GORENSTEIN LADDER DETERMINANTAL RINGS

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ABSTRACT

Ladder determinantal rings are rings associated with ideals of minors of certain subsets of a generic matrix of indeterminates. By results of Abhyankar, Narasimhan, Herzog and Trung, and Conca, they are known to be Cohen–Macaulay normal domains. In this paper we characterize the Gorenstein property of ladder determinantal rings in terms of the shape of the ladder.

Introduction

Let K be a field, and let $X = (X_{ij})$ be a matrix of indeterminates over K. A subset Y of X is called a *ladder* if whenever the main diagonal of a minor of X is in Y then the minor is in Y. Given a ladder Y one defines $I_t(Y)$ to be the ideal of $K[Y]$ generated by all the t-minors of X which involve only indeterminates of Y. Here $K[Y]$ denotes the polynomial ring over K whose indeterminates are the elements of Y. The ring $R_t(Y) = K[Y]/I_t(Y)$ is called a *ladder determinantal ring*. Ladder determinantal rings have been introduced and studied by Abhyankar [**1**], and subsequently by Abhyankar and Kulkarni [**2**], Narasimhan [**14**], Mulay [**13**], Herzog and Trung [**10**], and Conca [**7**]. They are known to be Cohen–Macaulay normal domains, see [**14**, **10**, **7**]. The aim of this paper is to give a characterization of the Gorenstein property of ladder determinantal rings in terms of the shape of the ladder and to determine the divisor class group. This has already been done for a ring defined by the 2-minors of a ladder by Hibi [**11**] and by Hashimoto, Hibi and Noma [**9**] and for the determinantal rings associated with one-sided ladders by the present author [**7**].

We now describe the content of each section. In the first section we recall the notions of Gröbner basis and initial ideal of an ideal of polynomials.

In the second section we recall the definition of ladder determinantal ring and some results from the above mentioned papers. Then we explain how, for our purposes, one may always assume that the ladder satisfies certain conditions. In order to determine the divisor class group of a ladder determinantal ring $R_t(Y)$, it suffices to treat the case of a ladder Y which is t-connected and satisfies Assumptions (a), (b), (c). To determine whether $R_t(Y)$ is Gorenstein or not, one may further assume that Y satisfies Assumption (d).

Section 3 is devoted to the study of some classes of ideals of $R_t(Y)$ generated by minors. These ideals play a role in the investigation of the divisor class group and canonical class of $R_t(Y)$.

In Section 4 we determine the divisor class group $\mathrm{Cl}(R_t(Y))$ of $R_t(Y)$. For technical reasons we consider first ladders which satisfy Assumption (d). At the end of the

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section we indicate briefly how to determine the divisor class group for ladders which do not satisfy Assumption (d). In order to determine the divisor class group, we show that after inversion of certain $(t-1)$-minors, say $f_1, \dots, f_{h+1}$, the ring $R_t(Y)$ becomes factorial. By Nagata's theorem it follows that $\mathrm{Cl}(R_t(Y))$ is generated by the classes of the minimal prime ideals of the elements f_i. It turns out that the ideals (f_i) are radical, and we are able to describe their minimal prime ideals. Then one shows that the only relations between the classes of the minimal primes of the elements f_i are those which arise from the primary decomposition of (f_i). It follows immediately that $\mathrm{Cl}(R_t(Y))$ is free. Further, we are able to determine a basis for $\mathrm{Cl}(R_t(Y))$. It appears that this basis is somehow the natural one, and for this reason we call it 'the basis' of $\mathrm{Cl}(R_t(Y))$.

A normal Cohen–Macaulay positively graded K-algebra R has a unique canonical module ω_R. The module ω_R can be identified with a divisorial ideal. The class $\mathrm{cl}(\omega_R)$ of ω_R in $\mathrm{Cl}(R)$ is called the canonical class of R. Recall that the ring R is Gorenstein if and only if its canonical module ω_R is principal, that is, the canonical class $\mathrm{cl}(\omega_R)$ vanishes in $\mathrm{Cl}(R)$.

To decide whether $R_t(Y)$ is Gorenstein or not we restrict our attention to a ladder which satisfies Assumption (d) and determine in Section 5 the canonical class of $R_t(Y)$ in terms of the basis of $\mathrm{Cl}(R_t(Y))$. The coefficients of the canonical class with respect to the basis of $\mathrm{Cl}(R_t(Y))$ depend only on the position of the so-called inside corners of Y. The Gorenstein property of $R_t(Y)$ is equivalent to the vanishing of these coefficients. It turns out that the ring $R_t(Y)$ is Gorenstein if and only if the minimal submatrix which contains Y is square and the inside corners of Y lie on certain diagonals.

To determine the canonical class of $R_t(Y)$ we use a 'divide and conquer method'. We pick an element of $R_t(Y)$, say h, such that the ring $R_t(Y)[h^{-1}]$ can be represented as a localization of a polynomial extension of a ladder determinantal ring $R_t(Y_1)$ associated with a one-sided ladder Y_1. This yields an epimorphism $\phi: \mathrm{Cl}(R_t(Y)) \to \mathrm{Cl}(R_t(Y_1))$ which behaves well with respect to the bases of $\mathrm{Cl}(R_t(Y))$ and $\mathrm{Cl}(R_t(Y_1))$. The expression of the canonical class of $R_t(Y_1)$ in terms of the basis of $\mathrm{Cl}(R_t(Y_1))$ is known [7, 4.9]. By means of ϕ we may determine the coefficients of the canonical class of $R_t(Y)$ which correspond to the basis elements which do not vanish under ϕ. But one needs to know that ϕ maps the canonical class to the canonical class. This problem is solved in Section 6 by showing that $R_t(Y)[h^{-1}]$ has a trivial Picard group. From this follows the fact that $R_t(Y)[h^{-1}]$ has a unique canonical module which in turn implies that ϕ maps the canonical class to the canonical class. In Section 7 we present some examples.

1. *Gröbner bases*

In this section we recall the definition and some properties of Gröbner bases and initial ideals. For more information on this theory we refer the reader to [**15**]. Let A be a polynomial ring over a field K and let τ be a monomial order on A, that is, a total order on the set of monomials of A which is compatible with the semigroup structure. Let $g \in A$ and let I be an ideal of A. The *initial monomial* $\mathrm{in}_\tau(g)$ of the polynomial g is the biggest monomial which appears in the representation of g as a linear combination of monomials. The *initial ideal* $\mathrm{in}_\tau(I)$ of I is the ideal generated by the initial monomials of the elements of I. Whenever there is no danger of confusion we shall use the shorter notation $\mathrm{in}(g)$ and $\mathrm{in}(I)$. Assume that I is a homogeneous ideal of A. Denote by I_i the ith homogeneous component of I and by $\dim I_i$ its dimension as K-vector space. It is well known that $\dim I_i = \dim \mathrm{in}(I)_i$ for all i. Further the set of the residue classes in A/I of the monomials not in $\mathrm{in}(I)$ is a K-basis of A/I. A *Gröbner basis* of the ideal I, with respect to τ, is a finite subset F of I such that the initial monomials of the elements in F generate $\mathrm{in}(I)$. If F is a Gröbner basis of I, then F generates I, but unfortunately a system of generators need not to be a Gröbner basis. The following three lemmas are well-known.

LEMMA 1.1. *Let $K[Z]$ be a polynomial ring over a field K, and let τ, σ be monomial orders. Let F be a finite subset of a homogeneous ideal I of $K[Z]$. Suppose that for all f in F one has $\mathrm{in}_\tau(f) = \mathrm{in}_\sigma(f)$. Then F is a Gröbner basis of I with respect to τ if and only if it is a Gröbner basis of I with respect to σ.*

Proof. Let J be the ideal generated by the monomials $\mathrm{in}_\tau(f) = \mathrm{in}_\sigma(f)$ with $f \in F$. We have $J \subseteq \mathrm{in}_\tau(I)$ and $J \subseteq \mathrm{in}_\sigma(I)$. One has $J = \mathrm{in}_\tau(I)$ if and only if J and I have the same Hilbert function if and only if $J = \mathrm{in}_\sigma(I)$.

LEMMA 1.2. *Let $K[Z]$ be the polynomial ring in the set of indeterminates Z, let Z_1 be a subset of Z, let I be an ideal of $K[Z]$, and let F be a Gröbner basis of I with respect to a monomial order τ. Assume that $f \in K[Z_1]$ for all $f \in F$ such that $\mathrm{in}(f) \in K[Z_1]$. It follows that $F \cap K[Z_1]$ is a Gröbner basis of the ideal $I \cap K[Z_1]$ with respect to the monomial order τ restricted to $K[Z_1]$. In particular $F \cap K[Z_1]$ generates $I \cap K[Z_1]$.*

Proof. Let $g \in I \cap K[Z_1]$. Then $\mathrm{in}(g) \in \mathrm{in}(I) \cap K[Z_1]$. Since F is a Gröbner basis of I, there exists $f \in F$ such that $\mathrm{in}(g) = M\,\mathrm{in}(f)$, where M is a monomial of $K[Z]$. Since $\mathrm{in}(g)$ is in $K[Z_1]$, then M and $\mathrm{in}(f)$ are in $K[Z_1]$. By assumption, $f \in F \cap K[Z_1]$. Therefore the initial monomials of the elements in $F \cap K[Z_1]$ generate the ideal $\mathrm{in}(I \cap K[Z_1])$. In other words, $F \cap K[Z_1]$ is a Gröbner basis of the ideal $I \cap K[Z_1]$ with respect to the monomial order τ restricted to $K[Z_1]$. A Gröbner basis of an ideal is a set of generators.

LEMMA 1.3. *Let $K[Z]$ be a polynormal ring over a field K, and let τ be a term order. Let I and J be homogeneous ideals of $K[Z]$. Then*

(a) $\mathrm{in}(I) + \mathrm{in}(J) \subseteq \mathrm{in}(I+J)$ *and* $\mathrm{in}(I \cap J) \subseteq \mathrm{in}(I) \cap \mathrm{in}(J)$,

(b) $\mathrm{in}(I) + \mathrm{in}(J) = \mathrm{in}(I+J)$ *if and only if* $\mathrm{in}(I \cap J) = \mathrm{in}(I) \cap \mathrm{in}(J)$,

(c) *let F be a Gröbner basis of I and let G be a Gröbner basis of J, then $F \cup G$ is a Gröbner basis of $I+J$ if and only if, for all $f \in F$ and $g \in G$, there exists $h \in I \cap J$ such that* $\mathrm{in}(h) = \mathrm{lcm}(\mathrm{in}(f), \mathrm{in}(g))$.

Proof. (*a*) is trivial. One has

$$\begin{aligned}
&\dim \mathrm{in}(I+J)_i - \dim [\mathrm{in}(I) + \mathrm{in}(J)]_i \\
&\quad = \dim [I+J]_i - \dim [\mathrm{in}(I) + \mathrm{in}(J)]_i \\
&\quad = \dim I_i + \dim J_i - \dim [I \cap J]_i - \dim \mathrm{in}(I)_i - \dim \mathrm{in}(J)_i + \dim [\mathrm{in}(I) \cap \mathrm{in}(J)]_i \\
&\quad = \dim [\mathrm{in}(I) \cap \mathrm{in}(J)]_i - \dim \mathrm{in}(I \cap J)_i.
\end{aligned}$$

Then (*b*) follows from (*a*). Finally (*c*) is just (*b*) rewritten in terms of the generators of the ideals $\mathrm{in}(I) + \mathrm{in}(J)$ and $\mathrm{in}(I) \cap \mathrm{in}(J)$.

