Hilbert-Kunz function $\mathrm{HK}_{R,x}(q)$ of R with respect to x to be the length of $R/x^{[q]}$ where $x^{[q]}$ is the ideal generated by the q-powers of the elements of the set x and $q \in \mathbf{N}$. We concentrate our attention on the Hilbert-Kunz function of rings of the form $R = K[X_1, \ldots, X_n]/I$ where I is a monomial ideal or a principal ideal generated by a homogeneous binomial form $X^a - X^b$ with $\gcd(X^a, X^b) = 1$.

If I is a monomial ideal, then we show that $\mathrm{HK}_{R,x}(q)$ agrees with a polynomial with integral coefficients for large q, see 2.1. The leading coefficient of this polynomial is exactly the multiplicity of R. As a corollary it follows the inequality

$$c(R) \leq e(R)$$

holds for any local ring or standard graded K-algebra R of prime characteristic.

When R is a binomial hypersurface of the type mentioned above, we show that there exists a polynomial $P(y,z) \in \mathbf{Q}[y,z]$ such that $\mathrm{HK}_{R,x}(q) = P(q,\varepsilon)$ for large q, where ε is the residue class of q modulo the highest exponent in X^a and X^b, see 3.1. The polynomial $P(y,z)$ is explicitly determined and its leading coefficient with respect to y is a rational number $c(R)$ which is expressed in terms of the exponent vectors a and b.

1 Notation

Let (R,m) denote a Noetherian local ring with maximal ideal m or a standard graded K-algebra with maximal homogeneous ideal m. Given a set of generators $x = x_1,\dots,x_n$ of m (not necessarily minimal) and a positive integer q we denote by $x^{[q]}$ the ideal $(x_1^q,\dots,x_n^q)$. We define the Hilbert-Kunz function of R with respect to x to be

$$\mathrm{HK}_{R,x}(q) = \ell(R/x^{[q]}).$$

Of course if the characteristic of R is prime and q is a power of the characteristic, then $\mathrm{HK}_{R,x}(q)$ coincides with the ordinary Hilbert-Kunz function $\mathrm{HK}_R(q)$ and it is independent of x. But in general $\mathrm{HK}_{R,x}(q)$ depends on x.

Denote by $H^1(R,q)$ the first iterated sum of the Hilbert function $H(R,q) = \dim_K(m^q/m^{q+1})$ of R, that is, $H^1(R,q) = \ell(R/m^{q+1})$. Since $m^q \supseteq x^{[q]} \supseteq m^{nq}$, one has

$$(1) \qquad H^1(R,q-1) \leq \mathrm{HK}_{R,x}(q) \leq H^1(R,nq-1).$$

Denote by d the dimension and by $e(R)$ the multiplicity of R. The function $H^1(R,q)$ agrees with a polynomial of degree d and leading coefficient $e(R)/d!$ for $q >> 0$. From inequality (1) it follows that $\mathrm{HK}_{R,x}(q)$ grows as a polynomial of degree d. For ordinary Hilbert-Kunz functions inequality (1) implies

$$e(R)/d! \leq c(R) \leq e(R)n^d/d! \tag{2}$$

It was shown by Kunz [5,3.2] that if (R,m) is a Cohen-Macaulay local ring of prime characteristic p, then

$$\mathrm{HK}_R(q) \leq e(R)q^d$$

for all powers q of p. It follows from this result that the constant $c(R)$ is bounded by the multiplicity for Cohen-Macaulay rings. Note that one cannot expect $\mathrm{HK}_R(q) \leq e(R)q^d$ to be true in general. For instance if R has multiplicity 1 and it is not regular (take for example $R = K[X,Y]/(X^2,XY)$), then by [5,3.3] one has $\mathrm{HK}_R(q) > q^d = e(R)q^d$ for all $q \geq p$.

