DIAGONAL SUBALGEBRAS OF BIGRADED ALGEBRAS AND EMBEDDINGS OF BLOW-UPS OF PROJECTIVE SPACES

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Abstract. Let V be closed subscheme of $\mathbb{P}^{n-1}$ defined by a homogeneous ideal $I \subseteq A = K[X_1, \ldots, X_n]$, and let X be the $(n-1)$-fold obtained by blowing-up $\mathbb{P}^{n-1}$ along V. If one embeds X in some projective space, one is led to consider the subalgebra $K[(I^e)_c]$ of A for some positive integers c and e. The aim of this paper is to study ring-theoretic properties of $K[(I^e)_c]$; this is achieved by developing a theory which enables us to describe the local cohomology of certain modules over generalized Segre products of bigraded algebras. These results are applied to the study of the Cohen-Macaulay property of the homogeneous coordinate ring of the blow-up of the projective space along a complete intersection. We also study the Koszul property of diagonal subalgebras of bigraded standard k-algebras.

Introduction. Let V be a smooth closed subscheme of $\mathbb{P}^{n-1}$ defined by a homogeneous ideal $I \subseteq A = K[X_1, \ldots, X_n]$, and let X be the $(n-1)$-fold obtained by blowing-up $\mathbb{P}^{n-1}$ along V.

If c is a positive integer, the c-graded part of I which we denote by I_c, corresponds to a complete linear system on X; for large c, this linear system is very ample and gives an embedding of X in $\mathbb{P}^{N-1}$, where $N = \dim_K I_c$. The homogeneous coordinate ring of this embedding is the subalgebra $K[I_c]$ of A which is generated by any set of generators of the K-vector space I_c.

More generally, we would like to embed X through more sophisticated very ample divisors; this leads us to consider, given the positive integers c and e, the subalgebra $K[(I^e)_c]$ of A.

The aim of this paper is to study ring-theoretic properties of $K[(I^e)_c]$, where e and c are positive integers and I is any homogeneous ideal of the polynomial ring $A = K[X_1, \ldots, X_n]$.

We are inspired by recent work of Geramita, et al. ([10], [11], [12]) who treated similar problems in the case X is the blow-up of $\mathbb{P}^{n-1}$ at a certain set of points.

Our main tool is an interesting relationship between $K[(I^e)_c]$ and the Rees algebra $A[It]$ of I, which is defined as the subring $\bigoplus_{j=0}^{\infty} I^j t^j$ of the polynomial ring $A[t]$.

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To describe this relationship we introduce the set

$$\Delta := \{(cs, es) \mid s \in \mathbb{Z}\},$$

which we call the (c, e)-*diagonal* of $\mathbb{Z}^2$.

For any $\mathbb{Z}^2$-graded algebra S, we will denote by $S_{(i,j)}$ the (i, j)-graded part of S. The *diagonal subalgebra* of S along Δ is defined as the $\mathbb{Z}$-graded algebra

$$S_\Delta := \bigoplus_{s\in\mathbb{Z}} S_{(cs,es)}.$$

Similarly we can define the Δ-submodule of a $\mathbb{Z}^2$-graded S-module L as

$$L_\Delta := \bigoplus_{s\in\mathbb{Z}} L_{(cs,es)}.$$

By construction L_Δ is an S_Δ-module.

The Rees algebra $A[It]$ has the natural $\mathbb{Z}^2$-grading $A[It]_{(i,j)} = (I^j)_i t^j$. We shall see that if I^e is generated by elements of degree $\leq c$, then

$$K[(I^e)_c] \simeq A[It]_\Delta.$$

This representation of $K[(I^e)_c]$ as a diagonal subalgebra of $A[It]$ was first discovered in the case I is a complete intersection generated by forms of the same degree d and Δ is the $(1, d + 1)$-diagonal of $\mathbb{Z}^2$ ([19]). Notice that a weaker version for Δ has been used there because in this case, $A[It]$ can be made a standard $\mathbb{N}^2$-graded algebra.

One main problem on diagonal subalgebras is to find suitable conditions on S such that certain algebraic properties of S are inherited by S_Δ. The operator Δ can be used to study the presentation and the normality of S_Δ as shown in [19]. Our main focus in this paper are the Cohen-Macaulay property and the Koszul property of S_Δ. We will mostly concentrate our interest on the diagonal subalgebras of the Rees algebra $A[It]$.

Assume that I is minimally generated by homogeneous polynomials $f_1, \ldots, f_r$. Let $S = A[Y_1, \ldots, Y_r]$ be a polynomial ring over A in r new indeterminates. By mapping Y_j to $f_j t$ we obtain a presentation of the Rees algebra $A[It]$ as a factor ring of S. In order for this map to be a homomorphism of $\mathbb{Z}^2$-graded algebras, we give the polynomial ring the natural $\mathbb{Z}^2$-grading $\deg X_i = (1, 0)$ and $\deg Y_j = (d_j, 1)$, where $d_j := \deg f_j$. Let

$$0 \to D_\ell \to \cdots \to D_1 \to D_0 = S \to A[It] \to 0$$

be a $\mathbb{Z}^2$-graded minimal free resolution of $A[It]$ over S. Then

$$0 \to (D_\ell)_\Delta \to \cdots \to (D_1)_\Delta \to S_\Delta \to A[It]_\Delta \to 0$$

is a graded resolution of $A[It]_\Delta$ over S_Δ. Since every free module D_p, $p = 1, \dots, l$, is a direct sum of modules of the form $S(a, b)$, where $S(a, b)$ denotes the twisted module S with shifting degree (a, b), we can deduce properties of $A[It]_\Delta$ from those of S_Δ and $S(a, b)_\Delta$.

For this reason it is of interest to study diagonal subalgebras of $\mathbb{Z}^2$-graded polynomial rings with such a $\mathbb{Z}^2$-grading and diagonal submodules of their twisted modules. We shall see that S_Δ is an affine semigroup ring for which we already have a well-developed theory ([16], [22]). To study $S(a, b)_\Delta$ we have to extend the notion of Segre products of $\mathbb{Z}$-graded algebras to $\mathbb{Z}^2$-graded algebras. It turns out that $S(a, b)_\Delta$ can be considered as a Segre product of two twisted $\mathbb{Z}^2$-graded polynomial rings whose local cohomology modules can be described in terms of the shift and the grading of S. Thus applying the diagonal operator to the minimal bigraded free resolution of a bigraded S-module L, the informations on $S(a, b)_\Delta$ yield results on L_Δ. For applications it is most important to understand the local cohomology of L_Δ. We have the following result:

THEOREM 3.6. *Let S be the polynomial ring with the bigrading as introduced above, and denote by R the ring S_Δ. Assume that $c \geq ed + 1$ where $d = \max\{d_1, \dots, d_r\}$. For any finitely generated $\mathbb{Z}^2$-graded S-module L, there exists a canonical homomorphism $\varphi_L^q \colon H^q_{\mathfrak{m}_R}(L_\Delta) \to H^{q+1}_{\mathfrak{m}_S}(L)_\Delta$ for all $q \geq 0$ such that φ_L^q is an isomorphism for $q > n$, and such that for $q \leq n$, φ_L^q induces an isomorphism of K-vector spaces between $H^q_{\mathfrak{m}_R}(L_\Delta)_s$ and $(H^{q+1}_{\mathfrak{m}_S}(L)_\Delta)_s$ for almost all s.*

From this theorem we deduce sufficient and necessary conditions for a $\mathbb{Z}^2$-graded S-module L to have a Cohen-Macaulay or Buchsbaum diagonal submodule L_Δ.

One of our main results deals with the algebra $K[(I^e)_c]$ when I is a complete intersection ideal. In this case, we can say exactly for which c, e this algebra is a Cohen-Macaulay ring, thereby solving an open problem of [19].

THEOREM 4.6. *Let $I \subset K[X_1, \dots, X_n]$ be a homogeneous complete intersection ideal minimally generated by r forms of degree $d_1, \dots, d_r$. Assume that $c \geq ed + 1$, $d = \max\{d_j \mid j = 1, \dots, r\}$. Then $K[(I^e)_c]$ is a Cohen-Macaulay ring if and only if $c > \sum_{j=1}^r d_j + (e-1)d - n$.*

As a corollary of this result we get the following interesting class of Gorenstein algebras.

COROLLARY 4.7. *Let $I \subset A = K[X_1, \dots, X_n]$ be a homogeneous complete intersection ideal minimally generated by two forms of degree $d_1 \leq d_2$. If $n \geq d_2 + 1$ then $K[I_n]$ is a Gorenstein ring with a-invariant -1.*

In the last two sections of the paper we study the Koszul property of diagonal subalgebras. Our results applied to the algebras of type $K[(I^e)_c]$ give the following

COROLLARY 6.9. *Let I be a homogeneous ideal of the polynomial ring $K[X_1, \dots, X_n]$. Denote by d the highest degree of a generator of I. Then there exist integers a, b such that the K-algebra $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq a$ and $e \geq b$.*

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1. Diagonal subalgebras of bigraded algebra. In this section we will collect some preliminary results. We will assume that the readers are familiar with the theory of multigraded rings (see e.g. [14]). Unless otherwise specified, Δ always denotes the (c, e)-diagonal of $\mathbb{Z}^2$ for a fixed pair of positive integers c, e.

A. Diagonal subalgebras of polynomial rings. Let $S = K[X_1, \dots, X_m]$ be a $\mathbb{N}^2$-graded polynomial ring with $\deg X_i = (a_i, b_i)$, $i = 1, \dots, m$, where a_i, b_i are given nonnegative integers. For convenience we assume that the matrix

$$\begin{pmatrix} a_1 & \dots & a_m \\ b_1 & \dots & b_m \end{pmatrix}$$

has rank 2. Otherwise, the $\mathbb{N}^2$-grading of S is actually an $\mathbb{N}$-grading.

For $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ we write X^α for the monomial $X_1^{\alpha_1} \cdots X_m^{\alpha_m}$. Then $\deg X^\alpha = (\sum_{i=1}^m \alpha_i a_i, \sum_{i=1}^m \alpha_i b_i)$. The monomial X^α belongs to S_Δ if and only if

$$\sum_{i=1}^m a_i \alpha_i = cs \qquad \text{and} \qquad \sum_{i=1}^m b_i \alpha_i = es$$

for some integer s. Let H denote the additive monoid of the solutions $\alpha \in \mathbb{N}^m$ of these systems of equations. Then $S_\Delta = K[X^\alpha \mid \alpha \in H]$, which is isomorphic to the affine semigroup ring $K[H]$ of H over K. See e.g. [5] or [22] for more information on the theory of affine semigroup rings.

PROPOSITION 1.1.

(i) $\dim S_\Delta = m - 1$.

(ii) *S_Δ is a normal Cohen-Macaulay domain.*

(iii) *$\omega_{S_\Delta} \simeq (\omega_S)_\Delta$, where ω_{S_Δ} and ω_S denote the canonical modules of S_Δ and S, respectively.*

Proof. Let G be the set of all integral solutions of the above systems of equations. Then G is a lattice of integral points in $\mathbb{Z}^m$ with $\operatorname{rank} G = m - 1$ and $H = G \cap \mathbb{N}^m$. Therefore $\dim K[H] = m - 1$ and $K[H]$ is a normal Cohen-Macaulay domain ([17]). Finally, by [14, Theorem 3.3.3 (2)] we have

$$\omega_{S_\Delta} = K[X^\alpha \mid \alpha \in G, \alpha > 0] = K[X^\alpha \mid \alpha > 0]_\Delta = (\omega_S)_\Delta,$$

where $\alpha > 0$ means that $\alpha_i > 0$ for all $i = 1, \ldots, m$. □

Remark. Every $\mathbb{N}$-graded affine semigroup ring $K[H]$ with $\dim K[H] = m - 1$ for which the corresponding convex polyhedral cone has exactly m facets arises as a diagonal subalgebra of an $\mathbb{N}^2$-graded polynomial ring.

B. Segre products of graded algebras. Let $S = A \otimes_K B$ be the tensor product of two $\mathbb{Z}$-graded algebras $A = \bigoplus_{i\in\mathbb{Z}} A_i$ and $B = \bigoplus_{j\in\mathbb{Z}} B_j$ over K. Then S is a $\mathbb{Z}^2$-graded algebra with $S_{(i,j)} = A_i \otimes_K B_j$. From this it follows that

$$S_\Delta = \bigoplus_{s\in\mathbb{Z}} A_{cs} \otimes_K B_{es},$$

which is the Segre product of order (c, e) of A and B over K ([7]).

We can extend the notion of Segre products of $\mathbb{Z}$-graded algebras to $\mathbb{Z}^2$-graded algebras as follows.

Definition. Let A and B be two $\mathbb{Z}^2$-graded algebras over a field K. The tensor product $A \otimes_K B$ is a $\mathbb{Z}^2$-graded algebra over K with

$$(A \otimes_K B)_{(i,j)} := \bigoplus_{\substack{(a_1,a_2),(b_1,b_2)\in\mathbb{Z}^2\\ (a_1,a_2)+(b_1,b_2)=(i,j)}} A_{(a_1,a_2)} \otimes_K B_{(b_1,b_2)}.$$

We have

$$(A \otimes_K B)_\Delta = \bigoplus_{s\in\mathbb{Z}} \bigoplus_{\substack{(a_1,a_2),(b_1,b_2)\in\mathbb{Z}^2\\ (a_1,a_1)+(b_1,b_2)=(cs,es)}} A_{(a_1,a_2)} \otimes_K B_{(b_1,b_2)},$$

which we call the *Segre product* of A and B along Δ. For convenience we denote it by $A \otimes_\Delta B$. Similarly, if M and N are $\mathbb{Z}^2$-graded modules over A and B, respectively, then the tensor product $M \otimes_K N$ is a $\mathbb{Z}^2$-graded $A \otimes_K B$-module with

$$(M \otimes_K N)_\Delta = \bigoplus_{s \in \mathbb{Z}} \bigoplus_{\substack{(a_1,a_2),(b_1,b_2) \in \mathbb{Z}^2 \\ (a_1,a_1)+(b_1,b_2)=(cs,es)}} M_{(a_1,a_2)} \otimes_K N_{(b_1,b_2)}.$$

We call $(M \otimes_K N)_\Delta$ the Segre product of M and N along Δ and denote it by $M \otimes_\Delta N$.

C. Embeddings of blow-ups of projective spaces. Let $A = K[X_1, \dots, X_n]$ be a polynomial ring over a field K and I a homogeneous ideal of A. For large c, the algebra $K[(I^e)_c]$ is isomorphic to the coordinate ring of some embedding of the blow-up of $\mathbb{P}_K^{n-1}$ along the ideal sheaf $\tilde{I}$ in a projective space.

Let $A[It] = \bigoplus_{j \geq 0} I^j t^j$ be the Rees algebra of I. Since the polynomial ring $A[t]$ is an $\mathbb{N}^2$-graded algebra with $A[t]_{(i,j)} = A_i t^j$, we may consider the Rees algebra $A[It]$ as an $\mathbb{N}^2$-graded subalgebra of $A[t]$ with $A[It]_{(i,j)} = (I^j)_i t^j$. Hence $A[It]$ has the diagonal subalgebra

$$A[It]_\Delta = \bigoplus_{i \geq 0} (I^{ei})_{ci} t^{ei}.$$

We note the following simple fact whose proof we leave to the reader:

LEMMA 1.2. *Assume that the ideal I^e is generated by forms of degree $\leq c$. Then*

$$K[(I^e)_c] \simeq A[It]_\Delta.$$

We will denote by $K((I^e)_c)$ the field of quotients of $K[(I^e)_c]$. One has:

LEMMA 1.3. *Assume that I^e is generated by forms of degree $\leq c - 1$. Then*

(i) $K((I^e)_c) = K\left(\frac{X_2}{X_1}, \dots, \frac{X_n}{X_1}, X_1 f\right)$ *for any nonzero element $f \in (I^e)_{c-1}$.*

(ii) $\dim K[(I^e)_c] = n$.