2. *Ladder determinantal rings, notation and preliminaries*

Let K be a field and $X = (X_{ij})$ an $m \times n$ matrix of indeterminates. Let $K[X]$ be the polynomial ring $K[X_{ij}: 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n]$. Given sequences of integers

$$1 \leqslant a_1 < \ldots < a_t \leqslant m \quad \text{and} \quad 1 \leqslant b_1 < \ldots < b_t \leqslant n,$$

we shall denote by $[a_1, \ldots, a_t \mid b_1, \ldots, b_t]$ the t-minor $\det(X_{a_i b_j})$ of X. The main diagonal of $[a_1, \ldots, a_t \mid b_1, \ldots, b_t]$ is the set $\{X_{a_1 b_1}, \ldots, X_{a_t b_t}\}$. A subset Y of X is called a *ladder* if whenever $X_{ij}, X_{hk} \in Y$ and $i \leqslant h, j \leqslant k$, then $X_{ik}, X_{hj} \in Y$.

Let Y be a ladder, and let $K[Y]$ be the polynomial ring $K[X_{ij}: X_{ij} \in Y]$. Denote by $R_t(Y)$ the quotient ring $K[Y]/I_t(Y)$, where $I_t(Y)$ is the ideal generated by all the t-minors of X which involve only indeterminates of Y. The ideal $I_t(Y)$ is called a *ladder determinantal ideal* and the ring $R_t(Y)$ a *ladder determinantal ring*.

We recall now a few facts from [**5, 7, 10, 14, 16**]. Let τ be the lexicographical monomial order on $K[X]$ induced by the total order

$$X_{11} > X_{12} > \ldots > X_{1n} > X_{21} > \ldots > X_{2n} > \ldots > X_{m-1n} > X_{m1} > \ldots > X_{mn}.$$

The initial monomial with respect to τ of a minor of X is the corresponding main diagonal. The following theorem was first proved by Narasimhan [**14**]. Subsequently Sturmfels, and Caniglia, Guccione and Guccione gave different proofs, see [**16, 5**].

THEOREM 2.1. *The set of all the t-minors of X is a Gröbner basis of $I_t(X)$ with respect to τ.*

As a consequence of this theorem Narasimhan proved that $I_t(Y) = I_t(X) \cap K[Y]$. The last equality implies that $R_t(Y) \subseteq R_t(X)$, and therefore that the ring $R_t(Y)$ is a domain since $R_t(X)$ is, see [**14**, 4.1]. Furthermore she deduced that the set of t-minors of Y is a Gröbner basis of $I_t(Y)$ with respect to τ, [**14**, 3.4]. Note that the ideal $\mathrm{in}(I_t(Y))$ of the initial monomials of $I_t(Y)$ is generated by the t-diagonals in Y. Hence $\mathrm{in}(I_t(Y))$ is a square free monomial ideal. The ring $K[Y]/\mathrm{in}(I_t(Y))$ is the Stanley–Reisner ring associated with the simplicial complex $\Delta_t(Y)$ of all the subsets of Y which do not contain t-diagonals. Studying this simplicial complex Herzog and Trung gave a characterization of the dimension and multiplicity of $R_t(Y)$ in terms of Y, and they proved that $R_t(Y)$ is Cohen–Macaulay, [**10**, 4.7, 4.8, 4.10]. Gröbner bases are also used in [**7**] to show that ladder determinantal rings are normal.

Throughout we identify the indeterminates of X with the points of the set $\{(i,j) \in \mathbb{N}^2: 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$. Similarly we identify ladders with subsets of points. In X we introduce two partial orders $\leqslant$ and $\preccurlyeq$. We define:

$$(i,j) \leqslant (h,k) \Leftrightarrow i \leqslant h \text{ and } j \leqslant k, \quad (i,j) \preccurlyeq (h,k) \Leftrightarrow i \geqslant h \text{ and } j \leqslant k.$$

It is clear that X is a distributive lattice with respect to both partial orders. Note that a subset Y of X is a ladder if and only if it is a sublattice of X with respect to $\preccurlyeq$. Since we are interested in the study of the ring $R_t(Y)$, without loss of generality we may assume that the following hold.

ASSUMPTION (a): $\max_{\preccurlyeq} Y = (1,n)$ and $\min_{\preccurlyeq} Y = (m,1)$. Otherwise we replace X with its smallest submatrix which contains Y.

ASSUMPTION (b): for all a with $1 \leqslant a \leqslant m$, there exists b with $1 \leqslant b \leqslant n$, such that $(a,b) \in Y$, and for all b with $1 \leqslant b \leqslant n$, there exists a with $1 \leqslant a \leqslant m$, such that

$(a, b) \in Y$. Otherwise we delete from X the rows and the columns which have an empty intersection with Y.

ASSUMPTION (c): all the entries of Y are involved in some t-minor of Y. Otherwise we get rid of the superfluous entries. It is easy to check that the resulting set is still a ladder.

Of course Assumptions (a) and (b) do not affect the ring $R_t(Y)$ at all. Assumption (c) does not affect the divisor class group and Gorensteinness since the old ring is just a polynomial extension of the new one.

We say that a ladder Y is t-disconnected if there exist two ladders Y_1, Y_2 such that $\emptyset \neq Y_1, Y_2 \subset Y$, $Y_1 \cap Y_2 = \emptyset$, $Y_1 \cup Y_2 = Y$ and every t-minor of Y is contained in Y_1 or in Y_2. If Y is t-disconnected we have $I_t(Y) = I_t(Y_1) + I_t(Y_2)$ and then

$$R_t(Y) = R_t(Y_1) \otimes_K R_t(Y_2).$$

The Cohen–Macaulay type of $R_t(Y)$ is equal to the Cohen–Macaulay type of $R_t(Y_1)$ times that of $R_t(Y_2)$. Thus, in order to characterize the Gorensteinness of $R_t(Y)$, it is not a restriction to consider only t-connected ladders Y. Also in the computation of the divisor class group of $R_t(Y)$, we may assume Y is t-connected. This is because the exact sequence that we shall use to compute the divisor class group splits when K is replaced by a normal domain.

So from now on we shall consider only t-connected ladders which satisfy Assumptions (a), (b) and (c). Furthermore, just to avoid trivial cases, we shall always assume that $t > 1$. It is easy to see that such ladders have the shape of the ladder in Figure 1.

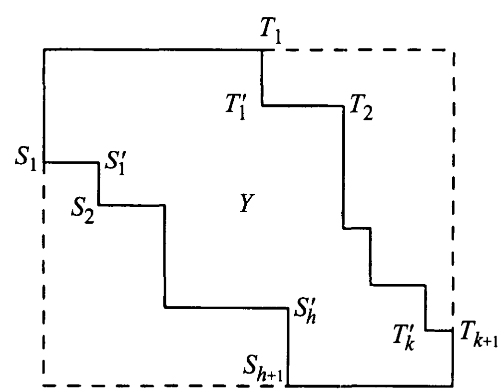

FIG. 1

We call the points $S_1', \dots, S_h'$ *inside lower corners*, $T_1', \dots, T_k'$ *inside upper corners*, $S_1, \dots, S_{h+1}$ *outside lower corners*, and $T_1, \dots, T_{k+1}$ *outside upper corners* of Y. For the following applications we fix the coordinates of these points:

$$S_i' = (a_i, b_i) \quad \text{for } i = 1, \dots, h, \qquad T_i' = (c_i, d_i) \quad \text{for } i = 1, \dots, k,$$
$$S_i = (a_{i-1}, b_i) \quad \text{for } i = 1, \dots, h+1, \qquad T_i = (c_i, d_{i-1}) \quad \text{for } i = 1, \dots, k+1.$$

For systematic reasons that will become apparent in Section 4, we set

$$a_0 = 1, \quad b_0 = n-t+2, \quad a_{h+1} = m-t+2, \quad b_{h+1} = 1,$$
$$c_0 = t-2, \quad d_0 = n, \quad c_{k+1} = m, \quad d_{k+1} = t-2$$

and introduce the points $S'_0 = (a_0, b_0)$, $S'_{h+1} = (a_{h+1}, b_{h+1})$, $T'_0 = (c_0, d_0)$, $T'_{k+1} = (c_{k+1}, d_{k+1})$.

We say that a ladder is a *one-sided ladder* if it has no inside lower corners (or no inside upper corners).

We define the *lower border* of Y to be the unique maximal chain with respect to $\preccurlyeq$ of Y which contains the points S_i for $i = 1, \dots, h+1$. Similarly one defines the upper border of Y.

The maximal elements under inclusion (facets) of the simplicial complex $\Delta_t(Y)$ associated with $\mathrm{in}(I_t(Y))$ are characterized in the following way. Set $Y_1 = Y$, and for all $i = 2, \dots, t-1$, define Y_i to be the ladder which is obtained from Y_{i-1} by deleting the lower border. The facets of $\Delta_t(Y)$ are of the form $G_1 \cup G_2 \cup \dots \cup G_{t-1}$, where G_i is a maximal chain with respect to $\preccurlyeq$ of Y_i, and $G_i \cap G_j = \emptyset$ if $i \neq j$, [**10**, 4.6].

The union of the lower borders of Y_i for $i = 1, \dots, t-1$, is said to be the *lower border with thickness* $(t-1)$ of Y. The dimension of $R_t(Y)$ is the cardinality of the lower border with thickness $(t-1)$ of Y.

The minor $[a, a+1, \dots, a+r-1 \mid b, b+1, \dots, b+r-1]$ is said to be the *r-minor based on* (a, b), while $[a-r+1, a-r+2, \dots, a \mid b-r+1, b-r+2, \dots, b]$ is said to be the *r-minor based under* (a, b).

For $i = 1, \dots, h$, consider the $(t-1)$-minor

$$[a_i, a_i+1, \dots, a_i+t-2 \mid b_i, b_i+1, \dots, b_i+t-2]$$

based on the inside lower corner S'_i of Y. Assume that either

$$[a_i, a_i+1, \dots, a_i+t-2 \mid b_i, b_i+1, \dots, b_i+t-2]$$

is not in Y or it contains a point of the upper border of Y. Then, because of Assumption (c) and since Y is t-connected, there exists a unique inside upper corner $T'_j = (c_j, d_j)$ such that $S'_i \leqslant T'_j \leqslant (a_i+t-2, b_i+t-2)$. Consider the ladders $Y_1 = \{(p, q) \in Y \colon (c_j, b_i) \preccurlyeq (p, q)\}$, $Y_2 = \{(p, q) \in Y \colon (p, q) \preccurlyeq (a_i, d_j)\}$, see Figure 2.

