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Gröbner bases of powers of ideals of maximal minors

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Abstract

We determine Gröbner bases of powers of ideals of maximal minors of generic matrices. Then we derive a formula for the Hilbert series of the rings defined by these ideals. © 1997 Elsevier Science B.V.

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0. Introduction

Let K be a field, let $X = (X_{ij})$ be an $m \times n$, $m \leq n$, matrix of indeterminates over K, and let I_m be the ideal generated by the m-minors of X. The aim of this paper is to show that the set of the elements of the form $M_1 \cdots M_i$, where the M_j are m-minors of X, is a Gröbner basis of the ideal I_m^i for all $i \in \mathbf{N}$ with respect to a diagonal monomial order.

The proof of this result is given in Section 2 and it is based on the fact that the powers of the ideal of maximal minors have a K-basis of standard monomials. Following the approach of Sturmfels [11], we employ the Knuth–Robinson–Schensted correspondence to compare the Hilbert function of I_m^i with the Hilbert function of the monomial ideal which is our candidate to be the initial ideal of I_m^i.

In Section 3 we present two applications of our result. First we determine a sequence of indeterminates which is regular over $K[X]/I_m^i$ for all i. Next we compute the Hilbert series of $K[X]/I_m^i$ in the case in which X has size $m \times m + 1$.

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1. Notation and generalities

Let R be a polynomial ring over the field K in the set of indeterminates X. A monomial of R is just a product of indeterminates. The set of all monomials of R form a semigroup which is isomorphic to $\mathbf{N}^X$. A monomial order τ on R is a total order in the set of monomials of R which is compatible with the semigroup structure. Given a polynomial $f \in R$, consider the unique expression of f as linear combination of monomials, say $f = \lambda_1 T_1 + \cdots + \lambda_s T_s$ where the T_i are monomials, $T_i \neq T_j$ if $i \neq j$, and $\lambda_j \in K \setminus \{0\}$. The initial monomial of f with respect to τ, denoted by $\mathrm{in}_\tau(f)$, is the biggest monomial in the set $\{T_1, \ldots, T_s\}$. Let now I be an ideal of R. One denotes by $\mathrm{in}_\tau(I)$ the ideal generated by the monomials $\mathrm{in}_\tau(f)$ with $f \in I$. The ideal $\mathrm{in}_\tau(I)$ is called the initial ideal of I with respect to τ. If the ideal I is generated by the polynomials $f_1, \ldots, f_r$, then the ideal $\mathrm{in}_\tau(I)$ need not to be generated by $\mathrm{in}_\tau(f_1), \ldots, \mathrm{in}_\tau(f_r)$. A set of polynomials $f_1, \ldots, f_r$ of I is called a Gröbner basis of I with respect to τ if $\mathrm{in}_\tau(f_1), \ldots, \mathrm{in}_\tau(f_r)$ generate $\mathrm{in}_\tau(I)$. Let I and J be ideals of R. It follows immediately from the definition of initial ideal that $\mathrm{in}_\tau(I)\mathrm{in}_\tau(J) \subseteq \mathrm{in}_\tau(IJ)$. In particular one has $\mathrm{in}_\tau(I)^i \subseteq \mathrm{in}_\tau(I^i)$. In general the ideal $\mathrm{in}_\tau(I)^i$ is smaller than $\mathrm{in}_\tau(I^i)$. For example if $R = K[X,Y]$, τ is any monomial order such that $X > Y$, $I = (X^2 + Y^2, XY)$, then $\mathrm{in}_\tau(I) = (X^2, XY, Y^3)$, $Y^5 \in I^2$, and $Y^5 \in \mathrm{in}_\tau(I^2) \setminus \mathrm{in}_\tau(I)^2$.

It is an interesting problem to find classes of ideals I for which the equality

$$(*) \qquad \mathrm{in}_\tau(I)^i = \mathrm{in}_\tau(I^i)$$

holds for all $i \in \mathbf{N}$.

