J. reine angew. Math. 474 (1996), 113—138

Journal für die reine und angewandte Mathematik

Sagbi bases with applications to blow-up algebras

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Introduction

In this paper we study blow-up algebras of certain rational normal scrolls. As the standard techniques to study these kind of algebras seem to fail, we applied Sagbi basis theory, which, in principle, is available whenever one wants to analyze the structure of a K-subalgebra A of a polynomial ring $R = K[x_1, \ldots, x_n]$.

Sagbi basis theory was introduced by Robbiano and Sweedler [16], and independently by Kapur and Madlener [15]. Given a term order τ on R, one associates to A an initial algebra $\mathrm{in}_\tau(A)$ which is generated over K by all initial monomials of elements in A. Just as in Gröbner basis theory, where one considers initial ideals, many properties of A are inherited by $\mathrm{in}_\tau(A)$. Thus the idea is to study $\mathrm{in}_\tau(A)$ instead of A, since the initial algebra has a simpler structure, provided it is finitely generated.

The first two sections of the paper are devoted to the general theory of Sagbi bases in which we give a concise survey on the algebraic aspects of the theory. The reader who is more interested in the computational aspects should consult [16] and [15].

In Section 1 we show (Theorem 1.2) that $\mathrm{in}_\tau(A)$, if it is finitely generated, may be viewed as the associated graded ring of suitable degree filtration on A. This implies that $\mathrm{in}_\tau(A)$ is the special fibre of a flat 1-parameter family whose general fibre is A. In [18] Sturmfels already introduced this deformation aspect, describing the deformation in terms of the defining ideals of the involved algebras.

As a consequence of this fact one can show that if $\mathrm{in}_\tau(A)$ is normal, Cohen-Macaulay, has rational singularities, or is F-rational, then A is so; see 2.3. Note that $\mathrm{in}_\tau(A)$ is always a semigroup ring, so that normality of $\mathrm{in}_\tau(A)$ has far-reaching consequences for A itself.

If $I \subset R$ is an ideal in the polynomial ring, then the Rees algebra $\mathscr{R}(I) = R[It]$ is a K-subalgebra of the polynomial ring $R[t]$. Moreover, if I is equi-generated, then the special fibre $\mathscr{R}(I)/(x_1, \ldots, x_n)\mathscr{R}(I)$ may be viewed as the K-subalgebra of R generated over K by a system of generators of I. Thus we may apply to these algebras the general theory of Sagbi bases, and to prove in some cases that they are normal or Cohen-Macaulay, but also to decide whether I is of linear type or to estimate the relation type of I. It also allows us to compare the ideals $\mathrm{in}_\tau(I)^i$ and $\mathrm{in}_\tau(I^i)$. It is clear that the first ideal is contained in the second. Equality holds for example if $\mathrm{in}_\tau(I)$ is of linear type; see 2.8 for a more precise statement.

In Section 3 we apply the Sagbi basis theory to study the Rees algebra and special fibre of the ideal $I(c,d)$ which is generated by the 2-minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & \cdots & \cdots & x_i & \cdots & \cdots & x_c \\ x_{1+d} & x_{2+d} & \cdots & \cdots & x_{i+d} & \cdots & \cdots & x_{c+d} \end{pmatrix}.$$

The ideals $I(c,d)$ define certain rational normal scrolls. Unfortunately we were not able to treat all rational normal scrolls with our method. Indeed, it turns out that the canonical generators of the ideals defining scrolls do not form a Sagbi basis in general, see Example 3.9. However, at least if the base field is of characteristic zero, we were able to compute by different methods the analytic spread of the defining ideal of all rational normal scrolls. This computation is presented in Section 5 and gives a positive answer to a problem stated in [8], p. 45, where it was conjectured that the analytic spread of the defining ideal of any rational normal curve in $\mathbb{P}^n$ is $n+1$, whenever $n \geqq 4$. This in fact was the starting point of our investigation.

In Section 3 (Theorem 3.3) we succeed to show that the natural generators of the Rees algebra and special fibre of the ideal $I(c,d)$ form a Sagbi-basis, to compute the relations of these algebras (3.4), and to show (Theorem 3.8) that these algebras are Cohen-Macaulay normal domains, and have rational singularities.

In Section 4 we study numerical data of the special fibre by using the fact, shown in Section 3, that with respect to a suitable term order, the initial ideal of the defining ideal of the special fibre is an ideal of square-free monomials. We then show (4.6) that the attached simplicial complex is shellable, which opens the possibility to compute the Hilbert function by determining the h-vector of the algebra (Theorem 4.7). From these informations one derives explicit formulas for the multiplicity (4.5) and a-invariant (4.8), and they enable us to determine those fibres which are Gorenstein (4.10).

1. Sagbi bases and filtrations

In this section we recall the notion of Sagbi bases, and list some of its basic properties. Most of the contents of this section can be found in the original article by Robbiano and Sweedler [16]. In this paper however we are more interested in applications to commutative algebra rather than to computer algebra. Therefore our presentation will be phrased in a more algebraic language.

The term *Sagbi* is an abbreviation of **S**ubalgebra **A**nalog to **G**röbner **B**ases for **I**deals, and indeed Sagbi basis theory is the counterpart to Gröbner basis theory. In both theories one constructs an initial object – the *initial algebra* in Sagbi theory, the *initial ideal* in

Gröbner basis theory – and in both theories the initial objects may be viewed as the special fibre of a flat 1-parameter family whose general fibre is the starting object. We now explain this in more details.

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K, and let $A \subset R$ be a finitely generated K-subalgebra. We fix a term order τ for the monomials in R, and let $\mathrm{in}_\tau(A)$ be the K-subalgebra of R generated by the initial monomials $\mathrm{in}_\tau(a)$, $a \in A$. We call $\mathrm{in}_\tau(A)$ the *initial algebra of A* (with respect to the term order τ). It is clear that $\mathrm{in}_\tau(A)$ is the semigroup $K[S]$ of a suitable subsemigroup S of $\mathbb{N}^n$.

A set of elements $a_i \in A$, $i \in I$, is called a *Sagbi basis* if $\mathrm{in}_\tau(A) = K[\mathrm{in}_\tau(a_i) : i \in I]$. Unfortunately, not every finitely generated K-subalgebra $A \subset R$ has a finite Sagbi basis, see [16]. Nevertheless, every Sagbi basis of A is a system of generators of A. The question arises under which conditions a (not necessarily minimal) system of generators of A is a Sagbi basis. The answer is given in

Proposition 1.1 (cf. [16]). *Let $a_1, \ldots, a_m$ be a system of generators of A, and let $f_1, \ldots, f_r$ be a system of binomial relations of the affine semigroup ring*

$$K[S] = K[\mathrm{in}(a_1), \ldots, \mathrm{in}(a_m)]\,,$$

that is, a system of binomial generators of the defining ideal I of $K[S]$. Then $a_1, \ldots, a_m$ is a Sagbi basis of A if and only if for $j = 1, \ldots, r$ there exist $\lambda_\nu^{(j)} \in K$ such that

$$f_j(a_1, \ldots, a_m) = \sum_\nu \lambda_\nu^{(j)} a^\nu \quad \textit{with} \quad \mathrm{in}_\tau(a^\nu) \leqq \mathrm{in}_\tau(f_j(a_1, \ldots, a_m))$$

for all $\lambda_\nu^{(j)} \neq 0$, where, as usual, $a^\nu = a_1^{\nu_1} \cdots a_m^{\nu_m}$ for a multi-index $\nu = (\nu_1, \ldots, \nu_m)$.

Note that the polynomials f_j in 1.1 play the same role as the S-polynomials in Gröbner basis theory.

Proposition 1.1 is a special case of a more general result about filtrations. Indeed we may view the initial algebra of A as the associated graded ring of a suitable ascending G-filtration where G is an ordered group. We first define a G-filtration F on the polynomial ring $R = K[x_1, \ldots, x_n]$ with $G = \mathbb{Z}^n$, where the total order $<$ on G is induced from the term order τ. In other words, $c \in G$ is positive, if and only if there exist monomials x^a and $x^b \in R$ such that $c = b - a$ and $x^a < x^b$. Now the filtration F on R is defined by

$$F_g R = \bigoplus_{h \leqq g} K x^h$$

for all $g \in \mathbb{Z}^n$. We denote again by F the induced filtration of A which is given by

$$F_g A = F_g R \cap A, \quad g \in \mathbb{Z}^n,$$

and call it the *filtration of A defined by the term order* τ.

It is clear that $\mathrm{gr}_F(R) = \bigoplus_{g \in \mathbb{Z}^n} F_g R / F_{<g} R$ with $F_{<g} R = \bigcup_{h<g} F_h R$, can be canonically identified with R, and that after this identification, the associated graded ring $\mathrm{gr}_F(A)$ is a K-subalgebra of R, and actually is equal to $\mathrm{in}_\tau(A)$.

Another filtration which plays a role in this connection is the so-called *degree-filtration*: Given a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ of positive integers, we assign to R the structure of a positively graded K-algebra by setting $\deg_\alpha x_i = \alpha_i$ for $i = 1, \ldots, n$, and define a $\mathbb{Z}$-filtration by setting

$$F_i^\alpha R = \bigoplus_{j \leq i} R_j$$

for all $i \in \mathbb{Z}$. Here R_j denotes the j-th homogeneous component of R with respect to the α-graduation. We denote by F^α the induced filtration on A, and call it a *degree filtration of A*.

One of the main results of this section is the following

Theorem 1.2. *Let A be a K-subalgebra of the polynomial ring $R = K[x_1, \ldots, x_n]$, and let τ be a term order such that $\mathrm{in}_\tau(A)$ is finitely generated. Then there exists a degree filtration F^α on A such that*

$$\mathrm{in}_\tau(A) = \mathrm{gr}_{F^\alpha}(A)\,.$$

Let us first generalize and prove Proposition 1.1. Let A be a K-algebra, and $(G, <)$ an ordered group. A *G-filtration F on A* is a family $F_g A$, $g \in G$, of K-submodules of A satisfying

(1) $F_g A \subset F_h A$ for $g < h$;

(2) $\bigcup_{g \in G} F_g A = A$;

(3) $(F_g A)(F_h A) \subset F_{g+h} A$ for all $g, h \in G$;

(4) $1 \in F_0 A \backslash F_{<0} A$.