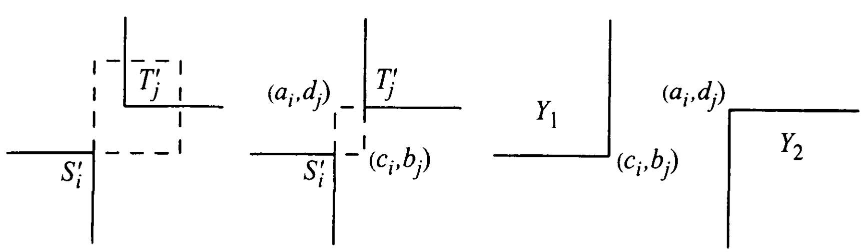

FIG. 2

The ring $R_t(Y)$ is (isomorphic to) $R_t(Y_1) \otimes_K R_t(Y_2)/I$, where I is generated by $(c_j - a_i + 1)(d_j - b_i + 1)$ linear forms. Comparing the dimensions of $R_t(Y)$, $R_t(Y_1)$, and $R_t(Y_2)$, one has that $\mathrm{height}\, I = (c_j - a_i + 1)(d_j - b_i + 1)$. The ring $R_t(Y_1) \otimes_K R_t(Y_2)$ is Cohen–Macaulay, so that I is generated by a regular sequence. Therefore the ring $R_t(Y)$ is Gorenstein if and only if both $R_t(Y_1)$, and $R_t(Y_2)$ are Gorenstein. Since we are interested on the Gorenstein property of $R_t(Y)$, without loss of generality we may assume the following.

ASSUMPTION (d). The $(t-1)$-minors based the inside lower corners of Y are in Y and do not contain points of the upper border of Y.

But in general one can not control the behaviour of the divisor class group under specialization.

3. *Some ideals of minors of* $R_t(Y)$

In this section we introduce and study certain ideals of the ring $R_t(Y)$ which are generated by residue classes of $(t-1)$-minors of certain subregions of Y. These ideals will play a fundamental role in the description of the divisor class group and of the canonical class of $R_t(Y)$.

From now on we fix τ to be the lexicographic monomial order induced by the total order $X_{11} > X_{12} > \dots > X_{1n} > X_{21} > \dots > X_{2n} > \dots > X_{m-1n} > X_{m1} > \dots > X_{mn}$. We consider a ladder Y and refer to the notation of Figure 1.

For a subset Z of the matrix X, we denote by $F_r(Z)$ the set of all the r-minors of X which involve only indeterminates of Z, and by $I_r(Z)$ the ideal generated by the elements of $F_r(Z)$.

The first class of ideal that we consider is the following. For all the inside upper corners T'_i of Y we define A_i to be the set $\{(a,b) \in Y: (a,b) \leqslant T'_i\}$.

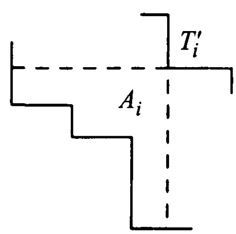

FIG. 3

Then we denote by $P_i(Y)$ the ideal $I_t(Y) + I_{t-1}(A_i)$ of $K[Y]$. Further set $\mathfrak{p}_i(Y) = P_i(Y)/I_t(Y)$. Whenever there is no danger of confusion we shall use the shorter notation P_i and $\mathfrak{p}_i$.

PROPOSITION 3.1. *For all the inside upper corners* T'_i *of* Y, *one has*
(a) $F_t(Y) \cup F_{t-1}(A_i)$ *is a Gröbner basis of* P_i *with respect to* τ;
(b) *the ideal* P_i *is prime, and* height P_i = height $I_t(Y) + 1$ *if* $P_i \neq I_t(Y)$.

Proof. Let Y_1 be the one-sided ladder with just one inside corner in $T'_i = (c_i, d_i)$. Let us denote by A the subset $\{(p,q) \in Y_1: (p,q) \leqslant (c_i, d_i)\}$ of Y_1. It is clear that $Y \subseteq Y_1$, and $A_i \subseteq A$, see Figure 4.

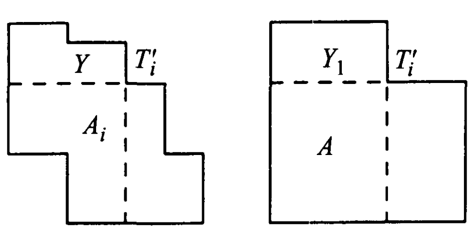

FIG. 4

Let us denote by P the ideal $P_1(Y_1) = I_t(Y_1) + I_{t-1}(A)$. By [7, 4.3], the set $F_t(Y_1) \cup F_{t-1}(A)$ is a Gröbner basis of P. Further if $M \in F_t(Y_1) \cup F_{t-1}(A)$, and $\mathrm{in}(M) \in K[Y]$, then $M \in K[Y]$. By 1.2, $(F_t(Y_1) \cup F_{t-1}(A)) \cap K[Y]$ is a Gröbner basis and a set of generators of $P \cap K[Y]$. But $(F_t(Y_1) \cup F_{t-1}(A)) \cap K[Y] = F_t(Y) \cup F_{t-1}(A_i)$, hence $P_i = P \cap K[Y]$. Therefore $F_t(Y) \cup F_{t-1}(A_i)$ is a Gröbner basis of P_i. Further P_i is prime, since P is [7, 4.6].

In order to compute the height of P_i, one considers the ladder which is obtained from Y by adding the point (c_i+1, d_i+1). Arguing as in the proof of [7, 4.4] one obtains the desired result.

Now we consider another class of ideals of $R_t(Y)$. Given a set Z of consecutive rows or columns of X, consider the ideal $I_t(Y)+I_{t-1}(Y \cap Z)$ generated by the set $F_t(Y) \cup F_{t-1}(Y \cap Z)$. We are mainly interested in the case when Z is a set of $(t-1)$ consecutive rows or columns. It is known that the ideal $I_t(X)+I_{t-1}(Z)$ is prime, see [**4**, 6.3]. We want to use 1.2 again in order to deduce that $I_t(Y)+I_{t-1}(Y \cap Z)$ is prime. But first we need to have a Gröbner basis of $I_t(X)+I_{t-1}(Z)$ with respect to τ.

PROPOSITION 3.2. *Let Z be a set of consecutive rows or columns of X. Then the set $F_t(X) \cup F_{t-1}(Z)$ is a Gröbner basis of $I_t(X)+I_{t-1}(Z)$ with respect to τ.*

Proof. Because of 1.1, it is enough to consider only the case in which Z is a set of consecutive rows. Let $Z = \{X_{ij}: a \leqslant i \leqslant b\}$, and set $X_1 = \{X_{ij}: a \leqslant i \leqslant m\}$, and $X_2 = \{X_{ij}: 1 \leqslant i \leqslant b\}$. By [**10**, 2.4], the sets $F_t(X)$, $F_{t-1}(Z)$, $F_t(X_1) \cup F_{t-1}(Z)$ are Gröbner bases with respect to τ. Because of 1.1 the same holds for $F_t(X_2) \cup F_{t-1}(Z)$. We apply 1.3(c), with $F = F_t(X)$ and $G = F_{t-1}(Z)$. Take $f = [\alpha_1, \ldots, \alpha_t \mid \beta_1, \ldots, \beta_t] \in F$, and $g = [\gamma_1, \ldots, \gamma_{t-1} \mid \delta_1, \ldots, \delta_{t-1}] \in G$. If $f \in F_t(X_1)$, since $F_t(X_1) \cup F_{t-1}(Z)$ is a Gröbner basis, there exists $h \in I_t(X_1) \cap I_{t-1}(Z)$, such that $\mathrm{in}(h) = \mathrm{lcm}(\mathrm{in}(f), \mathrm{in}(g))$. But

$$I_t(X_1) \cap I_{t-1}(Z) \subseteq I_t(X) \cap I_{t-1}(Z),$$

and therefore $h \in I_t(X) \cap I_{t-1}(Z)$ as desired. One argues similarly if $f \in F_t(X_2)$. So we may assume $f \notin F_t(X_1) \cup F_t(X_2)$, that is, $\alpha_1 < a$, and $\alpha_t > b$. If $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$, then $\mathrm{lcm}(\mathrm{in}(f), \mathrm{in}(g)) = \mathrm{in}(f)\mathrm{in}(g)$, and we may take $h = fg$. Hence we may also assume that $\gcd(\mathrm{in}(f), \mathrm{in}(g)) \neq 1$. In other words, the main diagonals of f and g share some element. Let V be one of these elements. Then there exist integers i and j, with $1 < i < t$, and $1 \leqslant j \leqslant t-1$, such that $V = (\alpha_i, \beta_i) = (\gamma_j, \delta_j)$. If $i < j$, then the first j points of the main diagonal of f, and the last $t-1-i$ of the main diagonal of g form a diagonal in X_2 with at least t elements, see Figure 5.

Therefore there exists a t-minor $f_1 \in F_t(X_2)$, such that $\mathrm{lcm}(\mathrm{in}(f_1), \mathrm{in}(g))$ divides $\mathrm{lcm}(\mathrm{in}(f), \mathrm{in}(g))$, say $\mathrm{lcm}(\mathrm{in}(f), \mathrm{in}(g)) = \mathrm{lcm}(\mathrm{in}(f_1), \mathrm{in}(g))\, v$. We know already that there exists $h_1 \in I_t(X) \cap I_{t-1}(Z)$, such that $\mathrm{in}(h_1) = \mathrm{lcm}(\mathrm{in}(f_1), \mathrm{in}(g))$. Hence $\mathrm{in}(vh_1) = \mathrm{lcm}(\mathrm{in}(f), \mathrm{in}(g))$, and $vh_1 \in I_t(X) \cap I_{t-1}(Z)$.

If $i \geqslant j$, then the first i points of the main diagonal of g and the last $t-j$ of the main diagonal of f form a diagonal of X_2 with at least t points, and we may proceed as before. This concludes the proof.

Now repeating the argument of 3.1 one obtains the following proposition.

PROPOSITION 3.3. *Let Y be a ladder and let Z be a set of consecutive rows or columns of Z. Then*

$$I_t(Y)+I_{t-1}(Y \cap Z) = [I_t(X)+I_{t-1}(Z)] \cap K[Y],$$

and $F_t(Y) \cup F_{t-1}(Y \cap Z)$ is a Gröbner basis of $I_t(Y)+I_{t-1}(Y \cap Z)$. In particular $I_t(Y)+I_{t-1}(Y \cap Z)$ is prime, since $I_t(X)+I_{t-1}(Z)$ is.

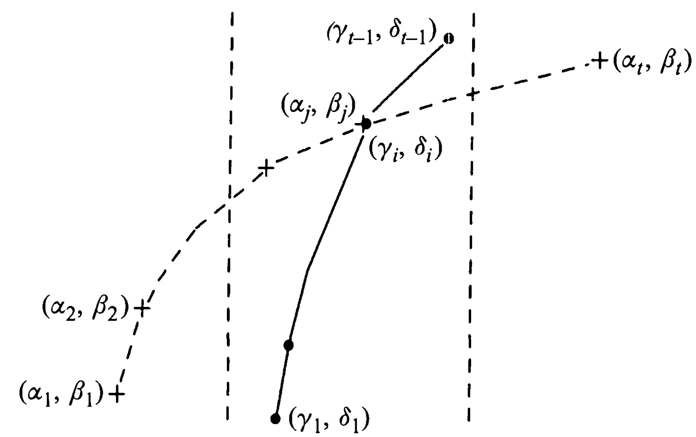

FIG. 5

We want to evaluate the height of $I_t(Y)+I_{t-1}(Y \cap Z)$. For our purpose we may restrict our attention to the case in which Z is a set of $(t-1)$ consecutive rows or columns.