Proof. Since $\frac{X_i}{X_1} = \frac{X_i f}{X_1 f} \in K((I^e)_c)$, we have

$$K\left(\frac{X_2}{X_1}, \dots, \frac{X_n}{X_1}, X_1 f\right) \subseteq K((I^e)_c).$$

Conversely, for every element $g \in (I^e)_c$, $\frac{g}{X_1 f} \in K\left(\frac{X_2}{X_1}, \dots, \frac{X_n}{X_1}\right)$ because $g, X_1 f$ have the same degree. Therefore $g = X_1 f \frac{g}{X_1 f} \in K\left(\frac{X_2}{X_1}, \dots, \frac{X_n}{X_1}, X_1 f\right)$. So we obtain (i). Now it is clear that the transcendent degree of $K((I^e)_c)$ is equal to n, which implies (ii). □

There have been some scattered results on the properties of $K[(I^e)_c]$, namely in the case I is the defining ideal of fat points ([10], [11], [12]) or when I is a complete intersection ideal generated by forms of the same degree ([19]).

Remark. If the ideal I is generated by forms of the same degree d, we can define another $\mathbb{Z}^2$-graded structure on $A[It]$ as follows. Let $R = \bigoplus_{(i,j)\in\mathbb{N}^2} R_{(i,j)}$ be the $\mathbb{N}^2$-graded algebra with

$$R_{(i,j)} := (I^j)_{i+dj}t^j$$

for all $(i,j) \in \mathbb{N}^2$. Since I^j is generated by forms of degree jd, we have $(I^j)_h = 0$ for $h < jd$. Therefore R covers all elements of $A[It] = \bigoplus_{j\geq 0} I^j t^j$, hence $R = A[It]$. We note that R is a standard $\mathbb{Z}^2$-graded K-algebra, i.e. $R_{(0,0)} = K$ and $R = K[R_{(1,0)}, R_{(0,1)}]$. This $\mathbb{Z}^2$-graded structure of $A[It]$ has been used successfully to study $K[I_{d+1}]$ in [19], and will also be used later in the paper.

Now let S be an arbitrary $\mathbb{Z}^2$-graded K-algebra which is an integral domain. Then its integral closure $\bar{S}$ in the field of fractions inherits a natural $\mathbb{Z}^2$-graded structure from S.

The following result was originally proved in [19] for the $(1,1)$-diagonal of $\mathbb{Z}^2$, but the proof there also holds for arbitrary Δ without any modification.

Proposition 1.4. *Let $Q(S_\Delta)$ denote the field of quotients of S_Δ. Then $\overline{(S_\Delta)} = (\overline{S})_\Delta \cap Q(S_\Delta)$.*

Now we would like to employ this relationship to study the integral closure of the algebra $K[(I^e)_c]$.

Corollary 1.5. *Assume that I^e is generated by forms of degree $\leq c$. Then*

$$\overline{K[(I^e)_c]} = K[(\overline{I^{es}})_{cs} \mid s \geq 0] \cap K((I^e)_c),$$

where $\overline{I^{es}}$ denotes the integral closure of I^{es}.

Proof. By Lemma 1.2 we have $K[(I^e)_c] \simeq A[It]_\Delta$. Hence

$$\overline{K[(I^e)_c]} = (\overline{A[It]})_\Delta \cap Q(A[It]_\Delta).$$

It is known that $\overline{A[It]} = \oplus_{i\geq 0}\overline{I^j}t^j$. Then $(\overline{A[It]})_\Delta = \oplus_{s\geq 0}(\overline{I^{es}})_{cs}t^{es} \simeq K[(\overline{I^{es}})_{cs} \mid s \geq 0]$. Since the latter isomorphism induces the isomorphism $Q(A[It]_\Delta) \simeq K((I^e)_c)$, we obtain the conclusion from the above formula for $\overline{K[(I^e)_c]}$. ◻

To study the Cohen-Macaulay property of diagonal submodules we will use local cohomology. We shall see that under certain conditions, the operator Δ commutes with local cohomology modules. For this we assume that S is an $\mathbb{N}^2$-

graded polynomial ring over K with $\dim S = m$. Let $R = S_\Delta$. We will denote by $\mathfrak{m}_S$ and $\mathfrak{m}_R$ the maximal graded ideal of S and R, respectively.

For any module L over a K-algebra T we will denote by $H^q_{\mathfrak{m}}(L)$ the qth local cohomology module of L with support in an ideal $\mathfrak{m}$ of T ([15]), and we put $L^* = \mathrm{Hom}_K(L, K)$.

Proposition 1.6. *Let L be a finitely generated $\mathbb{Z}^2$-graded S-module. For all $q \geq 0$, there is a canonical graded homomorphism $\varphi^q_L \colon H^q_{\mathfrak{m}_R}(L_\Delta) \to H^{q+1}_{\mathfrak{m}_S}(L)_\Delta$.*

Proof. We have

$$\underline{\mathrm{Hom}}_S(L, \omega_S)_\Delta = \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_S(L, \omega_S(cs, es)),$$

and by Proposition 1.2

$$\begin{aligned}\underline{\mathrm{Hom}}_R(L_\Delta, \omega_R) &= \underline{\mathrm{Hom}}_R(L_\Delta, (\omega_S)_\Delta)\\ &= \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_R(L_\Delta, (\omega_S)_\Delta(s))\\ &= \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_R(L_\Delta, \omega_S(cs, es)_\Delta).\end{aligned}$$

Here $\underline{\mathrm{Hom}}$ denotes the "graded Hom."

Since for each s there is a natural homomorphism from $\mathrm{Hom}_S(L, \omega_S(cs, es))$ to $\mathrm{Hom}_R(L_\Delta, \omega_S(cs, es)_\Delta)$, we get an induced natural graded homomorphism from $\underline{\mathrm{Hom}}_S(L, \omega_S)_\Delta$ to $\underline{\mathrm{Hom}}_R(L_\Delta, \omega_R)$, and hence canonical graded homomorphisms ψ^i_L from $\underline{\mathrm{Ext}}^i_S(L, \omega_S)_\Delta$ to $\underline{\mathrm{Ext}}^i_R(L_\Delta, \omega_R)$ for $i \geq 0$. Since S and R are Cohen-Macaulay rings with $\dim S = m$ and $\dim R = m - 1$ (Proposition 1.1), we have

$$\begin{aligned}H^{q+1}_{\mathfrak{m}_S}(L) &= \underline{\mathrm{Ext}}^{m-q-1}_S(L, \omega_S)^*,\\ H^q_{\mathfrak{m}_R}(L_\Delta) &= \underline{\mathrm{Ext}}^{m-q-1}_R(L_\Delta, \omega_R)^*\end{aligned}$$

for $q \geq 0$ [14, Theorem 2.2.2]. It is easy to check that

$$(\underline{\mathrm{Ext}}^i_S(L, \omega_S)^*)_\Delta = (\underline{\mathrm{Ext}}^i_S(L, \omega_S)_\Delta)^*.$$

Therefore, ψ^{m-q-1}_L yields a canonical homomorphism φ^q_L from $H^q_{\mathfrak{m}_R}(L_\Delta)$ to $H^{q+1}_{\mathfrak{m}_S}(L)_\Delta$. ◻

We will denote by $[\varphi^q_L]_s \colon H^q_{\mathfrak{m}_R}(L_\Delta)_s \to (H^{q+1}_{\mathfrak{m}_S}(L)_\Delta)_s$ the component of degree s of the map φ^q_L.

LEMMA 1.7. *Let L be a finitely generated $\mathbb{Z}^2$-graded S-module. Let*

$$0 \to D_\ell \to \cdots \to D_1 \to D_0 \to L \to 0$$

be a $\mathbb{Z}^2$-graded minimal free resolution of L over S. Let s and $i \geq 0$ be integers such that $[\varphi^m_{D_p}]_s$ is an isomorphism and $H^q_{\mathfrak{m}_R}((D_p)_\Delta)_s = 0$ for $i < q < m-1$, $p = 0, \ldots, \ell$. Then $[\varphi^q_L]_s$ is an isomorphism for all $q \geq i$.

Proof. If $\ell = 0$, $L = D_0$ is a free S-module with $\dim L = m$. Hence $H^{q+1}_{\mathfrak{m}_S}(L) = 0$ for $q \neq m$. By the assumption, $[\varphi^m_L]_s$ is an isomorphism and $H^q_{\mathfrak{m}_R}(L_\Delta)_s = 0$ for $i \leq q < m-1$. Since $H^{q+1}_{\mathfrak{m}_S}(L) = 0$ for $i \leq q < m-1$, $[\varphi^q_L]_s$ is an isomorphism for all $q \geq i$.

For $\ell \geq 1$ we consider first the kernel C of the map $D_0 \to L$. Since $0 \to D_\ell \to \cdots \to D_1 \to C$ is a $\mathbb{Z}^2$-graded minimal free resolution of C, we may assume, by induction on ℓ, that $[\varphi^q_C]_s$ is an isomorphism for $q \geq i$. The short exact sequence $0 \to C \to D_0 \to L \to 0$ implies $H^q_{\mathfrak{m}_S}(L) \simeq H^{q+1}_{\mathfrak{m}_S}(C)$ for $q \neq m-1, m$ and the exact sequence

$$0 \to H^{m-1}_{\mathfrak{m}_S}(L) \to H^m_{\mathfrak{m}_S}(C) \to H^m_{\mathfrak{m}_S}(D_0) \to H^m_{\mathfrak{m}_S}(L) \to 0.$$

Applying the functor Δ we get $H^q_{\mathfrak{m}_S}(L)_\Delta \simeq H^{q+1}_{\mathfrak{m}_S}(C)_\Delta$ for $q \neq m-1, m$ and the exact sequence

$$0 \to H^{m-1}_{\mathfrak{m}_S}(L)_\Delta \to H^m_{\mathfrak{m}_S}(C)_\Delta \to H^m_{\mathfrak{m}_S}(D_0)_\Delta \to H^m_{\mathfrak{m}_S}(L)_\Delta \to 0.$$

On the other hand, since $H^q_{\mathfrak{m}_R}((D_p)_\Delta)_s = 0$ for $i < q < m-1$, from the short exact sequence $0 \to C_\Delta \to (D_0)_\Delta \to L_\Delta \to 0$ we get $H^{q-1}_{\mathfrak{m}_R}(L_\Delta)_s \simeq H^q_{\mathfrak{m}_R}(C_\Delta)_s$ for $i+1 \leq q < m-1$ and the exact sequence

$$0 \to H^{m-2}_{\mathfrak{m}_R}(L_\Delta)_s \to H^{m-1}_{\mathfrak{m}_R}(C_\Delta)_s \to H^{m-1}_{\mathfrak{m}_R}((D_0)_\Delta)_s \to H^{m-1}_{\mathfrak{m}_R}(L_\Delta)_s \to 0.$$

Now consider the commutative diagrams

$$\begin{array}{ccc} H^{q-1}_{\mathfrak{m}_R}(L_\Delta)_s & \xrightarrow{\sim} & H^q_{\mathfrak{m}_R}(C_\Delta)_s \\ {\scriptstyle [\varphi^{q-1}_L]_s}\downarrow & & \downarrow{\scriptstyle [\varphi^q_C]_s} \\ (H^q_{\mathfrak{m}_S}(L)_\Delta)_s & \xrightarrow{\sim} & (H^{q+1}_{\mathfrak{m}_S}(C)_\Delta)_s \end{array}$$

for $i+1 \leq q < m-1$ and

$$\begin{array}{ccccccc} 0 \to H^{m-2}_{\mathfrak{m}_R}(L_\Delta)_s & \longrightarrow & H^{m-1}_{\mathfrak{m}_R}(C_\Delta)_s & \longrightarrow & H^{m-1}_{\mathfrak{m}_R}((D_0)_\Delta)_s & \longrightarrow & H^{m-1}_{\mathfrak{m}_R}(L_\Delta)_s \to 0 \\ {\scriptstyle [\varphi^{m-2}_L]_s}\downarrow & & {\scriptstyle [\varphi^{m-2}_C]_s}\downarrow & & {\scriptstyle [\varphi^{m-1}_{D_0}]_s}\downarrow & & {\scriptstyle [\varphi^{m-1}_L]_s}\downarrow \\ 0 \to (H^{m-1}_{\mathfrak{m}_S}(L)_\Delta)_s & \longrightarrow & (H^m_{\mathfrak{m}_S}(C)_\Delta)_s & \longrightarrow & (H^m_{\mathfrak{m}_S}(D_0)_\Delta)_s & \longrightarrow & (H^m_{\mathfrak{m}_S}(L)_\Delta)_s \to 0. \end{array}$$

Since $[\varphi_C^q]_s$ is an isomorphism for $q \geq i$, we can conclude that $[\varphi_L^q]_s$ is an isomorphism for $q \geq i$. □

In the following we say that φ_L^q is almost an isomorphism if there exists a positive integer s_0 such that $[\varphi_L^q]_s$ is an isomorphism for $|s| \geq s_0$. Recall that L_Δ is called a generalized Cohen-Macaulay module if $H^q_{\mathfrak{m}_R}(L_\Delta)$ is of finite length for $q \neq \dim L$.

PROPOSITION 1.8. *Let L be a finitely generated $\mathbb{Z}^2$-graded S-module and*

$$0 \to D_\ell \to \cdots \to D_1 \to D_0 \to L \to 0$$

a $\mathbb{Z}^2$-graded minimal free resolution of L over S. Assume that $\varphi_{D_p}^{m-1}$ is an isomorphism for $p = 0, \ldots, \ell$. Then

(i) *φ_L^q is an almost isomorphism for all $q \geq 0$ if $(D_p)_\Delta$ is a generalized Cohen-Macaulay module with* $\dim (D_p)_\Delta = m - 1$ *for $p = 0, \ldots, \ell$.*

(ii) *φ_L^q is an isomorphism for all $q \geq 0$ if $(D_p)_\Delta$ is a Cohen-Macaulay module with* $\dim (D_p)_\Delta = m - 1$ *for $p = 0, \ldots, \ell$.*

Proof. If $(D_p)_\Delta$ is a generalized Cohen-Macaulay module with $\dim (D_p)_\Delta = m - 1$, $p = 0, \ldots, \ell$, there exists an integer $s_0 \geq 0$ such that $H^q_{\mathfrak{m}_R}((D_p)_\Delta)_s = 0$ for $|s| \geq s_0$, $q \neq m-1$. Therefore, the assumptions of Proposition 1.7 are satisfied for $i = 0$ and $|s| \geq s_0$, hence $[\varphi_L^q]_s$ is an isomorphism for $|s| \geq s_0$ and $q \geq 0$. Similarly, if $(D_p)_\Delta$ is a Cohen-Macaulay module with $\dim (D_p)_\Delta = m - 1$, $p = 0, \ldots, r - 1$, then $H^q_{\mathfrak{m}_R}((D_p)_\Delta) = 0$ for $q \neq m - 1$. Therefore, the assumptions of Proposition 1.7 are satisfied for $i = 0$ all all integers s, hence φ_L^q is an isomorphism for $q \geq 0$. □

Note that every $\mathbb{Z}^2$-graded free S-module is a direct sum of free summands of the form $S(a, b)$. In studying $A[It]_\Delta$ we may put $S = K[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ with $\deg X_i = (1, 0)$, $i = 1, \ldots, n$, and $\deg Y_j = (d_j, 1)$, $j = 1, \ldots, r$, where $d_1, \ldots, d_r$ are the degree of the elements of a homogeneous basis of I. In this case we can compute the local cohomology modules of $S(a, b)_\Delta$ using the theory of Segre products of $\mathbb{N}^2$-graded algebras. This will be done in the next sections.