Let $F = f_1, \ldots, f_r$ be a sequence of polynomials. Given a vector $a = (a_1, \ldots, a_r) \in \mathbf{N}$ we denote by F^a the polynomial $f_1^{a_1} \cdots f_r^{a_r}$. Further we denote by F^i, $i \in \mathbf{N}$, the set of the power products of degree i of the elements of F, that is, $\{F^a : a_1 + \cdots + a_r = i\}$. Let $F = f_1, \ldots, f_r$ be a Gröbner basis of an ideal I. Then equality $(*)$ holds if and only if F^i is a Gröbner basis. Here is an example of a class of ideals for which equality $(*)$ holds:

Proposition 1.1. *Let $F = f_1, \ldots, f_r$ be a set of polynomials of R and τ a monomial order such that* $\gcd(\mathrm{in}_\tau(f_i), \mathrm{in}_\tau(f_j)) = 1$ *for all $i \neq j$. Then F^i is a Gröbner basis for all $i \geq 1$. In particular, if I denotes the ideal generated by F, then* $\mathrm{in}_\tau(I)^i = \mathrm{in}_\tau(I^i)$.

Proof. We may assume that $f_1, \ldots, f_r$ are monic polynomials. Denoted by $\mathrm{in}_\tau(F)$ the sequence $\mathrm{in}_\tau(f_1), \ldots, \mathrm{in}_\tau(f_r)$. Let F^a and F^b be elements of F^i, and consider the S-polynomial $S(F^a, F^b) = [\mathrm{in}_\tau(F)^b F^a - \mathrm{in}_\tau(F)^a F^b] / \gcd(\mathrm{in}_\tau(F)^a, \mathrm{in}_\tau(F)^b)$. By Buchberger's criterion [3, 6.2], it is enough to show that $S(F^a, F^b)$ can be expressed as $\sum p_d F^d$, where $F^d \in F^i$, $p_d \in R$, and $\mathrm{in}_\tau(p_d F^d) \leq \mathrm{in}_\tau(S(F^a, F^b))$ for all d such that $p_d \neq 0$. Let us denote by c the vector $(\min\{a_1, b_1\}, \ldots, \min\{a_r, b_r\})$. We may write

$$S(F^a, F^b) = (\mathrm{in}_\tau(F)^{b-c} - F^{b-c})F^a + (-\mathrm{in}_\tau(F)^{a-c} + F^{a-c})F^b$$

and we have to check that this expression satisfies the above mentioned condition. To this end one has just to note that from $\gcd(\mathrm{in}_\tau(F)^{a-c}, \mathrm{in}_\tau(F)^{b-c}) = 1$ it follows $\mathrm{in}_\tau(\mathrm{in}_\tau(F)^{b-c} - F^{b-c})\mathrm{in}_\tau(F)^a \neq \mathrm{in}_\tau(\mathrm{in}_\tau(F)^{a-c} - F^{a-c})\mathrm{in}_\tau(F)^b$. □

Remark 1.2. The proof of the proposition shows something more, namely, any subsets of F^i is a Gröbner basis. Sets with this property are called super G-bases, see [5].

2. Maximal minors

Let K be a field, and let $X = (X_{ij})$ be an $m \times n$ matrix of indeterminates, with $m \leq n$. We denote by $K[X]$ the polynomial ring over K in the indeterminates X_{ij}, and by I_t the ideal of $K[X]$ generated by the t-minors of X. A monomial order τ on $K[X]$ is said to be a diagonal monomial order if the initial monomial of any minor of X is the product of the elements of its main diagonal.

Narasimhan showed in [9] that the t-minors of X form a Gröbner basis of I_t with respect to a diagonal monomial order. Other proofs of this result can be found in [4, 7, 11]. Our goal is to show:

Theorem 2.1. *The set $\{M_1 \cdots M_i : M_1, \ldots, M_i$ are m-minors of $X\}$ is a Gröbner basis of I_m^i for all $i \in \mathbf{N}$ with respect to a diagonal monomial order τ. In particular* $\mathrm{in}_\tau(I_m^i) = \mathrm{in}_\tau(I_m)^i$ *for all $i \in \mathbf{N}$.*

We denote by $[a_1, \ldots, a_r \mid b_1, \ldots, b_r]$, $1 < a_1 < \cdots < a_r \leq m$, $1 < b_1 < \cdots < b_r \leq n$, the minor $\det(X_{a_i b_j})$, $i,j = 1, \ldots, r$, of X. Further, we define $[a_1, \ldots, a_r \mid b_1, \ldots, b_r] \leq [c_1, \ldots, c_s \mid d_1, \ldots, d_s]$ if $r \geq s$ and $a_i \leq c_i$, $b_i \leq d_i$ for $i = 1, \ldots, s$. A product of minors $M_1 \cdots M_s$ is called a standard monomial if $M_1 \leq M_2 \leq \cdots \leq M_s$.