PROPOSITION 3.4. *Let Y be a ladder, and let Z be a set of $(t-1)$ consecutive rows (respectively, columns) of X, say $Z=\{X_{ij}\colon a<i<a+t\}$, with $0 \leqslant a \leqslant m+1-t$ (respectively, $Z=\{X_{ij}\colon b<j<b+t\}$, with $0 \leqslant b \leqslant n+1-t$). One has the following.*

(a) *If $a=0$, or $a=m+1-t$, (respectively $b=0$ or $b=n+1-t$) then* $\text{height}\,[I_t(Y)+I_{t-1}(Y \cap Z)] = \text{height}\,I_t(Y)+1$.

(b) *If $0<a<m+1-t$, and there exists a', such that $(a,a'),(a+t,a'+t-2) \in Y$ (respectively if $0<b<n+1-t$, and there exists b', such that $(b',b),(b'+t-2,b+t) \in Y$), then* $\text{height}\,[I_t(Y)+I_{t-1}(Y \cap Z)] = \text{height}\,I_t(Y)+1$.

Proof. Assume that Z is the set of the first or last $(t-1)$ rows or columns. Then $I_t(Y)+I_{t-1}(Y \cap Z)$ can be interpreted as an ideal cogenerated by a minor in a ladder, as introduced by Herzog and Trung in **[10]**. So (a) is a consequence of **[10**, 4.2], and 1.1.

Under the hypothesis (b), the shape of the ladder locally looks like Figure 6.

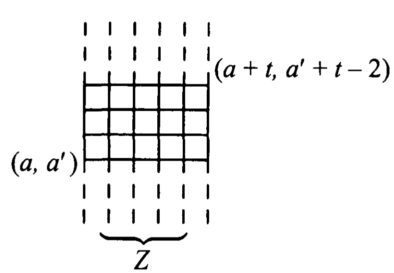

FIG. 6

Since $I_{t-1}(Y \cap Z) \neq 0$, then $\text{height}\,[I_t(Y)+I_{t-1}(Y \cap Z)] \geqslant \text{height}\,I_t(Y)+1$. In order to show that $\text{height}\,[I_t(Y)+I_{t-1}(Y \cap Z)] \leqslant \text{height}\,I_t(Y)+1$, we show that

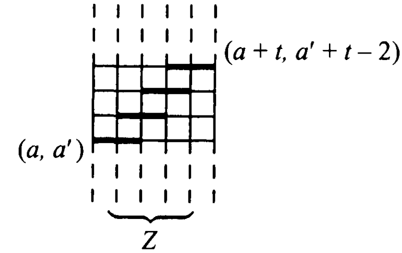

FIG. 7

$\operatorname{height} \operatorname{in}(I_t(Y)+I_{t-1}(Y \cap Z)) \leqslant \operatorname{height} \operatorname{in}(I_t(Y))+1$. Because of 3.3, the ideal $\operatorname{in}(I_t(Y)+I_{t-1}(Y \cap Z))$ is associated with the simplicial complex Δ of the subsets of Y which do not contain t-diagonals of Y and $(t-1)$-diagonals of $Y \cap Z$. Note that $\Delta \subset \Delta_t(Y)$, and $\Delta_t(Y)$ is pure. So it is enough to show that Δ has a facet which differs from a facet of $\Delta_t(Y)$ in exactly one point. This facet is constructed in the following way. Take a facet G of $\Delta_t(Y)$ which contains the points marked in Figure 7. The unique $(t-1)$-diagonal of points in $Z \cap G$ is

$$\{(a+1, a'), (a+2, a'+1), \ldots, (a+t-1, a'+t-2)\}.$$

Then consider $G' = G \backslash \{(a+t-1, a'+t-2)\}$. By construction $G' \in \Delta$.

In the case in which Z is a set of consecutive columns one argues similarly.

4. *The divisor class group of $R_t(Y)$*

We deal first with ladders which satisfy Assumption (d). At the end of this section we shall briefly indicate how to determine the divisor class group of $R_t(Y)$ when Y does not satisfy Assumption (d).

One of the simplifications that we get from Assumption (d) is that the height of the ideals $I_t(Y)+I_{t-1}(Y \cap Z)$, with Z a set of $(t-1)$ consecutive rows or columns of X, and the height of the ideals P_i, is always one more than that of $I_t(Y)$.

For all $i = 1, \ldots, h+1$, denote by F_i the $(t-1)$-minor based on the outside lower corner S_i of Y. Similarly, denote by G_i the $(t-1)$-minor based under the outside upper corner T_i of Y. Further denote by f_i and g_i the residue classes of F_i and G_i in $R_t(Y)$. Set $f = f_1 f_2 \ldots f_{h+1}$, and $F = F_1 F_2 \ldots F_{h+1}$. We begin with the following lemma.

LEMMA 4.1. *Let B be the set of points of the lower border with thickness $(t-1)$ of Y, and let C be the set of the points of the upper border with thickness $(t-1)$ of Y. Let $Y_1 = \{P \in Y: P \preccurlyeq (a_1, b_1+t-2)\}$, let $Y_2 = \{P \in Y: P \preccurlyeq (c_1-t+2, d_1)\}$, let $B_1 = B \backslash Y_1$, and let $C_1 = C \backslash Y_2$. Then Y_1 and Y_2 are subladders of Y, and one has*

$$R_t(Y)[f_1^{-1}] \simeq R_t(Y_1)[B_1][f_{11}^{-1}], \tag{1}$$

$$R_t(Y)[g_1^{-1}] \simeq R_t(Y_2)[C_1][g_{11}^{-1}], \tag{2}$$

$$R_t(Y)[f^{-1}] \simeq K[B][F^{-1}], \tag{3}$$

where $K[B]$ is the polynomial ring over the field K in the set of indeterminates B. The rings $R_t(Y_1)[B_1]$ and $R_t(Y_2)[C_1]$ are the polynomial extensions of $R_t(Y_1)$ and $R_t(Y_2)$ with the indeterminates in the sets B_1 and C_1. Further f_{11} is the residue class in $R_t(Y_1)[B_1]$ of the minor F_1, and g_{11} is the residue class in $R_t(Y_2)[C_1]$ of the minor G_1.

Proof. Clearly Y_1 and Y_2 are ladders. Consider $K[B_1 \cup Y_1]$, the K-subalgebra of $R_t(Y)$ generated by the residue classes of the elements in the set $B_1 \cup Y_1$. For all $(a,b) \in Y \setminus B_1 \cup Y_1$, the t-minor which is obtained from F_1 by adding the row with index a and the column with index b is in Y. Expanding this minor with respect to the last row we get $X_{ab} F_1 = H \bmod I_t(Y)$, where H is a polynomial which involves only indeterminates in $B_1 \cup Y_1$. It follows that $f_1 x_{ab} \in K[B_1 \cup Y_1]$. Therefore $K[B_1 \cup Y_1][f_1^{-1}] = R_t(Y)[f_1^{-1}]$. By the dimension formula of Herzog and Trung [**10**, 4.7], one has $\dim R_t(Y_1) + |B_1| = \dim R_t(Y)$. Therefore the only relations in $K[B_1 \cup Y_1]$ are the t-minors of the ladder Y_1. One concludes that $K[B_1 \cup Y_1][f_1^{-1}] \simeq R_t(Y_1)[B_1][f_{11}^{-1}]$. Similarly one proves (2).

Since $R_t(Y)[f^{-1}] = R_t(Y)[f_1^{-1}, f_2^{-1}, \dots, f_{h+1}^{-1}]$, (3) follows from (1) by induction on h. One has only to note that the outside lower corners of Y_1 are $S_2, \dots, S_{h+1}$, and that f_i, as an element of $K[B]$, is the determinant of the corresponding matrix of indeterminates, that is, F_i.

We have seen that, after inversion of $f_1, \dots, f_{h+1}$, the ring $R_t(Y)$ becomes a factorial ring. Hence, by Nagata's theorem [**8**, 7.2], the divisor class group of $R_t(Y)$ is generated by the classes of the minimal prime ideals of $f_1, \dots, f_{h+1}$.

We know already some minimal prime ideals of f_i. Denote by Z_i the set of the rows, and by Z_i' the set of columns of the minor F_i. Consider the ideals

$$Q_i(Y) = I_t(Y) + I_{t-1}(Y \cap Z_i) \quad \text{and} \quad Q_i'(Y) = I_t(Y) + I_{t-1}(Y \cap Z_i').$$

Denote by $\mathfrak{q}_i(Y)$ and $\mathfrak{q}_i'(Y)$ the ideals $Q_i(Y)/I_t(Y)$ and $Q_i'(Y)/I_t(Y)$ of $R_t(Y)$. Further set $J_i = \{j: (a_i + t - 2, b_i + t - 2) \leqslant T_j'\}$. By definition, $J_i = \{j: f_i \in \mathfrak{p}_j\}$, and it is clear that J_i is a set of consecutive indices.

Because of 3.1, 3.3 and 3.4, the ideals $\mathfrak{q}_i$, $\mathfrak{q}_i'$, and $\mathfrak{p}_j$ for $j \in J_i$, are height 1 prime ideals of $R_t(Y)$, and they contain f_i. We want to show that they are the only minimal prime ideals of f_i.

In order to be more flexible, and to use inductive arguments, we enlarge the class of ideals under consideration. Let S be a point of the lower border of Y which has first coordinate 1, say $S = (1, b)$. Assume that $b_1 \leqslant b \leqslant n - t + 2$. Let W be the $(t-1)$-minor based on S, and denote by w its residue class in $R_t(Y)$. Define J_S to be set $\{j: w \in \mathfrak{p}_j\}$. For all $j \in J_S$, let $A_j' = \{P \in Y: S_1 \leqslant P \leqslant T_j'\}$, and let $I_j = I_t(Y) + I_{t-1}(A_j')$. Then denote by $\mathscr{I}_j$ the ideal $I_j/I_t(Y)$ of $R_t(Y)$. Further denote by Z_S the set of rows, and by Z_S' the set of columns of W. Let $Q_S(Y) = I_t(Y) + I_{t-1}(Y \cap Z_S)$ and $Q_S'(Y) = I_t(Y) + I_{t-1}(Y \cap Z_S')$. Finally set $\mathfrak{q}_S(Y) = Q_S(Y)/I_t(Y)$, and $\mathfrak{q}_S'(Y) = Q_S'(Y)/I_t(Y)$.

Whenever there is no danger of confusion we shall use the shorter notation Q_i, $\mathfrak{q}_i$, Q_S, $\mathfrak{q}_S$, Q_i', etc....

We show that the ideals $\mathfrak{q}_i$, $\mathfrak{q}_i'$, $\mathfrak{p}_j$ behave well with respect to the isomorphisms (1) and (2) of 4.1.