2. Segre products of bigraded algebras. Let $A = K[X_1, \ldots, X_n]$ and $B = K[Y_1, \ldots, Y_r]$ be two $\mathbb{N}^2$-graded polynomial rings with $\deg X_i = (1, 0)$, $i = 1, \ldots, n$, and $\deg Y_j = (d_j, 1)$, $j = 1, \ldots, r$, where $d_1, \ldots, d_r$ are fixed nonnegative integers. Then A and B have only one maximal graded ideal which we denote by $\mathfrak{m}_A$ and $\mathfrak{m}_B$, respectively. Let

$$R = A \otimes_\Delta B.$$

Then R is an $\mathbb{N}$-graded algebra with $R_0 = k$. Hence R has only a maximal graded ideal which we denote by $\mathfrak{m}_R$.

The reason for choosing the above $\mathbb{N}^2$-graded polynomial rings is that the tensor product $A \otimes_K B = K[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$ appears in the presentation of the Rees algebra of a homogeneous ideal or of a standard $\mathbb{N}^2$-graded K-algebra ($d_1 = \cdots = d_r = 0$). We shall prove the following lemma which will play a crucial role in the computation of local cohomology modules of Segre products of $\mathbb{Z}^2$-graded modules over A and B.

LEMMA 2.1. *Assume that $c \geq ed + 1$, $d = \max\{d_1, \ldots, d_r\}$. For any pair of homogeneous elements $f \in \mathfrak{m}_A$ and $g \in \mathfrak{m}_B$, there exist positive integers ℓ and m such that $f^\ell \otimes_K g^m \in \mathfrak{m}_R$.*

Proof. Let $\deg f = (\alpha, 0)$ and $\deg g = (\gamma, \beta)$. Put $l = c\beta - e\gamma$ and $m = e\alpha$, and note that $l > 0$, $m > 0$ and further that $f^l \otimes_K g^m \in S_{(cs,es)}$ with $s = \alpha\beta$. □

First we will study the left derived functors of the $\mathfrak{m}_R$-transform on Segre products of $\mathbb{Z}^2$-graded modules. Recall that for any ideal $\mathfrak{m}$ of a Noetherian commutative ring T and any T-module L, the left derived functors of the $\mathfrak{m}$-transform on L ([4]) are defined as

$$D^q_{\mathfrak{m}}(L) := \varinjlim \operatorname{Hom}^q_T(\mathfrak{m}^n, L),$$

$q \geq 0$. Note that the relationship between $D^q_{\mathfrak{m}}(L)$ and the local homology modules $H^q_{\mathfrak{m}}(L)$ is described by the exact sequence

$$0 \to H^0_{\mathfrak{m}}(L) \to L \to D^0_{\mathfrak{m}}(L) \to H^1_{\mathfrak{m}}(L) \to 0$$

and the isomorphisms $H^q_{\mathfrak{m}}(L) \simeq D^{q-1}_{\mathfrak{m}}(L)$, $q \geq 2$.

Let M and N be two finitely generated bigraded modules over A and B, respectively.

THEOREM 2.2. *For any $q \geq 0$,*

$$D^q_{\mathfrak{m}_R}(M \otimes_\Delta N) = \bigoplus_{i+j=q} D^i_{\mathfrak{m}_A}(M) \otimes_\Delta D^j_{\mathfrak{m}_B}(N).$$

For Segre products of $\mathbb{Z}$-graded modules, this formula was already proved by Stückrad and Vogel [20, Lemma 1] and implicitly also by Goto and Watanabe [13, Theorem (4.1.5) and Remark (4.1.6)].

We consider first the case of graded injective modules.

LEMMA 2.3. *Let E and F be graded injective modules over A and B, respectively. Then*

$$D^0_{\mathfrak{m}_R}(E \otimes_\Delta F) = D^0_{\mathfrak{m}_A}(E) \otimes_\Delta D^0_{\mathfrak{m}_B}(F),$$

$$D^q_{\mathfrak{m}_R}(E \otimes_\Delta F) = 0, \ q \geq 1.$$

Proof. By the structure theorem for injective modules (see e.g. [13, Theorem (1.2.1)]) we may write $E = E_1 \oplus E_2$ with $\operatorname{Ass}_A(E_1) = \mathfrak{m}_A$ and $\mathfrak{m}_A \notin \operatorname{Ass}_A(E_2)$, and $F = F_1 \oplus F_2$ with $\operatorname{Ass}_B(F_1) = \mathfrak{m}_B$ and $\mathfrak{m}_B \notin \operatorname{Ass}_A(F_2)$.

We have $D^q_{\mathfrak{m}_A}(E_1) = 0$ for $q \geq 0$. Hence $D^q_{\mathfrak{m}_A}(E) = D^q_{\mathfrak{m}_A}(E_2)$ for $q \geq 0$. Moreover, there exists a homogeneous element $f \in \mathfrak{m}_A$ such that the multiplication map $E_2 \xrightarrow{f} E_2$ is bijective [13, Lemma (2.2.3)]. The induced map $H^q_{\mathfrak{m}_A}(E_2) \xrightarrow{f} H^q_{\mathfrak{m}_A}(E_2)$ must be bijective. Hence $H^q_{\mathfrak{m}_A}(E_2) = 0$ for $q \geq 0$ because every element of $H^q_{\mathfrak{m}_A}(E_2)$ is annihilated by a large power of x. From this it follows that $D^0_{\mathfrak{m}_A}(E_2) = E_2$. Hence

$$D^0_{\mathfrak{m}_A}(E) = E_2.$$

Similarly, there exists a homogeneous element $g \in \mathfrak{m}_A$ such that the multiplication map $F_2 \xrightarrow{g} F_2$ is bijective, and it follows that

$$D^0_{\mathfrak{m}_B}(F) = F_2.$$

Put $C_1 := (E_1 \otimes_\Delta F_1) \oplus (E_1 \otimes_\Delta F_2) \oplus (E_2 \otimes_\Delta F_1)$ and $C_2 := E_2 \otimes_\Delta F_2$. Then $E \otimes_\Delta F = C_1 \oplus C_2$. It is easy to check that $\operatorname{Ass}_S(C_1) = \mathfrak{m}_R$. From this it follows that $D^q_{\mathfrak{m}_R}(C_1) = 0$ for $q \geq 0$. Hence

$$D^q_{\mathfrak{m}_R}(E \otimes_\Delta F) = D^q_{\mathfrak{m}_R}(C_2).$$

Let $h = f \otimes g$. By Lemma 2.1 we may assume that $h \in m_R$. Then we have a multiplication map $C_2 \xrightarrow{h} C_2$ which is bijective. The induced map $H^q_{\mathfrak{m}_R}(C_2) \xrightarrow{h} H^q_{\mathfrak{m}_R}(C_2)$ must be bijective. Simlilarly as above, this implies $H^q_{\mathfrak{m}_R}(C_2) = 0$ for $q \geq 0$. Hence

$$\begin{aligned} D^0_{\mathfrak{m}_R}(C_2) &= C_2 = D^0_{\mathfrak{m}_A}(E) \otimes_\Delta D^0_{\mathfrak{m}_B}(F), \\ D^q_{\mathfrak{m}_R}(C_2) &= 0, \ q \geq 1. \end{aligned}$$ □

Proof. [Proof of 2.2.] Let E and F be minimal injective resolutions of M and N, respectively. It is known that

$$\begin{aligned} D^i_{\mathfrak{m}_A}(M) &= H^i(D^0_{\mathfrak{m}_A}(E)), \ i \geq 0, \\ D^j_{\mathfrak{m}_B}(N) &= H^j(D^0_{\mathfrak{m}_B}(F)), \ j \geq 0. \end{aligned}$$

Define canonical complexes C and D of R-modules with

$$\begin{aligned} C^q &:= \bigoplus_{i+j=q} E^i \otimes_\Delta F^j, \\ D^q &:= \bigoplus_{i+j=q} D^0_{\mathfrak{m}_A}(E^i) \otimes_\Delta D^0_{\mathfrak{m}_B}(F^j) \end{aligned}$$

for $q \geq 0$. It is clear that C is a resolution of $M \otimes_\Delta N$. By Lemma 2.3, $D^0_{\mathfrak{m}_R}(C) = D$ and $D^q_{\mathfrak{m}_R}(C) = 0$ for $q \geq 1$. Hence

$$D^q_{\mathfrak{m}_R}(M \otimes_\Delta N) = H^q(D)$$

for $q \geq 0$. Using the Künneth formula for tensor products of complexes over a field [8, Theorem 3.1, p. 113] we get

$$\begin{aligned} H^q(D) &= \bigoplus_{i+j=q} H^i(D^0_{\mathfrak{m}_A}(E)) \otimes_\Delta H^j(D^0_{\mathfrak{m}_B}(F)) \\ &= \bigoplus_{i+j=q} D^i_{\mathfrak{m}_A}(M) \otimes_\Delta D^j_{\mathfrak{m}_B}(N). \end{aligned}$$ □

As a consequence of Theorem 2.2 we obtain the following formula for the local cohomolgy modules of $M \otimes_\Delta N$.

COROLLARY 2.4. *For any $q \geq 2$,*

$$\begin{aligned} H^q_{\mathfrak{m}_R}(M \otimes_\Delta N) &= \Big(D^0_{\mathfrak{m}_A}(M) \otimes_\Delta H^q_{\mathfrak{m}_B}(N)\Big) \oplus \Big(H^q_{\mathfrak{m}_A}(M) \otimes_\Delta D^0_{\mathfrak{m}_B}(N)\Big) \\ &\quad \oplus \bigoplus_{\substack{i+j=q+1 \\ i,j\geq 2}} H^i_{\mathfrak{m}_A}(M) \otimes_\Delta H^j_{\mathfrak{m}_B}(N). \end{aligned}$$

Proof. For $q \geq 2$, we have

$$\begin{aligned} H^q_{\mathfrak{m}_R}(M \otimes_\Delta N) &= D^{q-1}_{\mathfrak{m}_R}(M \otimes_\Delta N) \\ &= \bigoplus_{i+j=q-1} D^i_{\mathfrak{m}_A}(M) \otimes_\Delta D^j_{\mathfrak{m}_B}(N). \end{aligned}$$

Now we only need to put $D^i_{\mathfrak{m}_A}(M) = H^{i+1}_{\mathfrak{m}_A}(M)$ for $i \geq 1$ and $D^j_{\mathfrak{m}_B}(M) = H^{j+1}_{\mathfrak{m}_A}(M)$ for $j \geq 1$ to get the conclusion. □

Example. The above formula does not hold for Segre products over arbitrary $\mathbb{Z}^2$-graded polynomial rings. Let $A = K[X_1]$ with $\deg X_1 = (1,0)$ and $B = K[Y_1, Y_2, Y_3]$ with $\deg Y_1 = (1,0)$, $\deg Y_2 = \deg Y_3 = (0,1)$. Let $M = A(2,0)$ and $N = B$ and Δ the $(1,2)$-diagonal. If Corollary 2.4 were true in this case too, then $H^2_{\mathfrak{m}_R}(M \otimes_\Delta N) = 0$. On the other hand, if we let $A' = K[X_1, Y_1]$ and $B' = K[Y_2, Y_3]$ with the same grading on the variables, then A' and B' satisfy the assumption of this section. We have $A' \otimes_\Delta B' = A \otimes_\Delta B$ and, for $M' = A'(1,2)$ and $N' = B'$, $M' \otimes_\Delta N' = M \otimes_\Delta N$. Applying Corollary 2.4 we get

$$H^2_{\mathfrak{m}_R}(M' \otimes_\Delta N') = (M' \otimes_\Delta H^2_{\mathfrak{m}_{B'}}(N')) \oplus (H^2_{\mathfrak{m}_{A'}}(M') \otimes_\Delta N').$$

It is easily seen that $M'_{(-2,0)} \neq 0$ and $H^2_{\mathfrak{m}_{B'}}(N')_{(0,2)} \neq 0$. Hence $M' \otimes_\Delta H^2_{\mathfrak{m}_{B'}}(N') \neq 0$, which is a contradiction to the assumed fact that $H^2_{\mathfrak{m}_R}(M' \otimes_\Delta N') = H^2_{\mathfrak{m}_R}(M \otimes_\Delta N) = 0$.

It is of interest to compare the local cohomology modules of $M \otimes_\Delta N$ with those of the tensor product $M \otimes_K N$. Let $S = A \otimes_K B$ and let m_S be the maximal graded ideal of S. By [13, Theorem (2.2.5)] we have, for $q \geq 0$,

$$H^q_{\mathfrak{m}_S}(M \otimes_K N)_\Delta = \bigoplus_{i+j=q} H^i_{\mathfrak{m}_A}(M) \otimes_\Delta H^j_{\mathfrak{m}_B}(N).$$

LEMMA 2.5. *Assume that* $H^q_{\mathfrak{m}_A}(M) \otimes_\Delta N = 0$ *and* $M \otimes_\Delta H^q_{\mathfrak{m}_B}(N) = 0$ *for some* $q \geq 1$. *Then*

$$H^q_{\mathfrak{m}_R}(M \otimes_\Delta N) = \bigoplus_{\substack{i+j=q+1 \\ i,j \geq 1}} H^i_{\mathfrak{m}_A}(M) \otimes_\Delta H^j_{\mathfrak{m}_B}(N).$$

Proof. Since $M \otimes_\Delta H^q_{\mathfrak{m}_B}(N) = 0$, applying the exact functor $- \otimes_\Delta H^q_{\mathfrak{m}_B}(N)$ to the exact sequence

$$0 \to H^0_{\mathfrak{m}_A}(M) \to M \to D^0_{\mathfrak{m}_A}(M) \to H^1_{\mathfrak{m}_A}(M) \to 0$$

we get $D^0_{\mathfrak{m}_A}(M) \otimes_\Delta H^q_{\mathfrak{m}_B}(N) = H^1_{\mathfrak{m}_A}(M) \otimes_\Delta H^q_{\mathfrak{m}_B}(N)$. Similarly, since $H^q_{\mathfrak{m}_A}(M) \otimes_\Delta N = 0$, one has $H^q_{\mathfrak{m}_A}(M) \otimes_\Delta D^0_{\mathfrak{m}_B}(N) = H^q_{\mathfrak{m}_A}(M) \otimes_\Delta H^1_{\mathfrak{m}_B}(N)$. Putting these relations into Corollary 2.4 we get the formula for $q \geq 2$.