2 The monomial case

Let S be a polynomial ring $K[X_1,\ldots,X_n]$ over an arbitrary field K. One has:

Theorem 2.1. *Let I be a monomial ideal of S and set $R = S/I$. Denote by x_i the residue class of X_i in R, and by x the sequence $x_1,\ldots,x_n$. Then there exists a polynomial $P_R(y) \in \mathbf{Z}[y]$ of degree the dimension $\dim R$ of R and leading coefficient the multiplicity $e(R)$ of R, such that $\mathrm{HK}_{R,x}(q) = P_R(q)$ for all integers q greater than or equal to the highest exponent which appear in the generators of I.*

PROOF. We argue by induction on n. The case $n = 1$ is trivial. Hence assume $n > 1$. Set $a = \max\{j : X_i^j$ divides a generator of I, for some i, $1 \leq i \leq n\}$ and $b = \max\{j : X_n^j$ divides a generator of $I\}$. Fix an integer $q \geq a$ and for $0 \leq i \leq q$ set

$$J_i = (I : X_n^i) + (X_1^q,\ldots,X_{n-1}^q,X_n^{q-i}).$$

Set $S_1 = K[X_1, \ldots, X_{n-1}]$. For $i \geq 0$ denote by K_i the monomial ideal of S_1 with the property $K_i + (X_n) = (I : X_n^i) + (X_n)$. Set $R_i = S_1/K_i$. Since $J_i : X_n = J_{i+1}$, one has an exact sequence:

$$(3) \quad 0 \longrightarrow S/J_{i+1} \xrightarrow{X_n} S/J_i \longrightarrow S_1/K_i + (X_1^q, \ldots, X_{n-1}^q) \longrightarrow 0$$

and then

$$\dim_K S/J_i = \dim_K S/J_{i+1} + \mathrm{HK}_{R_i,x'}(q)$$

where $x' = x_1, \ldots, x_{n-1}$ are the residue classes of $X_1, \ldots, X_{n-1}$. By construction $J_0 = I + X^{[q]}$ and $J_q = S$. It follows that

$$\mathrm{HK}_{R,x}(q) = \sum_{i=0}^{q-1} \mathrm{HK}_{R_i,x'}(q).$$

But $K_i = K_b$ for all $i \geq b$, and then

$$\mathrm{HK}_{R,x}(q) = (q-b)\,\mathrm{HK}_{R_{b,x'}}(q) + \sum_{i=0}^{b-1} \mathrm{HK}_{R_i,x'}(q).$$

One notes that the exponents in the generators of the monomial ideals K_i are less than or equal to a. By induction $\mathrm{HK}_{R_i,x'}(q)$ agrees with a polynomial $P_{R_i}(y) \in \mathbf{Z}[y]$ for $q \geq a$. Hence $\mathrm{HK}_{R,x}(q)$ agrees with the polynomial

$$(4) \qquad P_R(y) = (y-b)P_{R_b}(y) + \sum_{i=0}^{b-1} P_{R_i}(y)$$

for $q \geq a$. Since $H^1(R,q)$ agrees with a polynomial of degree $\dim R$ for large q, from (1) it follows that the degree of $P_R(y)$ is $\dim R$. We show now that the leading coefficient of $P_R(y)$ is $e(R)$. Denote by $c(R), c(R_0), \ldots, c(R_b)$ the leading coefficients of $P_R(y), P_{R_0}(y), \ldots, P_{R_b}(y)$ and by $d, d_0, \ldots, d_b$ the dimensions of $R, R_0, \ldots, R_b$. By (4) one has

$$(5) \qquad c(R) = \delta(d, d_b+1)c(R_b) + \sum_{i=0}^{b-1} \delta(d, d_i)c(R_i)$$

where $\delta(i,j)=0$ if $i\neq j$ and $\delta(i,i)=1$. Consider now the exact sequence

$$(6)\qquad 0\longrightarrow S/I:X_n^{i+1}[-1]\xrightarrow{X_n} S/I:X_n^i\longrightarrow R_i\longrightarrow 0$$

