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Ladder Determinantal Rings Have Rational Singularities

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INTRODUCTION

In this paper we use tight closure and Gröbner basis theory to prove that ladder determinantal rings have rational singularities.

We show that the ladder determinantal rings of a certain class of ladders, which we call wide ladders, are F-rational. Though F-rationality is only defined in positive characteristic, recent results of Smith [26] imply that these ladder determinantal rings are pseudorational in the sense of Lipman and Tessier [20], and in characteristic 0 are of F-rational type which in turn implies that they have rational singularities.

We will further show that an arbitrary ladder determinantal ring is an algebra retract of the determinantal ring of a wide ladder. Thus we may apply Boutot's theorem [3] to conclude that all ladder determinantal rings defined over an algebraically closed field of characteristic 0 have rational singularities. With some more effort one probably could avoid Boutot's theorem and prove instead that all ladder determinantal rings are F-rational; see Remark 4.5. For this we have to check that certain simplicial complexes which arise from the ladder are shellable. Assuming the ladders are wide simplifies the arguments considerably, and, as we hoped, makes the proof more readable.

Ladder determinantal rings were introduced by Abhyankar [1] in his studies of singularities of Schubert varieties of flag manifolds. An important subclass of general ladders are the one-sided ladders. In [22] Mulay showed that one-sided ladder determinantal rings occur as coordinate rings of certain affine sets in Schubert varieties, and Ramanathan [24] showed that all Schubert varieties have rational singularities. So their results cover a special case of our Theorem 1.7. Another special case, namely that of ladder determinantal rings which are complete intersections, has been treated by Glassbrenner and Smith [12].

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Let X be a matrix of indeterminates. A ladder Y is a subset of X with the property that $X_{ij'}$ and $X_{i'j}$ belong to Y whenever $X_{ij}, X_{i'j'} \in Y$ for $i \leqslant i'$ and $j \leqslant j'$. The ideal $I_t(Y)$ of all minors of size t which belong to Y defines the ladder determinantal ring $R_t(Y)$. These large classes of rings provide new and interesting examples of Cohen–Macaulay normal domains. Narasimhan [23] showed that they are domains, Cohen–Macaulayness was proved by Herzog and Trung [13] (for even more general ladders), and normality by Conca [5], who also determined their divisor class group [7] and used this information to characterize the Gorenstein ladders. All these results were proved using Gröbner bases, a technique which in this paper will be an essential tool as well. As mentioned in Mulay's paper [22], normality, Cohen–Macaulayness, and other properties also follow, at least for one-sided ladders, from his approach through Schubert varieties.

The other cornerstone for our proof are characteristic p-methods. Fedder and Watanabe [11] call a ring of prime characteristic F-rational if all its parameter ideals are tightly closed. The notion of tight closure was introduced by Huneke and Hochster. The reader is referred to their papers [14–16] in which the theory of tight closure is developed and applied to solve various outstanding problems or to give new and strikingly simple proofs of some of the homological conjectures. The contributions of this paper will show that the concepts related to tight closure are equally useful for solving very concrete problems.

We now outline the contents of each section. In Section 1 we describe the general strategy of our proof. It is based on a slightly modified criterion for F-rationality due to Fedder and Watanabe [11, Proposition 2.13]. Roughly, the criterion (Theorem 1.2) says that if the localization of a ring R with respect to a suitable element c is F-rational, then R itself is F-rational. In this criterion it is required that R/cR is F-injective.

We show in Section 2 that F-injectivity can be checked in many cases by considering Gröbner bases. Suppose $R = K[X_1, ..., X_n]/I$ is a finitely generated positively graded K-algebra, K a field, $I \subseteq (X)$, and let us denote by $\mathfrak{m}$ the maximal ideal of R which is generated by the residues of the variables X_i. Suppose further that for a suitable monomial order the ideal of initial terms $\operatorname{in}(I)$ is square-free and is the defining ideal of the Stanley–Reisner ring of a shellable simplicial complex. Then the main result 2.2 of this section asserts that $R_{\mathfrak{m}}$ is an F-injective Cohen–Macaulay ring.

We introduce the basic concepts concerning ladders in Section 3. We also show that if we are given two ladders Y and Z where Z is obtained from Y by adding a row or column, then $R_t(Y)$ is an algebra retract of $R_t(Z)$.

In Section 4 we verify the hypotheses of the Fedder–Watanabe criterion for determinantal rings of wide ladders. In order to do so we pick a suitable

$(t-1)$-minor $c \in R_t(Y)$, compute the ideal $\operatorname{in}(I)$ of initial terms of $I = c + I_t(Y)$, and show that it defines a shellable simplicial complex. The proof depends heavily on the analysis of the minimal prime ideals of I which have been studied by the first author in [5]. Indeed, it will be shown that the ideal of initial terms of any minimal prime ideal of I defines a shellable simplicial complex, and then it will be shown that these shellings glue together to yield a shelling of the simplicial complex defined by $\operatorname{in}(I)$.

We close our paper with Section 5 where we apply the results of Section 2 to show that all ladder determinantal rings are F-injective. We also observe that the determinantal rings of the wide Gorenstein and the so-called chain ladders are F-pure and F-regular. The class of chain ladders includes all one-sided ladders. Metha and Ramanathan [21] showed already that all Schubert varieties have Frobenius splitting which in our terminology is F-purity. Thus their theorem implies that determinantal rings of one-sided ladders are F-pure which is a special case of our result. It is still open whether all ladders determinantal rings are F-pure and F-regular.

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1. AN OUTLINE OF THE PROOF THE MAIN THEOREM

We first recall a few notions and results from tight closure theory. Let R be a commutative Noetherian ring of prime characteristic p. The complement of all minimal prime ideals of R will be denoted by R^0.

Let I be an ideal in R. Hochster and Huneke [14] define the *tight closure* I^* of I to be the set of elements $z \in R$ for which there exists $c \in R^0$ such that $cz^{p^e} \in I^{[p^e]}$ for all $e \gg 0$. Here $I^{[p^e]}$ denotes as usual the ideal generated by all elements a^{p^e}, $a \in I$. It is immediate from the definition that I^* is an ideal containing I. If $I^* = I$, then I is called *tightly closed.*

The ring R is called *F-regular* if every ideal in every localization of R is tightly closed, and R is called *F-rational* if every parameter ideal is tightly closed. Here we call an ideal a parameter ideal if it is generated by parameters, that is, a sequence of elements $x_1, \ldots, x_n$ whose images generate an ideal of height n in any localization R_P of R such that the prime ideal P contains them.

We shall need the following results about F-rationality:

THEOREM 1.1. (a) [17; 27, Proposition 1.4.3]. *Suppose* $(R, \mathfrak{m})$ *is an excellent local ring. Then R is F-rational if and only if its* $\mathfrak{m}$*-adic completion* $\hat{R}$ *is F-rational.*

(b) [16, Theorem 1.4]. *Suppose K is a perfect field, and R is a positively graded K-algebra with graded maximal ideal* $\mathfrak{m}$. *Then R is F-rational if and only if* $R_{\mathfrak{m}}$ *is F-rational.*

(c) [11, Proposition 2.2; 16, Theorem 1.4]. *Assuming* (a) *or* (b), *if a single* (*homogeneous*) *system of parameters is tightly closed, then R is F-rational.*

Thus if we want to check F-rationality for a graded ring as in 1.1(b), we only have to show that $\widehat{R_{\mathfrak{m}}}$ is F-rational. For complete local Cohen–Macaulay rings we have the following modified Fedder–Watanabe criterion:

THEOREM 1.2. *Suppose R is a complete local Cohen–Macaulay ring for which there exists a non-zero divisor* $c \in R$ *such that* (a) R_c *is F-rational, and* (b) R/cR *is F-injective. Then R is F-rational.*

Recall that R is *F-injective*, if the natural action $F\colon H^i_{\mathfrak{m}}(R) \to H^i_{\mathfrak{m}}(R)$ on the local cohomology modules induced by the Frobenius endomorphism of R is injective for all i. In case R is Cohen–Macaulay there is only one non-vanishing local cohomology module, namely $H^d_{\mathfrak{m}}(R)$, $d = \dim R$, and F-injectivity can be characterized by the property that some (all) system(s) of parameters $y_1, ..., y_d$ of R is (are) *F-contracted*, that is, $a^p \in (y_1^p, ..., y_d^p)$ implies $a \in (y_1, ..., y_d)$ for all $a \in R$; see [10, Proposition 1.4].

There is an obvious graded version of F-injectivity: In the definition we replace R by a positively graded K-algebra, where K is a field of positive characteristic, and $\mathfrak{m}$ by the unique graded maximal ideal. Note that R is F-injective if and only if the local ring $R_{\mathfrak{m}}$ is F-injective. Furthermore, if R is Cohen–Macaulay then R is F-injective if and only if some (all) homogeneous system(s) of parameters is (are) F-contracted.

The proof of 1.2 is exactly the same as that of the original Fedder–Watanabe criterion. Since Theorem 1.2 is so central to our arguments we repeat here its simple proof for which we only need one extra new ingredient: Smith in [28] (see also [25]) calls an element $c \in R^0$ a *parameter test element* if $cI^* \subset I$ for all parameter ideals, and proves the following:

THEOREM 1.3. *Let R be complete local Cohen–Macaulay ring, and let c be a non-zero divisor of R such that* R_c *is F-rational. Then some power of c is a parameter test element.*

Proof of 1.2. We complete c to a system of parameters $c, y_2, ..., y_d$. It suffices to show (see 1.1(c)) that $I=(c, y_2, ..., y_d)$ is tightly closed. So let $z\in I^*$. By 1.3 there exists a power of c, say c^n, such that c^n is a parameter test element. Hence for $q=p^e \gg 0$ we have $c^n z^q \in (c^q, y_2^q, ..., y_d^q)$. Since $c, y_2, ..., y_d$ is a regular sequence we conclude that $z^q\in(c^{q-n}, y_2^q, ..., y_d^q)$ for $q>n$. If $^-$ denotes reduction modulo c, then $\bar{z}^q\in(\bar{y}_2^q, ..., \bar{y}_d^q)$. Thus, since R/cR is F-injective, it follows that $\bar{z}\in(\bar{y}_2, ..., \bar{y}_d)$. In other words, $z\in I$. ▮

We apply 1.2 to get

THEOREM 1.4. *Let K be a perfect field of positive characteristic, and let Y be a wide ladder. Then the ladder determinantal ring $R_t(Y)=K[Y]/I_t(Y)$ defined over K is F-rational.*

Proof. It will be shown in Section 4 that there exists a suitable $(t-1)$-minor c such that $R_t(Y)_c \cong (K[Y']/I_t(Y''))_c$ where $Y''\subseteq Y'\subseteq Y$ are sub-ladders of each other, and Y'' is a wide ladder which is strictly smaller ladder than Y. Inducting on the size of the ladder, we may assume that $R_t(Y'')$ is F-rational.