For $q = 1$ we have to consider the commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
M \otimes_\Delta N & \longrightarrow & M \otimes_\Delta D^0_{\mathfrak{m}_B}(N) & \longrightarrow & M \otimes_\Delta H^1_{\mathfrak{m}_B}(N) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
D^0_{\mathfrak{m}_A}(M) \otimes_\Delta N & \longrightarrow & D^0_{\mathfrak{m}_A}(M) \otimes_\Delta D^0_{\mathfrak{m}_B}(N) & \longrightarrow & D^0_{\mathfrak{m}_A}(M) \otimes_\Delta H^1_{\mathfrak{m}_B}(N) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1_{\mathfrak{m}_A}(M) \otimes_\Delta N & \longrightarrow & H^1_{\mathfrak{m}_A}(M) \otimes_\Delta D^0_{\mathfrak{m}_B}(N) & \longrightarrow & H^1_{\mathfrak{m}_A}(M) \otimes_\Delta H^1_{\mathfrak{m}_B}(N) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0. & &
\end{array}$$

It is easy to check that if $M \otimes_\Delta H^1_{\mathfrak{m}_B}(N) = 0$ and $H^1_{\mathfrak{m}_A}(M) \otimes_\Delta N = 0$, then

$$\begin{aligned} H^1_{\mathfrak{m}_A}(M) \otimes_\Delta H^1_{\mathfrak{m}_B}(N) &= \operatorname{Coker}\left(M \otimes_\Delta N \to D^0_{\mathfrak{m}_A}(M) \otimes_\Delta D^0_{\mathfrak{m}_B}(N)\right) \\ &= H^1_{\mathfrak{m}_R}(M \otimes_\Delta N). \end{aligned}$$

Indeed, note that by Theorem 2.2, $D^0_{\mathfrak{m}_A}(M) \otimes_\Delta D^0_{\mathfrak{m}_B}(N) = D^0_{\mathfrak{m}_A}(M \otimes_\Delta N)$, and that the map $M \otimes_\Delta N \to D^0_{\mathfrak{m}_A}(M) \otimes_\Delta D^0_{\mathfrak{m}_B}(N) = D^0_{\mathfrak{m}_A}(M \otimes_\Delta N)$ in the diagram is the canonical map, that is, the map which appears in the exact sequence

$$M \otimes_\Delta N \longrightarrow D^0_{\mathfrak{m}_A}(M \otimes_\Delta N) \longrightarrow H^1_{\mathfrak{m}_R}(M \otimes_\Delta N) \longrightarrow 0. \qquad \square$$

COROLLARY 2.6. *Assume that* $v = \dim M \geq 2$ *and* $w = \dim N \geq 2$. *Then*

$$H^{v+w-1}_{\mathfrak{m}_R}(M \otimes_\Delta N) = H^v_{\mathfrak{m}_A}(M) \otimes_\Delta H^w_{\mathfrak{m}_B}(N) = H^{v+w}_{\mathfrak{m}_S}(M \otimes_K N)_\Delta.$$

Proof. We have $H^i_{\mathfrak{m}_A}(M) = 0$ for $i \neq v$ and $H^j_{\mathfrak{m}_B}(N) = 0$ for $j \neq w$. Since $v+w-1 > v, w$, putting this into Lemma 2.5 and the formula for the cohomology modules of $M \otimes_K N$ we get

$$H^{v+w-1}_{\mathfrak{m}_R}(M \otimes_\Delta N) = H^v_{\mathfrak{m}_A}(M) \otimes_\Delta H^w_{\mathfrak{m}_B}(N)$$
$$H^{v+w}_{\mathfrak{m}_S}(M \otimes_K N) = H^v_{\mathfrak{m}_A}(M) \otimes_K H^w_{\mathfrak{m}_B}(N).$$

Hence the conclusion is obvious. $\square$

Now we will apply the above results to estimate the dimension and to study the Cohen-Macaulay property of $M \otimes_\Delta N$.

LEMMA 2.7. *Assume that* $v = \dim M \geq 1$ *and* $w = \dim N \geq 1$. *Then*

$$\dim M \otimes_\Delta N \leq v + w - 1.$$

Equality holds if $H^v_{\mathfrak{m}_A}(M) \otimes_\Delta H^w_{\mathfrak{m}_B}(N) \neq 0$.

Proof. We have $H^i_{\mathfrak{m}_A}(M) = 0$ for $i > v$ and $H^j_{\mathfrak{m}_B}(N) = 0$ for $j > w$. Applying Corollary 2.4 we get $H^q_{\mathfrak{m}_R}(M \otimes_\Delta N) = 0$ for $q \geq v + w$. Hence $\dim M \otimes_\Delta N \leq v + w - 1$. Moreover, equality holds if $H^{v+w-1}_{\mathfrak{m}_R}(M \otimes_\Delta N) \neq 0$. If $v + w = 2$, then $v = w = 1$. Using the commutative diagram in the proof of Corollary 2.5 we get an acyclic sequence

$$M \otimes_\Delta N \to D^0_{\mathfrak{m}_R}(M \otimes_\Delta N) = D^0_{\mathfrak{m}_A}(M) \otimes_\Delta D^0_{\mathfrak{m}_B}(N) \to H^1_{\mathfrak{m}_A}(M) \otimes_\Delta H^1_{\mathfrak{m}_B}(N).$$

Hence there is a surjective map

$$H^1_{\mathfrak{m}_R}(M \otimes_\Delta N) \to H^1_{\mathfrak{m}_A}(M) \otimes_\Delta H^1_{\mathfrak{m}_B}(N).$$

For $v + w \geq 3$, applying Corollary 2.4 we get an injective map

$$H^v_{\mathfrak{m}_A}(M) \otimes_\Delta H^w_{\mathfrak{m}_B}(N) \to H^{v+w-1}_{\mathfrak{m}_A}(M \otimes_\Delta N).$$

In any case, we conclude that $H^{v+w-1}_{\mathfrak{m}_R}(M \otimes_\Delta N) \neq 0$ if $H^v_{\mathfrak{m}_A}(M) \otimes_\Delta H^w_{\mathfrak{m}_B}(N) \neq 0$. ◻

THEOREM 2.8. *Let M and N be $\mathbb{Z}^2$-graded Cohen-Macaulay modules over A and B, respectively. Assume that $v = \dim M \geq w = \dim N \geq 1$, and $\dim M \otimes_\Delta N = v + w - 1$. Then $M \otimes_\Delta N$ is a Cohen-Macaulay module if and only if one of the following conditions is satisfied:*

(i) $v = w = 1$.

(ii) $v > w = 1$ *and* $M \otimes_\Delta H^1_{\mathfrak{m}_B}(N) = 0$.

(iii) $w \geq 2$ *and* $H^v_{m_A}(M) \otimes_\Delta N = 0$ *and* $M \otimes_\Delta H^w_{\mathfrak{m}_B}(N) = 0$.

Proof. It is well-known that $M \otimes_\Delta N$ is a Cohen-Macaulay module if and only if $H^q_{\mathfrak{m}_R}(M \otimes_\Delta N) = 0$ for $q < v + w - 1$. Since M and N are Cohen-Macaulay modules, we have $H^i_{m_A}(M) = 0$ for $i \neq v$ and $H^j_{m_B}(N) = 0$ for $j \neq w$. In particular, the maps $M \to D^0_{\mathfrak{m}_A}(M)$ and $N \to D^0_{\mathfrak{m}_B}(N)$ are injective. Hence the map $M \otimes_\Delta N \to D^0_{\mathfrak{m}_R}(M \otimes_\Delta N)$ is injective. From this it follows that $H^0_{\mathfrak{m}_R}(M \otimes_\Delta N) = 0$.

(i) If $v = w = 1$, then $\dim M \otimes_\Delta N = 1$. Hence $M \otimes_\Delta N$ is Cohen-Macaulay.

(ii) If $v > w = 1$, then $H^i_{\mathfrak{m}_A}(M) = 0$ for $i = 0, 1$. Hence $D^0_{\mathfrak{m}_A}(M) = M$. By Theorem 2.2, $D^0_{\mathfrak{m}_R}(M \otimes_\Delta N) = M \otimes_\Delta D^0_{\mathfrak{m}_B}(N)$. Using the exact sequence

$$M \otimes_\Delta N \longrightarrow M \otimes_\Delta D^0_{\mathfrak{m}_B}(N) \longrightarrow M \otimes_\Delta H^1_{\mathfrak{m}_B}(N) \longrightarrow 0$$

we get $H^1_{\mathfrak{m}_R}(M \otimes_\Delta N) = M \otimes_\Delta H^1_{\mathfrak{m}_B}(N)$. By Corollary 2.4 we already have $H^q_{\mathfrak{m}_R}(M \otimes_\Delta N) = 0$ for $2 \leq q \leq v - 1$. Hence $M \otimes_\Delta N$ is Cohen-Macaulay if and only if $M \otimes_\Delta H^1_{\mathfrak{m}_B}(N) = 0$.

(iii) Now we assume that $v, w \geq 2$. Then $H^i_{\mathfrak{m}_A}(M) = 0$ for $i = 0, 1$ and $H^j_{m_B}(N) = 0$ for $j = 0, 1$. From this it follows that $D^0_{\mathfrak{m}_A}(M) = M$ and $D^0_{\mathfrak{m}_B}(N) = N$. Therefore, $D^0_{\mathfrak{m}_R}(M \otimes_\Delta N) = D^0_{\mathfrak{m}_A}(M) \otimes_\Delta D^0_{\mathfrak{m}_B}(N) = M \otimes_\Delta N$, which implies $H^1_{\mathfrak{m}_R}(M \otimes_\Delta N) = 0$. By Corollary 2.4 we have, for $q \geq 2$,

$$H^q_{m_S}(M \otimes_\Delta N) = \begin{cases} 0, & q \neq v, w, v + w - 1 \\ M \otimes_\Delta H^v_{\mathfrak{m}_B}(N), & q = w \neq v, \\ H^v_{\mathfrak{m}_A}(M) \otimes_\Delta N, & q = v \neq w, \\ \left(M \otimes_\Delta H^q_{\mathfrak{m}_B}(N)\right) \oplus \left(H^q_{\mathfrak{m}_A}(M) \otimes_\Delta N\right), & q = v = w. \end{cases}$$

Hence $M \otimes_\Delta N$ is a Cohen-Macaulay module if and only if $M \otimes_\Delta H^w_{\mathfrak{m}_B}(N) = 0$ and $H^v_{\mathfrak{m}_A}(M) \otimes_\Delta N = 0$. ◻

Remark. According to Lemma 2.7 and Theorem 2.8 we will need to check the condition $E \otimes_\Delta F = 0$ for some $\mathbb{Z}^2$-graded modules E and F. This can be easily done in terms of the supports of E and F. For any $\mathbb{Z}^2$-graded module L over a $\mathbb{Z}^2$-graded algebra we define

$$\operatorname{supp} L := \{(a_1, a_2) \in \mathbb{Z}^2 \mid L_{(a_1,a_2)} \neq 0\}.$$

Given two subsets V and W of $\mathbb{Z}^2$, let $V + W$ be the set of all elements of $\mathbb{Z}^2$ of the form $x + y$ with $x \in V$ and $y \in W$. Then $E \otimes_\Delta F = 0$ if and only if $(\operatorname{supp} E + \operatorname{supp} F) \cap \Delta = \emptyset$.

3. Existence of Cohen-Macaulay diagonal subalgebras. In this section we consider the polynomial ring

$$S = K[X_1, \dots, X_n, Y_1, \dots, Y_r]$$

with bigraded structure given by $\deg X_i = (1, 0)$, $i = 1, \dots, n$, and $\deg Y_j = (d_j, 1)$, $j = 1, \dots, r$, where $d_1, \dots, d_r$ are fixed nonnegative integers. For convenience we assume that $n \geq r \geq 2$.

Let $R = S_\Delta$, where Δ is a (c, e)-diagonal of $\mathbb{Z}^2$. Given a Cohen-Macaulay S-module L, we would like to know whether L has a Cohen-Macaulay diagonal submodule L_Δ.

First we will consider the case $L = S(a, b)$, where $S(a, b)$ denotes the $\mathbb{Z}^2$-graded module S with shifting degree (a, b). For this we shall need some notations.

Given a vector γ of integers, we will say that $\gamma \geq 0$ (or $\gamma > 0$ or $\gamma \leq 0$ or $\gamma < 0$) if all the components of γ satisfy this condition. For $s \in \mathbb{Z}$, let U_s (resp. V_s resp. W_s) be the K-vector space generated by the monomials $X^\alpha Y^\beta$ with $\alpha < 0$, $\beta < 0$ (resp. $\alpha \geq 0$, $\beta < 0$ resp. $\alpha < 0$, $\beta \geq 0$) and

$$\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{r} d_j \beta_j = cs + a, \tag{1}$$

$$\sum_{j=1}^{r} \beta_j = es + b. \tag{2}$$

Put $U = \bigoplus_{s \in \mathbb{Z}^2} U_s$, $V = \bigoplus_{s \in \mathbb{Z}^2} V_s$, and $W = \bigoplus_{s \in \mathbb{Z}^2} W_s$.

With these notations we are able to describe the local cohomology modules of $S(a, b)_\Delta$ as follows.

LEMMA 3.1. *For arbitrary integers a, b,*

$$H^q_{\mathfrak{m}_R}(S(a,b)_\Delta) \simeq \begin{cases} 0, & q \neq n, r, n + r - 1, \\ V, & q = n \neq r, \\ W, & q = r \neq n, \\ V \oplus W, & q = n = r \\ U, & q = n + r - 1. \end{cases}$$

Moreover, the canonical map $\varphi^{n+r-1}_{S(a,b)}\colon H^{n+r-1}_{\mathfrak{m}_R}(S(a,b)_\Delta) \to H^{n+r}_{\mathfrak{m}_S}(S(a,b))_\Delta$ *is an isomorphism.*

Proof. Put $A = K[X_1, \dots, X_n]$ and $B = K[Y_1, \dots, Y_r]$. As subalgebras of S, A and B are $\mathbb{N}^2$-graded. We have $S = A \otimes_K B$. The grading of $A \otimes_K B$ implies that $S(a,b) = A(a,b) \otimes_K B$. Hence $S(a,b)_\Delta = A(a,b) \otimes_\Delta B$. Note that $A(a,b)$ and B are Cohen-Macaulay modules with $\dim A(a,b) = n \geq 2$ and $\dim B = r \geq 2$. Then using the same argument as in the proof of Theorem 2.8 (iii) and Corollary 2.6 we get

$$H^q_{m_R}(A(a,b) \otimes_\Delta B) = \begin{cases} 0, & q \neq r, n, n+r-1, \\ A(a,b) \otimes_\Delta H^r_{\mathfrak{m}_B}(B), & q = r \neq n, \\ H^n_{\mathfrak{m}_A}(A(a,b)) \otimes_\Delta B, & q = n \neq r, \\ (A(a,b) \otimes_\Delta H^q_{\mathfrak{m}_B}(B)) \oplus (H^q_{\mathfrak{m}_A}(A(a,b)) \otimes_\Delta B), & q = n = r \\ H^n_{\mathfrak{m}_A}(A(a,b)) \otimes_\Delta H^r_{\mathfrak{m}_B}(B), & q = n+r-1. \end{cases}$$

It is known that $H^n_{\mathfrak{m}_A}(A) \cong 7_{\alpha<0} KX^\alpha$ and $H^r_{\mathfrak{m}_B}(B) \cong \bigoplus_{\alpha<0} KX^\alpha$. A monomial $X^\alpha Y^\beta$ in $S(a,b)$ has degree $(\sum_{i=1}^n \alpha_i + \sum_{j=1}^r \beta_j d_j - a, \sum_{j=1}^r \beta_j - b)$. Hence it belongs to $S(a,b)_\Delta$ if and only if α and β satisfy the system of equations (1) and (2) for some integer s.