The powers of the ideal I_m of maximal minors of X have a K-basis of standard monomials. The set of all standard monomials $M_1 \cdots M_s$ such that $s \geq i$ and $M_1, \ldots, M_i$ are m-minors is a K-basis of I_m^i, see [2, 9.3].

This fact gives us the possibility to use the Knuth–Robinson–Schensted correspondence to determine a Gröbner basis of I_m^i. The original Knuth–Robinson–Schensted correspondence is a one-to-one correspondence between standard bitableaux and matrices of non-negative integers, see [8, 11]. Different versions of this correspondence were used by Sturmfels [11], Herzog and Trung [7], and Conca [6] to determine Gröbner bases of classes of determinantal rings.

We will employ the correspondence described in [7, Section 1]. We denote this correspondence by KRS. Recall that KRS is a degree preserving bijection between the set of standard monomials of the matrix X and the set of ordinary monomials of the polynomial ring $K[X]$. Let J be the ideal generated by the initial monomials with respect to a diagonal monomial order τ of the m-minors of X. One has

Lemma 2.2. *Let $M = M_1 \cdots M_s$ be a standard monomial of X, and assume that $M_1, \ldots, M_i$ are m-minors. Then $KRS(M)$ belongs to J^i.*

First we indicate how Theorem 2.1 follows from Lemma 2.2.

Proof of Theorem 2.1. From Lemma 2.2 it follows that the standard monomial K-basis of I_m^i is mapped by KRS injectively to the monomial K-basis of J^i. The dimension argument of [11, Lemma 6] or [7, 2.4] implies immediately that J^i coincides with the initial ideal of I_m^i. $\square$

Before starting the proof of Lemma 2.2, let us introduce a piece of notation. An m-diagonal of X is a monomial of the form $X_{1a_1} \cdots X_{ma_m}$ such that $a_1 < a_2 < \cdots < a_m$. The m-diagonals are exactly the initial monomials of the m-minors of X. The partial order on the set of minors induces a partial order on the set of m-diagonals: $X_{1a_1} \cdots X_{ma_m} \leq X_{1b_1} \cdots X_{mb_m}$ if $a_i \leq b_i$ for all $i = 1, \ldots, m$. Let T be monomial of $K[X]$. Set $T_1 = T$. If T_1 is divisible by an m-diagonal, then there is a unique m-diagonal D_1 which divides T_1 and it is minimal with respect to this property. In this case one may write $T_1 = D_1 T_2$. If T_2 is divisible by an m-diagonal, then let D_2 be the minimal m-diagonal with this property, and so on. Therefore T has a unique decomposition $T = D_1 \cdots D_r T_{r+1}$, where D_j is the unique minimal diagonal which divides $T/D_1 \cdots D_{j-1}$, $j = 1, \ldots, r$, and T_{r+1} is not divisible by m-diagonals. In this case we say that T has m-diagonal type equal to r. Note that T has m-diagonal type at least r if and only if T is divisible by the product of r m-diagonals. Now we are ready to prove the lemma.