We call *F orderly* if for all non-zero elements $a \in A$ there is a smallest element $g \in G$ such that $a \in F_g A$. This smallest element is then called the *order of a*. Note that an orderly filtration F is separated if G has no smallest element, that is, satisfies $\bigcap_{g \in G} F_g A = 0$. It is convenient to set $\operatorname{ord} 0 = -\infty$.

The associated graded ring

$$\mathrm{gr}_F(A) = \bigoplus_{g \in G} F_g A / F_{<g} A$$

has a natural structure of a G-graded K-algebra. Suppose F is orderly, and a is a non-zero element of A of order g. Then we define the *initial form of a* to be the residue class $\mathrm{in}_F(a) = a + F_{<g} A$ of in $\mathrm{gr}_F(A)_g$. Note that $\deg \mathrm{in}_F(a) = g$, and that

$$\mathrm{in}_F(ab) = \mathrm{in}_F(a)\,\mathrm{in}_F(b) \quad \text{if} \quad \mathrm{in}_F(a)\,\mathrm{in}_F(b) \neq 0\,.$$

We let

$$O_F(A) = \{g \in G : \mathrm{gr}_F(A)_g \neq 0\} \cup \{-\infty\}.$$

The set $O_F(A)$ is the subset of G consisting of all orders of elements of A. It is clear that this set inherits a total order from G with $-\infty$ as the smallest element. For inductive arguments it is useful to assume that $O_F(A)$ is *well-ordered*, that is, that each non-zero subset of $O_F(A)$ has a minimal element. Note that this condition is satisfied for degree filtrations (since $\mathbb{Z}$ itself is well-ordered), but also for any filtration induced from a term order.

Proposition 1.3. *Let K be a field. Suppose A is a K-algebra with an orderly G-filtration F, for which any strictly descending sequence of elements of $O_F(A)$ terminates, and such that $\mathrm{gr}_F(A)$ is a domain. Let $a_1, \dots, a_m$ be a system of generators of A, let I be the kernel of the K-algebra homomorphism $\varphi : K[y_1, \dots, y_m] \to \mathrm{gr}_F(A)$ with $\varphi(y_i) = \mathrm{in}_F(a_i)$ for $i = 1, \dots, m$, and let $f_1, \dots, f_r$ be a system of generators of I. Suppose that for all $j = 1, \dots, r$ there exist $\lambda_\nu^{(j)} \in K$ such that*

$$(*) \qquad f_j(a_1, \dots, a_m) = \sum_\nu \lambda_\nu^{(j)} a^\nu \quad \text{with} \quad \deg(\mathrm{in}_F(a^\nu)) \leqq \deg(\mathrm{in}_F(f_j(a_1, \dots, a_m)))$$

for all ν with $\lambda_\nu^{(j)} \neq 0$. Then $\mathrm{gr}_F(A) = K[\mathrm{in}_F(a_1), \dots, \mathrm{in}_F(a_m)]$.

Conversely, if $\mathrm{gr}_F(A)$ is generated by the elements $\mathrm{in}_F(a_i)$, $i = 1, \dots, m$, then all elements $b \in A$ have a presentation as in $(*)$.

Observing that the defining ideal of an affine semigroup ring is generated by binomials, Proposition 1.1 follows from the above discussions and 1.3.

Proof of 1.3. Let $y^\nu = y_1^{\nu_1} \cdots y_m^{\nu_m}$ be a monomial in $K[y_1, \dots, y_m]$. Then we set $\deg y^\nu = \sum_{i=1}^m \nu_i g_i$ where $g_i = \mathrm{ord}\, a_i$ for $i = 1, \dots, m$, and for $f \in K[y_1, \dots, y_m]$, $f = \sum_\nu \lambda_\nu y^\nu$, we set $\deg f = \max\{\deg y^\nu : \lambda_\nu \neq 0\}$.

We first note that for any $f \in I$, $f \neq 0$, the following two facts hold:

(i) $\deg \mathrm{in}_F(f(a_1, \dots, a_m)) < \deg f$.

(ii) There exist $\lambda_\nu \in K$ such that $f(a_1, \dots, a_m) = \sum_\nu \lambda_\nu a^\nu$ with

$$\deg(\mathrm{in}_F(a^\nu)) \leqq \deg(\mathrm{in}_F(f(a_1, \dots, a_m)))$$

for all ν.

(i) follows immediately from the definitions, and (ii) follows from the fact that the same conditions hold by assumption for the generators $f_1, \dots, f_r$ of I.

Now let $b \in A$, and suppose that $g = \mathrm{ord}\, b$. We want to show that

$$\mathrm{in}_F(b) \in K[\mathrm{in}_F(a_1), \dots, \mathrm{in}_F(a_m)].$$

We choose $p = \sum_\nu \lambda_\nu y^\nu$ with $p(a_1, \ldots, a_m) = b$. Say, $\deg p = h$; then $h \geqq g$. If $h = g$, we are done. Thus suppose that $h > g$. We may assume that all λ_ν occurring in p are non-zero. Then we have

$$\sum_{\nu \in T} \lambda_\nu \mathrm{in}_F(a_1)^{\nu_1} \cdots \mathrm{in}_F(a_m)^{\nu_m} = 0\,,$$

where $T = \{\nu : \mathrm{ord}(a^\nu) = h\}$. Here we used that $\mathrm{in}_F(a^\nu) = \mathrm{in}_F(a_1)^{\nu_1} \cdots \mathrm{in}_F(a_m)^{\nu_m}$ which is true since by assumption $\mathrm{gr}_F(A)$ is a domain. It follows that $f = \sum_{\nu \in T} \lambda_\nu y^\nu$ belongs to I, and that $\deg f = h$. Therefore, by (i) and (ii), there exist $\kappa_\nu \in K$ such that

$$f(a_1, \ldots, a_m) = \sum_\nu \kappa_\nu a^\nu$$

with $\deg(\mathrm{in}_F(a^\nu)) < h$ for all ν with $\kappa_\nu \neq 0$. Therefore, if we set $\tilde{f} = \sum_\nu \kappa_\nu y^\nu$, and let $\tilde{p} = p - f + \tilde{f}$, then $\tilde{p}(a_1, \ldots, a_m) = b$ and $\deg \tilde{p} < h$. If we still have $\deg \tilde{p} > g$, we may repeat this argument and construct a polynomial of even smaller degree, which evaluated at $a_1, \ldots, a_m$, gives again b. By our assumption on $O_F(A)$ this process must terminate.

The remaining statement of the proposition is proved similarly. □

Now we are ready for

Proof of 1.2. Let F be the filtration of A which is defined by the term order τ. Then as remarked before the theorem we have $\mathrm{in}_\tau(A) = \mathrm{gr}_F(A)$. By our hypothesis this graded K-algebra is finitely generated, and hence we can find $a_1, \ldots, a_m \in A$ such that

$$\mathrm{gr}_F(A) = K[\mathrm{in}_F(a_1), \ldots, \mathrm{in}_F(a_m)]\,.$$

Let, as in Proposition 1.3, I be the kernel of the K-algebra homomorphism

$$\varphi : K[y_1, \ldots, y_m] \to \mathrm{gr}_F(A)$$

sending the variables y_i to $\mathrm{in}_F(a_i)$ for $i = 1, \ldots, m$, and let $f_1, \ldots, f_r$ be a set of generators of I. Then by 1.3, there exist $\lambda_\nu^{(j)}$ such that $f_j(a_1, \ldots, a_m) = \sum_\nu \lambda_\nu^{(j)} a^\nu$ with

$$\deg(\mathrm{in}_F(a^\nu)) \leqq \deg(\mathrm{in}_F(f_j(a_1, \ldots, a_m)))\,.$$

Suppose there exists a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ of positive integer such that for the corresponding degree filtration F^α we have

(a) $\mathrm{in}_F(a_i) = \mathrm{in}_{F^\alpha}(a_i)$ for $i = 1, \ldots, m$;

(b) $\deg(\mathrm{in}_{F^\alpha}(a^\nu)) \leqq \deg(\mathrm{in}_{F^\alpha}(f_j(a_1, \ldots, a_m)))$ for $j = 1, \ldots, r$.

Then 1.3 implies that

$$\mathrm{gr}_{F^\alpha}(A) = K[\mathrm{in}_{F^\alpha}(a_1), \ldots, \mathrm{in}_{F^\alpha}(a_m)] = K[\mathrm{in}_F(a_1), \ldots, \mathrm{in}_F(a_m)] = \mathrm{gr}_F(A),$$

as desired.

Let S be the set of all monomials which appear in either the a_i, the a^ν or the $f_j(a_1, \ldots, a_m)$. Then it is clear that (a) and (b) are satisfied for α if for all pairs of monomials $u, v \in S$ with $u > v$ one has $\deg_\alpha u > \deg_\alpha v$. In other words, for a finite set of pairs of monomials $\{(u_1, v_1), \ldots, (u_s, v_s)\}$ with $u_i > v_i$ we need to find α such that $\deg_\alpha u_i > \deg_\alpha v_i$ for $i = 1, \ldots, s$. We leave this last step to the reader. □

2. The study of A via $\mathrm{in}_\tau(A)$

Many informations on A can be obtained from $\mathrm{in}_\tau(A)$. Let us first show how the defining relations of these algebras are related to each other.

Corollary 2.1. *With the assumptions and notation of* 1.1, *let J be the kernel of the K-algebra homomorphism $\psi : K[y_1, \ldots, y_m] \to A$ with $\psi(y_i) = a_i$ for $i = 1, \ldots, m$. Then J is generated by the 'lifted' polynomials*

$$F_j(y_1, \ldots, y_m) = f_j(y_1, \ldots, y_m) - \sum_\nu \lambda_\nu^{(j)} y^\nu, \quad j = 1, \ldots, r.$$

Proof. We choose a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ of positive integers satisfying the conditions (a) and (b) in the proof of 1.2. Then $\mathrm{in}_\tau(A) = \mathrm{gr}_{F^\alpha}(A)$; hence, if we give $K[y_1, \ldots, y_m]$ a $\mathbb{Z}$-graded structure by setting $\deg y_i = \deg_\alpha(\mathrm{in}_{F^\alpha}(a_i))$ for $i = 1, \ldots, m$, then

$$\mathrm{in}_\tau(A) \cong K[y_1, \ldots, y_m]/\mathrm{in}(J)$$

where as usual $\mathrm{in}(J)$ is the ideal generated by the initial forms of the elements in J (with respect to the $\mathbb{Z}$-graduation on $K[y_1, \ldots, y_m]$). As $\deg f_j > \deg_\alpha(f_j(a_1, \ldots, a_m))$ for all j, it follows from (b) that $\mathrm{in}(F_j) = f_j$ for $j = 1, \ldots, r$. Finally since the f_j generate the defining ideal I of $\mathrm{in}_\tau(A)$, and since the filtration is separating, we see that the F_j generate J. □

The previous result can be complemented by the following.