Denote by $H_A(\lambda)$ the Hilbert series $\sum_{j\geq 0}\dim A_i\lambda^j$ of a $\mathbf{N}$-graded K-algebra $A=\bigoplus_{j\geq 0}A_j$. By virtue of (6) one has

$$(7)\ H_R(\lambda)=\sum_{i\geq 0}\lambda^i H_{R_i}(\lambda)=\lambda^b(1-\lambda)^{-1}H_{R_b}(\lambda)+\sum_{i=0}^{b-1}\lambda^i H_{R_i}(\lambda).$$

It follows from (7) that:

$$(8)\qquad e(R)=\delta(d,d_b+1)e(R_b)+\sum_{i=0}^{b-1}\delta(d,d_i)e(R_i)$$

By induction one has $c(R_i)=e(R_i)$ for all $i=0,\ldots,b$. From (5) and (8) it follows that $c(R)=e(R)$. □

Note that the polynomial $P_R(q)$ does not depend on the characteristic of the field K. We do not know in general how to relate the coefficients of $P_R(q)$ to other combinatorial or algebraic invariants of the ring R except that for the square free case. In the square free case the polynomial $P_R(y)$ has the following combinatorial interpretation:

Remark 2.2. Let $I\subset(X_1,\ldots,X_n)^2$ be a square free monomial ideal with associated simplicial complex Δ. Denote by $f=(f_{-1},f_0,\ldots,f_{d-1})$ the f-vector of Δ, $\dim\Delta=d-1$. The number $\mathrm{HK}_{K[\Delta],x}(q)$ is the cardinality of the set

$$B=\{b=(b_1,\ldots,b_n)\in\mathbf{N}^n: 0\leq b_i<q \text{ and } \operatorname{supp} b\in\Delta\}.$$

It follows that:

$$\mathrm{HK}_{K[\Delta],x}(q)=\sum_{F\in\Delta}|\{b\in B:\operatorname{supp} b=F\}=$$

$$\sum_{F\in\Delta}(q-1)^{|F|}=\sum_{i=0}^{d}f_{i-1}(q-1)^i$$

Using the known relation between the f-vector and the h-vector of a simplicial complex one has also:

$$\mathrm{HK}_{K[\Delta],x}(q) = \sum_{i=0}^{d} h_i (q-1)^i q^{d-i}.$$

For another interpretation of the Hilbert-Kunz function of a square free monomial ideal see also [1]. It follows from the remark that the Hilbert-Kunz function of a the ring defined by a square free monomial ideal depends only on the Hilbert function. This is not true for general monomial ideals. For instance if I is generated by a single monomial, say $I = (\prod_{i=1}^n X_i^{a_i})$, then the Hilbert function of the ring R defined by I depends only on the degree of the generator of I while its Hilbert-Kunz function $\mathrm{HK}_{R,x}(q)$ is equal to

$$q^n - \prod_{i=1}^{n}(q-a_i) = \sum_{i=0}^{n-1}(-1)^{n-1-i} s_{n-i,n}(a) q^i$$

for $q \geq \max\{a_i\}$ where $s_{j,n}$ denotes the elementary symmetric function of degree j in n indeterminates. Hence two monomials define rings with the same Hilbert-Kunz function if and only if they are the same up to a permutation of the indeterminates.