A polynomial extension $S[T]$ of a positively graded F-rational ring S is again F-rational. (Complete a homogeneous system of parameters $\mathbf{y}$ of S by T. Then show that $(\mathbf{y}, T)$ is tightly closed if $\mathbf{y}$ is so, and apply 1.1(c).) Thus we see that $R_t(Y'')[Y'\backslash Y'']$ is F-rational and hence, by 1.1(a), $\widehat{R_t(Y)''}[[Y'\backslash Y'']]$ is F-rational. Since $\widehat{R_t(Y)_c} \cong (\widehat{R_t(Y)''}[[Y'\backslash Y'']])_c$, and since any localization of an F-rational ring is again F-rational (see [27, Proposition 1.4.3]), we conclude that $\widehat{R_t(Y)_c}$ is F-rational.

For our choice of c, $R_t(Y)/cR_t(Y)$ is F-injective, as will be shown in Section 4. This is the combinatorial part of our paper. Since F-injectivity passes to the completion, all hypotheses of 1.2 are satisfied. Hence $\widehat{R_t(Y)}$, and so $R_t(Y)$, are F-rational. ▮

The next corollaries and our main theorem now all follows from the results of Smith [26] and from 1.4.

A d-dimensional local ring $(R, \mathfrak{m})$ is *pseudo-rational* if it is normal, Cohen–Macaulay, analytically unramified, and if for any proper, birational map π: $W\to X=\operatorname{Spec} R$ with closed fibre $E=\pi^{-1}(\mathfrak{m})$, the canonical map

$$H^d_{\mathfrak{m}}(\pi_* \mathcal{O}_W)=H^d_{\mathfrak{m}}(R)\to H^d_E(\mathcal{O}_W)$$

is injective. Observing that F-rationality localizes, Theorem 3.1 of [26] yields.

COROLLARY 1.5. *Let K be a perfect field of positive characteristic, Y a wide ladder, and $R_t(Y)$ the ladder determinantal ring defined over K. Then $R_t(Y)_{\mathfrak{P}}$ is pseudo-rational for all $\mathfrak{P} \in \operatorname{Spec} R_t(Y)$.*

Let K be a field of characteristic 0. According to Smith [26, Definition 4.1], a K-algebra R is of *F-rational type*, if there exist a finitely generated $\mathbb{Z}$-algebra A contained in K, a finitely generated A-algebra R_A, and a flat map $A \hookrightarrow R_A$ such that:

(i) $(A \hookrightarrow R) \otimes_A K$ is isomorphic to $K \hookrightarrow R$;

(ii) the ring $R_A \otimes_A A/\mathfrak{m}$ is F-rational for all maximal ideals $\mathfrak{m}$ in a dense open subset of $\operatorname{Spec} A$.

The ideal $I_t(Y)$ describing a ladder determinantal ring is defined over the integers, and $\mathbb{Z}[Y]/I_t(Y)$ is a free $\mathbb{Z}$-module; see [5, Lemma 1.1]. Thus we get

COROLLARY 1.6. *Let Y be a wide ladder, and $R_t(Y)$ the ladder determinantal ring defined over a field of characteristic* 0. *Then $R_t(Y)$ is of F-rational type.*

Two important classes of ladders, the one-sided ladders and the chain ladders, are wide; see Section 3. Therefore Corollaries 1.5 and 1.6 are valid for these classes of ladders as well.

Now we apply the result [26, Theorem 4.3] of Smith which says that singularities of F-rational type are rational; Boutot's theorem [3], according to which a direct summand of a finitely generated K-algebra (K an algebraically closed field of characteristic 0) with rational singularities has again rational singularities; and the fact, shown in Section 3, that an arbitrary ladder determinantal ring is an algebra retract (and hence a direct summand) of a ladder determinantal ring defined by a wide ladder. Thus we finally obtain

THEOREM 1.7. *Let $R_t(Y)$ be a latter determinantal ring defined over an algebraically closed field of characteristic* 0. *Then $R_t(Y)$ has rational singularities.*

2. F-INJECTIVITY AND GRÖBNER BASES

In this section we develop a technique to check F-injectivity which will applied in subsequent sections to ladder determinantal ideals.

Let $R = K[X_1, ..., X_n]/I$ be an affine K-algebra, where K is a field of characteristic $p > 0$ and $I \subset (X_1, ..., X_n)$. We denote by x_i the residue class of X_i for $i=1, ..., n$, and set $\mathfrak{m}=(x_1, ..., x_n)$. Our concern is to find conditions based on Gröbner bases guaranteeing that $R_{\mathfrak{m}}$ is F-injective. So let $<$ be an order of the monomials in $X_1, ..., X_n$. We denote by $\operatorname{in}(f)$ the initial monomial of a polynomial $f \in K[X_1, ..., X_n]$, and by $\operatorname{in}(I)$ the ideal generated by all $\operatorname{in}(f)$, $f \in I$. We further set $\operatorname{in}(R)=K[X_1, ..., X_n]/\operatorname{in}(I)$. Note that $\operatorname{in}(R)$ can be given the structure of a positively graded K-algebra by assigning arbitrary positive degrees to the variables.

THEOREM 2.1. *With the notation and hypotheses just introduced we have that $R_{\mathfrak{m}}$ is Cohen–Macaulay and F-injective, if* $\operatorname{in}(R)$ *is so.*

For the proof of the theorem we will use deformation theoretic arguments as explained in [8, Chapter 15]: In a first step one interprets $\operatorname{in}(I))$ as the ideal $\operatorname{in}_\lambda(I)$ of initial forms of a certain weight function λ, that is, a linear function $\lambda\colon \mathbb{Z}^n \to \mathbb{Z}$. Indeed, suppose $g_1, ..., g_m$ is a Gröbner basis for I. According to [8, Proposition 15.26], there is a finite set $\{(m_1, n_1), ..., (m_s, n_s)\}$ of pairs of monomials with $m_i > n_i$ for all i, and such that if λ is a weight function with $m_i >_\lambda n_i$ for all i, then $g_1, ..., g_m$ is a Gröbner basis for I with respect to $>_\lambda$; in particular, $\operatorname{in}_\lambda(I)=\operatorname{in}(I)$.

Note that for any finite set $\{(u_i, v_i)\colon i=1, ..., r\}$ of pairs of monomials with $u_i > v_i$ for all i, one always finds a weight function λ such that the induced weight order $>_\lambda$ satisfies $u_i >_\lambda v_i$ for all i. Thus we may choose λ such that Proposition 15.26 of [8] is satisfied and such that $X_i >_\lambda 1$ for $i=1, ..., n$.

Choosing a weight function is equivalent to assigning to each variable X_i a degree $a_i \in \mathbb{Z}$. Our choice of λ implies that the a_i are positive integers. This gives $K[X_1, ..., X_n]$ the structure of positively graded K-algebra. We denote the degree of a polynomial f with respect to this grading by $\deg_\lambda f$.

If $f=\sum_i \kappa_i m_i$ is a monomial expansion of an element in $K[X_1, ..., X_n]$ with all $\kappa_i \neq 0$ and $\deg_\lambda m_1 \geqslant \deg_\lambda m_i$ for all i, we denote by $\tilde{f}$ the polynomial

$$\sum_i \kappa_i t^{\deg_\lambda m_1 - \deg_\lambda m_i} m_i$$

in $K[X_1, ..., X_n, t]$.

We assign to t the degree 1; then $\tilde{f}$ is a homogeneous polynomial.

Let $\tilde{I}$ be the ideal in $K[X_1, ..., X_n, t]$ generated by $\tilde{f}$, $f \in I$, and set $\tilde{R}=K[X_1, ..., X_n, t]/\tilde{I}$. Then $\tilde{R}$ is a positively graded K-algebra with the following properties (see [8, Theorem 15.27]):

(i) $\tilde{R}$ is a free $K[t]$-algebra, and thus $K[t]$-flat;

(ii) $\tilde{R}/t\tilde{R}=\operatorname{in}(R)$ and $\tilde{R}/(t-1)\tilde{R}=R$.

In other words, we have a flat one-parameter family with a special fibre $\mathrm{in}(R)$ and a general fibre R.

Proof of 2.1. Property (i) of $\tilde{R}$ implies that t is a (homogeneous) non-zero divisor of $\tilde{R}$. As we assume that $\mathrm{in}(R)$ is Cohen–Macaulay, say of dimension d, we see that $\tilde{R}$ is Cohen–Macaulay (of dimension $d+1$). But then (i) and (ii) imply that $R_{\mathfrak{m}}$ is Cohen–Macaulay of dimension d.

We choose homogeneous elements $u_1, ..., u_d \in K[X_1, ..., X_n]$ (homogeneous with respect to the λ-grading) such that their images in $R_{\mathfrak{m}}$ form a system of parameters, and simultaneously form a homogeneous system of parameters of $\mathrm{in}(R)$. Such elements exist. In fact, let $\mathfrak{P}_1, ..., \mathfrak{P}_m$ be the minimal prime ideals of $\mathrm{in}(I)$ and $\mathfrak{P}_{m+1}, ..., \mathfrak{P}_r$ the minimal prime ideals of I contained in $\mathfrak{M} = (X_1, ..., X_n)$. Since $\mathrm{in}(R)$ and $R_{\mathfrak{m}}$ are Cohen–Macaulay of the same dimension, all $\mathfrak{P}_i$ have the same height. We may assume $d > 0$, then $\mathfrak{M} \neq \mathfrak{P}_i$ for all i, and thus, since the λ-grading is positive, there exists a homogeneous element $u_1 \in \mathfrak{M}$ such that $u_1 \notin \mathfrak{P}_i$ for all i; see for instance [2, Lemma 1.5.10]. One constructs u_2 by applying the same arguments to the ideals $(\mathrm{in}(I), u_1)$ and (I, u_1) which now define Cohen–Macaulay rings of dimension $d-1$. Therefore, the assertion follows by induction on d.