Corollary 2.2 (cf. [18]). *If in addition to the assumptions of* 2.1, *the binomials $f_1, \ldots, f_r$ form a Gröbner basis of I with respect to a term order σ' of $K[y_1, \ldots, y_m]$, then there exists a term order σ of $K[y_1, \ldots, y_m]$ such that the lifted equations $F_1, \ldots, F_r$ form a Gröbner basis of J with respect to σ, and $\mathrm{in}_{\sigma'}(I) = \mathrm{in}_\sigma(J)$.*

Proof. We define the term order σ as follows: Let u and v be two monomials in $K[y_1, \ldots, y_m]$. Then we set $u <_\sigma v$ if $\deg u < \deg v$ (with respect to the $\mathbb{Z}$-grading introduced in the proof of 2.1), or $\deg u = \deg v$ and $u <_{\sigma'} v$. It follows immediately that σ is indeed a term order, and that the initial term of F_j with respect to σ is the same as the initial term of f_j with respect to σ'.

Now let $F \in J$ be an arbitrary polynomial, and let $f = \mathrm{in}(F)$ denote the initial polynomial of F with respect to the $\mathbb{Z}$-graduation introduced in the proof of 2.1. There we have shown that $f \in (f_1, \ldots, f_r)$. It follows from the definition of σ that $\mathrm{in}_\sigma(F) = \mathrm{in}_{\sigma'}(f)$. Hence since $f_1, \ldots, f_r$ is a Gröbner basis of I we see that

$$\mathrm{in}_\sigma(F) \in \mathrm{in}_{\sigma'}(I) = (\mathrm{in}_{\sigma'}(f_1), \ldots, \mathrm{in}_{\sigma'}(f_r)) = (\mathrm{in}_\sigma(F_1), \ldots, \mathrm{in}_\sigma(F_r)),$$

as desired. □

As a consequence of Theorem 1.2 there exists a flat 1-parameter family of K-algebras with general fibre A and special fibre $\mathrm{in}_\tau(A)$. In fact, let F^α be a degree filtration such that $\mathrm{gr}_{F^\alpha}(A) \cong \mathrm{in}_\tau(A)$, and consider the Rees algebra $\mathscr{R} = \bigoplus_{i \in \mathbb{Z}} (F^\alpha)_i A t^i$ associated with the filtration F^α. Then $\mathscr{R}$ is a flat $K[t]$-algebra (since t is a non-zero divisor of $\mathscr{R}$), and we have

(1) $\mathscr{R}[t^{-1}] \cong A[t, t^{-1}]$;

(2) $\mathscr{R}/t\mathscr{R} \cong \mathrm{gr}_{F^\alpha}(A) \cong \mathrm{in}_\tau(A)$.

Note that $\mathscr{R}$ and $\mathrm{in}_\tau(A)$, are $\mathbb{Z}$-graded K-algebras, and t is homogeneous of degree 1. In this situation numerous properties of A are inherited by $\mathrm{in}_\tau(A)$. We list the most common ones.

Corollary 2.3. (a) *If* $\mathrm{in}_\tau(A)$ *is Cohen-Macaulay of dimension d and type r, then A is Cohen-Macaulay of dimension d and type* $\leqq r$. *In particular, if* $\mathrm{in}_\tau(A)$ *is Gorenstein, then so is A.*

(b) *If* $\mathrm{in}_\tau(A)$ *is normal, then A has rational singularities if K is of characteristic* 0, *and A is F-rational if K is of positive characteristic. In particular, normality of* $\mathrm{in}_\tau(A)$ *implies that A is a normal Cohen-Macaulay domain.*

Proof. The assertions of (a) follow from standard properties of Cohen-Macaulay and Gorenstein rings.

Recall that $\mathrm{in}_\tau(A)$ is an affine semigroup ring. By Hochster [11], Proposition 1, such a ring is normal if and only if it is a direct summand of a polynomial ring. Thus by a theorem of Boutot [5], $\mathrm{in}_\tau(A)$ has rational singularities if the characteristic of K is zero and by a theorem of Hochster and Huneke [12], $\mathrm{in}_\tau(A)$ is strongly F-regular, and in particular is F-rational, if K has positive characteristic.

It is a theorem of Elkik [6] that the general fibre of a 1-parameter flat family has rational singularities if this is so for the special fibre. Concerning F-rationality we note that by a result [13] of Hochster and Huneke, $\mathscr{R}$ is F-rational if $\mathscr{R}/t\mathscr{R} = \mathrm{in}_\tau(A)$ has this property. Furthermore, since F-rationality localizes, which follows from results of [14], it remains to show that A is F-rational if $A[t]$ is so. But this is straightforward from the definition of F-rationality.

The remaining statements of (b) can be seen directly, but also follow from rationality or F-rationality. □

Suppose now that A is generated by homogeneous polynomials of $K[x_1, \ldots, x_n]$. Then A, as well as $\mathrm{in}_\tau(A)$, are a positively graded K-algebra. Let A_i be a homogeneous component of A. We denote by $\mathrm{in}_\tau(A_i)$ the K-vector space spanned by the elements $\mathrm{in}_\tau(a)$ with $a \in A_i$. It is clear that this vector space is finite dimensional. Let $a_1, \ldots, a_r$ be elements of A_i such that $\mathrm{in}_\tau(a_1), \ldots, \mathrm{in}_\tau(a_r)$ is a basis of $\mathrm{in}_\tau(A_i)$. Then it is immediately seen that $a_1, \ldots, a_r$ is a K-basis of A_i. Thus

Proposition 2.4. *Suppose A is generated by homogeneous polynomials. Then*

$$\mathrm{in}_\tau(A) = \bigoplus_{i \geqq 0} \mathrm{in}_\tau(A_i),$$

and the Hilbert functions of A and $\mathrm{in}_\tau(A)$ *coincide.*

Note that in the previous proposition we do not require that $\mathrm{in}_\tau(A)$ is finitely generated.

We now assume that A has a finite Sagbi basis consisting of elements which are all of same degree. Normalizing the graduation of A and $\mathrm{in}_\tau(A)$ we may assume that both algebras are homogeneous, that is, are generated by elements of degree one.

For a homogeneous K-algebra S, the Hilbert series $H_S(\lambda) = \sum_{i \geqq 0} \dim_K S_i \lambda^i$ can be written as $Q_S(\lambda)/(1-\lambda)^d$, where $Q_S(\lambda) = \sum_{i=0}^{s} h_i \lambda^i \in \mathbb{Z}[\lambda]$, $h_s \neq 0$ and $d = \dim S$. Recall that $e(S) = Q_S(1) = \sum_{i=0}^{s} h_i$ is the multiplicity of S. Furthermore, $a(S) = s - d$ is called the a-invariant, and $h_0, \ldots, h_s$ the h-vector of S. We will study these numerical invariants for special classes of algebras in Section 4.

Here we note as a consequence of 2.4

Corollary 2.5. *Assume that A has a finite Sagbi basis consisting of elements which are all of same degree. Normalizing the graduation of A and* $\mathrm{in}_\tau(A)$, *we may assume that both algebras are homogeneous. Then the h-vectors of A and* $\mathrm{in}_\tau(A)$ *coincide and in particular one has*

$$e(A) = e(\mathrm{in}_\tau(A)) \quad \textit{and} \quad a(A) = a(\mathrm{in}_\tau(A)).$$

Retaining the assumptions of 2.5, the defining ideals J and I of A and $\mathrm{in}_\tau(A)$, respectively, are graded, and we may consider their *Castelnuovo-Mumford regularity*.

Recall, that if B is a finitely generated homogeneous K-algebra and M is a finitely generated graded B-module of finite projective dimension, one defines the Castelnuovo-Mumford regularity $\mathrm{reg}(M)$ of M to be the number

$$\mathrm{reg}(M) = \sup\{t_i(M) - i : i \geqq 0\}$$

where

$$t_i(M) = \sup\{j : \mathrm{Tor}_i^B(K, M)_j \neq 0\}\,.$$

We shall also consider another invariant, the so-called *rate*, which was introduced by Backelin in [1]:

$$\mathrm{rate}(B) = \sup\{(t_i(K) - 1)/(i-1) : i \geqq 1\}\,.$$

The algebra B is a called a *Koszul algebra*, if K has a linear B-resolution. It is clear that B is a Koszul algebra if and only if $\mathrm{rate}(B) = 1$.

Corollary 2.6. *Suppose A has a Sagbi basis consisting of polynomials all of same degree, and the term order τ respects degrees. Then after normalizing the graduation of A and $\mathrm{in}_\tau(A)$ one has*

$$\mathrm{reg}(J) \leqq \mathrm{reg}(I), \quad \textit{and} \quad \mathrm{rate}(A) \leqq \mathrm{rate}(\mathrm{in}_\tau(A))\,.$$

In particular, if $\mathrm{in}_\tau(A)$ is a Koszul algebra, then A is a Koszul algebra.

Proof. We use the following fact whose proof can be essentially found in [3]: Let B and M be as above, but equipped with filtrations, such that M is a filtered B-module. We assume that the filtrations on B and M, which we both denote by F, are refinements of the corresponding degree-filtrations. Then $\mathrm{gr}_F(B)$ may again be considered as $\mathbb{Z}$-graded K-algebra, if we set $\mathrm{gr}_F(B)_i = \mathrm{gr}_F(B_i)$ for all $i \in \mathbb{Z}$. Similarly $\mathrm{gr}_F(M)$ has the structure of a $\mathbb{Z}$-graded $\mathrm{gr}_F(B)$-module. Now for all i and j one has the inequality

$$\dim_K \mathrm{Tor}_i^B(M, K)_j \leqq \dim_K \mathrm{Tor}_i^{\mathrm{gr}_F(B)}(\mathrm{gr}_F(M), K)_j\,.$$

In order to get $\mathrm{reg}(J) \leqq \mathrm{reg}(I)$ we let $B = K[y_1, \ldots, y_m]$, and $M = J$, and to get

$$\mathrm{rate}(A) \leqq \mathrm{rate}(\mathrm{in}_\tau(A))$$

we let $B = A$ and $M = K$. We leave it to the reader to choose in both cases the appropriate filtration F. □

We close this section with some general remarks related to blow-up algebras. Suppose $I = (a_1, \ldots, a_m)$ is an ideal in the polynomial ring $R = K[x_1, \ldots, x_n]$. We denote by $\mathfrak{m}$ the maximal ideal $(x_1, \ldots, x_n)$. We will consider the *Rees algebra*

$$\mathcal{R}(I) = R[It] = \bigoplus_{i \geqq 0} I^i t^i \subset R[t]$$

of I, and its *special fibre* $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$.