Proof of Lemma 2.2. Let $M = M_1 \cdots M_s$ be a standard monomial such that $M_1, \ldots, M_i$ are m-minors. We have to show that the m-diagonal type of KRS(M) is at least i. We may assume M_{i+1} is not an m-minor. We argue by induction on m and i. The case $m = 1$ is trivial and the case $i = 1$ is covered by [7, 1.2]. So let $m > 1$ and $i > 1$. Let us denote by A and B respectively the standard tableaux associated with the row and column indices of the standard monomial M. Let A_1 and B_1 be respectively the standard tableaux associated with the row indices and column indices of the standard monomial $M_{i+1} \cdots M_s$. Denote by u_j, $1 \leq j \leq i$, the element of B which is in the position (j, m), that is, u_j is the mth column index of the minor M_j.

$$A = \begin{array}{l} \left.\begin{array}{cccc} 1 & 2 & \cdots & m \\ 1 & 2 & \cdots & m \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 2 & \cdots & m \end{array}\right\} i \text{ rows} \\ \left.\begin{array}{cc} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \end{array}\right\} A_1 \end{array} \qquad B = \begin{array}{l} \left.\begin{array}{cccc} \cdots & \cdots & \cdots & u_1 \\ \cdots & \cdots & \cdots & u_2 \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & u_i \end{array}\right\} i \text{ rows} \\ \left.\begin{array}{cc} \cdots & \cdots \\ \vdots & \vdots \\ \cdots & \end{array}\right\} B_1 \end{array}$$

If the tableau A_1 contains the index m, then the first output of the algorithm which defines KRS is an indeterminate X_{mj} and a pair of standard tableaux that correspond to a standard monomial $M' = M_1' \cdots M_r'$ such that $M_1', \dots, M_i'$ are m-minors. Note that, by definition, $\mathrm{KRS}(M) = \mathrm{KRS}(M')X_{mj}$. So we may assume from the very beginning that A_1 does not contain the index m.

We want now to show that we may assume that B_1 does not contain an index greater than or equal to u_i. Suppose the contrary is true, and consider w the largest index of B_1. By virtue of [7, 1.1], one has

$$\mathrm{KRS}(M) = \mathrm{KRS}(A,B) = [\mathrm{KRS}(B,A)]^{\top} = [\mathrm{KRS}(B',A')X_{wh}]^{\top} = \mathrm{KRS}(A',B')X_{hw}.$$

By the definition of KRS follows that the tableau A_1' does not contain m, the tableau B' differs from B exactly in w. Therefore it is not a restriction to assume that B_1 does not contain an index greater than or equal to u_i. Under these assumptions, again by [7, 1.1], one has

$$\mathrm{KRS}(M) = \mathrm{KRS}(M')X_{mu_i} = \mathrm{KRS}(M'')X_{mu_1} \cdots X_{mu_i},$$

where M' is the standard monomial associated with the tableaux:

$$\begin{array}{ll}
\left.\begin{array}{cccc} 1 & \cdots & \cdots & m \\ 1 & \cdots & \cdots & m \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \cdots & \cdots & m \\ 1 & \cdots & m-1 & \end{array}\right\} i \text{ rows} &
\left.\begin{array}{cccc} \cdots & \cdots & \cdots & u_1 \\ \cdots & \cdots & \cdots & u_2 \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & u_{i-1} \\ \cdots & \cdots & \cdots & \end{array}\right\} i \text{ rows} \\
\left.\begin{array}{ccc} \cdots & \cdots & \cdots \\ \vdots & \vdots & \\ \cdots & & \end{array}\right\} A_1 &
\left.\begin{array}{ccc} \cdots & \cdots & \cdots \\ \vdots & \vdots & \\ \cdots & & \end{array}\right\} B_1
\end{array}$$

and M'' is the standard monomial associated with the tableaux A'', B'' which are obtained from the tableaux A and B by deleting the column with index m.

By induction on m, $\mathrm{KRS}(M'')$ has $(m-1)$-diagonal type greater than or equal to i. Let $D_1, \dots, D_i$ denote the first i $(m-1)$-diagonals which appear in the decomposition of $\mathrm{KRS}(M'')$. By induction on i, $\mathrm{KRS}(M')$ has m-diagonal type equal $i-1$. Since $\mathrm{KRS}(M') = \mathrm{KRS}(M'')X_{mu_1} \cdots X_{mu_{i-1}}$, the m-diagonals of the decomposition of $\mathrm{KRS}(M')$ are $D_1X_{mu_1}, \dots, D_{i-1}X_{mu_{i-1}}$. Finally note that $D_iX_{mu_i}$ is an m-diagonal since the elements of B'' are all smaller than u_i. It follows that the m-diagonal type of $\mathrm{KRS}(M)$ is at least i. □

Remark 2.3. One can easily see that $\mathrm{in}_\tau(I_t)^2$ is a proper subset of $\mathrm{in}_\tau(I_t^2)$ if $1 < t < m$. This shows that Theorem 2.1 cannot be true for non-maximal minors. We believe that the ideal $\mathrm{in}_\tau(I_t^i)$ should be somehow related with the primary decomposition of I_t^i [2, Section 10], but we do not know exactly how.