Fix now a term order τ on $S = K[X_1, \dots, X_n]$. Let I be a homogeneous ideal of S. Denote by $\mathrm{in}(I)$ the initial ideal of I, and set $R = S/I$. Since

$$\mathrm{HK}_{R,x}(q) = \dim_K S/(I, X^{[q]}) = \dim_K S/\mathrm{in}(I, X^{[q]}),$$

and $\mathrm{in}(I, X^{[q]}) \supseteq \mathrm{in}(I) + X^{[q]}$, one has:

$$\mathrm{HK}_{R,x}(q) \leq \mathrm{HK}_{S/\mathrm{in}(I),x}(q). \tag{9}$$

It follows:

Proposition 2.3. *Let R be a local ring or a standard graded K-algebra. Let $x = x_1, \dots, x_n$ be a system of generators of the maximal (homogeneous) ideal of R. Then there exists a polynomial $P(y) \in \mathbf{Z}[y]$ depending on R and x with degree* $\dim R$ *and leading coefficient* $e(R)$ *such that* $\mathrm{HK}_{R,x}(q) \leq P(q)$ *for all* $q \in \mathbf{N}$.

PROOF. Assume first that R is local with maximal ideal m. Denote by G its associated graded ring $\bigoplus_{i\geq 0} m^i/m^{i+1}$, by f^* the initial form in G of an element $f \in R$ and by x^* the sequence $x_1^*, \ldots, x_n^*$. The associated graded ring G_q of $R/x^{[q]}$ is isomorphic to G/J, where J is generated by the initial forms of the elements of the ideal $x^{[q]}$. Note that if $(x_i^*)^q \neq 0$, then $(x_i^*)^q = (x_i^q)^*$. It follows that $(x_i^*)^q \in J$ and hence

$$\mathrm{HK}_{R,x}(q) = \ell(G_q) \leq \mathrm{HK}_{G,x^*}(q).$$

The rings R and G have the same dimension and multiplicity. Hence we may assume that R is standard graded K-algebra. Then we take a presentation $R \simeq K[X_1, \ldots, X_n]/I$, such that the residue class of X_i corresponds to x_i, and fix a term order τ on $K[X_1, \ldots, X_n]$. Since the dimension and multeplicity of R and $K[X_1, \ldots, X_n]/\operatorname{in}(I)$ are equal, it follows now from (9) and from 2.1 that there exists a polynomial $P_1(y) \in \mathbf{Z}[y]$ of degree $\dim R$ and leading coefficient $e(R)$ such that $\mathrm{HK}_{R,x}(q) \leq P_1(q)$ for all $q >> 0$. Then $P(y)$ is obtained from $P_1(y)$ by adding a suitable positive integer. □

The proposition has two corollaries. The first is the generalization of Kunz's result [5,3.2] to the non Cohen-Macaulay case:

Corollary 2.4. *Let R be a local ring or a standard graded K-algebra and assume that the characteristic of R is prime. Then $c(R) \leq e(R)$.*

The second is a partial extension of Monsky's result [7,3.10] to generalized Hilbert-Kunz functions:

Corollary 2.5. *Let R be a local ring or a standard graded K-algebra of dimension one and arbitrary characteristic. Let $x = x_1, \ldots, x_n$ be a system of generators of the maximal (homogeneous) ideal of R. Then there exists a constant α such that $|\mathrm{HK}_{R,x}(q) - e(R)q| \leq \alpha$ for all $q \in \mathbf{N}$.*

PROOF. By 2.3 there exists a constant β such that $H_{R,x}(q) \leq e(R)q + \beta$ for all $q \in \mathbf{N}$. On the other hand, by (1), $H_{R,x}(q) \geq H^1(R, q-1)$ and there exists a constant γ such that $H^1(R,q) = e(R)q + \gamma$ for $q >> 0$. Hence the claim follows. □

It follows from the corollary that for a one dimensional ring R the limit $\lim_{q\to\infty} \mathrm{HK}_{R,x}(q)/q$ exists and it is equal to the multiplicity $e(R)$ of R. We do not know whether $\lim_{q\to\infty} \mathrm{HK}_{R,x}(q)/q^{\dim R}$ exists in general, that is, if $\dim R > 1$. For one dimensional ring we do not know whether the difference $\mathrm{HK}_{R,x}(q) - e(R)q$ is periodic for large q.