LEMMA 4.2. (a) *Let $Y_1 = \{P \in Y: P \preccurlyeq (a_1, b_1 + t - 2)\}$, and let j' be the integer such that $J_1 = \{1, \dots, j'\}$. Consider the isomorphism $R_t(Y)[f_1^{-1}] \simeq R_t(Y_1)[B_1][f_{11}^{-1}]$. Then we have the following.*

(i) *For all $i = 2, \dots, h+1$, the ideals $\mathfrak{q}_i(Y) R_t(Y)[f_1^{-1}]$, and $\mathfrak{q}_i'(Y) R_t(Y)[f_1^{-1}]$ are mapped to the extensions of the ideals $\mathfrak{q}_{i-1}(Y_1)$, and $\mathfrak{q}_{i-1}'(Y_1)$.*

(ii) *For all $j > j'$, the ideal $\mathfrak{p}_j(Y) R_t(Y)[f_1^{-1}]$ is mapped to the extension of the ideal $\mathfrak{p}_{j-j'}(Y_1)$.*

(b) *Let* $Y_2 = \{P \in Y\colon P \preccurlyeq (c_1 - t + 2, d_1)\}$, *and let* $i_1 = \min\{i\colon a_i > c_1 + t - 2\}$. *Consider the isomorphism* $R_t(Y)[g_1^{-1}] \simeq R_t(Y_2)[C_1][g_{11}^{-1}]$. *Then we have the following.*

(i) *For all* $i = 1, \dots, i_1$, *the ideal* $\mathfrak{q}_i(Y) R_t(Y)[g_1^{-1}]$ *is mapped to the extension of the principal ideal generated by the* $(t-1)$*-minor of* C_1 *based under* $(a_{i-1} + t - 2, n)$, *and the ideal* $\mathfrak{q}'_i(Y) R_t(Y)[g_1^{-1}]$ *is mapped to the extension of the ideal* $\mathfrak{q}'_{U_i}(Y_2)$, *where* U_i *is the point of the lower border of* Y_2 *with coordinates* $(c_1 - t + 2, b_i)$.

(ii) *For all* $i = i_1 + 1, \dots, h+1$, *the ideals* $\mathfrak{q}_i(Y) R_t(Y)[g_1^{-1}]$ *and* $\mathfrak{q}'_i(Y) R_t(Y)[g_1^{-1}]$ *are mapped to the extensions of the ideals* $\mathfrak{q}_{i+1-i_1}(Y_2)$, *and* $\mathfrak{q}'_{i+1-i_1}(Y_2)$.

(iii) *The ideal* $\mathfrak{p}_1(Y) R_t(Y)[g_1^{-1}]$ *is mapped to the extension of the ideal* $\mathfrak{q}_1(Y_2)$, *and for all* $i = 2, \dots, k$, *the ideal* $\mathfrak{p}_i(Y) R_t(Y)[g_1^{-1}]$ *is mapped to the extension of* $\mathfrak{p}_{i-1}(Y_2)$.

Proof. For all $i = 2, \dots, h+1$, the image of the ideal $\mathfrak{q}_i(Y) R_t(Y)[f_1^{-1}]$ contains the ideal $\mathfrak{q}_{i-1}(Y_1) R_t(Y_1)[B_1][f_{11}^{-1}]$. Both are height 1 prime ideals, and so equality holds. The same argument works in all the other cases of (a) and (b), except one. In the statement (i) of (b), if $a_{i_1-1} = c_1 - t + 2$, then $g_1 \in \mathfrak{q}_{i_1}(Y)$ and therefore

$$\mathfrak{q}_{i_1}(Y) R_t(Y)[g_1^{-1}] = R_t(Y)[g_1^{-1}].$$

On the other hand, the minor of C_1 based under $(a_{i_1-1} + t - 2, n) = (c_1, n) = T_1$ is g_{11}, so that the assertion holds in this special case, too.

For later applications we need the following.

LEMMA 4.3. *Let* A *be a commutative ring with* 1, *and let* U *be a* $p \times q$ *matrix with entries in* A *and* $\operatorname{rank} U < t$. *Then for all* $1 \leqslant \gamma_{11} < \dots < \gamma_{1t-1} \leqslant p$, $1 \leqslant \gamma_{21} < \dots < \gamma_{2t-1} \leqslant p$, $1 \leqslant \delta_{11} < \dots < \delta_{1t-1} \leqslant q$, $1 \leqslant \delta_{21} < \dots < \delta_{2t-1} \leqslant q$, *one has:*

$$\begin{aligned}&[\gamma_{11}, \dots, \gamma_{1t-1} \mid \delta_{11}, \dots, \delta_{1t-1}]_U [\gamma_{21}, \dots, \gamma_{2t-1} \mid \delta_{21}, \dots, \delta_{2t-1}]_U \\ &\qquad = [\gamma_{11}, \dots, \gamma_{1t-1} \mid \delta_{21}, \dots, \delta_{2t-1}]_U [\gamma_{21}, \dots, \gamma_{2t-1} \mid \delta_{11}, \dots, \delta_{1t-1}]_U\end{aligned}$$

Proof. We may assume A is $\mathbb{Z}[X]/I_t(X)$ with X a generic $p \times q$ matrix and U is the residue class of X in A. After rows and columns permutations, we may also assume that $\gamma_{1i} = \delta_{2i} = i$ for all $i = 1, \dots, t-1$. Then the desired equality is obtained applying the straightening law, [**4**, 4]. Alternatively, one may note that the equation says that the 2-minors of $\bigwedge^{t-1} U$ vanish, and that holds since $\operatorname{rank} \bigwedge^{t-1} U \leqslant 1$.

LEMMA 4.4. *Let* $S = (1, b)$ *be a point of the lower border of* Y *with* $b_1 \leqslant b \leqslant n - t + 2$ *and let* w *be the residue class in* $R_t(Y)$ *of the* $(t-1)$*-minor based on* S. *Then*

$$\mathfrak{q}_S \mathfrak{q}'_S \prod_{j \in J_S} \mathcal{I}_j \subset (w).$$

Proof. If $J_S = \varnothing$, then by 4.3, $\mathfrak{q}_S \mathfrak{q}'_S \subset (w)$. If $J_S \neq \varnothing$, then there exists j' such that $J_S = \{j\colon 1 \leqslant j \leqslant j'\}$. The sets $Z_S \cap Y$ and A'_1 are contained in the submatrix $\{P \in Y\colon S_1 \leqslant P \leqslant T'_1\}$ of Y. Furthermore for all $j = 1, \dots, j'-1$, the sets $Z_S \cap A'_j$ and A'_{j+1} are contained in $\{P \in Y\colon S_1 \leqslant P \leqslant T'_{j+1}\}$ which is a submatrix of Y, too.

By induction on j for $j = 1, \dots, j'$, and using 4.3, one shows that $\mathfrak{q}_S \mathcal{I}_1 \cdots \mathcal{I}_j \subset I_{t-1}(Z_S \cap A'_j) + I_t(Y)/I_t(Y)$. Finally, since the set of the points of $Z'_S \cap Y$ which are involved in some $(t-1)$-minor of $Z'_S \cap Y$, and the set $Z_S \cap A'_{j'}$, are both contained in $\{P \in Y\colon S_1 \leqslant P \leqslant T'_{j'}\}$, one may again use 4.3, and one obtains $\mathfrak{q}_S \mathfrak{q}'_S \prod_{j \in J_S} \mathcal{I}_j \subset (w)$.

Of course the lemma holds also if we consider $(t-1)$-minors based under the upper border of Y. As a consequence we have the following corollary.

COROLLARY 4.5. *Let $S = (1, b)$ be a point of the lower border of Y with $b_1 \leqslant b \leqslant n-t+1$, and let w be the residue class in $R_t(Y)$ of the $(t-1)$-minor based on S. Then* $\operatorname{height}(w, g_1) = 2$.

Proof. Let $\mathfrak{p} = P/I_t(Y)$ be a prime ideal of $R_t(Y)$ which contains w and g_1. Then, by 4.4, $\mathfrak{p}$ contains one of the ideals $\mathfrak{q}_S$, $\mathfrak{q}'_S$, $\mathscr{I}_j$ for $j \in J_S$. Suppose that $\mathfrak{p}$ contains $\mathfrak{q}_S$ or $\mathfrak{q}'_S$. Then $\operatorname{height} \mathfrak{p} \geqslant 2$, since $\mathfrak{q}_S$ and $\mathfrak{q}'_S$ are prime ideals and do not contain g_1. Therefore we may assume that $J_S \neq \emptyset$ and $\mathscr{I}_\alpha \subset \mathfrak{p}$ for some $\alpha \in J_S$. We may argue in the same way also with respect to g_1, and hence we may assume that P contains the ideal $I_t(Y) + I_{t-1}(D_{\alpha\beta})$, where the region $D_{\alpha\beta}$ is contained in Y and has shape as in Figure 8.

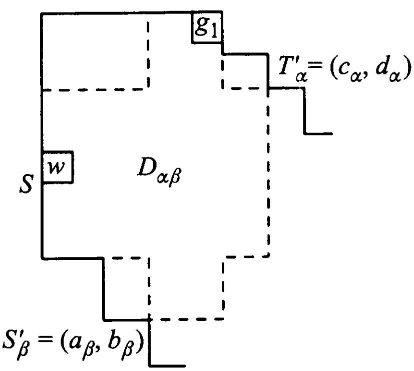

FIG. 8

It suffices to show that $\operatorname{height} I_t(Y) + I_{t-1}(D_{\alpha\beta}) > \operatorname{height} I_t(Y) + 1$. To this end let us consider the ideal L generated by the leading monomials of the elements in $F_t(Y) \cup F_{t-1}(D_{\alpha\beta})$. Then $L \subseteq \operatorname{in}(I_t(Y) \cup I_{t-1}(D_{\alpha\beta}))$, and it is enough to show that $\operatorname{height} L > \operatorname{height} \operatorname{in}(I_t(Y)) + 1$. Let Δ be the simplicial complex associated with L, and let G be a facet of Δ. Since Δ is a subcomplex of the simplicial complex $\Delta_t(Y)$ associated with $\operatorname{in}(I_t(Y))$, there exists a facet H of $\Delta_t(Y)$ which contains G. We claim that $|H \backslash G| > 1$. The claim implies that $\operatorname{height} L > \operatorname{height} \operatorname{in}(I_t(Y)) + 1$.