Remark 2.4. Let R be a polynomial ring over a field K, and τ be a monomial order. Let $f_1,\dots,f_r \in R$, and let $S = K[f_1,\dots,f_r]$ be the K-subalgebra of R generated by $f_1,\dots,f_r$. One defines the initial algebra $\mathrm{in}_\tau(S)$ of S to be the K-subalgebra of R generated by $\mathrm{in}_\tau(f)$, $f \in S$. The set $f_1,\dots,f_r$ is said to be a Sagbi basis of S if $\mathrm{in}_\tau(S)$ is generated as K-algebra by $\mathrm{in}_\tau(f_1),\dots,\mathrm{in}_\tau(f_r)$. For the theory of Sagbi bases we refer the reader to the paper of Robbiano and Sweedler [10]. In the case in which the f_i are all homogeneous of the same degree d, it is easy to see that $f_1,\dots,f_r$ are a Sagbi basis if and only if the components of degree id of the ideals $\mathrm{in}_\tau((f_1,\dots,f_r)^i)$ and $(\mathrm{in}_\tau(f_1),\dots,\mathrm{in}_\tau(f_r))^i$ coincide for all $i \in \mathbf{N}$.

Therefore from Theorem 2.1, it follows that the maximal minors form a Sagbi basis with respect to a diagonal monomial order, a result which was first proved by Sturmfels, see [12, 3.2.9].

By a result of Bernstein and Zelevinsky [1, Theorem 1] the maximal minors are known to be a universal Gröbner basis, that is, they form a Gröbner basis with respect to all monomial orders. The following question arises naturally:

Question. Is the set $\{M_1 \cdots M_i : M_1,\dots,M_i \text{ are } m\text{-minors of } X\}$ a universal Gröbner basis of I_m^i ? In other words: Does the equation $\mathrm{in}_\tau(I_m)^i = \mathrm{in}_\tau(I_m^i)$ hold for all the monomial orders τ?

3. Applications

This section is devoted to present two applications of Theorem 2.1 to the study of the ring $K[X]/I_m^i$. We start with a general fact:

Lemma 3.1. *Let I be an ideal of a polynomial ring $R = K[X_1,\dots,X_r]$. Let τ be a monomial order, and assume that the indeterminates $X_1,\dots,X_s$ do not appear in the generators of $\mathrm{in}_\tau(I)$. Then the residue classes of $X_1,\dots,X_s$ form a regular sequence of the ring R/I.*

Proof. Since $\mathrm{in}_\tau(I+X_1) = \mathrm{in}_\tau(I)+(X_1)$, it is enough to prove the assertion for $s = 1$. Suppose that the contention is false. Let f be an element of $R \setminus I$ such that $X_1 f \in I$, and we may assume that f has the smallest initial monomial among all the elements with these properties. From $X_1\,\mathrm{in}_\tau(f) \in \mathrm{in}_\tau(I)$, it follows $\mathrm{in}_\tau(f) \in \mathrm{in}_\tau(I)$. Let $g \in I$ such that $\mathrm{in}_\tau(g) = \mathrm{in}_\tau(f)$, and set $f' = f - g$. One has $f' \in R \setminus I$, $X_1 f' \in I$, and $\mathrm{in}_\tau(f') < \mathrm{in}_\tau(f)$. This is a contradiction. □

From Theorem 2.1 and Lemma 3.1, it follows immediately:

Proposition 3.2. *Let X be an $m \times n$ matrix of indeterminates over a field K. Then the residue classes of the indeterminates $X_{21}, X_{31}, X_{32},\dots,X_{m1}\dots,X_{mm-1}, X_{1n-m+2}$,*

$X_{1n-m+3},\ldots,X_{1n},X_{2n-m+3},\ \ldots,X_{2n},\ldots,X_{m-1n}$ *form a regular sequence of the ring* $K[X]/I_m^i$ *for all* $i\in\mathbf{N}$.