3 Binomial hypersurfaces

Given a homogeneous ideal I of $S = K[X_1, \ldots, X_n]$, in order to study the Hilbert-Kunz function of S/I one may try to understand the initial ideal of the ideal $I + X^{[q]}$. The information that one has on the behaviour of $\mathrm{in}(I + X^{[q]})$ with respect to q can be used to have some control on the Hilbert-Kunz function of S/I. As we already noticed $\mathrm{in}(I+X^{[q]}) \supseteq \mathrm{in}(I)+X^{[q]}$, and one cannot expect equality unless I is a monomial ideal. For a general I it is difficult to understand which monomials one has to add to fill the gap between $\mathrm{in}(I+X^{[q]})$ and $\mathrm{in}(I)+X^{[q]}$. If I happens to be a binomial ideals, then the situation is slightly simpler because the S-reductions $S(f,n)$ of a binomial f with monomial n with respect to a set of binomials produce only monomials. We are able to determine $\mathrm{in}(I + X^{[q]})$ explicitly when $I = (X^a - X^b)$ where X^a and X^b are monomials of the same degree and with $\gcd(X^a, X^b) = 1$. This allows us to determine the Hilbert-Kunz function for the hypersurface defined by $X^a - X^b$.

One has:

Theorem 3.1. *Let $S = K[X_1, \ldots, X_{m+n}]$ be a polynomial ring over an arbitrary field K. Let F be the binomial equation*

$$F = X_1^{a_1} \cdots X_m^{a_m} - X_{m+1}^{b_{m+1}} \cdots X_{m+n}^{b_{m+n}}.$$

Assume that F is homogeneous and denote by R the ring S/F and by x the set of the residue classes in R of the X_i's. Let u be the maximum of the integers $a_1, \ldots, a_m, b_{m+1}, \ldots, b_{m+n}$.

There exist a polynomial $P(y,z) \in \mathbf{Q}[y,z]$ of the form

$$P(y,z) = c(R)y^{m+n-1} + c_1(z)y^{m+n-2} + \ldots + c_{m+n-1}(z)$$

and an integer α *such that*

$$\mathrm{HK}_{R,x}(q) = P(q,\varepsilon) \quad \textit{for all} \quad q \geq \alpha, \quad q \in \mathbf{N}$$

where $\varepsilon = q \bmod u$ *and* $0 \leq \varepsilon < u$. *Furthermore*

$$c(R) = \sum_{h,k=1}^{m,n} (-1)^{h+k} s_{hm}(a) s_{kn}(b) \frac{hk}{(h+k-1)u^{h+k-1}}$$

where s_{ji} *denotes the elementary symmetric polynomial of degree* j *in* i *indeterminates.*

PROOF. It is not restrictive to assume that $a_1 \geq a_i > 0$ for $i = 1,\dots,m$, $b_{m+1} \geq b_{m+i} > 0$ for $i = 1,\dots,n$ and that $a_1 \geq b_{m+1}$, so that $u = a_1$. Set $X^a = X_1^{a_1} \cdots X_m^{a_m}$ and $X^b = X_{m+1}^{b_{m+1}} \cdots X_{m+n}^{b_{m+n}}$. We fix a term order on S such that X^a is bigger than X^b. Let q be a positive integer. We determine a Gröbner basis of the ideal

$$I_q = (F, X_1^q, \dots X_{m+n}^q)$$

by means of Buchberger algorithm. The first S-pair to be considered is $S(F, X_i^q)$, $1 \leq i \leq m$, and it produces the element $X_i^{(q-a_i)_+} X^b$, where

$$(v)_+ = \max\{0, v\}.$$

If $q \geq a_i$, then $S(F, X_i^{q-a_i} X^b)$ produces $X_i^{(q-2a_i)_+} X^{2b}$, and so on. In this way we obtain the elements $X_i^{q-ja_i} X^{jb}$ for $j = 0,\dots,j_i$, where