Using the above presentations of A, B, $H^n_{\mathfrak{m}_A}(A)$, $H^r_{\mathfrak{m}_B}(B)$ we get

$$\begin{aligned} A(a,b) \otimes_\Delta H^r_{\mathfrak{m}_B}(B) &\simeq V, \\ H^n_{\mathfrak{m}_A}(A(a,b)) \otimes_\Delta B &\simeq W, \\ H^n_{m_A}(A(a,b)) \otimes_\Delta H^r_{m_B}(B) &\simeq U. \end{aligned}$$

Finally, by Corollary 2.6 we have

$$\begin{aligned} H^{n+r}_{\mathfrak{m}_S}(S(a,b))_\Delta &= H^{n+r}_{\mathfrak{m}_S}(A(a,b) \otimes_K B)_\Delta = H^n_{m_A}(A(a,b)) \otimes_\Delta H^r_{m_B}(B) \\ &= H^{n+r-1}_{\mathfrak{m}_R}(A(a,b) \otimes_\Delta B) = H^{n+r-1}_{\mathfrak{m}_R}(S(a,b)_\Delta). \end{aligned}$$

Hence $\varphi^{n+r-1}_{S(a,b)}$ is an isomorphism. ◻

COROLLARY 3.2. *Assume that* $c \geq ed_1 + 1$, $d_1 = \min\{d_1, \dots, d_r\}$. *Then*

$$\dim S(a,b)_\Delta = n + r - 1.$$

Proof. We have $\dim S(a,b)_\Delta = n + r - 1$ if $H^{n+r-1}_{\mathfrak{m}_R}(S(a,b)_\Delta) \neq 0$. By Lemma 3.1, this condition is satisfied if $U_s \neq 0$, that is, if the system of equations (1) and (2) has a solution with $\alpha < 0$ and $\beta < 0$ for some integer s. For this we may choose

$$s \le \min\left\{-\frac{b+r}{e}, \frac{(b+r)d_1 - u - a - n}{c - ed_1}\right\},$$

where $u = \sum_{j=1}^{r} d_j$. Then $es + b + r \le 0$. Put $\beta_1 = es + b + r - 1$ and $\beta_i = -1$, $i = 2, \ldots, r$. Then

$$\begin{aligned} cs + a - \sum_{j=1}^{r} d_j\beta_j &= cs + a - (es + b + r - 1)d_1 + \sum_{j=2}^{r} d_j \\ &= (c - ed_1)s + a - (b + r)d_1 - u \le -n. \end{aligned}$$

Hence there exist $\alpha \in \mathbb{Z}$, $\alpha < 0$, such that $\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{r} d_j\beta_j = cs + a$. □

In the following lemma we determine exactly the nonvanishing graded pieces of V, W.

LEMMA 3.3. *Let $d_1 \le \cdots \le d_r = d$ and $u = \sum_{j=1}^{r} d_j$. Assume that $c \ge ed + 1$. Then*

(i) $V_s \neq 0$ *if and only if* $\frac{(b+r)d - u - a}{c - ed} \le s \le -\frac{b+r}{e}$.

(ii) $W_s \neq 0$ *if and only if* $-\frac{b}{e} \le s \le \frac{bd - a - n}{c - ed}$.

Proof. (i) We have $V_s \neq 0$ if and only if the system of equations (1) and (2) has a solution with $\alpha \ge 0$ and $\beta < 0$. Assume that this condition is satified. Then $es + b = \sum_{j=1}^{r} \beta_j \le -r$. Hence $s \le -\frac{r+b}{e}$. Moreover, $cs + a - \sum_{j=1}^{r} d_j\beta_j = \sum_{i=1}^{n} \alpha_i \ge 0$. Since

$$\begin{aligned} cs + a - \sum_{j=1}^{r} d_j\beta_j &= cs + a - \sum_{j=1}^{r-1} d_j\beta_j - \left(es + b - \sum_{j=1}^{r-1} \beta_j\right) d \\ &= (c - ed_1)s + a - bd + \sum_{j=1}^{r-1} \beta_j(d - d_j) \\ &\le (c - ed_1)s + a - bd - \sum_{j=1}^{r-1} (d - d_j) \\ &= (c - ed_1)s + a - (b + r)d + u, \end{aligned}$$

we get $(c - ed)s + a - (b + r)d + u \ge 0$. Hence $s \ge \frac{(b+r)d - u - a}{c - ed}$. Conversely, assume that $\frac{(b+r)d - u - a}{c - ed} \le s \le -\frac{r+b}{e}$. Then $es + b + r \le 0$. Put $\beta_r =$

$es+b+r-1$ and $\beta_i = -1$, $i = 1, \dots, r-1$. Then

$$\begin{aligned} cs + a - \sum_{j=1}^{r} d_j\beta_j &= cs + a - (es + b + r - 1)d - \sum_{j=1}^{r-1} d_j \\ &= (c - ed)s + a - (b + r)d + u \ \geq \ 0. \end{aligned}$$

Hence there exist $\alpha < 0$ such that $\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{r} d_j\beta_j = cs + a$.

(ii) We have $W_s \neq 0$ if and only if the system of equations (1) and (2) has a solution with $\alpha < 0$ and $\beta \geq 0$. Assume that this condition is satified. Then $es+b = \sum_{j=1}^{r} \beta_j \geq 0$. Hence $s \geq -\dfrac{b}{e}$. Moreover, replacing β_r by $es+b-\sum_{j=1}^{r-1} \beta_j$ in $\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{r} \beta_j d_j = cs + a$ one has

$$s = \frac{\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{r-1} \beta_j(d_j - d) + bd - a}{c - ed} \leq \frac{-u + bd - a}{c - ed}.$$

Conversely, assume $-\frac{b}{e} \leq \frac{bd-u-a}{c-ed}$, then set $\beta_i = 0$ for $1 \leq i \leq r-1$, and $\beta_{r-1} = es+b$. By assumption, $\beta_r \geq 0$. Then $cs+a-\sum_{j=1}^{r} \beta_j d_j = cs+a-d(es+b) = s(c-ed)+a-db \geq u$, by assumption. Hence there exists $\alpha \in \mathbb{Z}^n$, $\alpha < 0$, with $\sum_{i=1}^{n} \alpha_i = cs + a - \sum_{j=1}^{r} \beta_j d_j$. □

PROPOSITION 3.4. *Let* $d_1 \leq \cdots \leq d_r = d$ *and* $u = \sum_{j=1}^{r} d_j$. *Assume that* $c \geq ed+1$. *Then*

(i) $S(a,b)_\Delta$ *is a generalized Cohen-Macaulay module with* $\dim S(a,b)_\Delta = n+r-1$.

(ii) $S(a,b)_\Delta$ *is a Cohen-Macaulay module if and only if*

$$\left[-\frac{r+b}{e}\right] < \frac{(b+r)d - u - a}{c - ed},$$
$$\left[\frac{bd - a - n}{c - ed}\right] < -\frac{b}{e},$$

where $[x]$ *denotes* $\max\{n \in \mathbb{Z}\colon\ n \leq x\}$.

Proof. By Lemma 3.1, the module $S(a,b)_\Delta$ is a generalized Cohen-Macaulay module if $\dim S(a,b)_\Delta = n+r-1$ and V, W have finite lengths. But these conditions are always satisfied by Corollary 3.2 and Lemma 3.3. Similarly, $S(a,b)_\Delta$ is a Cohen-Macaulay module if and only if $V = 0$ and $W = 0$, which is equivalent to the conditions of (ii). □

In the following we say that a property holds for $c \gg 0$ *relatively to* $e \gg 0$ if there exists e_0 such that for all $e \geq e_0$ there exists a positive integer $c(e)$ depending on e such that this property holds for all (c,e) with $c \geq c(e)$.

COROLLARY 3.5. *Let* $d_1 \leq \cdots \leq d_r = d$ *and* $u = \sum_{j=1}^{r} d_j$.

(i) *For* $c \gg 0$ *relatively to* $e \gg 0$, $H^q_{\mathfrak{m}_R}(S(a,b)_\Delta)_s = 0$ *for* $s \neq 0$, $q < n+r-1$.

(ii) $S(a,b)_\Delta$ *is a Cohen-Macaulay module for* $c \gg 0$ *relatively to* $e \gg 0$ *if and only if* a, b *satisfy one of the following conditions:*

(1) $b \leq -r$ *and* $(b+r)d - u - a > 0$,
(2) $-r < b < 0$,
(3) $b \geq 0$ *and* $bd - a - n < 0$.

Proof. For $e > -(b+r)$ and $c > u + a - (b+r-e)d$, we have

$$-\frac{b+r}{e} < 1 \quad \text{and} \quad -1 < \frac{(b+r)d - u - a}{c - ed}.$$

In this case, $V_s = 0$ for all $s \neq 0$ by Lemma 3.3. Similarly, for $e > b$ and $c > (e+b)d - n - a$, we have

$$-1 < -\frac{b}{e} \quad \text{and} \quad \frac{bd - a - n}{c - ed} < 1,$$

hence $W_s = 0$ for all $s \neq 0$. Therefore (i) follows from Lemma 3.1.

To prove (ii) we may assume that $c \geq ed+1$. Assume that $S(a,b)_\Delta$ is a Cohen-Macaulay module. If $b \leq -r$, then $-\dfrac{b+r}{e} \geq 0$. Hence $0 < \dfrac{(b+r)d - u - a}{c - ed}$ by Proposition 3.4 (ii). From this it follows that $(b+r)d - u - a > 0$. If $b \geq 0$, then $-\dfrac{b}{e} \leq 0$. Hence $\dfrac{bd - a - n}{c - ed} < 0$ by Proposition 3.4 (ii). From this it follows that $bd - a - n < 0$. Conversely, for $c \gg 0$ relatively to $e \gg 0$ one easily checks that

$$\left\lceil -\frac{b+r}{e} \right\rceil = 0 < \frac{(b+r)d - u - a}{c - ed} \quad \text{if } b \leq -r, \text{ and } (b+r)d - u - a > 0,$$

$$\left\lceil \frac{bd - a - n}{c - ed} \right\rceil \leq 0 < -\frac{b}{e} \quad \text{if } b < 0,$$

$$\left\lceil -\frac{b+r}{e} \right\rceil \leq -1 < \frac{(b+r)d - u - a}{c - ed} \quad \text{if } -r < b,$$

$$\left\lceil \frac{bd - a - n}{c - ed} \right\rceil \leq -1 < -\frac{b}{e} \quad \text{if } b > 0, \text{ and } bd - a - n < 0.$$

From this it follows that the conditions of Proposition 3.4 (ii) are satisfied for (1), (2), (3). Hence $S(a,b)_\Delta$ is Cohen-Macaulay in all these cases. □

Now we will use the above information on the modules $S(a,b)_\Delta$ to study the diagonal submodule L_Δ of a finitely generated $\mathbb{Z}^2$-graded S-module L. The following result shows that the local cohomology modules of L_Δ are closely related to those of L.

THEOREM 3.6. *Let S be a $\mathbb{N}^2$-graded polynomial ring as above. Assume that $c \geq ed + 1$, $d = \max\{d_1, \dots, d_r\}$. For any finitely generated $\mathbb{Z}^2$-graded S-module L, the canonical homomorphism $\varphi_L^q \colon H_{\mathfrak{m}_R}^q(L_\Delta) \to H_{\mathfrak{m}_S}^{q+1}(L)_\Delta$ is an isomorphism for $q > n$ and almost an isomorphism for $q \leq n$.*

Proof. Let $0 \to D_\ell \to \cdots \to D_1 \to D_0 \to L \to 0$ be a $\mathbb{Z}^2$-graded minimal free resolution of L over S. By Lemma 3.1 and Proposition 3.4, $\varphi_{D_p}^{n+r-1}$ is an isomorphism, $H_{\mathfrak{m}_R}^q((D_p)_\Delta) = 0$ for $n < q < n + r - 1$, and L_p is a generalized Cohen-Macaulay module with $\dim L_p = n + r - 1$, $p = 0, \dots, \ell$. Therefore, φ_L^q is an isomorphism for $q > n$ by Lemma 1.7 and almost an isomorphism for $q \leq n$ by Proposition 1.8. □

It would be interesting if the above theorem could be extended to arbitrary $\mathbb{Z}^2$-graded polynomial rings

Definition 3.7. We say that L has a good $\mathbb{Z}^2$-graded minimal free resolution

$$0 \to D_\ell \to \cdots \to D_1 \to D_0 \to L \to 0$$

if every free module D_p is a direct sum of modules $S(a, b)$ such that a, b satisfy the conditions of Corollary 3.5 (ii).

LEMMA 3.8. *Let L be a finitely generated $\mathbb{Z}^2$-graded S-module. Then the following properties hold for $c \gg 0$ relatively to $e \gg 0$:*

(i) $[\varphi_L^q]_s$ *is an isomorphism for all $s \neq 0$ and $q \geq 0$,*

(ii) φ_L^q *is an isomorphism for all $q \geq 0$ if L has a good $\mathbb{Z}^2$-graded minimal free resolution.*

Proof. By Lemma 3.1, $\varphi_{D_p}^{n+r-1}$ is an isomorphism for $c \gg 0$ relatively to $e \gg 0$. Therefore, using Corollary 3.5 we will obtain (i) from Lemma 1.7 and (ii) from Proposition 1.8 (ii). □

THEOREM 3.9. *Let L be a finitely generated $\mathbb{Z}^2$-graded S-module which has a good $\mathbb{Z}^2$-graded minimal free resolution. Assume that $\dim L_\Delta = \dim L - 1$ for $c \gg 0$ relatively to $e \gg 0$. Then the following conditions are equivalent:*

(i) L_Δ *is a Cohen-Macaulay module for $c \gg 0$ relatively to $e \gg 0$.*

(ii) $H_{\mathfrak{m}_S}^q(L)_{(0,0)} = 0$ *and* $H_{\mathfrak{m}_S}^q(L)_{(-i,-j)} = 0$ *for $i \gg 0$ relatively to $j \gg 0$, $0 < q < \dim L$.*

Proof. By Lemma 3.8 (ii), $H_{\mathfrak{m}_R}^q(L_\Delta) = H_{\mathfrak{m}_S}^{q+1}(L)_\Delta$ for $q \geq 0$, $c \gg 0$ relatively to $e \gg 0$. If (i) is satisfied, then $H_{\mathfrak{m}_R}^q(L_\Delta) = 0$ for $q \neq \dim L - 1$. Hence $H_{\mathfrak{m}_S}^{q+1}(L)_{(cs,es)} = 0$ for all $s \in \mathbb{Z}$. Putting $s = 0, -1$, we see that $H_{\mathfrak{m}_S}^q(L)_{(0,0)} = 0$ and $H_{\mathfrak{m}_S}^q(L)_{(-i,-j)} = 0$ for $i \gg 0$ relatively to $j \gg 0$, $0 < q < \dim L$. For the converse we first note that for $q \neq \dim L$, $H_{\mathfrak{m}_S}^q(L)$ is an artinian module, hence $H_{\mathfrak{m}_S}^q(L)_{(i,j)} = 0$ for $i \gg 0$

relatively to $j \gg 0$. This together with (ii) implies that $H^q_{\mathfrak{m}_S}(L)_{(cs,es)} = 0$ for all integers s, $0 < q < \dim L$, $c \gg 0$ relatively to $e \gg 0$. So we have $H^q_{\mathfrak{m}_S}(L)_\Delta = 0$ and therefore $H^{q-1}_{\mathfrak{m}_R}(L_\Delta) = 0$ for $0 < q < \dim L$. Hence L_Δ is a Cohen-Macaulay module. □

Remark. Theorem 3.9 does not hold without the assumption on the minimal free resolution of L. The condition (ii) alone does not imply (i), as one may expect. In fact, by Corollary 3.5, there exist modules $S(a,b)$ which satisfy (ii) but not (i).