In order to complete the proof of the theorem it remains to show that $(u_1, ..., u_d)\, R_{\mathfrak{m}}$ is F-contracted. So let $f/g \in R_{\mathfrak{m}}$ such that $(f/g)^p \in (u_1^p, ..., u_d^p)\, R_{\mathfrak{m}}$. Then there exists $h \in R \backslash \mathfrak{m}$ such that $h^p f^p \in (u_1^p, ..., u_d^p) R$. It follows that $t^q \tilde{h}^p \tilde{f}^p \in (\tilde{u}_1^p, ..., \tilde{u}_d^p)\, \tilde{R}$ for some positive integer q. Of course, we may assume that $q = p^e$ for some $e \geqslant 0$. Since the u_i are homogeneous we have $\tilde{u}_i = u_i$ for all i, so that for all $\ell \in \mathbb{N}$ we get $t^{p^e} \tilde{h}^p \tilde{f}^p \in (u_1^p, ..., u_d^p, (t^\ell)^p)\, \tilde{R}$.

Since we assume that $\mathrm{in}(R)$ is F-injective and Cohen–Macaulay, and since t is a homogeneous non-zero divisor of $\tilde{R}$ with $\tilde{R}/t\tilde{R} \cong \mathrm{in}(R)$ we may apply the graded version of [9, Theorem 3.4] to conclude that $\tilde{R}$ is F-injective. Hence since $u_1, ..., u_d, t^\ell$ is a homogeneous system of parameters, we get $t^{p^{e-1}} \tilde{h}\tilde{f} \in (u_1, ..., u_d, t^\ell)\, \tilde{R}$ for all ℓ, and therefore $t^{p^{e-1}} \tilde{h}\tilde{f} \in (u_1, ..., u_d)\, \tilde{R}$. Substituting t by 1 we see that $hf \in (u_1, ..., u_d)\, R$, and so $f/g \in (u_1, ..., u_d)\, R_{\mathfrak{m}}$, as desired. ∎

Observe that $\mathrm{in}(R)$ is a ring with monomial relations. In the case all the generating monomial relations are square-free, $\mathrm{in}(R)$ is the Stanley–Reisner ring of a certain simplicial complex Δ, see [29] or [2], and there exist geometric or combinatorial conditions on Δ to make sure that $\mathrm{in}(R)$ is Cohen–Macaulay. One of these conditions is the shellability of Δ; see [2, Theorem 5.1.13]. On the other hand, any Stanley–Reisner ring is F-pure. Indeed, suppose $J \subset K[X_1, ..., X_n]$ is an ideal generated by square-free monomials. Then it is easy to see that $J^{[p]} : J \not\subset (X_1^p, ..., X_n^p)$. This implies F-purity of $K[X_1, ..., X_n]/J$ by Fedder [9, Theorem 1.12]; see also [19, Proposition 5.38]. As F-purity implies F-injectivity [19, Lemma 2.2], we obtain

COROLLARY 2.2. *If* in(R) *is the Stanley–Reisner ring of a shellable simplicial complex, then* $R_{\mathfrak{m}}$ *is an F-injective Cohen–Macaulay ring.*

3. GENERALITIES ABOUT LADDERS

Let $X=(X_{ij})$ be an $m\times n$ matrix of distinct indeterminates over a field K, and denote by $K[X]$ the polynomial ring $K[X_{ij}\colon 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n]$. Given sequences of integers $1\leqslant a_1<\cdots<a_t\leqslant m$ and $1\leqslant b_1<\cdots<b_t\leqslant n$ we denote by $[a_1, ..., a_t \mid b_1, ..., b_t]$ the t-minor $\det(X_{a_ib_j})$ of X.

The main diagonal of $[a_1, ..., a_t \mid b_1, ..., b_t]$ is defined to be the set of indeterminates $\{X_{a_1b_1}, ..., X_{a_tb_t}\}$. The main diagonal of a t-minor or the product of its elements is called a t-diagonal of X. The minor $[i, i+1, ..., i+t-1 \mid j, j+1, ..., j+t-1]$ is said to be the t-minor based on the indeterminate X_{ij} or on the point (i, j).

A subset of indeterminates Y of X is called a *ladder* if whenever $X_{ij}, X_{hk}\in Y$ and $i\leqslant h$, $j\leqslant k$, then $X_{ik}, X_{hj}\in Y$. In other words, a subset Y of X is a ladder if whenever the main diagonal of a minor is contained in Y then all the entries of the minor are in Y.

Let Y be a ladder, and let $K[Y]$ be the polynomial ring $K[X_{ij}\colon X_{ij}\in Y]$. Throughout we fix a positive integer t bigger than 1. Denote by $R_t(Y)$ the ring $K[Y]/I_t(Y)$, where $I_t(Y)$ is the ideal generated by all the t-minors of X which only involve indeterminates of Y. The ideal $I_t(Y)$ is called a *ladder determinantal ideal* and the ring $R_t(Y)$ a *ladder determinantal ring*.

Throughout we identify the indeterminates of X with the points of the set $\{(i, j)\in\mathbb{N}^2\colon 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n\}$. This identification motivates us to define the ith row (resp. column) of X to be the set of points (a, b) of X with $b=i$ (resp., $a=i$). Here a confusion is possible because by the ith row of a matrix one usually understands the set of the entries of this matrix with first index equal to i.

The set X is equipped with the partial order $\preccurlyeq$:

$$(i, j)\preccurlyeq(h, k) \qquad \Leftrightarrow \qquad i\geqslant h \quad \text{and} \quad j\leqslant k.$$

X is clearly a distributive lattice, and a ladder Y is just a sublattice of X.

A ladder Y is said to be *t-disconnected* if there exist two ladders $\theta\neq Y_1$, $Y_2\subset Y$ such that $Y_1\cap Y_2=\varnothing$, $Y_1\cup Y_2=Y$, and every t-minor of Y is contained in Y_1 or in Y_2. If Y is t-disconnected we get $I_t(Y)=I_t(Y_1)+I_t(Y_2)$ and then $R_t(Y)=R_t(Y_1)\otimes_K R_t(Y_2)$. A ladder is *t-connected* if it is not t-disconnected. Any ladder is the disjoint union of its maximal t-connected components.

Deleting from X the rows and columns which do not intersect a t-connected ladder Y, one may assume that Y is the set of the points enclosed between two maximal chains of X as illustrated in Fig. 1.

It is clear that a t-connected ladder Y is uniquely determined by the points S_i, T_i (Fig. 2). The points $S_1, ..., S_h$ are called *inside lower corners* of the ladder Y, while the points $T_1, ..., T_l$ are called *inside upper corners.* A t-connected ladder Y is called a *chain ladder* if the inside corners of Y form a chain with respect to $\preccurlyeq$, and no column or row of Y contains two of them.

A t-connected ladder is said to be a *one-sided ladder* if it has no inside lower corners (Fig. 3).

The set of the t-minors of Y is a Gröbner basis of the ideal $I_t(Y)$ with respect to a diagonal monomial order (i.e., a monomial order such that the leading monomial of any minor of Y is the product of the elements of its main diagonal); see [23, Corollary 3.4]. Hence the ideal $\mathrm{in}(I_t(Y))$ of the leading monomial of $I_t(Y)$ is the square-free monomial ideal generated by the t-diagonals in Y. Herzog and Trung [13, Section 4] described the facets of the simplicial complex $\Delta_t(Y)$ associated with $\mathrm{in}(I_t(Y))$ in terms of families of non intersecting paths in the poset Y and they show that $\Delta_t(Y)$ is shellable. The dimension of $R_t(Y)$ is equal to the dimension of the ring defined by $\mathrm{in}(I_t(Y))$ and is the number of elements of any facet of $\Delta_t(Y)$ [13, Corollary 4.7]. It turns out that the dimension of $R_t(Y)$ is the cardinality of the lower border of Y with thickness $(t-1)$.

Fix two consecutive columns of a maximal t-connected component of the ladder Y, say the rth and the $(r+1)$th column. Let (r, a) and (r, b) be the bottom and top points of the rth column and $(r+1, c)$ and $(r+1, d)$ those of the $(r+1)$th column. One has $c \leqslant a \leqslant d \leqslant b$. Let a' and b' be integers such that $c \leqslant a' \leqslant a$ and $d \leqslant b' \leqslant b$. Let W be the ladder with the same

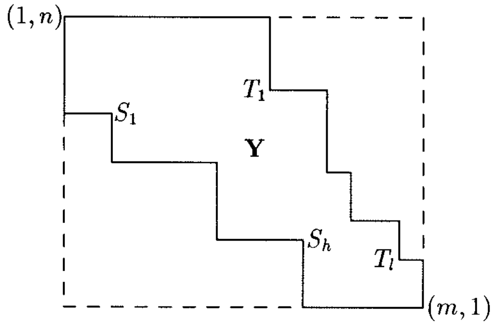

FIG. 1. A t-connected ladder.

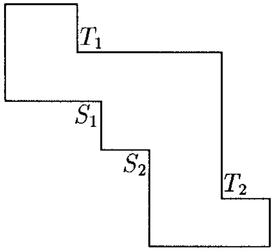

FIG. 2. A chain ladder.

shape as Y, except that we have added a new column between the rth and $(r+1)$th columns of Y. This new column has bottom point $(r+1, a')$ and top point $(r+1, b')$.

We define a K-algebra homomorphism $\Phi: K[Y] \to K[W]$ by setting $\Phi(Y_{ij}) = W_{ij}$ if $i \leqslant r$ and $\Phi(Y_{ij}) = W_{i+1j}$ if $i \geqslant r+1$. It is clear that Φ induces a K-algebra homomorphism $\varphi: R_t(Y) \to R_t(W)$. Let J be the ideal in $R_t(W)$ generated by the elements of the $(r+1)$th column of W. Then φ composed with the canonical epimorphism $R_t(W) \to R_t(W)/J$ is an isomorphism. Therefore $\varphi: R_t(Y) \to R_t(W)$ is an algebra retract. The addition of a row is defined alike.