It is known that $\min_{i\in\mathbf{N}}\operatorname{depth} K[X]/I_m^i = m^2-1$, see [2, 9.27]. The regular sequence given in Proposition 3.2 has $m(m-1)$ elements. So it is not a maximal regular sequence even for $i \gg 0$.

The second application we present is the computation of the Hilbert series of $K[X]/I_m^i$. We restrict our attention to the case of an $m\times m+1$ matrix. Our result is

Theorem 3.3. *Let X be an $m\times m+1$ matrix of indeterminates over a field K. Denoted by $H_{mi}(\lambda)$ the Hilbert series of the ring $K[X]/I_m^i$. Then*

$$H_{mi}(\lambda)=\frac{\sum_{j=0}^{im-1}(j+1)\lambda^j+\sum_{j\ge 0}(-1)^{j+1}\left[\sum_{k\ge j+2}\binom{i}{k}\binom{m}{k}\binom{k-2}{j}\right]\lambda^{im+j}}{(1-\lambda)^{m(m+1)-2}}.$$

Proof. It is well-known that the Hilbert series of $K[X]/I_m^i$ coincides with that of $K[X]/\mathrm{in}_\tau(I_m^i)$. Hence, by virtue of Theorem 2.1, H_{mi} is the Hilbert series of $K[X]/\mathrm{in}_\tau(I_m)^i$, where τ is a diagonal monomial order.

The proof of the theorem is based on the following simple observation. Suppose we may find a class of homogeneous ideals, say I_γ with $\gamma\in\Gamma$, such that the ideals $\mathrm{in}_\tau(I_m)^i$ belong to this class, and for all $\gamma\in\Gamma$ there exists an homogeneous element f_γ of degree d_N such that $I_\gamma+(f_\gamma)$ and $I_\gamma : f_\gamma$ are again elements of our class and they are somehow better than I_γ. Then the exact sequence

$$0\longrightarrow K[X]/(I_\gamma : f_\gamma)[-d_\gamma]\xrightarrow{f_\gamma} K[X]/I_\gamma\longrightarrow K[X]/I_\gamma+(f_\gamma)\longrightarrow 0 \qquad (1)$$

together with some sort of induction gives the possibility to determine the Hilbert series of the rings defined by the ideals of the class. Of course one cannot expect to be able to do this if the class I_γ is improperly chosen. For instance, the class of all homogeneous ideals, or the class of the monomial ideals are certainly too big.

In order to define the right class of ideals, we need to introduce a piece of notation. We set $X_i = X_{ii}$ and $Y_i = X_{ii+1}$ for $i=1,\ldots,m$, $M_k = X_1\cdots X_kY_{k+1}\cdots Y_m$ for $k=0,\ldots,m$, $J=(M_0,\ldots,M_m)$. Further set $d=m(m+1)=\dim K[X]$. Note that the initial monomial of the minor $[1,\ldots,m\mid 1,\ldots,k,k+2,\ldots,m+1]$ is M_k, and that the ideal J is equal to $\mathrm{in}_\tau(I_m)$. We define a class of ideals $I_{n,k,i}$ with $0\le n\le m$, $n-1\le k\le m$, and $i\in\mathbf{N}$, setting

$$I_{n,k,i}=(M_n,\ldots,M_k)+(M_{k+1},\ldots,M_m)^i.$$

Denote by $H_{n,k,i}(\lambda)$ the Hilbert series of $K[X]/I_{n,k,i}$ In particular one has

$$I_{n,n-1,i}=(M_n,\ldots,M_m)^i,\ I_{0,-1,i}=(M_0,\ldots,M_m)^i=J^i,$$

$$I_{n,m,i}=(M_n,\ldots,M_m)=I_{n,m,1}.$$

As element associated with the ideal $I_{n,k,i}$, with $k < m$, we consider M_{k+1}. Note that