Conjecture. If $A[It]$ is a Cohen-Macaulay ring, then there exist c, e such that $A[It]_\Delta$ is a Cohen-Macaulay ring.

In the case of standard bigraded K-algebras we can give reasonable conditions guaranteeing that high diagonal subalgebras are Cohen-Macaulay. Indeed, let $S = K[X_1, \dots, X_n, Y_1, \dots, Y_r]$ be standard bigraded with $\deg X_i = (1,0)$ and $\deg Y_i = (0,1)$.

Given integers $a, b \in \mathbb{Z}$ we consider the bishifted free S-module $S(a,b)$. Let Δ be the diagonal associated with $c, e \in \mathbb{N} \setminus \{0\}$. From 3.2 and 3.4 we immediately get

LEMMA 3.10. (i) $\dim S(a,b)_\Delta = n + r - 1$*; and* (ii) $S(a,b)_\Delta$ *is Cohen-Macaulay if and only if* $\left[-\frac{r+b}{e}\right] < -\frac{a}{c}$ *and* $\left[-\frac{n+a}{c}\right] < -\frac{b}{e}$.

In particular $S(a,b)_\Delta$ *is Cohen-Macaulay for large* Δ *if and only if one of the following conditions is satisfied:*

(1) $-r < b < 0$ *or* $-n < a < 0$*;*

(2) $a \geq 0$ *and* $b \geq 0$*;*

(3) $a \leq -n$ *and* $b \leq -r$.

More precisely, if one of the conditions (1), (2) *or* (3) *is satisfied, then* $S(a,b)_\Delta$ *is Cohen-Macaulay if* $c > \max\{a, -n-a\}$ *and* $e > \max\{b, -r-b\}$.

Now let R be a bigraded standard K-algebra, and let $R_1 = \bigoplus_{i \geq 0} R_{(i,0)}$ and $R_2 = \bigoplus_{i \geq 0} R_{(0,i)}$. Assume $\operatorname{emb\,dim} R_1 = n$ and $\operatorname{emb\,dim} R_2 = r$, so that we have a minimal presentation $S \to R$. If $S(a,b)$ appears in the minimal free resolution of R an S-module; then $a \leq 0$ and $b \leq 0$. Hence, by Lemma 3.10, $S(a,b)_\Delta$ is Cohen-Macaulay for large Δ unless $a = 0$ and $b \leq -r$, or $b = 0$ and $a \leq -n$.

Notice that the shifts $(a,0)$ and $(0,b)$ in the resolution of R are exactly the shifts of R_1 and R_2 over $K[X_1, \dots, X_n]$ and $K[Y_1, \dots, Y_r]$, respectively. Indeed, $R_1 = R_{\Delta'}$ where $\Delta' = \{(i,0)\colon\ i \in \mathbb{Z}\}$. Applying the exact functor $(\cdots)_{\Delta'}$ to the bigraded resolution of R we see that $S(a,b)_{\Delta'} = 0$ if $b < 0$ and that $S(a,b)_{\Delta'} = S_1(a)$ if $b = 0$ where $S_1 = K[X_1, \dots, X_n]$. Similarly one argues for the shifts $(0,b)$.

The above discussions now yield the following result:

THEOREM 3.11. *Suppose the standard bigraded K-algebra is Cohen-Macaulay, and that for R_1 and R_2 the shifts in the resolution are strictly greater than $-n$ and $-r$, respectively. Then R_Δ is Cohen-Macaulay for large Δ.*

More explicitly, one has under these assumptions that R_Δ is Cohen-Macaulay if for all shifts (a,b) in the resolution one has $c > -a-n$ and $e > -b-r$.

In particular, R_Δ is Cohen-Macaulay if $c \geq a(R)+r$ and $e \geq a(R)+n$. Here $a(R)$ denotes the a-invariant of R where R is equippped with the natural $\mathbb{Z}$-graded structure given by $R_i = \bigoplus_{k+l=i} R_{(k,l)}$.

COROLLARY 3.12. *Let R be a standard bigraded Cohen-Macaulay K-algebra. Suppose that R_1 and R_2 are Cohen-Macaulay with $a(R_1) < 0$ and $a(R_2) < 0$. Then R_Δ is Cohen-Macaulay for large Δ.*

Note that, in a more special case, the previous result has a converse ([13]): if $R = R_1 \otimes_K R_2$, and R_1 and R_2 are Cohen-Macaulay, then R_Δ is Cohen-Macaulay if and only if $a(R_1) < 0$ and $a(R_2) < 0$.

For Rees rings our arguments yield the following:

COROLLARY 3.13. *Suppose $I \subseteq R = K[X_1,\dots,X_n]$ is an equigenerated ideal, say of degree d, such that $R[It]$ and $K[I_d]$ are Cohen-Macaulay. Suppose further that the relation type $r(I)$ of I is less than the analytic spread $l(I)$ of I (i.e. $a(K[I_d]) < 0$). Then $R[It]_\Delta$ is Cohen-Macaulay for large Δ.*

COROLLARY 3.14. *If $I \subseteq R = K[X_1,\dots,X_n]$ is equigenerated and of linear type, and $R[It]$ is Cohen-Macaulay, then $R[It]_\Delta$ is Cohen-Macaulay for large Δ.*

As a last application of 3.11 we have

COROLLARY 3.15. *Let $I \subset R = K[X_1,\dots,X_n]$ be a perfect ideal of codimension 2. Suppose that I has a linear presentation matrix of size $d \times d+1$, that $d+1 > n$ and that I satisfies G_n, that is, $\mu(I_P) \leq \operatorname{height} P$ for all prime P with $P \supseteq I$ and $\operatorname{height} P \leq d-1$. Then $R[It]_\Delta$ is Cohen-Macaulay for large Δ.*

Proof. By [18] one has that $R[It]$ is Cohen-Macaulay, and that the fibre $K[I_d]$ is Cohen-Macaulay with a-invariant -1. Then the claim follows from Corollary 3.13. □

THEOREM 3.16. *Let L be a finitely generated $\mathbb{Z}^2$-graded S-module. Assume that $\dim L_\Delta = \dim L - 1$ for $c \gg 0$ relatively to $e \gg 0$. Then the following conditions are equivalent:*

(i) *L_Δ is a Buchsbaum module with $H^q_{\mathfrak{m}_R}(L_\Delta)_s = 0$ for $s \neq 0$, $0 < q < \dim L - 1$, $c \gg 0$ relatively to $e \gg 0$.*

(ii) *$H^q_{\mathfrak{m}_S}(L)_{(-i,-j)} = 0$ for $i \gg 0$ relatively to $j \gg 0$, $0 < q < \dim L$.*

Proof. By Lemma 3.8, $H^q_{\mathfrak{m}_R}(L_\Delta)_s = (H^{q+1}_{\mathfrak{m}_S}(L)_\Delta)_s$ for $s \neq 0$, $q \geq 0$, $c \gg 0$ relatively to $e \gg 0$. If (i) is satisfied, then $H^{q+1}_{\mathfrak{m}_S}(L)_{(cs,es)} = H^q_{\mathfrak{m}_R}(L_\Delta)_s = 0$ for $s \neq 0$,

$q < \dim L - 1$. Putting $s = -1$ we see that $H^{q+1}_{\mathfrak{m}_S}(L)_{(-i,-j)} = 0$ for $i \gg 0$ relatively to $j \gg 0$. Conversely, assume that (ii) is satisfied. Using the same argument as in the proof of Theorem 3.5 we can show that for c, e large enough, $H^q_{\mathfrak{m}_S}(L)_{(cs,es)} = 0$ for all integers $s \neq 0$, $q \leq \dim L - 1$. Therefore $H^{q-1}_{\mathfrak{m}_R}(L_\Delta)_s = H^q_{\mathfrak{m}_S}(L)_{(cs,es)} = 0$ for all integers $s \neq 0$. By [21], this implies that L_Δ is a Buchsbaum module. □

Conjecture. For $L = A[It]$, 3.16 (ii) is equivalent to the property that $A[It]_\Delta$ is a generalized Cohen-Macaulay module for $c \gg 0$ relatively to $c \gg 0$.

COROLLARY 3.17. *Assume that $A[It]_\mathfrak{m}$ is a generalized Cohen-Macaulay ring, where $\mathfrak{m}$ denotes the maximal graded ideal of $A[It]$. Then $K[(I^e)_c]$ is a Buchsbaum ring for $c \gg 0$ relatively to $c \gg 0$.*

Proof. For $c \gg 0$ relatively to $e \gg 0$, we may assume that $c \geq ed + 1$. Then $K[(I^e)_c] = A[It]_\Delta$ with $\dim A[It]_\Delta = n = \dim A[It] - 1$ by Lemma 1.2 and Lemma 1.3. The assumption means that $H^q_{\mathfrak{m}_R}(A[It])$ is of finite length for $q \neq n$. Hence $H^q_{\mathfrak{m}_R}(A[It])_{(-i,-j)} = 0$ for $i \gg 0$ relatively to $j \gg 0$. The conclusion now follows from Theorem 3.16. □

4. Blow-ups of projective spaces at complete intersections. Let $A = K[X_1, \ldots, X_n]$, $n \geq 2$, and I a complete intersection ideal in A generated by a regular sequence of r forms $f_1, \ldots, f_r$ of degree $d_1, \ldots, d_r$, $r \geq 2$. Put $d := \max\{d_1, \ldots, d_r\}$.

Let X be the blow-up of $\mathbb{P}^{n-1}_K$ along the ideal sheaf $\tilde{I}$. Fix a positive integer e. It is well-known that for $c \geq de + 1$, the forms of degree c of the ideal I^e define an embedding of X in the projective space $\mathbb{P}^{N-1}_K$, $N = \dim_K (I^e)_c$. The aim of this section is to study the Cohen-Macaulay property of the homogeneous coordinate ring $K[(I^e)_c]$ of such an embedding in terms of c and e.

By Lemma 1.2, we may replace $K[(I^e)_c]$ by the diagonal subalgebra $A[It]_\Delta$. Let $S = K[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ be a $\mathbb{N}^2$-graded polynomial ring with $\deg X_i = (1, 0)$, $i = 1, \ldots, n$, and $\deg Y_j = (d_j, 1)$, $j = 1, \ldots, r$. By mapping Y_j to $f_j t$ we obtain a presentation for the Rees algebra of I: $A[It] = S/P$, where P is the ideal generated by the 2-minors of the matrix

$$\begin{pmatrix} f_1 & \cdots & f_r \\ Y_1 & \cdots & Y_r \end{pmatrix}.$$

$A[It]$ is a Cohen-Macaulay ring with $\dim A[It] = n + 1$. Therefore P is a perfect ideal of S with height $P = r - 1$. Hence $A[It]$ has a minimal free resolution over S of length $r - 1$:

$$0 \to D_{r-1} \to \cdots \to D_1 \to D_0 = S \to A[It] \to 0.$$

LEMMA 4.1. *For* $p = 1, \ldots, r-1$,

$$D_p = \bigoplus_{m=1}^{p} \bigoplus_{1 \le j_1 < \cdots < j_{p+1} \le r} S(-(d_{j_1} + \cdots + d_{j_{p+1}}), -m).$$

Proof. It is well-known that the Eagon-Northcott complex gives a minimal free resolution for S/P. Hence we may assume that

$$D_p = \wedge^{r-p+1}(G) \otimes_S S_{p-1}(F),$$

where $F = Sf_1 \oplus Sf_2$ and $G = \oplus_{i=1}^r S_i g_i$ are free S-modules with $\deg f_i = (u, i)$, $i = 1, 2$ and $u = \sum_{j=1}^r d_j$, and $\deg g_i = (u - d_i, 1)$, $i = 1, \ldots, r$. From this it follows that

$$\wedge^{r-p+1}(S^2) \otimes_S S_{p-1}(S^r) = \bigoplus_{m=1}^{p-1} \bigoplus_{1 \le j_1 < \cdots < j_{p+1} \le r} S(-(d_{j_1} + \cdots + d_{j_{p+1}}), -m). \quad \square$$

Lemma 4.1 implies that $A[It]$ has a good minimal free resolution over S in the sense of 3. By Lemma 1.7, $K[(I^e)_c] = A[It]_\Delta$ is a Cohen-Macaulay ring for large c, e. The question here is for which c and e is $K[(I^e)_c]$ a Cohen-Macaulay ring? To solve this question we need to compute the local cohomology modules of the free module $S(a, b)$ for all the shifts

$$(a, b) = (-(d_{j_1} + \cdots + d_{j_{p+1}}), -m),$$

$1 \le j_1 < \cdots < j_{p+1} \le r$ and $1 \le m \le p$, $p = 1, \ldots, r-1$.

LEMMA 4.2. *Let* (a, b) *be the shift of a free summands of* D_p, $p = 1, \ldots, r-1$. *Assume that* $c \ge ed + 1$. *Then*

(i) $H^q_{\mathfrak{m}_R}(S(a, b)_\Delta) = 0$ *for* $q \ne n, n+r-1$.

(ii) $\dim_K H^n_{\mathfrak{m}_R}(S(a, b)_\Delta) = \sum_{s \ge 1} \sum_{\substack{\beta \ge 0 \\ \sum \beta_j = es+b}} \binom{\sum_{j=1}^r d_j \beta_j - cs - a - 1}{n-1}$.

(iii) $S(a, b)_\Delta$ *is a Cohen-Macaulay module if* $c > (e+b)d - a - n$.

Proof. By Lemma 3.1 we have $H^q_{\mathfrak{m}_R}(S(a, b)_\Delta) = 0$ for $q \ne n, r, n+r-1$. Let U and V be defined as in Lemma 3.1. First, we shall see that $V = 0$. By Lemma 3.3 it suffices to show that $-\frac{r+b}{e} < \frac{(b+r)d - a - u}{c - ed}$, where $u = \sum_{j=1}^r d_j$. Let $\{j_{p+2}, \ldots, j_r\}$ be the complement of the set $\{j_1, \ldots, j_{p+1}\}$ in the set of indices $\{1, \ldots, r\}$. Then $u + a = d_{j_{p+2}} + \cdots + d_{j_r}$. Since $-p \le b \le -1$ and $d_j \le d$,

$j = 1, \dots, r$, we have

$$-\frac{r+b}{e} \le \frac{p-r}{e} < 0 < \frac{(r-p)d - d_{j_{p+2}} - \cdots - d_{j_r}}{c-ed} \le \frac{(b+r)d-a-u}{c-ed}.$$

(ii) By Lemma 3.1, $V = 0$ implies $H^q_{\mathfrak{m}_R}(S(a,b)_\Delta) = 0$ for $q \ne n, n+r-1$. and $H^n_{\mathfrak{m}_R}(S(a,b)_\Delta) = W$. Hence $\dim_K H^n_{\mathfrak{m}_R}(S(a,b)_\Delta)$ is the number of solutions of the systems of equations

$$\begin{aligned}\sum_{i=1}^n \alpha_i + \sum_{j=1}^r \beta_j d_j &= cs + a\\ \sum_{j=1}^r \beta_j &= es + b\end{aligned}$$

with $\alpha < 0$, $\beta \ge 0$. Put $\gamma_i = -(\alpha_i + 1)$, $i = 1, \dots, n$. Then $\alpha < 0$ if and only if $\gamma \ge 0$. Rewriting the first equation as $\sum_{i=1}^n \gamma_i = \sum_{j=1}^r d_j\beta_j - cs - a - n$, we see that the number of the solutions $\gamma \ge 0$ is equal $\binom{\sum_{j=1}^r d_j\beta_j - cs - a - 1}{n-1}$. Note that if the second equation has a solution $\beta \ge 0$, $es + b \ge 0$. Hence $s \ge 1$ because $b = -m \le -1$. Now we only need to sum up the above binomial over all $s \ge 1$ and $\beta \ge 0$ with $\sum_{j=1}^r \beta_j = es + b$ to obtain the number of solutions of the above system of equations with $\alpha < 0$, $\beta \ge 0$.