PROPOSITION 3.1. *Whenever a ladder Z is obtained from Y by a sequence of row and column additions, then $R_t(Y)$ is an algebra retract of $R_t(Z)$. Hence if we study properties of $R_t(Y)$ which behave well with respect to the algebra retract we may add to Y rows and columns as we like.*

Let Y be a t-connected ladder. The ladder Y is said to be a *wide ladder* if for all inside lower corners $S_i = (a_i, b_i)$ of Y and for all $j = 1, ..., t-1$, the set $\{(x, y) \in Y: y = b_i + j - 1\}$ contains no point of the horizontal part of the upper border of Y.

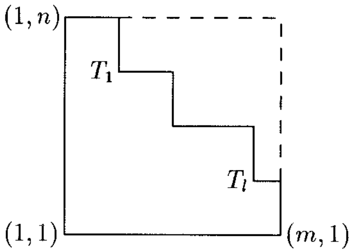

FIG. 3. A one-sided ladder.

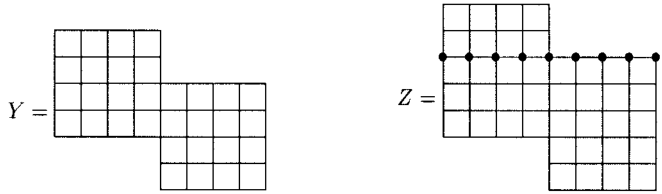

FIGURE 4

A general ladder (i.e. not necessarily t-connected) is said to be a chain ladder (resp. a one-sided ladder or a wide ladder) if its maximal t-connected components enjoy this property.

Note that a one-sided ladder is a chain ladder, and that a chain ladder is a wide ladder.

The reader may find the definition of wide ladders somewhat artificial. However, it turns out that this class of ladders is combinatorially simple enough to be treated without the need of many case-by-case discussions, but general enough for our purposes. Indeed, one has

PROPOSITION 3.2. *Let Y be a ladder. Then there exists a wide ladder Z such that $R_t(Y)$ is an algebra retract of $R_t(Z)$.*

It suffices to construct the wide ladder Z from Y by a sequence of row additions. It is clear that it is enough to do this for each maximal t-connected component, and so we may assume from the very beginning that Y is t-connected. If Y is not wide, proceed as follows: For all i for which the wide condition is violated, take j minimal such that $\{(x, y) \in Y : y = b_i + j - 1\}$ contains points of the horizontal part of the upper border of Y. Then add between the $(b_i + j - 1)$th row and the $(b_i + j)$th row of Y a set of $t - j$ copies of the $(b_i + j - 1)$th row of Y. The resulting ladder is the desired Z.

Figure 4 illustrates this construction in one example where t is 4. The line with bold points has been added to Y to get Z.

4. F-RATIONALITY OF WIDE LADDER DETERMINANTAL RINGS

The goal of this section is to complete the proof of 1.4. We show that for any wide ladder determinantal ring $R_t(Y)$ there exists a $(t-1)$-minor c of Y such that:

(a) $R_t(Y)_c$ is isomorphic to a localization of a polynomial extension of $R_t(Y'')$, where Y'' is a proper wide subladder of Y.

(b) $R_t(Y)/cR_t(Y)$ is F-injective.

For simplicity of notation we will assume that the ladder Y is t-connected. The reader may easily check that this is not a restriction, since all the constructions and proofs we are going to present only involve points of a maximal t-connected component of Y.

Let Y be a t-connected wide ladder. We refer the reader to the notation in Fig. 1. If a point p of Y is not involved in some t-minor of Y, then $\{p\}$ is a t-connected component of Y, and then $Y=\{p\}$. So it is not a restriction to assume that all indeterminates of Y appear in some t-minor of Y. Hence the points $(1, n), ..., (t, n)$ and $(m, 1), ..., (m, t)$ belong to Y.

For all $i=1, ..., l$, let (c_i, d_i) be the coordinates of the point T_i and for all $i=1, ..., h$, let (a_i, b_i) be the coordinates of the point S_i.

Throughout this section we denote by τ the lexicographic monomial order induced by the variable order $X_{11} > X_{12} > \cdots > X_{1n} > X_{21} > \cdots > X_{mn}$ and by $\operatorname{in}(I)$ the ideal of the leading monomials with respect to τ of an ideal I of the ring $K[Y]$.

First of all we have to define c. Let c be the residue class in $R_t(Y)$ of the $(t-1)$-minor based on the point $(1, b_1)$, that is, $[1, ..., t-1 \mid b_1, ..., b_1+t-2]$. In the case Y is a one-sided ladder, c is just the residue class of $[1, ..., t-1 \mid 1, ..., t-1]$.

PROPOSITION 4.1. *If Y is a one-sided ladder, then the ring $R_t(Y)_c$ is isomorphic to a localization of a polynomial ring. Otherwise, the ring $R_t(Y)_c$ is isomorphic to a localization of a polynomial extension of $R_t(Y'')$ where Y'' is the wide ladder which is the intersection of Y with the set $\{(i, j) \in X: i \geqslant a_1, j \leqslant b_1+t-2\}$.*

Proof. Suppose Y is not a one-sided ladder. It is easy to see that Y'' is wide since Y is. Consider the set $B=\{(i, j) \in Y: i \leqslant t-1, j \geqslant b_1+t-1\} \cup \{(i, j) \in Y: 1 \leqslant i \leqslant a_1, b_1 \leqslant j \leqslant b_1+t-2\}$. Denote by $K[B \cup Y'']$ the K-subalgebra of $R_t(Y)$ generated by the residue classes of the elements of $B \cup Y''$, and by $R_t(Y'')[B]$ the polynomial extension of the ring $R_t(Y'')$ with the indeterminates in B. The lower border with thickness $(t-1)$ of Y is the union of lower border with thickness $(t-1)$ of Y'' with B, and therefore by virtue of [13, Corollary 4.7], $\dim R_t(Y'')[B] = \dim R_t(Y)$.

For all $(i, j) \in Y \backslash (Y'' \cup B)$ the minor $[1, ..., t-1, i \mid b_1, ..., b_1+t-2, j]$ is in Y and it involves only entries of $Y'' \cup B \cup \{(i, j)\}$. Since $[1, ..., t-1, i \mid b_1, ..., b_1+t-2, j]=0$ in $R_t(Y)$, $x_{ij}c$ belongs to $K[B \cup Y'']$. In other words, $R_t(Y)_c = K[B \cup Y'']_c$. Since $R_t(Y)$ and $K[B \cup Y'']$ have the same field of fraction, they have the same dimension.

The canonical map $\psi: R_t(Y'')[B] \to K[B \cup Y'']$ is a surjection between domains of the same dimension. Hence ψ has to be an isomorphism. This concludes the proof for a general ladder. The proof for a one-sided ladder is similar. ∎

It remains to show that $R_t(Y)/cR_t(Y)$ is F-injective. By virtue of 2.2 it is enough to prove that the ideal $\mathrm{in}(I_t(Y)+(c))$ is generated by square-free monomials and that the associated simplicial complex is shellable.

Square-freeness. The ideal $I_t(Y)+(c)$ is radical and its minimal prime ideals are described in [7, Proposition 3.4.7]. Let us introduce some more notation to explain the results we need. Let k be the maximum integer such that $b_1+t-2 \leqslant d_k$ or $k=0$ if $b_1+t-2>d_1$. Note that in the one-sided case $k=l$, and that in general $T_1, ..., T_k$ are exactly the inside upper corners lying northeast of c. For $i=1, ..., k$, let A_i denote the set $\{(p,q)\in Y: p\leqslant c_i$ and $q\leqslant d_i\}$, and further set $A_0=\{(p,q)\in Y: p\leqslant t-1\}$ and $A_{k+1}=\{(p,q)\in Y: b_1\leqslant q\leqslant b_1+t-2\}$, (Fig. 5).

For $i=0, ..., k+1$ let $I_{t-1}(A_i)$ be the ideal generated all the $(t-1)$-minors of the region A_i and set $P_i=I_t(Y)+I_{t-1}(A_i)$. The ideals $P_0, P_1, ..., P_{k+1}$ are the minimal prime ideals of $I_t(Y)+(c)$, and therefore $I_t(Y)+(c)=P_0\cap P_1\cap\cdots\cap P_{k+1}$, see [7, Proposition 3.4.7]. Note that the ideal P_0 is the ideal $P_\delta(Y)$ cogenerated by the minor $\delta=[1, ..., t-2, t\,|\,1, ..., t-1]$ in Y as it is defined by Herzog and Trung in [13, Section 4]. It is proved in [13, Theorem 4.2] (for $i=0$) and in [7, Propositions 3.3.4 and 3.3.5] (for $i=1, ..., k+1$) that the union J_i of set of the t-minors of Y with the set of the $(t-1)$-minors of A_i is a Gröbner basis of the ideal P_i with respect to τ. Hence $\mathrm{in}(P_i)$ is generated by the t-diagonals of Y and by the $(t-1)$-diagonals of A_i. In particular, $\mathrm{in}(P_i)$ is a square-free monomial ideal. From

$$I_t(Y)+(c)=P_0\cap P_1\cap\cdots\cap P_{k+1}$$

we would like to deduce that

$$\mathrm{in}(I_t(Y)+(c))=\mathrm{in}(P_0)\cap\mathrm{in}(P_1)\cap\cdots\cap\mathrm{in}(P_{k+1}). \tag{1}$$

This will be a consequence of the following general criterion:

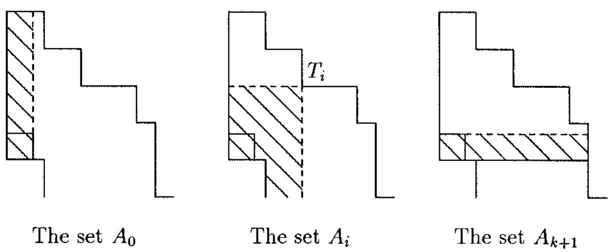

FIG. 5. (left) The set A_0, (middle) the set A_i, and (right) the set A_{k+1}.