Now we prove the claim. We know that H has a unique decomposition as $H_1 \cup H_2 \cup \ldots \cup H_{t-1}$, where H_i is a maximal chain of Y_i and $H_i \cap H_j = \emptyset$ if $i \neq j$. Set $p_i = \min\{p : (p, d_\alpha + i + 1 - t) \in H_i\}$, and $E_i = (p_i, d_\alpha + i + 1 - t)$ for all $i = 1, \ldots, t-1$. Note that p_i is well defined since every maximal chain in Y_i contains at least one point in every row and column of Y_i. Further $p_1 < p_2 < \ldots < p_{t-1}$, and $E_i = (p_i, d_\alpha + i + 1 - t) \in D_{\alpha\beta}$. The points $E_1, \ldots, E_{t-1}$ form a $(t-1)$-diagonal in $D_{\alpha\beta}$. Therefore $\{E_1, \ldots, E_{t-1}\} \not\subset G$. Then there exists i such that $E_i \notin G$. Note that the point E_i in the chain H_i precedes the point $(p_i, d_\alpha + i + 2 - t)$. Repeating this argument with respect to S_β, and exchanging the role between rows and columns, we find a point $E'_j = (a_\beta + j, q_j) \in H_j \backslash G$ and such that E'_j in the chain H_j precedes $(a_\beta + j - 1, q_j)$. Hence $E_i \neq E'_j$ and $|H \backslash G| > 1$.

Then we obtain the following proposition.

PROPOSITION 4.6. *Let $S = (1, b)$ be a point of the lower border of Y with $b_1 \leqslant b \leqslant n-t+2$, and let w be the residue class in $R_t(Y)$ of the $(t-1)$-minor based on S. Then the ideal (w) is radical and its minimal prime ideals are $\mathfrak{q}_S$, $\mathfrak{q}'_S$, and $\mathfrak{p}_j$ with $j \in J_S$.*

Proof. First assume that $J_S = \emptyset$. By virtue of 4.4, $\mathfrak{q}_S \mathfrak{q}'_S \subset (w)$ and the desired conclusion follows from the normality of $R_t(Y)$.

Now assume that $J_S \neq \emptyset$, say $J_S = \{j\colon 1 \leqslant j \leqslant j'\}$. In this case $b < n-t+2$, and w, g_1 is a regular sequence. To simplify the notation, set $A = R_t(Y)[g_1^{-1}]$. So it is enough to show that $(w)A$ is radical and that its minimal prime ideals are $\mathfrak{q}_S A$, $\mathfrak{q}'_S A$ and $\mathfrak{p}_j A$ with $j \in J_S$.

Because of 4.1(2), we have $A \simeq R_t(Y_2)[C_1][g_{11}^{-1}]$, where

$$Y_2 = \{P \in Y\colon P \preccurlyeq (c_1 - t + 2, d_1)\},$$

C is the upper border with thickness $(t-1)$, and $C_1 = C \setminus Y_2$. By 4.3, we have $wg_1 = w'v$, where w' is the $(t-1)$-minor based on the point $S' = (c_1 + 2 - t, b)$ of the lower border of Y_2, and v is the $(t-1)$-minor of Y based on $(1, n+2-t)$, see Figure 9.

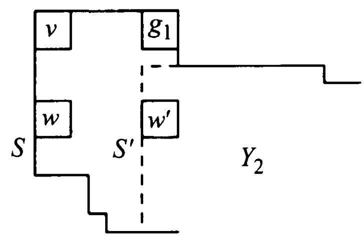

FIG. 9

In the ring A, we have $w = g_1^{-1} v w'$. The element v is a prime element in $R_t(Y_2)[C_1][g_{11}^{-1}]$ since it is the determinant of a matrix of indeterminates. By induction on the number of the inside upper corners, we may assume that (w') is radical in $R_t(Y_2)$, and that its minimal prime ideals are $\mathfrak{q}_{S'}(Y_2)$, $\mathfrak{q}'_{S'}(Y_2)$ and $\mathfrak{p}'_i(Y_2)$ with $j \in J_{S'} = \{j\colon 1 \leqslant j \leqslant j'-1\}$. Since $(w')A$ and $(v)A$ are radical ideals with no common minimal primes, it follows that $(w)A$ is radical, and its minimal prime ideals are $(v)A$ and those of $(w')A$. As in the proof of 4.2 one shows that the ideals $(v)A$, $\mathfrak{q}_{S'}(Y_2)A$, $\mathfrak{q}'_{S'}(Y_2)A$, $\mathfrak{p}_1(Y_2)A, \ldots, \mathfrak{p}_{j'-1}(Y_2)A$ coincide with the ideals $\mathfrak{q}_S A$, $\mathfrak{q}'_S A$, $\mathfrak{p}_1 A, \ldots, \mathfrak{p}_{j'} A$, respectively.

PROPOSITION 4.7. *For all $i = 1, \ldots, h+1$, the ideal (f_i) of $R_t(Y)$ is radical and its minimal prime ideals are $\mathfrak{q}_i$, $\mathfrak{q}'_i$ and $\mathfrak{p}_j$ with $j \in J_i$.*

Proof. We argue by induction on the number of inside lower corners h. If $h = 0$, the statement is a particular case of 4.6. Let $h > 0$, and let $1 \leqslant i \leqslant h+1$. For $i = 1$, the statement is again a particular case of 4.6. So we may assume that $i > 1$. First of all we determine the minimal prime ideals of f_i. The minimal prime ideals of f_i in $R_t(Y)$ are the minimal prime ideals of f_i in $R_t(Y)[f_1^{-1}]$, together with the minimal prime ideals of f_1 which contain f_i. We know the minimal prime ideals of f_1; to determine those of $f_i R_t(Y)[f_1^{-1}]$, we use the isomorphism 4.1(1). By induction and using 4.2, we obtain the desired result.

Since $R_t(Y)$ is normal, the ring $R_t(Y)_\mathfrak{p}$ is a DVR for all height 1 prime ideal $\mathfrak{p}$ of $R_t(Y)$. Let us denote by $v_\mathfrak{p}$ the discrete valuation on $R_t(Y)_\mathfrak{p}$. In order to show that (f_i) is radical, it is enough to show that $v_\mathfrak{p}(f_i) = 1$ for all the minimal prime ideals $\mathfrak{p}$ of f_i. By induction, (f_i) is radical in $R_t(Y)[f_1^{-1}]$ and in $R_t(Y)[g_1^{-1}]$. But height$(f_1, g_1) = 2$, and then $v_\mathfrak{p}(f_i) = 1$, for all the minimal prime ideals $\mathfrak{p}$ of f_i.

We are ready to determine the divisor class group of $R_t(Y)$.

THEOREM 4.8. *Assume that Y satisfies Assumption* (d). *Then the divisor class group* $\mathrm{Cl}(R_t(Y))$ *of* $R_t(Y)$ *is free of rank* $h+k+1$. *Furthermore*

$$\mathrm{cl}(\mathfrak{q}_1), \ldots, \mathrm{cl}(\mathfrak{q}_{h+1}), \mathrm{cl}(\mathfrak{p}_1), \ldots, \mathrm{cl}(\mathfrak{p}_k)$$

form a basis of $\mathrm{Cl}(R_t(Y))$.

Proof. By virtue of Nagata's theorem [**8**, 7.2] and 4.1(3), $\mathrm{Cl}(R_t(Y))$ is generated by the classes of the minimal prime ideals of $f = f_1 \ldots f_{h+1}$. We know already the minimal prime ideals of f: they are $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_{h+1}, \mathfrak{q}'_1, \ldots, \mathfrak{q}'_{h+1}, \mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$. Since (f_i) is radical in $R_t(Y)$, we have

$$\mathrm{cl}(\mathfrak{q}_i) + \mathrm{cl}(\mathfrak{q}'_i) + \sum_{j \in J_i} \mathrm{cl}(\mathfrak{p}_j) = 0. \qquad (i)$$

Repeating the argument of [**4**, pp. 94] one shows that the relations between the classes of the minimal prime ideals of f are linear combinations of the relations (i). Then using the relation (i), we get rid of $\mathrm{cl}(\mathfrak{q}'_i)$ and the desired result follows.

The basis of $\mathrm{Cl}(R_t(Y))$ which is given in 4.8 is in some sense the most natural. We will call it 'the basis' of $\mathrm{Cl}(R_t(Y))$.

Now we want to indicate briefly how to determine the divisor class group of $R_t(Y)$ in the general case, that is, when Assumption (d) need not to be satisfied. We classify the inside corners of the ladder Y in two types.

We say that the inside lower corner $S'_i = (a_i, b_i)$ is of type 1 if the $(t-1)$-minor based on S'_i is contained in the ladder Y and contains at most one point of the upper border (note that, if this is the case, this point must be (a_i+t-2, b_i+t-2)).

Moreover we say an inside lower corner is of type 2 if it is not of type 1. Similarly one defines inside upper corners of type 1, and 2. Because of Assumption (c) and since Y is t-connected, the inside lower corners of type 2 are in one-to-one correspondence with the inside upper corners of type 2.

Figure 10 illustrates two examples of corners of type 1, and two examples of corners of type 2.

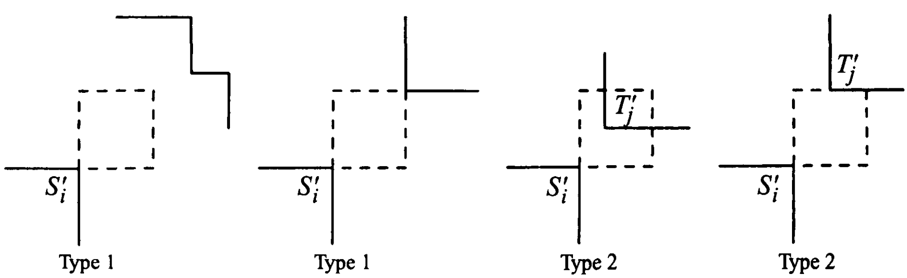

FIG. 10

Let h^* be the number of inside lower corners of type 1, and let k^* be the number of inside upper corners of type 1, of Y.

The isomorphisms of 4.1 do not depend on Assumption (d). One can determine the minimal prime ideals of f_i, and prove that (f_i) is radical using the same sort of arguments as in 4.4, 4.5, 4.6, and 4.7. It turns out that each inside upper corner of type 1 determines one minimal prime ideal of $f = f_1 \ldots f_{h+1}$, while each inside lower corner of type 1 determines two minimal prime ideals of f. Further each pair of corresponding inside lower corner and inside upper corner of type 2 determines two minimal prime ideals of f. The set of the minimal prime ideals of f is complete if one considers also the ideal generated by the $(t-1)$-minors of the first $(t-1)$ rows, and the ideal generated by the $(t-1)$-minors of the first $(t-1)$ columns. So f has $2h^*+k^*+2(h-h^*)+2$ minimal prime ideals. Since (f_i) is radical, one has $h+1$ relations between the classes of these ideals in the divisor class group. Again these relations generate all the relations. Therefore the divisor class group is free of rank $h+k^*+1$.

5. *Canonical class and Gorenstein property of* $R_t(Y)$

Let Y be a ladder which satisfies Assumption (d). Let $\mathrm{cl}(\omega)$ be the canonical class of $R_t(Y)$ and let $\mathrm{cl}(\omega)=\sum_{i=1}^{h+1}\lambda_i\,\mathrm{cl}(\mathfrak{q}_i)+\sum_{j=1}^{k}\delta_j\,\mathrm{cl}(\mathfrak{p}_j)$ be the unique representation of $\mathrm{cl}(\omega)$ with respect to the basis of $\mathrm{Cl}(R_t(Y))$. Our goal is to express the coefficients λ_i and δ_j in terms of the shape of the ladder. This has already been done by the present author if Y is a one-sided ladder or if $t=2$ [7, 2.4, 4.9]. As we shall see, the general case can be reduced to the case of a one-sided ladder by means of suitable localizations.