By its very definition, $\mathcal{R}(I)$ is the K-subalgebra of the polynomial ring $R[t]$ generated over R by the elements $a_1 t, \ldots, a_m t$. One considers the R-algebra epimorphism

$$\varphi : R[T_1, \ldots, T_m] \to \mathcal{R}(I) = R[a_1 t, \ldots, a_m t]$$

defined by setting $\varphi(T_i) = a_i t$. The ideal $J = \mathrm{Ker}\,\varphi$ is R-homogeneous, and we denote by J_i its R-homogeneous component of degree i. The *relation type* $\mathrm{reltyp}(I)$ of I is defined to be the highest T-degree of a minimal R-homogeneous generator of J. The ideal I is said to be of *linear type* if its relation type is 1.

Note that, if I is equi-generated, that is, the generators of I are homogeneous and all of same degree, then the special fibre of the Rees algebra may be identified with the K-subalgebra $K[a_1, \ldots, a_m]$ of R. Thus we may study the Rees algebra of an ideal I, and, provided I is equi-generated, also study its special fibre by Sagbi basis methods.

We fix a term order τ on R which we extend to a term order τ' on $R[t]$. For example one may define τ' as follows: Given two monomials u and v in R, and non-negative integers i and j. Then we set

$$ut^i <_{\tau'} vt^j \Leftrightarrow i < j \quad \text{or} \quad i = j \quad \text{and} \quad u <_\tau v .$$

The following observations follow immediately from the definitions:

Theorem 2.7. *One has* $\text{in}_\tau(I)^i \subseteqq \text{in}_\tau(I^i)$ *for all* i, *and*

$$\mathscr{R}(\text{in}_\tau(I)) = \bigoplus_{i \geqq 0} \text{in}_\tau(I)^i t^i ,$$

as well as

$$\text{in}_{\tau'}(\mathscr{R}(I)) = \bigoplus_{i \geqq 0} \text{in}_\tau(I^i) t^i .$$

In particular, $\mathscr{R}(\text{in}_\tau(I)) = \text{in}_{\tau'}(\mathscr{R}(I))$ *if and only if* $\text{in}_\tau(I)^i = \text{in}_\tau(I^i)$ *for all* i.

As a consequence of 1.1 and 2.7 one has

Corollary 2.8. *Suppose* $\text{in}_\tau(I)^i = \text{in}_\tau(I^i)$ *for all* $1 \leqq i \leqq \text{reltyp}(\text{in}_\tau(I))$. *Then*

$$\text{in}_\tau(I)^i = \text{in}_\tau(I^i)$$

for all $i \geqq 1$ *and* $\text{reltyp}(I) \leqq \text{reltyp}(\text{in}_\tau(I))$. *In particular, if* $\text{in}_\tau(I)$ *is of linear type, then* $\text{in}_\tau(I)^i = \text{in}_\tau(I^i)$ *for all* $i \geqq 1$ *and* I *is of linear type.*

The following example shows that the smallest exponent for which $\text{in}_\tau(I)^i$ and $\text{in}_\tau(I^i)$ differ may be arbitrarily high.

Example 2.9. For a given integer $n > 1$, we define an ideal I such that $\text{in}_\tau(I)^i = \text{in}_\tau(I^i)$ if and only if $1 \leqq i \leqq n-1$. Consider a set of $2n+2$ indeterminates $x_1, \ldots, x_{2n}, y, z$ over the field K, and let $R = K[x_1, \ldots, x_{2n}, y, z]$. Let

$$\begin{aligned} &f_1 = x_1x_2 - yz, \quad f_2 = x_2x_3, \qquad f_3 = x_3x_4, \qquad \cdots \quad f_{2n-1} = x_{2n-1}x_{2n}, \\ &f_{2n} = x_{2n}x_1, \qquad f_{2n+1} = x_3y, \quad f_{2n+2} = x_{2n}y. \end{aligned}$$

Denote by I the ideal of R generated by $f_1, \ldots, f_{2n+2}$, and fix a term order τ such that $\text{in}_\tau(f_1) = x_1x_2$. It is easy to see that $f_1, \ldots, f_{2n+2}$ form a Gröbner basis of I with respect to τ, so that

$$\text{in}_\tau(I) = (x_1x_2, x_2x_3, \ldots, x_{2n-1}x_{2n}, x_{2n}x_1, x_3y, x_{2n}y).$$

The kernel of the presentation:

$$R[T_1, \ldots, T_{2n+2}] \to \mathscr{R}(\text{in}_\tau(I)) = R[x_1x_2t, \ldots, x_{2n-1}x_{2n}t, x_{2n}x_1t, x_3yt, x_{2n}yt]$$

is generated by the linear relations and by the pure relation

$$F(T) = T_1T_3 \cdots T_{2n-1} - T_2T_4 \cdots T_{2n}$$

of degree n. The linear relations are 'liftable' because $f_1, \ldots, f_{2n+2}$ form a Gröbner basis (in fact they correspond to the S-pairs). Hence $\text{in}_{\tau'}(\mathscr{R}(I))$ and $\mathscr{R}(\text{in}_\tau(I))$ coincide up to degree $n-1$, that is, $\text{in}(I)^i = \text{in}(I^i)$ if $1 \leqq i \leqq n-1$. Furthermore one has

$$-(x_1x_2)^{i-n}yzx_3 \cdots x_{2n} = (x_1x_2)^{i-n}F(f) \in \text{in}_\tau(I^i) \backslash \text{in}_\tau(I)^i$$

for all $i \geqq n$. To see that it is enough to note that z does not appear among the generators of $\text{in}_\tau(I)$.

3. The Rees algebra and special fibre of the defining ideal of certain rational normal scrolls

This section is devoted to the study of the Rees algebras associated with ideals of definition of certain rational normal scrolls and their special fibers. We use the techniques developed in the first two sections to show that these algebras are Cohen-Macaulay normal domains, and to determine their defining equations.

Let c, d be positive integers with $c \geqq d$, and let $X = \{x_1, x_2, \ldots, x_{c+d}\}$ be a set of indeterminates over a field K. Denote by $K[X]$ the polynomial ring $K[x_1, x_2, \ldots, x_{c+d}]$. Consider the following matrix:

$$S = \begin{pmatrix} x_1 & x_2 & \cdots & \cdots & x_i & \cdots & \cdots & x_c \\ x_{1+d} & x_{2+d} & \cdots & \cdots & x_{i+d} & \cdots & \cdots & x_{c+d} \end{pmatrix}.$$

Let $I(c,d)$ be the ideal of $K[X]$ generated by the 2-minors of S. For all $1 \leqq i,j \leqq c$, let m_{ij} be the 2-minor of S with column indices i, j, that is, $m_{ij} = x_ix_{j+d} - x_jx_{i+d}$. Denote by M the set of polynomials $\{m_{ij} : 1 \leqq i < j \leqq c\}$. One may rearrange the columns of S in the following way:

$$\left(\begin{array}{cccc|cccc|c|cccc} x_1 & x_{1+d} & x_{1+2d} & \cdots & x_2 & x_{2+d} & x_{2+2d} & \cdots & \cdots & x_d & x_{2d} & x_{3d} & \cdots \\ x_{1+d} & x_{1+2d} & \cdots & \cdots & x_{2+d} & x_{2+2d} & \cdots & \cdots & \cdots & x_{2d} & x_{3d} & \cdots & \cdots \end{array}\right).$$

In this decomposition there are d blocks, the first r blocks have $q+1$ columns and the last $d-r$ have q columns, where $c = qd + r$ with $r < d$. According to [10], p. 108, the ideal $I(c,d)$ defines a rational normal scroll of dimension d in $\mathbb{P}_K^{c+d+1}$. To date we are able to apply Sagbi basis methods only to the ideals of type $I(c,d)$ which are a subclass of the class of the defining ideals of rational normal scrolls. Here we depart from the standard notation for these ideals, which however will be used in the last section. The reason is that the combinatoric of this special class of ideals is better describable by means of the presentation given here.

Throughout this section we denote by I the ideal $I(c,d)$. We want to study the Rees algebra $\mathscr{R}(I) = \bigoplus_{i\geq 0} I^i t^i$ associated with the ideal I. The Rees algebra $\mathscr{R}(I)$ can be identified with the K-subalgebra of $K[X][t]$ generated by the polynomials in the set $X \cup Mt$, where Mt denotes the set $\{m_{ij}t : m_{ij} \in M\}$. Since I is equigenerated, the special fibre of $\mathscr{R}(I)$ can be identified with the K-subalgebra $K[M]$ of $K[X]$ generated by the polynomials of the set M.

Our first goal is to determine Sagbi bases of $\mathscr{R}(I)$ and $K[M]$. Let τ be the lexicographic monomial order on $K[X]$ induced by the total order $x_1 > x_2 > \cdots > x_{c+d}$. We choose the monomial order τ' on $K[X][t]$ as follows: for $x^a t^i, x^b t^j$ monomials of $K[X][t]$, set $x^a t^i < x^b t^j$ if $i<j$ or $i=j$ and $x^a < x^b$ with respect to τ. We will show that the set $X \cup Mt$ is a Sagbi basis $\mathscr{R}(I)$ with respect to τ', and that M is a Sagbi basis of $K[M]$ with respect to τ.