LEMMA 4.2. *Let S be a polynomial ring over an arbitrary field K and let σ be a monomial order on the monomials of S.*

(a) *Let I and J be homogeneous ideals of S, then* $\mathrm{in}_\sigma(I+J) = \mathrm{in}_\sigma(I) + \mathrm{in}_\sigma(J)$ *if and only if* $\mathrm{in}_\sigma(I \cap J) = \mathrm{in}_\sigma(I) \cap \mathrm{in}_\sigma(J)$.

(b) *Let $I_1, ..., I_p$ be homogeneous ideals of S and assume that* $\mathrm{in}_\sigma(I_i + I_j) = \mathrm{in}_\sigma(I_i) + \mathrm{in}_\sigma(I_i)$ *for all $1 \leqslant I \leqslant j \leqslant n$. Then* $\mathrm{in}_\sigma(I_1 \cap \cdots \cap I_p) = \mathrm{in}_\sigma(I_1) \cap \cdots \cap \mathrm{in}_\sigma(I_p)$.

Proof. (a) Denote by $\dim[I]_i$ the K-dimension of the homogeneous component of degree i of a homogeneous ideal I of S. It is well-known that $\dim[I]_i = \dim[\mathrm{in}_\sigma(I)]_i$ for all i. It follows easily from the definition that $\mathrm{in}_\sigma(I) + \mathrm{in}_\sigma(J) \subseteq \mathrm{in}_\sigma(I+J)$ and that $\mathrm{in}_\sigma(I \cap J) \subseteq \mathrm{in}_\sigma(I) \cap \mathrm{in}_\sigma(J)$. Further,

$$\begin{aligned}
&\dim[\mathrm{in}_\sigma(I) \cap \mathrm{in}_\sigma(J)]_i - \dim[\mathrm{in}_\sigma(I \cap J)]_i \\
&\quad = \dim[\mathrm{in}_\sigma(I) \cap \mathrm{in}_\sigma(J)]_i - \dim[I \cap J]_i \\
&\quad = \dim[\mathrm{in}_\sigma(I)]_i + \dim[\mathrm{in}_\sigma(J)]_i - \dim[\mathrm{in}_\sigma(I) \\
&\qquad + \mathrm{in}_\sigma(J)]_i - \dim[I]_i - \dim[J]_i + \dim[I+J]_i \\
&\quad = \dim[\mathrm{in}_\sigma(I+J)]_i - \dim[\mathrm{in}_\sigma(I) + \mathrm{in}_\sigma(J)]_i,
\end{aligned}$$

which implies the desired conclusion.

(b) We argue by induction on p. The case $p = 2$ is treated in (a). So we may assume $p > 2$, and set $J = I_{p-1} \cap I_p$. First we show that the ideals $I_1, ..., I_{p-2}, J$, satisfy the assumption of (b). It is enough to show that $\mathrm{in}_\sigma(I_i + J) = \mathrm{in}_\sigma(I_i) + \mathrm{in}_\sigma(J)$. Indeed,

$$\begin{aligned}
&\mathrm{in}_\sigma(I_i + J) \supseteq \mathrm{in}_\sigma(I_i) + \mathrm{in}_\sigma(J) \\
&\quad = \mathrm{in}_\sigma(I_i) + [\mathrm{in}_\sigma(I_{p-1}) \cap \mathrm{in}_\sigma(I_p)] \\
&\quad = [\mathrm{in}_\sigma(I_i) + \mathrm{in}_\sigma(I_{p-1})] \cap [\mathrm{in}_\sigma(I_i) + \mathrm{in}_\sigma(I_p)] \\
&\quad = \mathrm{in}_\sigma(I_i + I_{p-1}) \cap \mathrm{in}_\sigma(I_i + I_p) \supseteq \mathrm{in}_\sigma([I_i + I_{p-1}] \cap [I_i + I_p]) \\
&\qquad \supseteq \mathrm{in}_\sigma(I_i + [I_{p-1} \cap I_p]) \\
&\quad = \mathrm{in}_\sigma(I_i + J).
\end{aligned}$$

The three inclusions $\supseteq$ are trivially true. The first equation holds by induction, the second holds because the ideals involved are generated by monomials, and the last holds by assumption. Thus $\mathrm{in}_\sigma(I_i + J) = \mathrm{in}_\sigma(I_i) + \mathrm{in}_\sigma(J)$. Then by induction,

$$
\begin{aligned}
\mathrm{in}_\sigma(I_1 \cap \cdots \cap I_p) &= \mathrm{in}_\sigma(I_1 \cap \cdots \cap I_{p-2} \cap J) \\
&= \mathrm{in}_\sigma(I_1) \cap \cdots \cap \mathrm{in}_\sigma(I_{p-2}) \cap \mathrm{in}_\sigma(J) \\
&= \mathrm{in}_\sigma(I_1) \cap \cdots \cap \mathrm{in}_\sigma(I_p)
\end{aligned}
$$

and the proof is complete. ▮

Now equality (1) will follow from 4.2 and

LEMMA 4.3. *For all i, j, $0 \leqslant i < j \leqslant k+1$, one has* $\mathrm{in}(P_i + P_j) = \mathrm{in}(P_i) + \mathrm{in}(P_j)$.

Proof. Let us treat first the cases $0 \leqslant i < j \leqslant k$ and $i = 0$, $j = k+1$. We show that $J_i \cup J_j$ is a Gröbner basis of $P_i + P_j$ with respect to τ. By virtue of Buchberger's criterion [4], it suffices to show that for each pair of elements h, g in $J_i \cup J_j$ there exists a subset J of $J_i \cup J_j$ which is a Gröbner basis with respect to τ and it contains h and g. Since J_i and J_j are Gröbner bases we may assume that h is a $(t-1)$-minor of A_i and g is a $(t-1)$-minor of A_j. The union of the set of the $(t-1)$-minors of A_i with the set of the $(t-1)$-minors of A_j is the set of the $(t-1)$-minors of the ladder $A_i \cup A_j$. Since the set of the $(t-1)$-minors of a ladder form a Gröbner basis with respect to a diagonal order, we may take as J the set of the $(t-1)$-minors of $A_i \cup A_j$.

In case $0 < i \leqslant k$ and $j = k+1$, we cannot use the argument above because $A_i \cup A_{k+1}$ is no longer a ladder (unless Y is a one-sided ladder). By virtue of 4.2(a), it suffices to show that $\mathrm{in}(P_i \cap P_{k+1}) = \mathrm{in}(P_i) \cap \mathrm{in}(P_{k+1})$. The inclusion $\mathrm{in}(P_i \cap P_{k+1}) \subseteq \mathrm{in}(P_i) \cap \mathrm{in}(P_{k+1})$ holds always. Further,

$$
\begin{aligned}
&\mathrm{in}(P_i) \cap \mathrm{in}(P_{k+1}) \\
&\quad = [\mathrm{in}(I_t(Y)) + \mathrm{in}(I_{t-1}(A_i))] \cap [\mathrm{in}(I_t(Y)) + \mathrm{in}(I_{t-1}(A_{k+1}))] \\
&\quad = \mathrm{in}(I_t(Y)) + [\mathrm{in}(I_{t-1}(A_i)) \cap \mathrm{in}(I_{t-1}(A_{k+1}))].
\end{aligned}
$$

Clearly, $\mathrm{in}(I_t(Y)) \subset \mathrm{in}(P_i \cap P_{k+1})$. So it remains to show that for every pair $g = [\alpha_1, ..., \alpha_{t-1} \mid \beta_1, ..., \beta_{t-1}]$, $h = [\gamma_1, ..., \gamma_{t-1} \mid \delta_1, ..., \delta_{t-1}]$, with g a $(t-1)$-minor of A_i and h a $(t-1)$-minor of A_{k+1}, there exists an element $m \in P_i \cap P_{k+1}$ such that $\mathrm{in}(m)$ divides the least common multiple of $\mathrm{in}(g)$ and $\mathrm{in}(h)$.

If g is contained in the region $B_i = \{(p, q) \in A_i \colon q \geqslant b_1\}$, then g and h are contained in the ladder $B_i \cup A_{k+1}$. Since $B_i \cup A_{k+1}$ is a ladder, $\mathrm{in}(I_{t-1}(B_i)) + \mathrm{in}(I_{t-1}(A_{k+1})) = \mathrm{in}(I_{t-1}(B_i) + I_{t-1}(A_{k+1}))$, and by 4.2(a) one has $\mathrm{in}(I_{t-1}(B_i)) \cap \mathrm{in}(I_{t-1}(A_{k+1})) = \mathrm{in}(I_{t-1}(B_i) \cap I_{t-1}(A_{k+1}))$. So we may find the element m with the desired property already in $I_{t-1}(B_i) \cap I_{t-1}(A_{k+1})$.

In the general case we note that if $\text{in}(g)$ and $\text{in}(h)$ do not share common indeterminates we may take $m = gh$. So we may assume that $\text{in}(g)$ and $\text{in}(h)$ share a common indeterminate, say $(\alpha_v, \beta_v) = (\gamma_u, \delta_u)$. Now consider the set M of the first u points of the main diagonal of h and the last $t-1-v$ of that of g. M is a $(u+t-1-v)$-diagonal in Y. Hence if $u+t-1-v \geqslant t$, then M contains a t-diagonal of Y and the corresponding t-minor is the element m we are looking for. However, if $u+t-1-v < t$, then the set N of the first v points of the main diagonal of g and the last $t-1-u$ of that of g is a diagonal in B_i with at least $t-1$ points. Then we may pick a $(t-1)$-minor g_1 of B_1 whose main diagonal divides N. For the pair g_1, h we know already that there exists $m \in P_i \cap P_{k+1}$ such that $\text{in}(m)$ divides the least common multiple of $\text{in}(g_1)$ and $\text{in}(h)$. But the least common multiple of $\text{in}(g_1)$ and $\text{in}(h)$ divides that of $\text{in}(g)$ and $\text{in}(h)$. ∎

Now since each $\text{in}(P_i)$ is generated by square-free monomials, the same holds true for $\text{in}(I_t(Y) + (c))$ by equality (1).

In order to prove that the simplicial complex associated with the ideal $\text{in}(I_t(Y) + (c))$ is shellable we recall some more facts from [13] and [7].