$$j_i = [q/a_i].$$

The S-reduction $S(F, X_i^{q-j_i a_i} X^{j_i b})$ produces the element $X^{(j_i+1)b}$ which is prime to X^a and hence the procedure stops. From this it follows that the elements

$$F, X_{m+1}^q, \dots, X_{m+n}^q, X^{(j_1+1)b}, \text{ and } X_i^{q-ja_i} X^{jb}$$

with $1 \leq i \leq m$ and $0 \leq j \leq j_i$ form a Gröbner basis of I_q. Then $\mathrm{in}(I_q)$ is generated by $X^a, X_{m+1}^q, \dots, X_{m+n}^q, X^{(j_1+1)b}$, and $X_i^{q-ja_i} X^{jb}$ with $1 \leq i \leq m$, and $0 \leq j \leq j_i$.

Now we have to compute the dimension of $S/\operatorname{in}(I_q)$. In order to do this we consider the ideals $K_r = \operatorname{in}(I_q) : X^{rb}$ for $r = 0,\dots,j_1+1$. Since $K_r : X^b = K_{r+1}$ we have the exact sequence:

$$0 \to S/K_{r+1} \stackrel{X^b}{\to} S/K_r \to\!\!\to S/(K_r, X^b) \to 0$$

Note that $K_0 = \operatorname{in}(I_q)$ and that $K_{j_1+1} = S$. It follows that

$$\dim_K S/\operatorname{in}(I_q) = \sum_{r=0}^{j_1} \dim_K S/(K_r, X^b).$$

But the ideal (K_r, X^b) is equal to

$$(X^a, X^b, X_i^{q-ra_i},\ \ 1 \le i \le m,\ \ X_{m+i}^{q-rb_{m+i}},\ 1 \le i \le n)$$

and hence $\dim_K S/(\operatorname{in}(I_q) : X^{rb}, X^b)$ is equal to the product

$$\left[\dim_K K[X_1,\dots,X_m]/(X^a, X_i^{q-ra_i},\ \ 1 \le i \le m)\right] \times$$

$$\left[\dim_K K[X_{m+1},\dots,X_{m+n}]/(X^b, X_{m+i}^{q-rb_{m+i}},\ \ 1 \le i \le n)\right].$$

It is easy to see that in general one has

$$\dim_K K[Y_1,\dots,Y_n]/(Y^\gamma, Y_1^{\delta_1},\dots,Y_n^{\delta_n}) = \prod_{i=1}^n \delta_i - \prod_{i=1}^n (\delta_i - \gamma_i)_+.$$

It follows that $\dim_K S/\operatorname{in}(I_q)$ can be written as the sum of the two terms

$$(i)\quad \sum_{r=0}^{j_1-1}\left[\prod_{i=1}^m (q - ra_i) - \prod_{i=1}^m (q-(r+1)a_i)\right] \times \left[\prod_{i=1}^n (q - rb_{m+i}) - \prod_{i=1}^n (q-(r+1)b_{m+i})\right]$$

and

$$(ii)\quad \left[\prod_{i=1}^m (q - j_1 a_i)\right]\left[\prod_{i=1}^n (q - j_1 b_{m+i}) - \prod_{i=1}^n (q-(j_1+1)b_{m+i})_+\right]$$

Using the elementary symmetric polynomials, the term (i) can be written as follows:

$$\sum_{h,k=1}^{m,n}\left[(-1)^{h+k}s_{hm}(a)s_{kn}(b)q^{m+n-h-k}\sum_{r=0}^{j_1-1}W_{hk}(r)\right]$$

where $W_{hk}(r)=(r^h-(r+1)^h)(r^k-(r+1)^k)$. Note that $W_{hk}(r)$ is a polynomial in $\mathbf{Q}[r]$ of degree $h+k-2$, and leading coefficient hk. Hence $\sum_{r=0}^{j_1-1}W_{hk}(r)$ is a polynomial in $\mathbf{Q}[j_1]$ of degree $h+k-1$, and leading coefficient $hk/(h+k-1)$. Since $j_1=(q-\varepsilon)/a_1$, where $\varepsilon=q \bmod a_1$ and $0\le\varepsilon<a_1$, we obtain that there exists a polynomial $P_1(y,z)\in\mathbf{Q}[y,z]$ of the form