To this end we consider the K-subalgebra $K[X, \mathrm{in}_\tau(M)t]$ of $K[X][t]$ generated by the elements of the set $X \cup \mathrm{in}_\tau(M)t$ of initial monomials of the elements of $X \cup Mt$. Further we consider the K-subalgebra $K[\mathrm{in}_\tau(M)]$ of $K[X]$ generated by the elements of the set $\mathrm{in}_\tau(M)$ of the initial monomials of the elements of M. For all $1 \leqq i < j \leqq c$, we denote by n_{ij} the initial monomial of m_{ij} with respect to τ, that is, $n_{ij} = x_i x_{j+d}$. One has

$$K[X, \mathrm{in}_\tau(M)t] = K[x_k, n_{it}t : k = 1, \ldots, c+d, 1 \leqq i < j \leqq c],$$

and $K[\mathrm{in}_\tau(M)] = K[n_{ij} : 1 \leqq i < j \leqq c]$. Consider the presentations of these algebras:

$$\varphi : K[X,T] = K[x_k, T_{ij}, k = 1, \ldots, c+d, 1 \leqq i < j \leqq c] \to K[X, \mathrm{in}_\tau(M)t],$$

$$\psi : K[T] = K[T_{ij}, 1 \leqq i < j < c] \to K[\mathrm{in}_\tau(M)]$$

defined by setting $\varphi(x_k) = x_k$, $\varphi(T_{ij}) = n_{ij}t$ and $\psi(T_{ij}) = n_{ij}$. We want to describe the kernels of the maps φ and ψ. It is easy to see that the following polynomials belong to $\operatorname{Ker}\varphi$:

(1) $x_h T_{ij} - x_i T_{hj}$, $\quad 1 \leqq h < i < j \leqq c$,

(2) $x_h T_{ij} - x_{j+d} T_{i,h-d}$, $\quad 1 \leqq i < h-d < j \leqq c$,

(3) $T_{ij} T_{hk} - T_{ik} T_{hj}$, $\quad 1 \leqq i < h < k < j \leqq c$,

(4) $T_{ij} T_{hk} - T_{i,h-d} T_{j+d,k}$, $\quad 1 \leqq i < j < h < k \leqq c, h-j > d$.

We call the polynomials of type (1) and (2) the linear relations, those of type (3) the Plücker relations, and those of type (4) the N-relations. Of course the Plücker relations and the N-relations belong to $\operatorname{Ker}\psi$, too. We would like to show that these polynomials generate $\operatorname{Ker}\varphi$ and $\operatorname{Ker}\psi$. To this end we need the following general criterion:

Lemma 3.1. *Let $K[Y]$ be a polynomial ring equipped with a term order σ. Let J be an ideal of $K[Y]$ and let $f_1, \ldots, f_s$ be polynomials of J. Assume that the monomials of the set $\Omega = \{Y^b : Y^b \notin (\mathrm{in}_\sigma(f_1), \ldots, \mathrm{in}_\sigma(f_s))\}$ are linearly independent in $K[Y]/J$. Then $f_1, \ldots, f_s$ is a Gröbner basis of J with respect to σ.*

Proof. Set $I = (\mathrm{in}_\sigma(f_1), \ldots, \mathrm{in}_\sigma(f_s))$. By definition $I \subseteq \mathrm{in}_\sigma(J)$. If $I \neq \mathrm{in}_\sigma(J)$, then there exists $f \in J$ such that $\mathrm{in}_\sigma(f) \notin I$. Note that $f = 0 \bmod J$ and $\mathrm{in}_\sigma(f) \in \Omega$. This is a contradiction because the elements of Ω are linearly independent in $K[Y]/J$. Hence $I = \mathrm{in}_\sigma(J)$. □

Now one has

Proposition 3.2. (i) *The relations* (1), (2), (3), *and* (4), *form a Gröbner basis of* $\mathrm{Ker}\,\varphi$ *with respect to a suitable monomial order. Furthermore, let Ω_1 be the set of all elements of the form $un_{i_1j_1}t \cdots n_{i_rj_r}t$ where u is a monomial in the x_i, such that the following conditions are satisfied*:

(c_1) *$k \geqq i_r$ for all x_k which appear in u.*

(c_2) *If x_k appears in u and $k > i_s + d$ for some $1 \leqq s \leqq r$, then $k \geqq j_s + d$.*

(c_3) $1 \leqq i_1 \leqq \cdots \leqq i_r \leqq c, \quad 1 \leqq j_1 \leqq \cdots \leqq j_r \leqq c.$

(c_4) $i_r - j_1 \leqq d.$

Then Ω_1 is a K-basis of the K-algebra $K[X, \mathrm{in}_\tau(M)t]$.

(ii) *The relations* (3), *and* (4), *form a Gröbner basis of* $\mathrm{Ker}\,\psi$ *with respect to a suitable monomial order. Furthermore, the set Ω_2 of elements of the form $n_{i_1j_1} \cdots n_{i_rj_r}$ which satisfy the conditions* (c_3) *and* (c_4) *is a K-basis of the K-algebra $K[\mathrm{in}_\tau(M)]$.*

Proof. (i) We first fix a monomial order on $K[X,T]$. Let σ be a monomial order on $K[X,T]$ such that the initial monomials of the polynomials (1), (2), (3), and (4) are those on the left hand side, that is, x_hT_{ij} for (1) and (2), and $T_{ij}T_{hk}$ for (3) and (4). We will show later that such a monomial order exists. We consider now the set Ω defined as in 3.1 associated with the monomials we have chosen. By construction, the elements of Ω correspond to the elements of Ω_1 under the isomorphism $K[X,T]/\mathrm{Ker}\,\varphi \cong K[X, \mathrm{in}_\tau(M)t]$. Hence in order to apply 3.1 it suffices to show that the elements of Ω_1 are linearly independent. Note that the elements of Ω_1 are monomials in the indeterminates $x_1, x_2, \ldots, x_{c+d}$ and t and therefore they are linearly independent provided they are distinct. The proof is complete if we show that an element f of the set Ω_1 has a unique representation as $un_{i_1j_1}t \cdots n_{i_rj_r}t$, where u is a monomial in the x_i and such that the conditions (c_1), (c_2), (c_3), and (c_4) are satisfied. To this end one notes that r is the exponent of t in the monomial f, and that (c_1), (c_3) and (c_4) imply that

$$i_s = \min\{v : x_v \text{ divides } f/x_{i_1}\cdots x_{i_{s-1}}\} \quad \text{for all } s = 1, \ldots, r.$$

Further from (c_3) it follows that

$$j_s = \min\{v : x_{v+d} \text{ divides } f/x_{i_1}\cdots x_{i_r}x_{j_1+d}\cdots x_{j_{s-1}+d}\} \quad \text{for all } s = 1, \ldots, r,$$

and finally

$$u = f/x_{i_1}\cdots x_{i_r}x_{j_1+d}\cdots x_{j_r+d}t^r.$$

It remains to define the monomial order σ. Given a monomial in the T_{ij}, say $T_{a_1b_1}\cdots T_{a_rb_r}$, we consider the associated sequence $e = e_1, \ldots, e_{2r}$ which is obtained by rearranging the sequence $a_1, b_1, \ldots, a_r, b_r$ in decreasing order. We define a monomial order σ_1 on $K[T]$ in the following way: Let f and f' be monomials in the T_{ij} whose associated sequences are e, and e'. Then we set $f >_{\sigma_1} f'$ if $\deg f > \deg f'$, or $\deg f = \deg f'$ and $e > e'$ in the lexicographic order, or $\deg f = \deg f'$, $e = e'$ and $f > f'$ in the lexicographic monomial order induced by the total order $T_{1c} > T_{1,c-1} > \cdots > T_{12} > T_{2c} > T_{2,c-1} \cdots > T_{c-1,c}$. Let σ_2 be the lexicographic monomial order on $K[X]$ induced by the total order

$$x_1 > x_2 > \cdots > x_{c+d}.$$

Then we define σ to be the product of σ_2 and σ_1, that is, for uf and u_1f_1 monomials of $K[X, T]$ where u, u_1 are monomials in the x's and f, f_1 are monomials in the T_{ij}, $uf >_\sigma u_1f_1$ if $u >_{\sigma_2} u_1$ or $u = u_1$ and $f >_{\sigma_1} f_1$. Now it is easy to see that σ selects the desired monomials from the relations (1), (2), (3), and (4).

(ii) The above argument works in this case, too. One has just to take the monomial order σ_1 to select the desired monomials from the relations (3) and (4). □

We are ready to prove:

Theorem 3.3. (i) *The polynomials of the set $X \cup Mt$ form a Sagbi basis of the Rees algebra $\mathcal{R}(I)$ with respect to the monomial order τ'. In particular*

$$\text{in}_\tau(\mathcal{R}(I)) = K[X, \text{in}_\tau(M)t] = \mathcal{R}(\text{in}_\tau(I)).$$

(ii) *The polynomials of the set M form a Sagbi basis of the K-algebra $K[M]$ with respect to the monomial order τ. In particular* $\text{in}_\tau(K[M]) = K[\text{in}_\tau(M)]$.

Proof. (i) By 3.2 we know a set of generators of $\text{Ker}\,\varphi$ of the presentation of $K[X, \text{in}_\tau(M)t]$. By virtue of 1.1, it is enough to show that for all elements $f(X, T)$ in this set, one may express $f(X, Mt)$ as a linear combination of elements of the form $\lambda x^\alpha(Mt)^b$, with $\lambda \in K\setminus\{0\}$ and $\text{in}_{\tau'}(x^\alpha(Mt)^b) \leq \text{in}_{\tau'}(f(X, Mt))$. The linear relation (1), with $1 \leq h < i < j \leq c$, lifts to $x_h m_{ij}t - x_i m_{hj}t = -x_j m_{hi}t$. The linear relation (2), with $1 \leq i < h - d < j \leq c$, lifts to $x_h m_{ij}t - x_{j+d}m_{i,h-d}t = x_{i+d}m_{h-d,j}t$. The Plücker relation (3), with $1 \leq i < h < k < j \leq c$, lifts to $m_{ij}tm_{hk}t - m_{ik}tm_{hj}t = -m_{ih}tm_{kj}t$. It remains to lift the N-relation (4). For all $1 \leq i < j < h < k \leq c$, $h - j > d$, we consider the following matrix:

$$\begin{pmatrix} x_i & x_{i+d} & x_j & x_{j+d} \\ x_{i+d} & x_{i+2d} & x_{j+d} & x_{j+2d} \\ x_{h-d} & x_h & x_{k-d} & x_k \\ x_h & x_{h+d} & x_k & x_{k+d} \end{pmatrix}.$$

Expanding the determinant of this matrix with respect to the first two rows and then with respect to the first two columns one obtains the following equality:

$$\begin{aligned} & m_{i,i+d}m_{k-d,k} - m_{ij}m_{hk} + m_{i,j+d}m_{h,k-d} + m_{i+d,j}m_{h-d,k} \\ & -m_{i+d,j+d}m_{h-d,k-d} + m_{j,j+d}m_{h-d,h} = m_{i,i+d}m_{k-d,k} - m_{i,h-d}m_{j+d,k} \\ & + m_{ih}m_{j+d,k-d} + m_{i+d,h-d}m_{jk} - m_{i+d,h}m_{j,k-d} + m_{h-d,h}m_{j,j+d}\,. \end{aligned}$$

It follows that

$$(*) \qquad \begin{aligned} & m_{ij}m_{hk} - m_{i,h-d}m_{j+d,k} = m_{i,j+d}m_{h,k-d} + m_{i+d,j}m_{h-d,k} \\ & -m_{i+d,j+d}m_{h-d,k-d} - m_{ih}m_{j+d,k-d} - m_{i+d,h-d}m_{jk} + m_{i+d,h}m_{j,k-d}\,, \end{aligned}$$

and then

$$(**) \qquad \begin{aligned} & m_{ij}tm_{hk}t - m_{i,h-d}tm_{j+d,k}t = m_{i,j+d}tm_{h,k-d}t + m_{i+d,j}tm_{h-d,k}t \\ & -m_{i+d,j+d}tm_{h-d,k-d}t - m_{ih}tm_{j+d,k-d}t - m_{i+d,h-d}tm_{jk}t + m_{i+d,h}tm_{j,k-d}t\,. \end{aligned}$$

The N-relation (4) lifts to (**), and it easy to check that the desired condition on the initial monomials is satisfied.