(iii) This follows immediately from Proposition 3.4. □

Corollary 4.3. *Assume that* $c \ge ed + 1$. *Then for* $p = 1, \dots, r-1$, *we have*

(i) $H^q_{\mathfrak{m}_R}((D_p)_\Delta) = 0$ *for* $q \ne n, n+r-1$.

(ii) $(D_p)_\Delta$ *is a Cohen-Macaulay module if* $c > \sum_{j=1}^r d_j + (e-1)d - n$.

Proof. The conclusions follow from Lemma 4.2, where for (ii) we note that for every free summand $S(a,b)$ of D_p, $b \le -1$ and $a \ge -\sum_{j=1}^r d_j$, hence $(e+b)d - a - n \le (e-1)d + \sum_{j=1}^r d_j - n < c$. □

Lemma 4.4. *Let* $c \ge ed + 1$ *and* $u = \sum_{j=1}^r d_j$. *Then*

$$\sum_{p=1}^{r-1} (-1)^{p+r-1} \dim_K H^n_{\mathfrak{m}_R}((D_p)_\Delta) = \sum_{s\ge 1}\sum_{m=1}^{r-1} \sum_{\substack{\beta\ge 0\\ \sum \beta_j = es-m}} \dim_K (A/I)_{(\sum_{j=1}^r d_j\beta_j + u - cs - n)}.$$

Proof. We first note that

$$\binom{\sum_{j=1}^r d_j\beta_j - cs + d_{j_1} + \cdots + d_{j_{p+1}} - 1}{n-1} = \dim_K A_{(\sum_{j=1}^r d_j\beta_j - cs + d_{j_1} + \cdots + d_{j_{p+1}} - n)}.$$

By Lemma 4.1 and Lemma 4.2 (ii) we get

$$\sum_{p=1}^{r-1}(-1)^{p+r-1}\dim_K H^n_{\mathfrak{m}_R}((D_p)_\Delta)$$
$$=\sum_{p=1}^{r-1}(-1)^{p+r-1}\sum_{m=1}^{p}\sum_{1\le j_1<\cdots<j_{p+1}\le r}\sum_{s\ge 1}$$
$$\sum_{\substack{\beta\ge 0\\ \sum\beta_j=es-m}}\dim_K A_{(\sum_{j=1}^r d_j\beta_j-cs+d_{j_1}+\cdots+d_{j_{p+1}}-n)}.$$

Let $d_r=d$. Since $\beta_r=es-m-\sum_{j=1}^{r-1}\beta_j$,

$$\sum_{j=1}^{r} d_j\beta_j-cs+d_{j_1}+\cdots+d_{j_{p+1}}-n$$
$$=\sum_{j=1}^{r-1}d_j\beta_j+d\left(es-m-\sum_{j=1}^{r-1}\beta_j\right)-cs+d_{j_1}+\cdots+d_{j_{p+1}}-n$$
$$=\sum_{j=1}^{r-1}(d_j-d)\beta_j+(de-c)s-dm+d_{j_1}+\cdots+d_{j_{p+1}}-n$$
$$< d_{j_1}+\cdots+d_{j_{p+1}}-dm.$$

For $1\le p\le r-1$ and $p+1\le m\le r-1$, or for $p=-1,0$ and $1\le m\le r$, we have $d_{j_1}+\cdots+d_{j_{p+1}}-dm<0$, hence $\dim_K A_{(\sum_{j=1}^r d_j\beta_j-cs+d_{j_1}+\cdots+d_{j_{p+1}}-n)}=0$. Therefore we may add these values of m and p to the above alternating sum. Changing the order of the summations we get

$$\sum_{p=1}^{r-1}(-1)^{p+r-1}\dim_K H^n_{\mathfrak{m}_R}((D_p)_\Delta)$$
$$=\sum_{s\ge 1}\sum_{m=1}^{r-1}\sum_{\substack{\beta\ge 0\\ \sum\beta_j=es-m}}\Big(\sum_{p=-1}^{r-1}(-1)^{p+r-1}\sum_{1\le j_1<\cdots<j_{p+1}\le r}$$
$$\dim_K A_{(\sum_{j=1}^r d_j\beta_j-cs+d_{j_1}+\cdots+d_{j_{p+1}}-n)}\Big).$$

Let $\{j_{p+2},\ldots,j_r\}$ denote the complement of the set $\{j_1,\ldots,j_{p+1}\}$ in the set of the indices $\{1\dots,r\}$. Then $d_{j_1}+\cdots+d_{j_{p+1}}=u-d_{j_{p+2}}-\cdots-d_{j_r}$. It is easy to see that

$$\begin{aligned}\sum_{p=-1}^{r-1}(-1)^{p+r-1}&\sum_{1\le j_1<\cdots<j_{p+1}\le r}\dim_K A_{(\sum_{j=1}^r d_j\beta_j-cs+d_{j_1}+\cdots+d_{j_{p+1}}-n)}\\&=\sum_{p=-1}^{r-1}(-1)^{p+r-1}\sum_{1\le d_{j_{p+2}}<\cdots<d_{j_r}\le r}\dim_K A_{(\sum_{j=1}^r d_j\beta_j-cs+u-d_{j_{p+2}}-\cdots-d_{j_r}-n)}\\&=\dim_K(A/I)_{(\sum_{j=1}^r d_j\beta_j-cs+u-n)}.\end{aligned}$$ □

Using the commuting property of Δ on local cohomology modules we obtain the following general information on the vanishing of the local cohomology modules of $A[It]_\Delta$.

PROPOSITION 4.5. *Assume that* $c\ge ed+1$. *Then*

(i) $H^q_{\mathfrak{m}_R}(A[It]_\Delta)=0$ *for* $q\le n-r$.

(ii) *For* $n-r<q<n$, $H^q_{\mathfrak{m}_R}(A[It]_\Delta)=0$ *if and only if the sequence*

$$H^n_{\mathfrak{m}_R}((D_{n-q+1})_\Delta)\to H^n_{\mathfrak{m}_R}((D_{n-q})_\Delta)\to H^n_{\mathfrak{m}_R}((D_{n-q-1})_\Delta)$$

is exact.

(iii) $\omega_{A[It]_\Delta}\simeq(\omega_{A[It]})_\Delta$.

Proof. Let $C_p:=\operatorname{Coker}(D_{p+1}\to D_p)$, $p=0,\dots,r-1$. Then $C_0=A[It]$ and there are the short exact sequences

$$0\to C_{p+1}\to D_p\to C_p\to 0,$$

$p=0,\dots,r-2$. By Lemma 4.3 (i), $H^q_{\mathfrak{m}_R}((D_p)_\Delta)=0$ for $q\ne n,n+r-1$. Using the short exact sequences

$$0\to(C_{p+1})_\Delta\to(D_p)_\Delta\to(C_p)_\Delta\to 0,$$

we get $H^q_{\mathfrak{m}_R}((C_p)_\Delta)\simeq H^{q+1}_{\mathfrak{m}_R}((C_{p+1})_\Delta)$ for $q<n-1$. Since $C_0=A[It]$, $C_{r-1}=D_{r-1}$, this implies $H^q_{\mathfrak{m}_R}(A[It]_\Delta)=H^{q+r-1}_{\mathfrak{m}_R}((C_{r-1})_\Delta)$ for $q\le n-r$. Since $C_{r-1}=D_{r-1}$, $H^{q+r-1}_{\mathfrak{m}_R}((C_{r-1})_\Delta)=0$ for $q\le n-r$, hence (i).

On the other hand, from the first exact sequences we get $H^q_{\mathfrak{m}_S}(C_p)\simeq H^{q+1}_{\mathfrak{m}_S}(C_{p+1})$ for $q<n+r-1$. Note that $H^q_{\mathfrak{m}_S}(C_0)=0$ for $q\le n$ because $A[It]$ is a Cohen-Macaulay ring with $\dim A[It]=n+1$. Then we can successively deduce that $H^q_{\mathfrak{m}_S}(C_{p+1})=0$ for $q\le n+p+1$. Hence $H^{r+2}_{\mathfrak{m}_S}(C_{p+1})=0$ for $p=1,\dots,r-2$. Applying Theorem 3.6 we obtain $H^{n+1}_{\mathfrak{m}_R}((C_{p+1})_\Delta)=H^{n+2}_{\mathfrak{m}_S}(C_{p+1})_\Delta=0$, hence the induced map $H^n_{\mathfrak{m}_R}((D_p)_\Delta)\to H^n_{\mathfrak{m}_R}((C_p)_\Delta)$ is surjective for $p=1,\dots,r-2$.

Now consider the commutative diagram

$$\begin{array}{ccccccc} H^n_{\mathfrak{m}_R}((D_{n-q+1})_\Delta) & & \longrightarrow & & H^n_{\mathfrak{m}_R}((D_{n-q})_\Delta) & & \longrightarrow & & H^n_{\mathfrak{m}_R}((D_{n-q-1})_\Delta) \\ & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & H^n_{\mathfrak{m}_R}((C_{n-q+1})_\Delta) & & & & H^n_{\mathfrak{m}_R}((C_{n-q})_\Delta) & & \end{array}$$

for $n-r<q<n$. Since the maps $\searrow$ are surjective, by chasing the trace of an element in the kernel of the map $H^n_{\mathfrak{m}_R}((D_{n-q})_\Delta) \to H^n_{\mathfrak{m}_R}((D_{n-q-1})_\Delta)$ we can easily see that the top sequence is exact if and only if the map $H^n_{\mathfrak{m}_R}((C_{n-q})_\Delta) \to H^n_{\mathfrak{m}_R}((D_{n-q-1})_\Delta)$ is injective or, equivalently, $H^{n-1}_{\mathfrak{m}_R}((C_{n-q-1})_\Delta) = 0$. Since we have that $H^q_{\mathfrak{m}_R}(A[It]_\Delta) = H^{n-1}_{\mathfrak{m}_R}((C_{n-q-1})_\Delta)$, this proves (ii).

For (iii) we first note that $H^q_{\mathfrak{m}_S}(S) = 0$ for $q \le n+1$ because S is a Cohen-Macaulay ring with $\dim S = n+r > n+1$. Then the exact sequence $0 \to C_1 \to S \to A[It] \to 0$ implies

$$H^{n+1}_{\mathfrak{m}_S}(A[It]) \simeq H^{n+2}_{\mathfrak{m}_S}(C_1).$$

Similarly, since S_Δ is a Cohen-Macaulay ring with $\dim S_\Delta = n+r-1$ by Lemma 1.1, we have

$$H^n_{\mathfrak{m}_R}(A[It])_\Delta) \simeq H^{n+1}_{\mathfrak{m}_R}((C_1)_\Delta).$$

Applying Proposition 1.8 to C_1 we get $H^{n+1}_{\mathfrak{m}_R}((C_1)_\Delta) \simeq H^{n+2}_{\mathfrak{m}_S}(C_1)_\Delta$. Therefore we have $H^n_{\mathfrak{m}_R}(A[It]_\Delta) \simeq H^{n+1}_{\mathfrak{m}_S}(A[It])_\Delta$. From this it follows that

$$\begin{aligned} \omega_{A[It]_\Delta} &= \operatorname{Hom}_K(K, H^n_{\mathfrak{m}_R}(A[It]_\Delta)) \\ &\simeq \operatorname{Hom}_K(K, H^{n+1}_{\mathfrak{m}_S}(A[It])_\Delta) \\ &\simeq \operatorname{Hom}_K(K, H^{n+1}_{\mathfrak{m}_S}(A[It]))_\Delta \simeq (\omega_{A[It]})_\Delta. \qquad \square \end{aligned}$$

Now we are able to determine exactly for which c, e the algebra $K[(I^e)_c]$ is a Cohen-Macaulay ring.

THEOREM 4.6. *Let $I \subset K[X_1, \ldots, X_n]$ be a homogeneous complete intersection ideal minimally generated by r forms of degree $d_1, \ldots, d_r$. Assume that $c \ge ed+1$, $d = \max\{d_j \mid j = 1, \ldots, r\}$. Then $K[(I^e)_c]$ is a Cohen-Macaulay ring if and only if $c > \sum_{j=1}^r d_j + (e-1)d - n$.*

Proof. By 1.2 and 1.3 (ii) we have $K[(I^e)_c] = A[It]_\Delta$ and $\dim A[It]_\Delta = n$. Put $u = \sum_{j=1}^r d_j$. Assume that $c > u + (e-1)d - n$. Then $(D_p)_\Delta$ is a Cohen-Macaulay module with $\dim (D_p)_\Delta = n+r-1$ by Corollary 3.2 and Corollary 4.3 (ii) for

$p = 0, \ldots, r-1$. Therefore from the resolution

$$0 \to (D_{r-1})_\Delta \to \cdots \to (D_1)_\Delta \to (D_0)_\Delta \to A[It]_\Delta \to 0$$

we can deduce that $A[It]_\Delta$ is a Cohen-Macaulay ring.

Conversely, assume that $A[It]_\Delta$ is a Cohen-Macaulay ring. Then $H^q_{\mathfrak{m}_R}(A[It]_\Delta) = 0$ for all $q \neq n$. By virtue of Lemma 4.5 this condition is satisfied only if the sequence

$$0 \to H^n_{\mathfrak{m}_R}((D_{r-1})_\Delta) \to \cdots \to H^n_{\mathfrak{m}_R}((D_1)_\Delta) \to 0 = H^n_{\mathfrak{m}_R}((D_0)_\Delta)$$

is exact. As a consequence we get

$$\sum_{p=1}^{r-1} (-1)^{p+r-1} \dim_K H^n_{\mathfrak{m}_R}((D_p)_\Delta) = 0.$$

By Lemma 4.4 this implies $\dim_K (A/I)_{((e-1)d+u-c-n)} = 0$ because for $s = m = 1$, $d = d_r$, and $\beta_1 = \cdots = \beta_{r-1} = 0$, $\beta_r = e-1$, we have $\sum_{j=1}^r d_j\beta_j + u - cs - n = (e-1)d + u - c - n$. If $\dim A/I > 0$, then $\dim_K (A/I)_{((e-1)d+u-c-n)} = 0$ only if $(e-1)d + u - c - n < 0$. If $\dim A/I = 0$, then $r = n$. In this case, $A_\ell \neq 0$ if $0 \leq \ell \leq u - n$ (the degree of the socle of the complete intersection ideal I). Since $(e-1)d - c < 0$, we have $(e-1)d + u - c - n < u - n$. Hence $\dim_K (A/I)_{((e-1)d+u-c-n)} = 0$ only if $(e-1)d + u - cs - n < 0$. In both cases, we get $c > u + (e-1)d - n$. The proof is now complete. □

Remark. The case $e = 1$ and $d_1 = \cdots = d_r = d$ was already handled in [19], where one could only show that $K[I_{d+1}]$ is a Cohen-Macaulay ring if $(r-1)d < n$ and that it fails to do so if $(r-1)d > n$. It was conjectured there that $K[I_{d+1}]$ is Cohen-Macaulay if and only if $(r-1)d \leq n$. But this follows from Theorem 4.6.