THEOREM 5.1. *Let* $\mathrm{cl}(\omega)$ *be the canonical class of* $R_t(Y)$, *and let*

$$\mathrm{cl}(\omega)=\sum_{i=1}^{h+1}\lambda_i\,\mathrm{cl}(\mathfrak{q}_i)+\sum_{j=1}^{k}\delta_j\,\mathrm{cl}(\mathfrak{p}_j)$$

be the unique representation of $\mathrm{cl}(\omega)$ *with respect to the basis of* $\mathrm{Cl}(R_t(Y))$. *Set* $i_j=\min\{i\colon 1\leqslant i\leqslant h+1, a_i>c_j-t+2\}$. *Then:*

$$\lambda_i=(a_i+b_i)-(a_{i-1}+b_{i-1}),$$
$$\delta_j=(a_{i_j}+b_{i_j}+t-2)-(c_j+d_j-t+2).$$

Proof. We first determine λ_i. Denote by B the lower border with thickness $(t-1)$ of Y. Consider the set $Y_1=\{p\in Y\colon (a_i+t-2,b_i)\preccurlyeq p\preccurlyeq(a_{i-1},b_{i-1}+t-2)\}$, and put $Z=B\cup Y_1$. Note that Y_1 is a one-side ladder and its outside lower corner is S_i. Figure 11 illustrates these sets by one example; here i is taken to be 2 and the shaded region represents $Z\setminus Y_1$.

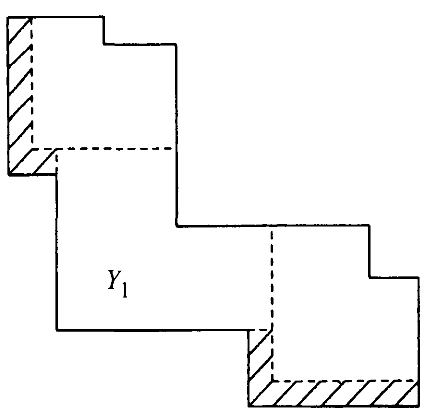

FIG. 11

Denote by $K[Z]$ the K-subalgebra of $R_t(Y)$ generated by the elements of Z. As in the proof of 4.1 one shows that $R_t(Y)[f_j^{-1}\colon j\neq i]=K[Z][f_j^{-1}\colon j\neq i]$. Again by dimension considerations, the only relations among the elements of Z are the t-minors of Y_1. Therefore we have an isomorphism

$$\alpha\colon R_t(Y)[f_j^{-1}: j \neq i] \longrightarrow R_t(Y_1)[Z\setminus Y_1][z_j^{-1}: j \neq i],$$

where the elements z_j are the images of the f_j. For simplicity of notation we set $A = R_t(Y_1)[Z\setminus Y_1][z_j^{-1}: j \neq i]$. The element z_j does not belong to any of the minimal prime ideals of f_i in $R_t(Y_1)[Z\setminus Y_1]$, so that f_i, z_j is a regular sequence in $R_t(Y_1)[Z\setminus Y_1]$. Since z_j is a matrix of indeterminates (hence a prime element) in $R_t(Y_1)[f_i^{-1}, Z\setminus Y_1]$, it follows that z_j is prime in $R_t(Y_1)[Z\setminus Y_1]$. By Nagata's theorem [**8**, 7.2 and 8.1] the canonical map

$$\psi\colon \mathrm{Cl}(R_t(Y_1)) \longrightarrow \mathrm{Cl}(A)$$

is an isomorphism. Composing the natural surjection

$$\mathrm{Cl}(R_t(Y)) \longrightarrow \mathrm{Cl}(R_t(Y)[f_j^{-1}: j \neq i]) \simeq \mathrm{Cl}(A)$$

with ψ^{-1}, we obtain a surjection

$$\phi\colon \mathrm{Cl}(R_t(Y)) \longrightarrow \mathrm{Cl}(R_t(Y_1)).$$

It is easy to control the behaviour of the elements of the basis of $\mathrm{Cl}(R_t(Y))$ under ϕ.

(1) If $j \neq i$ then $\mathfrak{q}_j R_t(Y)[f_j^{-1}: j \neq i] = R_t(Y)[f_j^{-1}: j \neq i]$ and the class $\mathrm{cl}(\mathfrak{q}_j)$ is mapped to 0.

(2) The (image under α of the) ideal $\mathfrak{q}_i R_t(Y)[f_j^{-1}, j \neq i]$ contains $\mathfrak{q}_0(Y_1)A$. Hence they coincide because both are prime ideals of height 1. It follows that ϕ maps $\mathrm{cl}(\mathfrak{q}_i)$ to $\mathrm{cl}(\mathfrak{q}_0(Y_1))$.

(3) If $\mathfrak{p}_s$ contains one of the f_j for $j \neq i$, then $\mathrm{cl}(\mathfrak{p}_s)$ is mapped 0. The other classes $\mathrm{cl}(\mathfrak{p}_s)$ are mapped bijectively to the $\mathrm{cl}(\mathfrak{p}_s(Y_1))$.

We claim that ϕ maps the canonical class of $R_t(Y)$ to the canonical class of $R_t(Y_1)$. From the claim it follows that λ_i is the coefficient of $\mathrm{cl}(\mathfrak{q}_0(Y_1))$ in the expression of the canonical class of $R_t(Y_1)$ with respect to the basis of $\mathrm{Cl}(R_t(Y_1))$. Then, by virtue of [7, 4.9], λ_i is equal to the difference between the number of rows and the number of columns of Y_1, that is, $\lambda_i = (a_i + b_i) - (a_{i-1} + b_{i-1})$.

Now we prove the claim. Canonical modules behave well with respect to localizations and polynomial extensions. So ψ maps the canonical class of $R_t(Y_1)$ to a canonical class of A. Further the canonical class of $R_t(Y)$ is mapped under the natural surjection to a canonical class of A. Therefore the claim follows if we show that the ring A has a unique canonical class. By [**3**, 3.3.17], it suffices to show that the Picard group of A is trivial. Note that the element z_j of the ring $R_t(Y_1)[Z\setminus Y_1]$ is a $(t-1)$-minor of a matrix which has (at least) a row or a column whose entries are indeterminates of the set $Z\setminus Y_1$. If the z_j involve only elements of $Z\setminus Y_1$, then it follows from 6.2 that $\mathrm{Pic}(A) = 0$. Otherwise denote by J the set $\{j: z_j$ involves elements of $Y_1\}$. For $j \in J$, consider the expression of z_j as a polynomial in the indeterminates $Z\setminus Y_1$ with coefficients in $R_t(Y_1)$. These coefficients are minors of size less than $(t-1)$ of the lower border with thickness $(t-1)$ of Y_1. For all $j \in J$, we pick c_j, one of the coefficients of z_j, and set $c = \prod_{j\in J} c_j$. Since the c_j are prime elements of $R_t(Y_1)$, see 6.3, they are prime elements of A. Hence, by 6.1, the canonical map $\mathrm{Pic}(A) \to \mathrm{Pic}(A[c^{-1}])$ is injective. Now note that $A[c^{-1}] = (R_t(Y_1)[c^{-1}])[Z\setminus Y_1][z_j^{-1}: j \neq i]$. It follows from 6.2 that $\mathrm{Pic}(A[c^{-1}]) = \mathrm{Pic}(R_t(Y_1)[c^{-1}])$, and further, by virtue of 6.4, one has $\mathrm{Pic}(R_t(Y_1)[c^{-1}]) = 0$. This shows that $\mathrm{Pic}(A) = 0$.

Finally we determine δ_j. The argument is similar to the one we used in the previous case, so we shall just indicate the main steps. Let Y_1 be the set

$$\{p \in Y : (a_{i_j}+t-2, b_{i_j}) \preccurlyeq p \preccurlyeq (c_j-t+2, d_j)\}.$$

Denote by B (respectively, C) the lower (respectively, upper) border with thickness $(t-1)$ of Y. Then set

$$Z = Y_1 \cup \{p \in B : p=(x,y), x \geqslant a_{i_j}\} \cup \{p \in C : p=(x,y), x \leqslant c_j\}.$$

The set Y_1 is a one-sided ladder with its outside lower corner in (c_j-t+2, b_{i_j}). Figure 12 illustrates these sets by one example; here j is taken to be 2, i_2 is 3, and the shaded region represents $Z \setminus Y_1$.

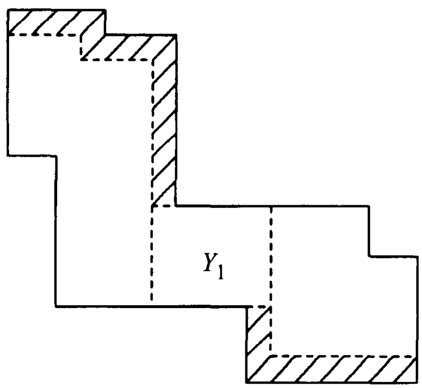

FIG. 12

One shows that

$$R_t(Y)[g_r^{-1}, f_s^{-1}, 1 \leqslant r \leqslant j, i_j < s \leqslant h+1] = K[Z][g_r^{-1}, f_s^{-1}, 1 \leqslant r \leqslant j, i_j < s \leqslant h+1],$$

where $K[Z]$ is the K-subalgebra of $R_t(Y)$ generated by the element of Z. It follows that

$$R_t(Y)[g_r^{-1}, f_s^{-1}, 1 \leqslant r \leqslant j, i_j < s \leqslant h+1]$$
$$\simeq R_t(Y_1)[Z \setminus Y_1][z_r^{-1}, w_s^{-1}, 1 \leqslant r \leqslant j, i_j < s \leqslant h+1],$$

where the z_r are the images of the g_r and the w_s are the images of the f_s. The elements z_r and w_s are prime in $R_t(Y_1)[Z \setminus Y_1]$. One obtains a surjective map $\phi: \mathrm{Cl}(R_t(Y)) \to \mathrm{Cl}(R_t(Y_1))$. The Picard group of $R_t(Y_1)[Z \setminus Y_1][z_r^{-1}, w_s^{-1}, 1 \leqslant r \leqslant j, i_j < s \leqslant h+1]$ is trivial. Hence ϕ maps the canonical class to the canonical class. Furthermore $\phi(\mathrm{cl}(\mathfrak{q}_i)) = 0$ for all $i = 0, \ldots, h+1$, $\phi(\mathrm{cl}(\mathfrak{p}_i)) = 0$ if $i < j$, $\phi(\mathrm{cl}(\mathfrak{p}_j)) = \mathrm{cl}(\mathfrak{q}_0(Y_1))$, and the other classes $\mathrm{cl}(\mathfrak{p}_i)$ which do not vanish under ϕ are mapped bijectively to the elements $\mathrm{cl}(\mathfrak{p}_i(Y_1))$. Then, by [7, 4.9], δ_j is equal to the difference between the number of rows and columns of Y_1, that is, $\delta_j = (a_{i_j}+b_{i_j}+t-2)-(c_j+d_j-t+2)$.

The following theorem is immediate.