Families of Non Intersecting Paths. A *path* from a point $(v_1, v_2) \in \mathbb{N}^2$ to a point $(s_1, s_2) \in \mathbb{N}^2$ is a sequence of points $(v_1, v_2) = (p_1, q_1), ..., (p_r, q_r) = (s_1, s_2)$ such that $(p_{i+1} - p_i, q_{i+1} - q_i)$ is equal to $(0, 1)$ or to $(-1, 0)$. A *family of non intersecting paths* from a set of starting points $V_1, ..., V_c$, to a set of ending points $S_1, ..., S_c$, is just a collection of paths from V_i to S_i, $i = 1, ..., c$, with no common points. We will say that a path is in Y if its points are in Y. By definition a point (a, b) of a path is a *right turn* (resp., *left turn*) of the path if the points $(a+1, b)$ and $(a, b+1)$ (resp., $(a, b-1)$ and $(a-1, b)$) belong to the path as well.

We now define a procedure to modify a family of non-intersection paths.

Definition 4.4. Let $E = E_1, ..., E_c$ be a family of non intersecting paths and let P be a right turn of E_i. Denote by x_i and y_i the coordinates of P and set $h = \max\{j : i \leqslant j \leqslant c,\ (x_i + j - i, y_i + j - i) \in E_j\}$ and $P_j = (x_i + j - i, y_i + j - i)$ for all $j = i, ..., h+1$. Then we define a new family of non intersecting paths $H = H_1, ..., H_c$, setting

$$H_j = \begin{cases} E_j & \text{if } \ 1 \leqslant j \leqslant i-1 \quad \text{or} \quad h+1 \leqslant j \leqslant c \\ E_j \backslash \{P_j\} \cup \{P_{j+1}\} & \text{if } \ i \leqslant j \leqslant h. \end{cases}$$

The family of non intersecting paths H has the following properties:

(i) H_j starts and ends where E_j does, unless $i \leqslant j \leqslant h$ and P_j is an extreme (starting or ending) point of E_j.

(ii) H_j is on the northeast side of E_j, for all j.

(iii) H differs from E only in the point P, that is, $(\bigcup_{j=1}^{c} E_j)\setminus(\bigcup_{j=1}^{c} H_j)=\{P\}$.

We will say that H is obtained from E by switching the right turn P. One defines similarly the procedure to switch a left turn of a family of non intersecting paths.

The families of non intersecting paths are ordered in a natural way. Given two paths G_1 and F_1 with the same extreme points we say that G_1 is smaller than F_1 in the *northeast order* (denotes by $G_1 \leqslant F_1$) if G_1 is on the northeast side of F_1. Given two families of non intersecting paths $G=\{G_1, ..., G_c\}$ and $F=\{F_1, ..., F_c\}$ with the same extreme points we say that G is smaller than F in the northeast order (denoted by $G \leqslant F$) if $G_i \leqslant F_i$ for $i=1, ..., c$. This is a partial order on the set of families of non intersecting paths with given extreme points.

Shellability. Let us denote by $\mathbb{F}(\Delta)$ the set of the facets (i.e., the maximal elements under inclusion) of a simplicial complex Δ. Recall that a simplicial complex Δ is said to be *shellable* if it is pure (i.e., its facets all have the same number of elements) and its facets can be given a linear order, called a *shelling*, in such a way that if $Z<Z_1$ are facets of Δ, then there exists a facet $Z_1<Z_1$ of Δ, then there exists a facet $Z_2<Z_1$ of Δ and an element $x \in Z_1$ such that $Z \cap Z_1 \subseteq Z_2 \cap Z_1 = Z_1 \setminus \{x\}$.

Since $\mathrm{in}(P_i)$ is the square-free monomial ideal generated by the t-diagonals of Y and the $(t-1)$-diagonals of A_i, $K[Y]/\mathrm{in}(P_i)$ is the Stanley–Reisner ring associated with the simplicial complex Δ_i of the subsets of Y which do not contain t-diagonals of Y and $(t-1)$-diagonals of A_i. It is clear that the simplicial complex associated with $\mathrm{in}(I_t(Y)+(c))$ is just $\Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{k+1}$. Our goal is to show that the simplicial complex $\Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{k+1}$ is shellable. We will see that each Δ_i is a shellable simplicial complex whose facets can be described in terms of families of non intersecting paths. Then we will glue together these shellings to get a shelling of $\Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{k+1}$.

The facets of Δ_0 are described in [13, Theorem 4.6] as families of non intersecting paths in Y. They are the families of non intersecting paths in Y from the points $(m, 1), ..., (m, t-1)$ to the points $(1, n), ..., (t-2, n)$, (t, n). Figure 6 illustrates a typical example of a facet of Δ_0.

The facets of Δ_i, $i=1, ..., k$, can be described as families of non intersecting paths as well. Let us consider the ladder Y_i which is obtained from Y by adding the point $K_i=(c_i+1, d_i+1)$ and let $\Delta_t(Y_i)$ be the simplicial complex of the subsets of Y_i which do not contain t-diagonals of Y_i. The link of K_i in $\Delta_t(Y_i)$ is, by definition, the simplicial complex of all the subsets G of $Y_i \setminus \{K_i\}$ such that $G \cup \{K_i\} \in \Delta_t(Y_i)$. It is easy to see that the link

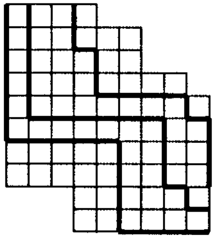

FIG. 6. A facet of Δ_0.

of K_i in $\Delta_t(Y_i)$ is exactly Δ_i. If we consider a facet F of $\Delta_t(Y_i)$ which contains K_i, then $F \setminus \{K_i\}$ is a facet of Δ_i and all the facets of Δ_i arise in this way. The facets of $\Delta_t(Y_i)$ are the families of non intersecting paths in Y_i with starting points $(m, 1), ..., (m, t-1)$ and ending points $(1, n), ...,$ $(t-1, n)$; see [13, Theorem 4.6]. Therefore a facet of Δ_i is a family of non intersecting paths in Y with starting points $(m, 1), ..., (m, t-1), (c_i, d_i+1)$ and ending points $(1, n), ..., (t-2, n), (c_i+1, d_i), (t-1, n)$. Figure 7 illustrates a facet of Δ_i, $1 \leqslant i \leqslant k$.

The simplicial complex Δ_0 is shellable [13, Theorem 4.9]. Also, the simplicial complex Δ_i, $i = 1, ..., k$, is shellable since it is a link of the shellable simplicial complex $\Delta_t(Y_i)$, so that any shelling on $\Delta_t(Y_i)$ induces canonically a shelling on Δ_i.

Let us recall how one defines the shelling of these simplicial complexes. The northeast order is a partial order on the facets of Δ_0 and $\Delta_t(Y_i)$. As slight modification of the argument in [13, Theorem 4.9] shows that by extending this partial order to a total order (it does not matter how) one gets shellings on Δ_0 and $\Delta_t(Y_i)$. Since any shelling on $\Delta_t(Y_i)$ induces a shelling on Δ_i, a shelling on Δ_i arises by extension of the northeast order.

We prefer to use these shellings instead of the original ones of [13] because they are more appropriate to our needs.

We still have to determine the facets of the simplicial complex Δ_{k+1} and to describe a shelling on it.

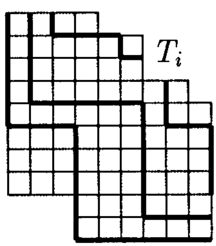

FIG. 7. A facet of Δ_i.

If Y is a on-sided ladder, then P_{k+1} is the ideal cogenerated by the minor $\gamma=[1, ..., t-1 \mid 1, ..., t-2, t]$ in Y [13, Section 4]. In this case the facets and the shelling of Δ_{k+1} are described in [13, Section 4]. A facet of Δ_{k+1} is a family of non intersecting paths in Y from the points $(m, 1), ..., (m, t-2), (m, t)$ to the points $(1, n), ..., (-1, n)$, and a shelling is given extending the northeast order.

Now we treat the case of a ladder which is not one-sided.

Remark 4.5. So far we did not use the fact that we are dealing with a wide ladder. It turns out that the description of the facets and of the shelling of Δ_{k+1} is easier for a wide ladder than for general ones. That is the reason why we have restricted our attention to wide ladders. In order to show that any ladder determinantal ring is F-rational one should extend 4.6, 4.7, and 4.8 to general (i.e., not necessarily wide) ladders.

For $j=1, ..., t-1$, denote by W_j the (b_1-1+j)th row of Y, that is, the set $\{(i, b_1-1+j): (i, b_1-1+j) \in Y\}$. Note that the set A_{k+1} is just the union of $W_1, ..., W_{t-1}$.

Since $\mathrm{in}(P_{k+1}) \supset \mathrm{in}(I_t(Y))$, the simplicial complex Δ_{k+1} is contained in $\Delta_t(Y)$. Any facet of Δ_{k+1} is contained in a facet of $\Delta_t(Y)$. A facet F of $\Delta_t(Y)$ is a family of non intersecting paths $F_1, ..., F_{t-1}$ in Y from the points $(m, 1), ..., (m, t-1)$ to $(1, n), ..., (t-1, n)$. The sets $F_i \cap W_i$ are all not empty, and the only $(t-1)$-diagonals of the region A_{k+1} which are contained in F are of the form $e_1, ..., e_{t-1}$, with $e_i \in F_i \cap W_i$.

In case $|F_i \cap W_i|=1$ for some i, the set $F \backslash (F_i \cap W_i)$ does not contain $(t-1)$-diagonals of A_{k+1}. Therefore $F \backslash (F_i \cap W_i)$ is a face of Δ_{k+1} and since it has maximal dimension it is actually a facet. We want to show that all the facets of Δ_{k+1} arise in this way.

Proposition 4.6. *Let G be a subset of Y. Then G is a facet of Δ_{k+1} if and only if there exist a facet $F=F_1, ..., F_{t-1}$ of $\Delta_t(Y)$ and an integer i, $1 \leqslant i \leqslant t-1$, such that $|F_i \cap W_i|=1$ and $G=F \backslash (F_i \cap W_i)$. Furthermore, the pair (F, i) is uniquely determined by G.*

Proof. We show first that there exist $F \in \mathbb{F}(\Delta_t(Y))$ and i, $1 \leqslant i \leqslant t-1$, such that $|F_i \cap W_i|=1$ and $G \subseteq F \backslash (F_i \cap W_i)$. Then equality follows since G and $F \backslash (F_i \cap W_i)$ are facets of Δ_{k+1}.