$$I_{n,k,i} + M_{k+1} = I_{n,k+1,i},$$

$$I_{n,k,i} : M_{k+1} = (Y_{k+1}) + I_{k+1,k,i-1} \quad \text{if } k \geq n,$$

$$I_{n,n-1,i} : M_n = I_{n,n-1,i-1} \qquad\qquad \text{if } k = n-1.$$

For the ideal $I_{n,m,1}$, $n \geq 0$, we choose the element M_{n-1}. One has

$$I_{n,m,1} + M_{n-1} = I_{n-1,m,1}, \quad I_{n,m,1} : M_{n-1} = (X_n).$$

The ideal $(Y_{k+1}) + I_{k+1,k,i-1}$ is not exactly one element of the class under consideration, but the indeterminate Y_{k+1} does not appear in the generators of $I_{k+1,k,i-1}$. Hence the Hilbert series of the ring defined by $(Y_{k+1}) + I_{k+1,k,i-1}$ is $(1-\lambda)H_{k+1,k,i-1}(\lambda)$.

Now from the exact sequence (1), with $I_\gamma = I_{n,k,i}$ and $f_\gamma = M_{k+1}$, it follows: For all $0 \leq n \leq k \leq m-1$,

$$H_{n,k,i}(\lambda) = H_{n,k+1,i}(\lambda) + (1-\lambda)\lambda^m H_{k+1,k,i-1}(\lambda) \quad \text{if } k \geq n,$$

$$H_{n,n-1,i}(\lambda) = H_{n,n,i}(\lambda) + \lambda^m H_{n,n-1,i-1}(\lambda) \qquad\qquad \text{if } k = n-1.$$

Summing up one obtains:

$$H_{n,n-1,i}(\lambda) = H_{n,m,1}(\lambda) + (1-\lambda)\lambda^m \sum_{j=n+1}^{m} H_{j,j-1,i-1}(\lambda) + \lambda^m H_{n,n-1,i-1}(\lambda). \tag{2}$$

Using the exact sequence (1), with $I_\gamma = I_{n,m,1}$ and $f_\gamma = M_{n-1}$, $n > 0$,

$$H_{n-1,m,1}(\lambda) = H_{n,m,1}(\lambda) - \lambda^m/(1-\lambda)^{d-1}. \tag{3}$$

Since $H_{m,m,1}(\lambda) = [\sum_{j=0}^{m-1} \lambda^j]/(1-\lambda)^{d-1}$, Eq. (3) yields

$$H_{n,m,1}(\lambda) = \left[\sum_{j=0}^{m-1} \lambda^j - (m-n)\lambda^m\right] \Big/ (1-\lambda)^{d-1}.$$

By substituting this expression in (2), one obtains

$$H_{n,n-1,i}(\lambda) = \left[\sum_{j=0}^{m-1} \lambda^j - (m-n)\lambda^m\right] \Big/ (1-\lambda)^{d-1}$$
$$+ (1-\lambda)\lambda^m \sum_{j=n+1}^{m} H_{j,j-1,i-1}(\lambda) + \lambda^m H_{n,n-1,i-1}(\lambda). \tag{4}$$

Eq. (4) involves only Hilbert series associated with triplets of the type $(j, j-1, i)$. By induction on i and using (4), one shows that

$$H_{n,n-1,i}(\lambda) = \frac{\sum_{j=0}^{im-1} \lambda^j + \sum_{j \geq 0} (-1)^{j+1} \left[\sum_{k \geq j+1} \binom{i}{k}\binom{m-n}{k}\binom{k-1}{j}\right] \lambda^{im+j}}{(1-\lambda)^{d-1}}.$$

In particular,

$$H_{mi}(\lambda) = H_{0,-1,i}(\lambda) = \frac{\sum_{j=0}^{im-1} \lambda^j + \sum_{j\geq 0} (-1)^{j+1} \left[\sum_{k\geq j+1} \binom{i}{k}\binom{m}{k}\binom{k-1}{j}\right] \lambda^{im+j}}{(1-\lambda)^{d-1}}.$$

One obtains the desired expression just by dividing the numerator and the denominator by $(1-\lambda)$. □

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