THEOREM 5.2. *Let Y be a ladder which satisfies Assumption* (d). *Then the ring $R_t(Y)$ is Gorenstein if and only if $m=n$, the lower inside corners of Y lie on the diagonal $\{(i,j) \in X : i+j = (m+1)-(t-2)\}$, and the upper inside corners of Y lie on the diagonal $\{(i,j) \in X : i+j = (m+1)+(t-2)\}$.*

6. *The Picard group of localizations of one-sided ladder determinantal rings*

The purpose of this section is to show that the Picard group of certain localizations of ladder determinantal rings associated with one-sided ladders is trivial. We need this fact to complete the proof of 5.1.

The Picard group $\mathrm{Pic}(R)$ of a normal domain R can be identified with the subgroup of $\mathrm{Cl}(R)$ of all the classes of invertible (that is, projective) fractionary ideals. It is well known that $\mathrm{Pic}(R) = 0$ if R is a local ring or if R is a finitely generated positively graded K-algebra, say $R = \bigoplus_{i \geqslant 0} R_i$ with $R_0 = K$ a field, see [**8**, 10.3].

We start with some preliminary results.

LEMMA 6.1. *Let R be a normal domain and S be a multiplicative subset of R. If the ideals of the set $\{P\colon P$ is prime of height 1 of R and $P \cap S \neq \varnothing\}$ are principal, then the natural map* $\mathrm{Pic}(R) \to \mathrm{Pic}(R_S)$ *is injective.*

Proof. By Nagata's theorem [**8**, 7.2] the map $\mathrm{Cl}(R) \to \mathrm{Cl}(R_S)$ is injective. It follows that the map $\mathrm{Pic}(R) \to \mathrm{Pic}(R_S)$ is injective too.

PROPOSITION 6.2. *Let R be a normal domain. Let $R[X_1, \dots, X_n]$ be a polynomial extension of R and let $z_1, \dots, z_p$ be prime elements of $R[X_1, \dots, X_n]$. Assume that the coefficients of the polynomial z_j generate the unit ideal for all $j = 1, \dots, p$. Then the natural map* $\mathrm{Pic}(R) \to \mathrm{Pic}(R[X_1, \dots, X_n][z_1^{-1}, \dots, z_p^{-1}])$ *is an isomorphism.*

Proof. Denote by A the ring $R[X_1, \dots, X_n][z_1^{-1}, \dots, z_p^{-1}]$. By [**8**, 7.2, 8.1] the natural map $\mathrm{Cl}(R) \to \mathrm{Cl}(A)$ is an isomorphism. Then the map $\mathrm{Pic}(R) \to \mathrm{Pic}(A)$ is injective and it remains to show that it is surjective. We claim that A is a faithfully flat R-algebra. Flatness is clear. By virtue of [**12**, 7.2] it suffices to show that any maximal ideal $\mathfrak{m}$ of R extends to a proper ideal of A. By contradiction, if $\mathfrak{m}A = A$, then $\mathfrak{m}R[X_1, \dots, X_n]$ contains one of the z_j. But this is impossible because of our assumption.

Now let J be a projective A-module of rank one. Since $\mathrm{Cl}(R) \to \mathrm{Cl}(A)$ is an isomorphism, there exists a divisorial ideal I of R such that $I \otimes_R A \simeq J$, and we have to show that I is projective. For any R-module N and any integer $i > 0$, one has $\mathrm{Ext}^i_R(I, N) \otimes_R A \simeq \mathrm{Ext}^i_A(J, N \otimes_R A) = 0$. Since the extension is faithfully flat, $\mathrm{Ext}^i_R(I, N) = 0$. It follows that I is projective.

LEMMA 6.3. *Let Y be a one-sided ladder and let B be its lower border with thickness $(t-1)$. Let z be a minor of B of size less than $(t-1)$. Then z is a prime element in $R_t(Y)$.*

Proof. Let f be the $(t-1)$-minor based on the outside lower corner of Y. The element z does not belong to the union of the minimal prime ideals of f, see 4.7. It follows that f, z is a regular sequence in $R_t(Y)$. By virtue of 4.1, z is a prime in $R_t(Y)[f^{-1}]$. It follows that z is a prime element in $R_t(Y)$.

Now we are ready to prove the following.

THEOREM 6.4. *Let Y be a one-sided ladder, and let B be its lower border with thickness $(t-1)$. Let $w_1, \dots, w_p$ be minors of B of size less than $(t-1)$. Then* $\mathrm{Pic}(R_t(Y)[w_1^{-1}, \dots, w_p^{-1}]) = 0$.

Proof. We argue by induction on t. For $t = 2$ there is nothing to prove. So we may assume that $t > 2$. Let w_0 be the 1-minor x_{11}. By virtue of 6.3, w_0 is a prime element in $R_t(Y)$. Therefore, by 6.1, it suffices to show that $\operatorname{Pic}(R_t(Y)[w_0^{-1}, \dots, w_p^{-1}]) = 0$. We have an isomorphism [7, 4.1]

$$R_t(Y)[x_{11}^{-1}] \simeq R_{t-1}(Z)[X_{11}, \dots, X_{1n}, X_{21}, \dots, X_{m1}][X_{11}^{-1}],$$

where Z is the one-sided ladder which is obtained from Y by deleting the lower border with thickness 1. So that

$$R_t(Y)[w_0^{-1}, \dots, w_p^{-1}] = R_{t-1}(Z)[X_{11}, \dots, X_{1n}, X_{21}, \dots, X_{m1}][X_{11}^{-1}, W_1^{-1}, \dots, W_p^{-1}],$$

where the W_i are the images of w_i for $i > 0$. The element W_i is a polynomial in $X_{11}, \dots, X_{1n}, X_{21}, \dots, X_{m1}, X_{11}^{-1}$, and among its coefficients there is always a minor c_i of the lower border with thickness $(t-2)$ of Z with size one less than the size of w_i (c_j is taken to be 1 if the size of w_i is 1), see the proof of [7, 4.1]. By virtue of 6.3, c_i is a prime element of $R_{t-1}(Z)$. By 6.1, it suffices to show that the Picard group of

$$R_{t-1}(Z)[c_1^{-1}, \dots, c_p^{-1}][X_{11}, \dots, X_{1n}, X_{21}, \dots, X_{m1}][X_{11}^{-1}, W_2^{-1}, \dots, W_p^{-1}]$$

is trivial. By induction we know that $\operatorname{Pic}(R_{t-1}(Z)[c_1^{-1}, \dots, c_p^{-1}]) = 0$. Then the desired conclusion follows from 6.2.

7. *Some examples*

We conclude the paper with some examples.

EXAMPLE 7.1. Consider the following ladders:

$$Y_1 = \begin{matrix} X_{15} & X_{25} & X_{35} & \\ X_{14} & X_{24} & X_{34} & \\ X_{13} & X_{23} & X_{33} & X_{43} \\ & X_{22} & X_{32} & X_{42} \\ & X_{21} & X_{31} & X_{41} \end{matrix} \qquad Y_2 = \begin{matrix} X_{14} & X_{24} & X_{34} & \\ X_{13} & X_{23} & X_{33} & X_{43} \\ X_{12} & X_{22} & X_{32} & X_{42} \\ & X_{21} & X_{31} & X_{41} \end{matrix}$$

The rings $R_3(Y_1)$, $R_3(Y_2)$ are both complete intersection, hence Gorenstein. The point $(3, 3)$ is an inside upper corner of type 2 of Y_1, while $(3, 3)$ is an inside upper corner of type 1 of Y_2. Therefore $\operatorname{Cl}(R_3(Y_1)) = \mathbb{Z}^2$, and $\operatorname{Cl}(R_3(Y_2)) = \mathbb{Z}^3$.

EXAMPLE 7.2. Consider the following ladder:

$$Y = \begin{matrix} X_{17} & X_{27} & X_{37} & X_{47} & & & \\ X_{16} & X_{26} & X_{36} & X_{46} & & & \\ X_{15} & X_{25} & X_{35} & X_{45} & & & \\ & X_{24} & X_{34} & X_{44} & X_{54} & X_{64} & X_{74} \\ & X_{23} & X_{33} & X_{43} & X_{53} & X_{63} & X_{73} \\ & X_{22} & X_{32} & X_{42} & X_{52} & X_{62} & X_{72} \\ & & & & X_{51} & X_{61} & X_{71} \end{matrix}$$

The ladder satisfies Assumption (d) with respect to $t = 3$. Then $\operatorname{Cl}(R_3(Y)) = \mathbb{Z}^4$. The ring $R_3(Y)$ is not Gorenstein because the inside corner $(4, 4)$ does not lie on the diagonal $\{(x,y): x+y=9\}$. By 5.1, the canonical class is $\mathrm{cl}(\mathfrak{p}_1)$. Thus $\mathfrak{p}_1$ is the canonical module of $R_3(Y)$. In this case $\mathfrak{p}_1$ is generated by the residue classes of the 2-minors of the matrix

$$\begin{matrix} X_{24} & X_{34} & X_{44} \\ X_{23} & X_{33} & X_{43} \\ X_{22} & X_{32} & X_{42} \end{matrix}$$

Hence $R_3(Y)$ has Cohen–Macaulay type equal to 9.

EXAMPLE 7.3. Consider the following ladder:

$$Y = \begin{matrix} X_{18} & X_{28} & X_{38} & & & & \\ X_{17} & X_{27} & X_{37} & X_{47} & X_{57} & & \\ X_{16} & X_{26} & X_{36} & X_{46} & X_{56} & & \\ X_{15} & X_{25} & X_{35} & X_{45} & X_{55} & & \\ & & X_{34} & X_{44} & X_{54} & X_{64} & \\ & & & X_{43} & X_{53} & X_{63} & X_{73} \\ & & & X_{42} & X_{52} & X_{62} & X_{72} \\ & & & X_{41} & X_{51} & X_{61} & X_{71} \end{matrix}$$

The inside upper corner $(5,4)$ is of type 2 with respect to $t=3$, while $(3,7)$ and $(6,3)$ are of type 1. Then $\mathrm{Cl}(R_3(Y)) = \mathbb{Z}^5$. If we want to decide whether $R_3(Y)$ is Gorenstein, we cannot apply 5.2 to Y directly since Assumption (d) is not satisfied. We have first to split Y in accordance with Figure 2 of Section 1:

$$Y_1 = \begin{matrix} X_{18} & X_{28} & X_{38} & & \\ X_{17} & X_{27} & X_{37} & X_{47} & X_{57} \\ X_{16} & X_{26} & X_{36} & X_{46} & X_{56} \\ X_{15} & X_{25} & X_{35} & X_{45} & X_{55} \\ & & X_{34} & X_{44} & X_{54} \end{matrix} \qquad Y_2 = \begin{matrix} X_{44} & X_{54} & X_{64} & \\ X_{43} & X_{53} & X_{63} & X_{73} \\ X_{42} & X_{52} & X_{62} & X_{72} \\ X_{41} & X_{51} & X_{61} & X_{71} \end{matrix}$$

Then by 5.2, $R_3(Y_1)$ and $R_3(Y_2)$ are Gorenstein. Therefore $R_3(Y)$ is Gorenstein.

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