$$P_1(y,z)=cy^{m+n-1}+d_1(z)y^{m+n-2}+\ldots+d_{m+n-1}(z)$$

with

$$c=\sum_{h,k=1}^{m,n}(-1)^{h+k}s_{hm}(a)s_{kn}(b)\frac{hk}{(h+k-1)a_1^{h+k-1}}$$

such that $P_1(q,\varepsilon)$ is equal to (i) for all $q\in\mathbf{N}$.

We analyze now the second term (ii). We have to distinguish two cases. If $a_1=b_{m+1}$ then (ii) is equal to

$$\prod_{i=1}^{m}(q-j_1a_i)\prod_{i=1}^{n}(q-j_1b_{m+i})$$

and replacing j_1 with $(q-\varepsilon)/a_1$ one obtains the following expression for (ii):

$$a_1^{2-m-n}\varepsilon^2\prod_{i=2}^{m}(q(a_1-a_i)+\varepsilon a_i)\prod_{i=2}^{n}(q(a_1-b_{m+i})+\varepsilon b_{m+i})$$

If $a_1>b_{m+1}$, then one has $q-(j_1+1)b_{m+i}\ge 0$ for $q\ge a_1b_{m+1}/(a_1-b_{m+1})$. Hence for $q\ge a_1b_{m+1}/(a_1-b_{m+1})$ the term (ii) is equal to:

$$\left[\prod_{i=1}^{m}(q-j_1a_i)\right]\left[\prod_{i=1}^{n}(q-j_1b_{m+i})-\prod_{i=1}^{n}(q-(j_1+1)b_{m+i})\right]$$

and again replacing j_1 with $(q-\varepsilon)/a_1$ one obtains the following expression for (ii):

$$a_1^{1-m-n}\varepsilon\prod_{i=2}^{m}(q(a_1-a_i)+\varepsilon\, a_i)\left[\prod_{i=1}^{n}(q(a_1-b_{m+i})+\varepsilon b_{m+i})-\right.$$

$$\left.\prod_{i=1}^{n}(q(a_1-b_{m+i})+(\varepsilon-a_1)b_{m+i})\right]$$

In both cases we have seen that there exists a polynomial $P_2(y,z)\in\mathbf{Q}[y,z]$ such that $P_2(y,z)$ has degree less than $m+n-1$ in y and $P_2(q,\varepsilon)$ is equal to the second term for all q or for all $q>a_1b_{m+1}/(a_1-b_{m+1})$ according to whether $a_1=b_{m+1}$ or $a_1>b_{m+1}$. Then we set $P(y,z)=P_1(y,z)+P_2(y,z)$ and $P(y,z)$ has the desired properties. □

The theorem has the following

Corollary 3.2. *Let K be a field of prime characteristic and let F be any homogeneous binomial form of $S=K[X_1,\dots,X_n]$. Then $c(S/F)$ is a rational number which is independent of the characteristic of K.*

PROOF. The binomial F can be written as $F=X^d(X^a-X^b)$ where $\gcd(X^a,X^b)=1$. Since $(X^d)\cap(X^a-X^b)=(F)$, one has the short exact sequence

$$0\to S/F\to S/X^a-X^b\oplus S/X^d\to S/(X^d,X^a-X^b)\to 0$$

Since $\text{height}(X^d,X^a-X^b)=2$, it follows from [9] that $c(S/F)=c(S/X^a-X^b)+c(S/X^d)$. By virtue of 2.1 and 3.1 we know that both $c(S/X^a-X^b)$ and $c(S/X^d)$ are rational and independent of the characteristic of K, and this completes the proof. □

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