(ii) The generators of $\operatorname{Ker}\varphi$ are the Plücker relations (3) and the N-relations (4). The Plücker relations lift to $m_{ij}m_{hk} - m_{ik}m_{hj} = -m_{ih}m_{kj}$ and the N-relations lift to (*). □

From 3.2, 3.3 and 2.2 it follows

Corollary 3.4. *Consider the presentations*:

$$\begin{aligned} & \alpha : K[X,T] = K[x_k, T_{ij},\ k = 1, \ldots, c+d,\ 1 \leqq i < j \leqq c] \to \mathcal{R}(I)\,, \\ & \beta : K[T] = K[T_{ij},\ 1 \leqq i < j \leqq c] \to K[M] \end{aligned}$$

defined by setting $\alpha(x_k) = x_k$, $\alpha(T_{ij}) = m_{ij}t$ *and* $\beta(T_{ij}) = m_{ij}$. *For systematic reason set* $T_{ii} = 0$ *and* $T_{ij} = -T_{ji}$ *if* $i > j$. *Then*:

(i) *The polynomials*

$$(5)\quad x_hT_{ij} - x_iT_{hj} + x_jT_{hi}, \qquad 1 \leqq h < i < j \leqq c,$$

$$(6)\quad x_hT_{ij} - x_{j+d}T_{i,h-d} + x_{i+d}T_{h-d,j}, \quad 1 \leqq i < h-d < j \leqq c,$$

$$(7)\quad T_{ij}T_{hk} - T_{ik}T_{hj} + T_{ih}T_{kj}, \qquad 1 \leqq i < h < k < j \leqq c,$$

$$(8)\quad T_{ij}T_{hk} - T_{i,h-d}T_{j+d,k} - T_{i,j+d}T_{h,k-d} - T_{i+d,j}T_{h-d,k} + T_{i+d,j+d}T_{h-d,k-d}$$
$$+ T_{ih}T_{j+d,k-d} + T_{i+d,h-d}T_{jk} - T_{i+d,h}T_{j,k-d}, \quad 1 \leqq i < j < h < k \leqq c, h-j > d,$$

form a Gröbner basis of $\operatorname{Ker}\alpha$ *with respect to a suitable monomial order. In particular, they generate* $\operatorname{Ker}\alpha$.

(ii) *The polynomials* (7) *and* (8) *form a Gröbner basis of* $\operatorname{Ker}\beta$ *with respect to a suitable monomial order. In particular, they generate* $\operatorname{Ker}\beta$.

From now on we set $\deg T_{ij} = 1$, so that the Rees algebra $\mathscr{R}(I)$ and its special fibre $K[M]$ are homogeneous K-algebras. We have seen that the ideals of presentation of $\mathscr{R}(I)$ and $K[M]$ are generated by Gröbner bases of quadrics. From [2], Theorem 2.2, it follows:

Corollary 3.5. *The Rees algebra* $\mathscr{R}(I)$ *and its special fibre* $K[M]$ *are Koszul algebras.*

As another consequence we have

Corollary 3.6. *The polynomials* m_{ij}, $1 \leqq i < j \leqq c$ *form a Gröbner basis of* I *with respect to* τ *and further* $\operatorname{in}_\tau(I^i) = \operatorname{in}_\tau(I)^i$ *for all* $i \in \mathbb{N}$.

In order to apply 2.3 to the algebras under consideration we prove:

Proposition 3.7. *The affine semigroup rings* $\operatorname{in}_{\tau'}(\mathscr{R}(I)) = K[X, \operatorname{in}_\tau(M)t]$ *and*

$$\operatorname{in}_\tau(K[M]) = K[\operatorname{in}_\tau(M)]$$

are normal.

Proof. By virtue of [11], Proposition 1, it suffices to show that the associated affine semigroups are normal. To this end let a, a_1, b be monomials of $K[X, \operatorname{in}_\tau(M)t]$ (resp. of $K[\operatorname{in}_\tau(M)]$) such that $a^p = a_1^p b$ with $p \in \mathbb{N}$. We have to show that there exists a monomial b_1 of $K[X, \operatorname{in}_\tau(M)t]$ (resp. of $K[\operatorname{in}_\tau(M)]$) such that $b = b_1^p$.

From $a^p = a_1^p b$ it follows immediately that the exponents of the indeterminates in the monomial b are always multiples of p. According to 3.2, the monomial b has a unique representation $b = un_{i_1j_1}t \cdots n_{i_rj_r}t$ such that u is a monomial in the x_i and the conditions (c_1), (c_2), (c_3), (c_4) are satisfied (resp. $b = n_{i_1j_1} \cdots n_{i_rj_r}$ and (c_3), (c_4) are satisfied). Since $i_s = \min\{v : x_v \text{ divides } b/x_{i_1} \cdots x_{i_{s-1}}\}$ for all $s = 1, \ldots, r$, it follows that

$$i_1 = i_2 = \cdots = i_p.$$

Now note that the exponents of the indeterminates in the monomial $b/x_{i_1}\cdots x_{i_p}=b/x_{i_1}^p$ are multiples of p. It follows that

$$\begin{gathered} i_{p+1}=i_{p+2}=\cdots=i_{2p}, \\ \vdots \\ i_{(m-1)p+1}=i_{(m-1)p+2}=\cdots=i_{mp} \end{gathered}$$

and $mp=r$. Since

$$j_s=\min\{v : x_{v+d} \text{ divides } b/x_{i_1}^p x_{p+1}^p\cdots x_{i_{(m-1)p+1}}^p x_{j_1+d}\cdots x_{j_{s-1}+d}\}$$

for all $s=1,\dots,r$, it follows that

$$\begin{gathered} j_1=j_2=\cdots=j_p, \\ j_{p+1}=j_{p+2}=\cdots=j_{2p}, \\ \vdots \\ j_{(m-1)p+1}=j_{(m-1)p+2}=\cdots=j_{mp}. \end{gathered}$$

Finally

$$u=b/x_{i_1}^p x_{i_{p+1}}^p\cdots x_{i_{(m-1)p+1}}^p x_{j_1}^p x_{j_{p+1}}^p\cdots x_{j_{(m-1)p+1}}^p t^{mp}.$$

Hence the exponents of the indeterminates that appear in u are multiples of p. Then there exists a monomial u_1 in the x's such that $u=u_1^p$. Now one has

$$b=(u_1 n_{i_1 j_1} t n_{i_{p+1} j_{p+1}} t\cdots n_{i_{(m-1)p+1} j_{(m-1)p+1}} t)^p$$

(resp. $b=(n_{i_1 j_1} n_{i_{p+1} j_{p+1}}\cdots n_{i_{(m-1)p+1} j_{(m-1)p+1}})^p$), and the proof is complete. □

Now, by virtue of 2.3, one has:

Theorem 3.8. *The Rees algebra $\mathscr{R}(I)$ and its special fibre $K[M]$ are Cohen-Macaulay normal domains. If* $\operatorname{char} K=0$ *then $\mathscr{R}(I)$ and $K[M]$ have rational singularities, and if* $\operatorname{char} K>0$, *then $\mathscr{R}(I)$ and $K[M]$ are F-rational.*

We conclude the section by showing that the canonical generators of the ideals defining scrolls do not form in general a Sagbi basis with respect to a diagonal order.

Example 3.9. Let M be the set of 2-minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_6 \\ x_2 & x_3 & x_4 & x_5 & x_7 \end{pmatrix}$$

and let $K[M]$ be the K-subalgebra of $R=K[x_1,\dots,x_7]$ generated by the elements of M. Let τ be the lexicographical order induced by $x_1>x_2>\cdots>x_7$. Denote by m_{ij} the minor with column indices i and j. One has

$$\operatorname{in}_\tau(m_{12}m_{45}-m_{13}m_{35})=x_1x_3x_5x_6\notin K[\operatorname{in}_\tau(m_{ij}) : 1\leqq i<j\leqq 5].$$

Hence M is not a Sagbi basis of $K[M]$ with respect to τ.

4. Numerical invariants

We keep the notation of the previous section. From now on we concentrate our attention on the special fibre $K[M]$ of $\mathscr{R}(I)$, $I = I(c,d)$. In particular we are interested in determining its dimension, its multiplicity and its h-vector. We have seen that the defining equations of $K[M]$ are the Plücker relations (7) and the N-relations (8), and that they form a Gröbner basis. The N-relations are present if an only if $d+4 \leqq c$. It is well known that the Plücker relations (7) define the coordinate ring $G(2,c)$ of the Grassmanian variety of indices 2, c, and that $G(2,c)$ is a Gorenstein and factorial domain of dimension $2c-3$, see for instance [4], Sections 4, 5, and 6. Hence $K[M]$ is a factor ring of $G(2,c)$ and $\dim K[M] \leqq \dim G(2,c) = 2c-3$. Furthermore $K[M]$ is isomorphic to $G(2,c)$ if and only if $d+4 > c$. In this case one has $\dim K[M] = 2c-3$, and $K[M]$ is factorial and Gorenstein.