Take $H \in \mathbb{F}(\Delta_t(Y))$ such that $G \subset H$. We argue by induction on $n(H)=\sum_{j=1}^{t-1} |H_j \cap W_j|$. If $n(H)=t-1$, then $|H_j \cap W_j|=1$, for all j, say $H_j \cap W_j=\{w_j\}$. Since $w_1, ..., w_{t-1}$ is a $(t-1)$-diagonal of A_{k+1}, $\{w_1, ..., w_{t-1}\} \not\subset G$. So $w_j \notin G$ for some j, and we may take $F=H$ and $i=j$. Now let $n(H)>t-1$. Let w_j be the element of $H_j \cap W_j$ with smallest x-coordinate. Again $w_1, ..., w_{t-1}$ is a $(t-1)$-diagonal of A_{k+1}, and so $w_j \notin G$ for some j. If $|H_j \cap W_j|=1$, we argue as before. If $|H_j \cap W_j|>1$, then w_j

is a right turn of H_i. We switch the right turn w_j of H and get a family of non intersecting paths H'. Since Y is a wide ladder, H' is still a family of non intersecting paths in Y. Furthermore, $n(H')=n(H)-h$, where h is the integer defined by the switching procedure. Since $G\subseteq H\setminus\{w_j\}\subset H'$ and $n(H')<n(H)$, the conclusion follows by induction.

For the uniqueness, assume that there exist $F, F'\in\mathbb{F}(\Delta_t(Y))$, and i, j such that $|F_i\cap W_i|=|F'_j\cap W_j|=1$ and $F\setminus(F_i\cap W_i)=F'\setminus(F'_j\cap W_j)$. One has to show that $F=F'$ and $i=j$. Clearly it suffices to show $F=F'$. By contradiction, if $F\neq F'$ we may assume that F precedes F' in the shelling of $\Delta_t(Y)$. Let w' be the unique point of $F'_j\cap W_j$. As $F'\cap F=F'\setminus\{w'\}$, w' has to be a right turn of F'; see [13, Theorems 4.6, 4.9]. Therefore the point of F'_j that precedes w_j is in W_j too, a contradiction to $|F'_j\cap W_j|=1$. ∎

Figure 8 illustrates a facet of Δ_{k+1}. By 4.6, an element of $G\in\mathbb{F}(\Delta_{k+1})$ is represented in a unique way by a pair (F, i) with $F\in\mathbb{F}(\Delta_t(Y))$, $|F_i\cap W_i|=1$. In particular, Δ_{k+1} is pure since $\Delta_t(Y)$ is. We consider the shelling on $\Delta_t(Y)$ which arises as extension of the northeast order. This shelling induces a shelling on Δ_{k+1}.

PROPOSITION 4.7. *The simplicial complex Δ_{k+1} is shellable, and a shelling is given by the following total order: Given two facets G, H of Δ_{k+1} represented by (F, i) and (E, j), we set $G<H$ if $F<E$ in the shelling of $\Delta_t(Y)$, or $F=E$ and $i<j$.*

Proof. Let G and H be facets of Δ_{k+1}, and assume $G<H$. We have to find a facet L of Δ_{k+1} and a point x of H, such that $L<H$ and $G\cap H\subseteq L\cap H=H\setminus\{x\}$. Let (F, i) and (E, j) be the pairs that represent G and H, and denote by w the point of the set $F_i\cap W_i$ and by z that of $E_j\cap W_j$. If $F=E$, then it suffices to take $L=G$ and x to be w. If $F\neq E$, then $F<E$ in the shelling of $\Delta_t(Y)$. Thus there exist a facet D of $\Delta_t(Y)$ and a point y of E, such that $D<E$ and $F\cap E\subseteq D\cap E=E\setminus\{y\}$. It appears in the proof of [13, Theorem 4.9] that y has to be a right turn of E and D

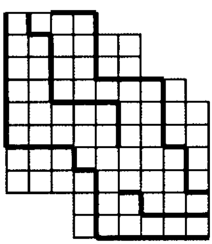

FIG. 8. A facet of Δ_{k+1}.

is obtained by switching the right turn y of E. If $D_j \cap W_j = \{z\}$, then we take L to be the facet represented by the pair (D, j), and x to be y.

The case $D_j \cap W_j \neq \{z\}$ arises only if the point which precedes z in the path E_j is a right turn and it is involved in the switching procedure. If this is the case, let E_i be the path which contains y, and let h the integer which is defined by the switching procedure. Then $i \leqslant j \leqslant h$.

Figure 9 illustrates this situation in an example where $i = j-2$ and $h = j+2$.

By the definition of the switching procedure, it follows immediately that $D \cap E = D \backslash \{u\}$, where u belongs to D_h. Since z is the unique point of $E_j \cap W_j$ and the switching procedure involves the point which precedes it, z becomes a right turn of D_j. So we may switch the right turn z of D and we get a family of non intersecting paths C. Note that our assumption on the shape of the ladder Y guarantees that C is contained in Y, that is, it is a facet of $\Delta_t(Y)$. By construction u is the only point of $C_h \cap W_h$, so that $C \backslash \{u\}$ is a facet of Δ_{k+1} represented by the pair (C, h). The next picture shows the facet C which is obtained from the facet D of Fig. 10 (here $h = j+2$).

Now we take L to be the facet (C, h) and $x = y$, and we show that they have the desired properties. Indeed $L < H$, since $C < D < E$, and further by construction $E \cap C = E \backslash \{z, y\}$. It follows that $G \cap H = F \backslash \{w\} \cap E \backslash \{z\} = F \cap E \backslash \{z, w\} \subseteq D \cap E \backslash \{z\} = E \backslash \{z, y\} = E \cap C = E \backslash \{z\} \cap C \backslash \{u\} = H \cap L = H \backslash \{y\}$. ∎

In order to avoid unnecessary distinctions we always write a facet F of D_i, $i = 0, ..., k+1$, as a disjoint union of sets $F_1, ..., F_{t-1}$. Here F_j is a path from (m, j) to (j, n), except in the following cases: If $i = 0$, then the path F_{t-1} ends in (t, n). If $0 < i < k+1$, then F_{t-1} is the union of a path F''_{t-1} from (c_i, d_i+1) to $(t-1, n)$ and a path F'_{t-1} from $(m, t-1)$ to (c_i+1, d_i). If $i = k+1$ and Y is a one-sided ladder, then the path F_{t-1} starts in (m, t). If $i = k+1$ and Y is not a one-sided ladder, then one of the F_j is of the form $F'_j \backslash \{x\}$ where F'_j is a path from (m, j) to (j, n), and $\{x\} = F'_j \cap W_j$.

For convenience we call F_i a path even in the cases in which it is just a union of two paths, or a path with a point deleted. Moreover, in the case F_i

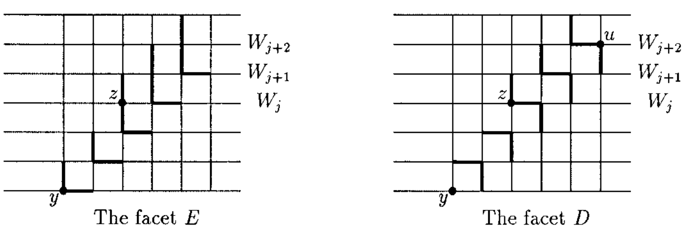

FIG. 9. (left) The facet E, and (right) the facet D.

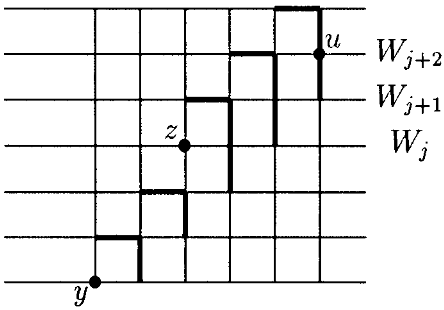

FIG. 10. The facet C.

is not really a path, we say that a point x is a right-(or left-) turn of F_i if it is a right-(or left-) turn of one of the path components of F_i.

We are ready to prove

THEOREM 4.8. *The simplicial complex Δ associated with the square-free monomial ideal* $\operatorname{in}(I_t(Y)+(c))$ *is shellable.*

Proof. The simplicial complex Δ associated with $\operatorname{in}(I_t(Y)+(c))$ is $\Delta_0 \cup \Delta_1 \cup \cdots \cup \Delta_{k+1}$. One has $\mathbb{F}(\Delta)=\mathbb{F}(\Delta_0)\cup\mathbb{F}(\Delta_1)\cup\cdots\cup\mathbb{F}(\Delta_{k+1})$, and hence Δ is pure since the Δ_i's are pure and they have all the same dimension. We need the following:

LEMMA 4.9. *For all $0\leqslant i<j\leqslant k+1$ and for all $F=\{F_1, ..., F_{t-1}\}\in\mathbb{F}(\Delta_j)$ there exist $Q_1\in F_1, ..., Q_{t-1}\in F_{t-1}$ such that $Q_1, ..., Q_{t-1}$ form a $(t-1)$-diagonal of A_i. The point Q_h can be chosen to be a right turn of F_h for $h=1, ..., t-2$. Further, the point Q_{t-1} can be chosen to be a right turn of F_{t-1} unless $i=0$ and $F_{t-1}\cap A_0=\{(t-1, n)\}$, in which case $Q_{t-1}=(t-1, n)$.*

Proof. We treat first the case $i=0$. In this case the path F_h ends in (h, n). We define Q_h to be the point of $F_h\cap\{(h, l)\colon (h, l)\in Y\}$ with smallest y-coordinate. It is clear that the points $Q_1, ..., Q_{t-1}$ have the desired properties.