Let us denote by A the set of points $\{(i,j) \in \mathbb{N}^2 : 1 \leqq i < j \leqq c\}$. We equip A with the partial order: $(i,j) \leqq (h,k)$ if $i \leqq h$ and $j \leqq k$. By virtue of 3.4, the initial ideal of $\operatorname{Ker}\beta$ is $J = (T_{ij}T_{hk}, 1 \leqq i < h < k < j \leqq c) + (T_{ij}T_{hk} : 1 \leqq i < j < h < k \leqq c, h-j > d)$. The ideal J is generated by square-free monomials. Therefore we may associate with J the simplicial complex $\Delta = \{H \subset A : T^H \notin J\}$. The ring $K[T]/J$ is called the Stanley-Reisner associated with Δ. For the theory of the Stanley-Reisner ring we refer the reader to [17] or [2], Chapter 5. Our next goal is to describe the facets of Δ, that is, the elements of Δ which are maximal under inclusion. In order to do this, let us introduce a piece of notation. Set $J_1 = (T_{ij}T_{hk} : 1 \leqq i < h < k < j \leqq c)$, and let Δ' be the simplicial complex associated with J_1. Since J_1 is generated by the monomials which correspond to the pairs of incomparable elements of A, the elements of Δ' are the chains of A. The complex Δ' is the so-called order complex of the poset A. If $d+4 \leqq c$, and $2 \leqq j \leqq c-d-1$, we denote by A_j the set $\{(i,h) \in A : (1,j) \leqq (i,h) \leqq (j+d, c)\}$. Clearly the saturated chains of A_i are maximal facets of Δ if $d+4 \leqq c$ and it is not difficult to see that

Lemma 4.1. (i) *If $d+4 > c$, then the facets of Δ are the saturated chains of A.*

(ii) *If $d+4 \leqq c$, then the facets of Δ are the saturated chains of A_j, $2 \leqq j \leqq c-d-1$.*

Recall that the analytic spread $\ell(J)$ of an ideal J of a local ring S (or positively graded K-algebra) is the dimension of the special fibre of the Rees algebra associated with J. We have $\ell(I) = \dim K[M] = \dim K[T]/J = \dim \Delta + 1$, and hence from the lemma it follows:

Corollary 4.2. *The simplicial complex Δ is pure of dimension* $\min\{2c-3, c+d\} - 1$, *and* $\ell(I) = \min\{2c-3, c+d\}$.

Remark 4.3. We will see in 5.2 that the formula

$$\ell(I) = \min\{2c-3, c+d\}$$

holds for any ideal I of definition of a rational normal scroll of dimension d in $\mathbb{P}_K^{c+d-1}$ which is not a cone, where K is a field of characteristic 0.

In order to carry further our investigation we introduce a piece of notation. Let a, b, k be positive integers with $k \leqq \min\{a, b\}$. Consider the sublattice

$$L(a,b,k) = \{(i,j) \in \mathbb{N}^2 : 1 \leqq i \leqq a, 1 \leqq j \leqq b, j \geqq i - a + k\}$$

of $\mathbb{N}^2$. Let H be a saturated chain of $L(a,b,k)$ and $(i,j) \in H$. The point (i,j) is said to be a left turn of H if $(i-1,j)$ and $(i,j+1)$ belong to H. Let us denote by $e(a,b,k)$ the number of saturated chains of $L(a,b,k)$ and by $h_i(a,b,k)$ the number of saturated chains of $L(a,b,k)$ with exactly i left turns. By induction on a and b one easily shows:

Lemma 4.4.

$$e(a,b,k) = \binom{a+b-2}{a-1} - \binom{a+b-2}{a+b-k},$$

$$h_i(a,b,k) = \binom{a-1}{i}\binom{b-1}{i} - \binom{a+b-k-2}{i-1}\binom{k}{i+1}.$$

The multiplicity $e(K[M])$ of $K[M]$ coincides with that of $K[\Delta]$ and it is equal to the number of facets of Δ. Hence by virtue of 4.1 and 4.4 we have

Corollary 4.5.

$$e(K[M]) = \begin{cases} \binom{2c-4}{c-2} - \binom{2c-4}{c-1} & \text{if } d+4 > c, \\ \sum_{j=2}^{c-d-1} \binom{c+d-1}{c-j} - \binom{c+d-1}{c-1}(c-d-2) & \text{if } d+4 \leqq c. \end{cases}$$

We would like to determine the h-vector of $K[M]$. It can be expressed in terms of the so-called f-vector of Δ [2], Theorem 5.1.7. In the case in which Δ is shellable it is possible to determine the h-vector by means of the invariants of the shelling [2], Corollary 5.1.14. Indeed one has

Proposition 4.6. *The simplicial complex Δ is shellable.*

Proof. For $2 \leqq j \leqq c-d-1$, denote by Δ_j the order complex of A_j, that is, the simplicial complex of the chains of A_j. By definition Δ' is the order complex of A. The order complex of a distributive lattice is shellable [2], Theorem 5.1.12. The posets A and A_j are distributive lattices, hence Δ' and Δ_j are shellable. For our purpose we need to describe explicitly a shelling on Δ' and Δ_j. Given two saturated chains H and H_1 of A or of A_j, we say that $H \leqq H_1$ if H is on the north-west side of H_1. This is just a partial order on the set of saturated chains. Nevertheless extending this partial order to a total order one obtains a shelling on Δ' and Δ_j. We call this shelling the north-west shelling. Note that if H and H_1 are saturated chains of A or of A_j such that $H < H_1$ with respect to the north-west shelling and $H_1 \backslash H = \{(x,y)\}$ then (x,y) is a left turn of H_1.

If $d+4>c$, then we have seen that Δ coincides with the order complex Δ' of A and therefore it is shellable.

Now we discuss the case $d+4 \leqq c$. We have seen that $\Delta = \bigcup_{j=2}^{c-d-1} \Delta_j$ and that the Δ_j are shellable simplicial complexes. There is a natural way to define a total order on the facets of Δ: Let H be a saturated chain of A_j and H_1 be a saturated chain of A_h, then we set $H \leqq H_1$ if $j<h$ or $j=h$ and $H \leqq H_1$ in the north-west order shelling. It is clear that this is a total order on the set of the facets of Δ and it is easy to see that it is indeed a shelling. □

Now we are ready to determine the h-vector of $K[M]$.

Theorem 4.7. *Let $h_0, h_1, \ldots, h_i, \ldots, h_s$ be the h-vector of $K[M]$.*

(i) *If $d+4>c$, then one has*:

$$h_i = \binom{c-2}{i}^2 - \binom{c-3}{i-1}\binom{c-1}{i+1}.$$

(ii) *If $d+4 \leqq c$, then one has $h_i = f_i + g_i$ with*:

$$f_i = \binom{d+1}{i}\binom{c-2}{i} - \binom{d+2}{i+1}\binom{c-3}{i-1} - \binom{d+1}{i}\binom{c-2}{i-1}(c-d-3),$$

$$g_i = \sum_{j=3}^{c-d-1} \binom{d-2+j}{i-1}\binom{c+1-j}{i}.$$

In particular

$$h_i = \binom{d+c}{2i} \quad \text{if } i>d+1.$$

Proof. By virtue of [2], Corollary 5.1.14, the number h_i is equal to the number of the facets H of Δ such that

$$|\{(x,y) \in H : \text{there exists a facet } W \text{ of } \Delta, W<H, H \setminus W = \{(x,y)\}\}| = i,$$

where $<$ is the shelling on Δ described in 4.6.

(i) Since $d+4>c$, h_i is the number of saturated chains of A with exactly i left turns. But $A \cong L(c-1, c-1, c-1)$, and hence the claim follows from 4.4.

(ii) Assume $d+4 \leqq c$. Let H be a facet of Δ; H is a saturated chain of A_j for some $j, j=2, \ldots, c-d-1$. In order to apply [2], Corollary 5.1.14 one needs to understand the set $\{(x,y) \in H : \text{there exists a facet } W \text{ of } \Delta, W<H, H \setminus W = \{(x,y)\}\}$. One observes that the elements of this set are the left turns of H and the end point $(j+d, c)$ if it is reached with a horizontal step and $j \geqq 3$. Since $A_2 \cong L(d+2, c-1, d+2)$, the contribution of the facets of Δ_2 to the i-th component of the h-vector of Δ is $h_i(d+2, c-1, d+2)$.

Now let $3 \leqq j \leqq c-d-1$. Note that $A_j \cong L(j+d, c-j+1, d+2)$. The number of the saturated chains of A_j which do not contain $(j+d-1, c)$ and have exactly i left turns is $h_i(j+d, c-j+1, d+2) - h_i(j+d-1, c-j+1, d+1)$. The number of those which contain $(j+d-1, c)$ and have exactly $i-1$ left turns is $h_{i-1}(j+d-1, c-j+1, d+1)$. Hence the contribution of the facets of Δ_j to the i-th component of the h-vector of Δ is

$$h_i(j+d, c-j+1, d+2) - h_i(j+d-1, c-j+1, d+1) + h_{i-1}(j+d-1, c-j+1, d+1).$$

Adding up and using 4.4, one obtains $h_i = f_i + g_i$ for all $i \geqq 0$. Finally one notes that $f_i = 0$ and $g_i = \binom{c+d}{2i}$ if $i > d+1$. □

Theorem 4.7 has two corollaries:

Corollary 4.8. *Denote by $s(K[M])$ the degree of the h-vector of $K[M]$ and by $a(K[M])$ its a-invariant. One has*:

$$s(K[M]) = \begin{cases} c-3 & \text{if } d+4 > c, \\ \left[\dfrac{c+d}{2}\right] & \text{if } d+4 \leqq c, \end{cases} \qquad a(K[M]) = \begin{cases} -c & \text{if } d+4 > c, \\ -\left[\dfrac{c+d+1}{2}\right] & \text{if } d+4 \leqq c. \end{cases}$$

Remark 4.9. Since $K[M]$ is Cohen-Macaulay, the degree $s(K[M])$ of its h-vector coincides with the so-called reduction number of I.

By virtue of [2], Corollary 4.3.8, $K[M]$ is Gorenstein if and only if its h-vector is symmetric. Hence from 4.7 it follows:

Corollary 4.10. *The ring $K[M]$ is Gorenstein if and only if $d+4 \geqq c$.*

5. The analytic spread of a rational normal scroll

In this section we determine the analytic spread of the defining ideal of a general rational normal scroll in $\mathbb{P}_K^n$, where K is a field of characteristic 0.