If $0<i<k+1$, note that the path F_{t-1} contains a point of the region A_i. Then we take Q_{t-1} to be the point with smallest x-coordinate among the points with smallest y-coordinate of $F_{t-1}\cap A_i$. Denote by L_{t-1} the set of the points of Y lying strictly to the southwest side of Q_{t-1}. The path F_{t-2} contains a point L_{t-1}. We take Q_{t-2} to be the point with smallest x-coordinate among the points with smallest y-coordinate of $F_{t-2}\cap L_{t-1}$. The points $Q_{t-3}, ..., Q_1$ are defined iterating the previous procedure. By construction Q_h is a right turn of F_h, and $Q_1, ..., Q_{t-1}$ form a $(t-1)$-diagonal of A_i. This concludes the proof. ∎

Now we continue with the proof of 4.8. Since a facet of Δ_i does not contain $(t-1)$-diagonals of the region A_i, it follows from 4.9 that the sets $\mathbb{F}(\Delta_{k+1})$ are pairwise disjoint. Recall that we have fixed on Δ_i a shelling which arises from the extension of the northeast order. In order to have a shelling on Δ we glue together the shellings of the Δ_i's in the following manner:

Let G, $G \in \mathbb{F}(\Delta)$; we define a total order on $\mathbb{F}(\Delta)$ setting $G \leqslant F$ if $G \in \mathbb{F}(\Delta_i)$ and $F \in \mathbb{F}(\Delta_j)$, with $i<j$, or if $i=j$ and $G \leqslant F$ with respect to the shelling on Δ_i. This definition makes sense because the sets $\mathbb{F}(\Delta_i)$ are pairwise disjoint.

We claim that the given total order is a shelling on Δ. The claim is a straightforward consequence of 4.9 and of the following statement:

($*$) Let $0 \leqslant i < j \leqslant k+1$ and let $F = \{F_1, ..., F_{t-1}\} \in \mathbb{F}(\Delta_j)$. Let $Q \in A_i$ be a right turn of F_s for some $s=1, ..., t-1$, or assume $i=0$ and $F_{t-1} \cap A_0 = \{(t-1, n)\}$ and let $Q=(t-1, n)$. Then there exists $H \in \mathbb{F}(\Delta_i) \cup \mathbb{F}(\Delta_j)$ such that $H<F$ and $H \cap F = F \backslash \{Q\}$.

In order to prove ($*$) we have to distinguish several cases.

Case 1: $0=i<j \leqslant k+1$. Assume first:

Subcase 1.1.: $F_{t-1} \cap A_0 \neq \{(t-1, n)\}$. Then Q has to be a right turn of F_s. We may switch the right turn Q of F and we get the desired H. Since $F_{t-1} \cap A_0 \neq \{(t-1, n)\}$, the family of non intersecting paths H is in Y, and so it belongs to $\mathbb{F}(\Delta_j)$. Further, $H<F$ because H is on the northeast side of F and by construction $F \cap H = F \backslash \{Q\}$.

Subcase 1.2.: $F_{t-1} \cap A_0 = \{(t-1, n)\}$. If Q is a right turn of F and the family of non intersecting paths which is obtained by switching it is contained in Y, then one proceeds as in Subcase 1.1

The case in which the point Q has coordinate $(s, n+s-t+1)$ is left. We construct $H \in \mathbb{F}(\Delta_0)$ such that $F \cap H = F \backslash \{Q\}$. Note that the condition $H<F$ is automatically satisfied. The facet H is defined by the following operations:

(1) Let L be the family of non intersecting paths of $\mathbb{F}(\Delta_j)$ which is determined as follows. If $j<k+1$ and T_j belongs to F, then T_j has to be a left turn of F_{t-2}, and L is the facet which is obtained from F by switching T_j. If $j=k+1$, Y is an one-sided ladder, and $(m, t-1)$ belongs to F; then (m, t) has to be a left turn of F_{t-2}, and L is the facet which is obtained from F by switching T_j. In the other cases L is defined to be F.

(2) Let H' be the family of non intersecting paths of $\mathbb{F}(\Delta_0)$ which is determined as follows: If $j<k+1$ or if $j=k+1$ and Y is an one-sided ladder set $H'_h = L_h$ for $h=1, ..., t-2$. Note that by construction the point T_j does not belong to L if $j<k+1$. The set L_{t-1} is the union of two paths L'_{t-1} and L''_{t-1}, where L'_{t-1} starts from $(m, t-1)$ and ends in (c_j+1, d_j), and L''_{t-1} by

means of the point T_j and we get a path from $(m, t-1)$ to $(t-1, n)$. Omitting from this path the point $(t-1, n)$, we get a path H'_{t-1} from $(m, t-1)$ to (t, m). (Recall that we are working under the assumption $F_{t-1} \cap A_0 = \{(t-1, n\}.)$

If $j=k+1$ and Y is a one-sided ladder, then L_{t-1} is a path from (m, t) to $(t-1, n)$, and the point $(m, t-1)$ is not in L. So we set $H'_{t-1} = L_{t-1} \backslash \{(t-1, n)\} \cup \{(m, t-1)\}$.

Finally, if Y is not a one-sided ladder and $j=k+1$, then L is of the form $C \backslash \{x\}$, where C is a family of non intersecting paths with starting points $(m, 1), ..., (m, t-1)$ and ending points $(1, n), ..., (t-1, n)$, and $\{x\} = C_r \cap W_r$ for some r. The facet H is obtained from L by adding the point x to L_r and omitting $(t-1, n)$ from L_{t-1}.

(3) If $s=t-1$, that is, $Q=(t-1, n)$, then $H=H'$. If $s<t-1$, that is $Q \neq (t-1, n)$, then Q is a right turn of F, and this property is not destroyed passing from F to H' (note that $c_1 > t-1$). Then H is obtained from H' by switching Q. Note that, since $(t-1, n)$ does not belong to H', H is contained in Y and therefore it is an element of Δ_0 as desired.

It is easy now to check that $F \backslash H = \{Q\}$ in all the cases. The following picture illustrates the construction of H in a case in which s is supposed to be 2, and $j<k+1$.

Case 2: $0<i<j \leqslant k+1$. One may essentially mimic the constructions given in Case 1. So let us just sketch the argument. Let first $j<k+1$. The point Q is a right turn of F_s. If the family of non intersection paths is obtained from F by switching Q we are done. If not, this means that F_{t-1} has a right turn in T_i and that Q has coordinate $(c_i+s-t+1, d_i+s-t+1)$. In this case we construct $H \in \mathbb{F}(\Delta_i)$. One defines L, H', and H as in (1), (2), and (3) except for H'_{t-1} which is now defined to be $L_{t-1} \backslash T_i \cup \{T_j\}$.

Now let $j=k+1$. If Y is one-sided ladder, one proceeds as before, the only difference being that H'_{t-1} is now defined to be $L_{t-1} \backslash \{T_i\} \cup \{(m, t-1)\}$.

Finally, assume that Y is not a one-sided ladder. If the family of non intersecting paths which is obtained from F by switching Q is in Y, then we may switch twice (if it is needed) as we did in the proof of 4.7 to get H with

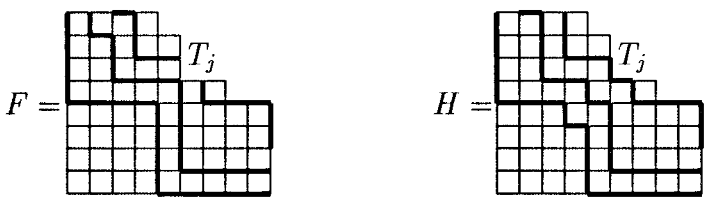

FIGURE 11

the desired properties. Otherwise, we proceed as in (2) and (3) but define $H' = L \backslash \{T_i\} \cup \{x\}$.

This concludes the proof of ($*$) and of the theorem. ∎

5. F-PURE AND F-REGULAR LADDER DETERMINANTAL RINGS

Let $R_t(Y)$ be a ladder determinantal ring. In [13, Theorem 4.9] it is shown that $\mathrm{in}(R_t(Y))$ is the Stanley–Reisner ring of a shellable simplicial complex. Hence our Corollary 2.2 in conjunction with 1.1(c) implies

THEOREM 5.1. *Any ladder determinantal ring which is defined over a perfect field of positive characteristic is F-injective.*

For a Gorenstein ring, F-rationality and F-regularity ([17, Corollary 4.7]) and F-injectivity and F-purity ([10, Corollary 1.5]) are the same. On the other hand, any subring of an F-regular (resp., F-pure) ring which is a direct summand is again F-regular (resp., F-pure); see [18, Theorem 3.4] and [19, Corollary 4.13]. Thus 1.4 and 5.2 imply

COROLLARY 5.2. *Let Y be a wide Gorenstein ladder, and let $R_t(Y)$ be the ladder determinantal ring of Y which is defined over a perfect field of positive characteristic. Let $R \subseteq R_t(Y)$ be a subring such that R is a direct summand of $R_t(Y)$ as an R-module. Then R is F-regular and F-pure.*

An explicit situation where 5.2 applies is described in

COROLLARY 5.3; *Let Y be a chain ladder, and let $R_t(Y)$ be the ladder determinantal ring of Y which is defined over a perfect field of positive characteristic. Then $R_t(Y)$ is F-regular and F-pure.*

As a consequence of 5.3 we also have that the ladder determinantal ring of any one-sided ladder is F-regular and F-pure, since one-sided ladders are special chain ladders.

For the proof of 5.3 we may assume that Y is t-connected, and we only have to note that any chain ladder Y is a subladder of a wide Gorenstein ladder W such that the natural inclusion map $R_t(Y) \subseteq R_t(W)$ is an algebra retract. Indeed, one constructs W from Y by adding suitable rows and columns to Y as described in Section 3. Then 3.1 implies that $R_t(Y)$ is an algebra retract of $R(W)$. Of course, we have to add the rows and columns in a such way that the resulting W is wide and Gorenstein. In [6] Conca describes all Gorenstein ladders. He shows that W is Gorenstein if $m = n$ (see Fig. 1) and if the inside upper corners of W lie on the line with equation $x + y = n + t - 1$, while the inside lower corners lie on the line with equation $x + y = n - t + 3$. Such a ladder is also wide. Now it is easy to see that one can move the first inside corner of Y to the right position by adding suitable rows or columns to Y which, referring to Fig. 1, lie north and west of this corner. By assumption the inside corners form a chain, and hence are naturally ordered. The next inside corner can again be moved to the right position by adding suitable rows and columns which now are east and south of the first corner (and thus do not affect the position of the first corner). Proceeding in this manner we finally obtain a wide Gorenstein ladder.

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