Let V be a rational normal scroll of dimension d in $\mathbb{P}_K^n$, $d \geqq 1$. It is clear that we need only to consider the case in which V is not a cone, that is, all the variables $x_0, \dots, x_n$ appear in the defining equations of V. With this assumption, after a suitable change of coordinates, we may assume that the defining ideal of V is generated by the 2×2 minors of a matrix which is obtained from the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \end{pmatrix}$$

by deleting $d-1$ columns. Since V is not a cone, V is defined as the rank one locus of a $2 \times (n-d+1)$ matrix

$$S = \begin{pmatrix} x_0 & \cdots & x_{j_1-1} & x_{j_1+1} & \cdots & x_{j_d-1} & x_{j_d+1} & \cdots & x_{n-1} \\ x_1 & \cdots & x_{j_1} & x_{j_1+2} & \cdots & x_{j_d} & x_{j_d+2} & \cdots & x_n \end{pmatrix}$$

where $j_1, \ldots, j_{d-1}$ are positive integers such that $1 \leqq j_1, j_{d-1} \leqq n-2, j_{k+1} \geqq j_k + 2$ for all $k = 1, \ldots, d-2$. A column $\begin{pmatrix} x_j \\ x_{j+1} \end{pmatrix}$ of S is said to be isolated if neither $\begin{pmatrix} x_{j-1} \\ x_j \end{pmatrix}$ nor $\begin{pmatrix} x_{j+1} \\ x_{j+2} \end{pmatrix}$ belong to S.

Remark 5.1. If $d \geqq 2$, then $n \geqq 2d-1$. If S has no isolated columns, then $n \geqq 3d-1$.

Denote by M the set of the 2-minors of the matrix S, by I and $K[M]$ respectively the ideal and the subalgebra of $K[x_0, \ldots, x_n]$ generated by the elements of M. The analytic spread $\ell(I)$ of I is the dimension of $K[M]$. Note that $K[M]$ is a specialization of the coordinate ring $G(2, n+1-d)$ of the Grassmannian variety of indices $2, n+1-d$ and hence $\ell(I) \leqq \dim G(2, n+1-d) = 2n-2d-1$. On the other hand one knows that $\ell(I) \leqq \dim(R) = n+1$, and hence $\ell(I) \leqq \min\{2n-2d-1, n+1\}$. The main result of this section is

Theorem 5.2. *Let I be the defining ideal of a rational normal scroll V of dimension d in $\mathbb{P}^n_K$, and assume that V is not a cone. Then* $\ell(I) = \min\{2n-2d-1, n+1\}$.

Proof. Since $\ell(I) \leqq \min\{2n-2d-1, n+1\}$, the result follows from:

Statement $\mathrm{S}_{(d,n)}$: At least $\min\{2n-2d-1, n+1\}$ elements in the set M are algebraically independent over K.

Let us consider the following claims.

Claim 1. Let $d \geqq 2$ and $n \geqq 4$. If the matrix has at least one isolated column and $\mathrm{S}_{(d-1,n-2)}$ holds, then $\mathrm{S}_{(d,n)}$ holds.

Claim 2. Let $n \geqq 2d+3$. If no column of the matrix is isolated and $\mathrm{S}_{(d,n-1)}$ holds, then $\mathrm{S}_{(d,n)}$ holds.

Claim 3. $\mathrm{S}_{(1,2)}$, $\mathrm{S}_{(1,3)}$, $\mathrm{S}_{(1,4)}$, $\mathrm{S}_{(2,3)}$, $\mathrm{S}_{(2,5)}$, $\mathrm{S}_{(2,6)}$ and $\mathrm{S}_{(3,8)}$ hold.

We show first that $\mathrm{S}_{(d,n)}$ follows from the above claims. We prove:

(i) $\mathrm{S}_{(d,2d-1)}$ holds for every $d \geqq 2$.

(ii) $\mathrm{S}_{(d,2d)}$ holds for every $d \geqq 1$.

(iii) $\mathrm{S}_{(d,2d+1)}$ holds for every $d \geqq 1$.

(iv) $S_{(d,2d+2)}$ holds for every $d \geqq 1$.

(v) $S_{(1,n)}$ holds for every $n \geqq 2$.

(vi) $S_{(d,n)}$ holds for every d and n.

(i) By Claim 3, $S_{(2,3)}$ holds; further $2d-1 \leqq 3d-2$, and hence by 5.1, we have at least one isolated column. We get the conclusion by induction on d since, by Claim 1, if $S_{(d-1,2(d-1)-1)} = S_{(d-1,2d-3)}$ holds then $S_{(d,2d-1)}$ holds. One uses similar arguments to show that (ii), (iii), (iv), and (v) follow from Claim 1, 2, 3.

(vi) By induction on d and n. Since by (v), $S_{(1,n)}$ holds for every n, we may assume that $d \geqq 2$ and $S_{(d-1,n)}$ holds for every n. Since $S_{(d,2d-1)}$, $S_{(d,2d)}$, $S_{(d,2d+1)}$ and $S_{(d,2d+2)}$ hold, we may assume $n \geqq 2d+3$. If we have an isolated column, we get the conclusion by Claim 1 because $S_{(d-1,n-2)}$ holds by induction. If we have no isolated column we get the conclusion by Claim 2 because $S_{(d,n-1)}$ holds by induction.

We now prove the claims. For this we recall that if K is a field of characteristic 0 and $g_1, \ldots, g_s$ are forms in the polynomial ring $K[x_0, \ldots, x_n]$, then they are algebraically independent over K if and only if the rank of the Jacobian matrix

$$J = \left(\frac{\partial g_i}{\partial x_j}\right)_{\substack{i=1,\ldots,s \\ j=0,\ldots,n}}$$

is s, [9], p. 36.

Proof of Claim 1. If we have an isolated column, say $\begin{pmatrix} x_i \\ x_{i+1} \end{pmatrix}$, we may delete it and obtain a matrix P whose 2×2 minors generate the defining ideal of a scroll W in $\mathbb{P}_K^{n-2}$ which is not a cone. Note that $\dim(W) = d-1$. Since $S_{(d-1,n-2)}$ holds, we can find $s = \min\{2n-2d-3, n-1\}$ minors of P, say $m_1, \ldots, m_s$, which are algebraically independent on K. Since $n \geqq 4$, if $d=2$ we have $n-d+1 \geqq 3$, while, if $d \geqq 3$, by 5.1, we have $n \geqq 2d-1$, hence $n-d+1 \geqq d \geqq 3$. In any case the matrix S has at least three columns and we can consider two more minors $m_{s+1} = x_t x_{i+1} - x_{t+1} x_i$, $m_{s+2} = x_v x_{i+1} - x_{v+1} x_i$. The Jacobian matrix of $m_1, \ldots, m_{s+2}$ has the following shape:

$$\begin{array}{c} \\ m_1 \\ \vdots \\ m_s \\ m_{s+1} \\ m_{s+2} \end{array} \left(\begin{array}{cccccccc} \dfrac{\partial}{\partial x_0} & \cdots & \dfrac{\partial}{\partial x_{i-1}} & \dfrac{\partial}{\partial x_{i+2}} & \cdots & \dfrac{\partial}{\partial x_n} & \dfrac{\partial}{\partial x_i} & \dfrac{\partial}{\partial x_{i+1}} \\ * & \cdots & * & * & \cdots & * & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * & 0 & 0 \\ * & \cdots & * & * & \cdots & * & -x_{t+1} & x_t \\ * & \cdots & * & * & \cdots & * & -x_{v+1} & x_v \end{array}\right).$$

Since $m_1, \ldots, m_s$ are algebraically independent, the submatrix involving the first s rows has rank s; this clearly implies that the matrix has rank $s+2$. This concludes the proof of Claim 1.

Proof of Claim 2. Let P be the matrix obtained by deleting from S the last column $\begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix}$. Since $\begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix}$ was not isolated, the last column of N is $\begin{pmatrix} x_{n-2} \\ x_{n-1} \end{pmatrix}$. Hence the 2×2 minors of P generate the defining ideal of a scroll W in $\mathbb{P}_K^{n-1}$ which is not a cone and $\dim(W) = d$. Since $S_{(d,n-1)}$ holds, we can find

$$\min\{2(n-1) - 2d - 1, n\} = \min\{2n - 2d - 3, n\}$$

minors of P which are algebraically independent over K. Since $n \geqq 2d + 3$, we get $2n - 2d - 3 \geqq n$, so that we can find n minors of P which are algebraically independent over K. If we consider the minor $x_0 x_n - x_1 x_{n-1}$, we get $n+1$ forms which are clearly algebraically independent over K. Since $n \geqq 2d + 3$, we get $2n - 2d - 1 \geqq n + 1$, and the conclusion follows.

Proof of Claim 3. The cases $S_{(1,2)}$, $S_{(1,3)}$, $S_{(1,4)}$ and $S_{(2,3)}$ are covered by 4.2. As for $S_{(2,5)}$, $S_{(2,6)}$, $S_{(3,8)}$, it is clear that we have to consider the following configurations:

$$(2,5) \quad \begin{pmatrix} x_0 & x_2 & x_3 & x_4 \\ x_1 & x_3 & x_4 & x_5 \end{pmatrix}, \qquad \begin{pmatrix} x_0 & x_1 & x_3 & x_4 \\ x_1 & x_2 & x_4 & x_5 \end{pmatrix},$$

$$(2,6) \quad \begin{pmatrix} x_0 & x_2 & x_3 & x_4 & x_5 \\ x_1 & x_3 & x_4 & x_5 & x_6 \end{pmatrix}, \qquad \begin{pmatrix} x_0 & x_1 & x_3 & x_4 & x_5 \\ x_1 & x_2 & x_4 & x_5 & x_6 \end{pmatrix},$$

$$(3,8) \quad \begin{pmatrix} x_0 & x_2 & x_4 & x_5 & x_6 & x_7 \\ x_1 & x_3 & x_5 & x_6 & x_7 & x_8 \end{pmatrix}, \begin{pmatrix} x_0 & x_1 & x_3 & x_4 & x_6 & x_7 \\ x_1 & x_2 & x_4 & x_5 & x_7 & x_8 \end{pmatrix},$$

$$\begin{pmatrix} x_0 & x_2 & x_3 & x_5 & x_6 & x_7 \\ x_1 & x_3 & x_4 & x_6 & x_7 & x_8 \end{pmatrix}.$$

The matrices on the right hand side correspond again to cases which are already covered by 4.2. Those on the left have an isolated column. Then the conclusion follows by Claim 1, since $S_{(1,3)}$ and $S_{(1,4)}$ imply $S_{(2,5)}$ and $S_{(2,6)}$ and $S_{(2,6)}$ implies $S_{(3,8)}$. This completes the proof of the theorem. □

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Eingegangen 20. Januar 1995, in revidierter Fassung 8. Mai 1995