COROLLARY 4.7. *Let $I \subset A = K[X_1, \ldots, X_n]$ be a homogeneous complete intersection ideal minimally generated by two forms f_1, f_2 of degree $d_1 \leq d_2$. If $n \geq d_2+1$ then $K[I_n]$ is a Gorenstein ring with a-invariant -1.*

Proof. For $c = n, e = 1$, it is easy to check that $c \geq ed_2+1$ and $c > d_1+ed_2-n$. By virtue of Theorem 4.6 and Proposition 4.5 (iii), $K[I_n]$ is a Cohen-Macaulay ring with $\omega_{K[I_n]} \simeq (\omega_{A[It]})_\Delta$.

Since $A[It] \simeq A[Y_1, Y_2]/(f_1Y_2 - f_2Y_1)$, $\omega_{A[Y_1,Y_2]} \simeq A[Y_1, Y_2](-n-d_1-d_2, -2)$ and the degree of the hypersurface $f_1Y_2 - f_2Y_1$ is $(d_1 + d_2, 1)$, it follows that

$$\omega_{A[It]} \simeq A[It](-n, -1).$$

This implies $(\omega_{A[It]})_\Delta \simeq (A[It]_\Delta)(-1) = K[I_n](-1)$. Hence $K[I_n]$ is a Gorenstein ring with a-invariant -1. □

5. Diagonal subalgebras of a bigraded polynomial ring In this section, motivated by our studies in the previous sections, we study the diagonal subalgebras of the polynomial ring

$$S = K[X, Y] = K[X_1, \dots, X_n, Y_1, \dots, Y_r]$$

with bigraded structure induced by the assignment

$$\deg X_i = (1, 0),\ i = 1, \dots, n, \quad \text{and} \ \deg Y_j = (d_j, 1),\ j = 1, \dots, r,$$

where $d_1, \dots, d_r$ are given nonnegative integers.

As before we let Δ be the (c, e)-diagonal of $\mathbb{Z}^2$. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $\beta = (\beta_1, \dots, \beta_r) \in \mathbb{N}^r$, we denote as before by X^α and Y^β the monomials $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and $Y_1^{\beta_1} \cdots Y^{\beta_r}$. Further we set $|\alpha| = \sum_{i=1}^n \alpha_i$ and $|\beta| = \sum_{i=1}^r \beta_i$. The degree of the monomial $X^\alpha Y^\beta$ in S is

$$(|\alpha| + \beta \cdot d, |\beta|)$$

where $\beta \cdot d$ denotes the scalar product of the vectors β and $d = (d_1, \dots, d_r)$. Hence $X^\alpha Y^\beta$ belongs to S_Δ if and only if there exists an integer s such that

$$|\alpha| + \beta \cdot d = sc \quad \text{and} \quad |\beta| = se.$$

It is easy to see that S_Δ is a standard K-algebra (i.e., it is generated as a K-algebra by its degree one component) provided

$$c \geq e \max\{d_1, \dots, d_r\}.$$

From now on we assume that this condition holds. Then the generators of S_Δ are the monomials $X^\alpha Y^\beta$ with $|\alpha| = c - \beta \cdot d$ and $|\beta| = e$. We set

$$F = \{(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^r \colon\ |\alpha| = c - \beta \cdot d \text{ and } |\beta| = e\},$$

and consider the presentation

$$\Phi\colon\ K[T_{(\alpha,\beta)}\colon\ (\alpha, \beta) \in F] \to S_\Delta$$

of S_Δ defined by setting $\Phi(T_{(\alpha,\beta)}) = X^\alpha Y^\beta$ for all $(\alpha, \beta) \in F$, where $T = \{T_{(\alpha,\beta)}\colon\ (\alpha, \beta) \in F\}$ is a set of indeterminates. Our goal is to prove the following

THEOREM 5.1. *The kernel of Φ has a Gröbner basis of quadrics.*

By virtue of [6, Theorem 2.2] follows

COROLLARY 5.2. *The algebra S_Δ is Koszul.*

Note that if $d_1 = d_2 = \cdots = d_r$, then S_Δ is the Segre product of Veronese rings $K[X]^{(c-d_1e)}$ and $K[Y]^{(e)}$, and in this case Theorem 5.1 was proved by Eisenbud, Reeves and Totaro [9, Proposition 17]. In order to prove Theorem 5.1 in general we use a slight modification of their argument.

Proof. [Proof of 5.1] We introduce a transitive relation $\prec$ on the nonzero vectors of $\mathbb{N}^m$. Let $a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_m) \in \mathbb{N}^m$, $a, b \neq 0$. We set

$$a \prec b \quad \text{if} \quad \max\{i\colon\ a_i \neq 0\} \leq \min\{i\colon\ b_i \neq 0\}.$$

Further denote by $\leq$ the partial order on $\mathbb{N}^m$ defined coefficientwise and by $\leq_{lex}$ the lexicographic order. The relation $\prec$ extends to $\mathbb{N}^n \times \mathbb{N}^r$ by setting $(\alpha_1, \beta_1) \prec (\alpha_2, \beta_2)$ if $\alpha_1 \prec \alpha_2$ and $\beta_1 \prec \beta_2$.

First note that for any monomial X^aY^b in $S_{(sc,se)}$, there exists a unique representation $X^aY^b = X^{\gamma_1}Y^{\delta_1} \cdots X^{\gamma_s}Y^{\delta_s}$ such that $(\gamma_i, \delta_i) \in F$ and $(\gamma_s, \delta_s) \prec \cdots \prec (\gamma_1, \delta_1)$. The representation exists because one can define δ_i and γ_i recursively by setting

$$\delta_i = \min_{\leq_{lex}} \left\{ \delta \in \mathbb{N}^r\colon\ |\delta| = e, \delta \leq b - \sum_{j=1}^{i-1} \delta_j \right\}$$

and

$$\gamma_i = \min_{\leq_{lex}} \left\{ \gamma \in \mathbb{N}^n\colon\ (\gamma, \delta_i) \in F, \gamma \leq a - \sum_{j=1}^{i-1} \gamma_j \right\}.$$

The representation is unique because the above recursive equations must be satisfied by all the $(\gamma_1, \delta_1), \ldots, (\gamma_s, \delta_s)$ with the desired properties. We call this representation the standard representation of X^aY^b.

For all the pairs $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ of elements of F such that $(\alpha_1, \beta_1) \not\prec (\alpha_2, \beta_2) \not\prec (\alpha_1, \beta_1)$, take the standard representation $X^{\gamma_1}Y^{\delta_1}X^{\gamma_2}Y^{\delta_2}$ of $X^{\alpha_1}Y^{\beta_1}X^{\alpha_2}Y^{\beta_2}$. By construction we obtain an element

$$T_{(\alpha_1,\beta_1)}T_{(\alpha_2,\beta_2)} - T_{(\gamma_1,\delta_1)}T_{(\gamma_2,\delta_2)}$$

of $\operatorname{Ker}\Phi$ that we call "straightening law."

For example let $n = r = 3$, $d_1 = 1, d_2 = d_3 = 2$, $e = 2, c = 5$. Then

$$X_2X_3Y_1Y_3, X_1X_3Y_1Y_2 \in S_{(c,e)}$$

and the standard representation of the product is $(X_1X_2X_3Y_1^2)(X_3Y_2Y_3)$. The associated straightening law is

$$T_{((0,1,1),(1,0,1))}T_{((1,0,1),(1,1,0))} - T_{((1,1,1),(2,0,0))}T_{((0,0,1),(0,1,1))}.$$

We claim that the straightening laws form a Gröbner basis of $\operatorname{Ker}\Phi$ with respect to any term order τ on $K[T]$ such that

$$\operatorname{in}_\tau\,(T_{(\alpha_1,\beta_1)}T_{(\alpha_2,\beta_2)} - T_{(\gamma_1,\delta_1)}T_{(\gamma_2,\delta_2)}) = T_{(\alpha_1,\beta_1)}T_{(\alpha_2,\beta_2)}.$$

We first prove the claim and then we show that there exists a term order τ with the above property. Consider the ideal J of $K[T]$ generated by all the monomials $T_{(\alpha_1,\beta_1)}T_{(\alpha_2,\beta_2)}$ such that $(\alpha_1,\beta_1) \not\leq (\alpha_2,\beta_2) \not\leq (\alpha_1,\beta_1)$. Since $J \subseteq \operatorname{in}_\tau(\operatorname{Ker}\Phi)$, to prove the claim it suffices to show that the monomials not in J are linearly independent in $K[T]/\operatorname{Ker}\Phi = S_\Delta$. But this is true because the standard representation is unique, and because a product $X^{\gamma_1}Y^{\delta_1}\cdots X^{\gamma_s}Y^{\delta_s}$ is standard if and only if all the pairs $X^{\gamma_i}Y^{\delta_i}X^{\gamma_j}Y^{\delta_j}$ with $i \neq j$ are standard. It remains to prove that there exists a term order τ as above. To this end consider the total order on the set T defined by $T_{(\alpha_1,\beta_1)} < T_{(\alpha_2,\beta_2)}$ if $\beta_1 <_{lex} \beta_2$ or $\beta_1 = \beta_2$ and $\alpha_1 <_{lex} \alpha_2$. Then let τ be the reverse lexicographic order on the monomials of $K[T]$ induced by the given total order. By the property of the standard representation it follows that $T_{(\gamma_1,\delta_1)} < T_{(\alpha_1,\beta_1)}, T_{(\alpha_2,\beta_2)}$, and hence τ has the desired property. ◻

6. Asymptotic Koszul property of diagonal subalgebras. Let R be a bigraded standard K-algebra. In this section we show that the (c,e)-diagonal algebra $\bigoplus_{s\in\mathbb{N}} R_{(sc,se)}$ of R is Koszul provided c and e are large enough. This result will be applied to study the Koszulness of algebras of type $K[(I^e)_c]$.

A bigraded K-algebra $R = \bigoplus_{(i,j)\in\mathbb{N}^2} R_{(i,j)}$ is *standard* if $R_{(0,0)} = K$ and if it is generated as K-algebra by $R_{(1,0)}$ and $R_{(0,1)}$. Let $m = \dim R_{(1,0)}$ and $n = \dim R_{(0,1)}$, and let $X = X_1,\ldots,X_m$, $Y = Y_1,\ldots,Y_n$ be two sets of indeterminates over K. Let $S = K[X,Y]$ be bigraded by setting $\deg X_i = (1,0)$, $\deg Y_i = (0,1)$. Then R is isomorphic to a factor ring S/J of S by a bihomogeneous ideal J. Let $f_1,\ldots,f_r$ be a minimal set of bihomogeneous generators of J, and let $\deg f_j = (a_j,b_j)$. Let c,e be positive integers. Denote by R_Δ the (c,e)-diagonal algebra $\bigoplus_{s\in\mathbb{N}} R_{(sc,se)}$ of R.

The presentation of R as S-module

$$\oplus_{j=1}^r S(-a_j,-b_j) \to S \to R \to 0$$

induces a presentation of R_Δ as S_Δ module

$$\oplus_{j=1}^r S(-a_j,-b_j)_\Delta \to S_\Delta \to R_\Delta \to 0.$$

The K-algebra S_Δ is nothing but the ordinary Segre product $K[X]^{(c)}\underline{\otimes}K[Y]^{(e)}$ of the cth Veronese subring of $K[X]$ and the eth Veronese subring of $K[Y]$. Denote by F the set $\{(\alpha,\beta) \in \mathbb{N}^m \times \mathbb{N}^n\colon\ |\alpha| = c, |\beta| = e\}$. We may present S_Δ and R_Δ as factor rings of the polynomial ring:

$$K[T] = K[T_{(\alpha,\beta)}\colon\ (\alpha,\beta) \in F] \to S_\Delta \to R_\Delta$$

by sending $T_{(\alpha,\beta)}$ to $X^\alpha Y^\beta$. The kernel of $K[T] \to S_\Delta$ is generated by quadrics (Theorem 5.1). It is easy to see that the S_Δ-module $S(-a,-b)_\Delta$ is generated by elements of degree $\max\{\lceil a/c\rceil, \lceil b/e\rceil\}$. Here $\lceil x\rceil$ denotes $\min\{n \in \mathbb{Z}\colon\ n \geq x\}$. From the above presentation it follows that the kernel of the map $S_\Delta \to R_\Delta$ is generated by elements of degree less than or equal to $\max\{\lceil a_j/c\rceil, \lceil b_j/e\rceil\colon\ j = 1,\ldots,r\}$. So we have shown that:

PROPOSITION 6.1. *The ideal of definition I of R_Δ as a quotient of the polynomial ring $K[T]$ is generated by polynomials of degree less than or equal to*

$$\max\{2, \max\{\lceil a_j/c\rceil, \lceil b_j/e\rceil\colon j = 1,\ldots,r\}\}.$$

In particular if $c \geq \max\{a_j\colon\ j = 1,\ldots,r\}/2$ and $e \geq \max\{b_j\colon\ j = 1,\ldots,r\}/2$, then I is generated by forms of degree less than or equal to 2.

Furthermore if $c \geq \max\{a_j\colon j = 1,\ldots,r\}$ and $e \geq \max\{b_j\colon j = 1,\ldots,r\}$, then the kernel of $S_\Delta \to R_\Delta$ is generated by linear forms.

We want to investigate the Koszul property of R_Δ. To this end it does not suffice to consider the first syzygy module of R over S. One has to consider the minimal bigraded free resolution

$$0 \to D_p \to D_{p-1} \to \cdots \to D_1 \to S \to R \to 0$$

of R as an S-module. The free S-modules D_i are direct sums of bishifted copies of S, say

$$D_i = \bigoplus_{(a,b)\in\mathbb{N}^2} S(-a,-b)^{\beta_{i,a,b}}.$$

The main goal of this section is to show the following

THEOREM 6.2. *Let c, e be positive integers such that*

$$\max\{a/c, b/e\colon\ \beta_{i,a,b} \neq 0\} \leq i+1$$

for all $i = 1,\ldots,p$. Then the (c,e)-diagonal algebra $R_\Delta = \bigoplus_{s\in\mathbb{N}} R_{(sc,se)}$ of R is Koszul.

Let us first introduce a piece of notation and prove some preliminary facts. Let A be a positively graded K-algebra. Denote by $\mathfrak{m}$ its maximal homogeneous ideal. For a finitely generated graded A-module M denote by M_i its homogeneous component of degree i, and set

$$t_i(M) = \sup\{j\colon\ \operatorname{Tor}_i^A(M,K)_j \neq 0\}$$

with $t_i(M) = -\infty$ if $\operatorname{Tor}_i^A(M,K) = 0$. The Castelnuovo-Mumford regularity $\operatorname{reg}_A M$ of an A-module M is defined to be

$$\operatorname{reg}_A M = \sup\{t_i(M) - i\colon\ i \geq 0\}.$$

The initial degree $\operatorname{indeg}(M)$ of M is the minimum of the i such that $M_i \neq 0$. The module M is said to have a linear A-resolution if

$$\operatorname{reg}_A M = \operatorname{indeg}(M).$$

Note that a module M with linear A-resolution is generated by elements of degree $\operatorname{indeg}(M)$. It is clear that a shifted copy $M(a)$ of a module M has a linear A-resolution if and only if M has a linear A-resolution. The K-algebra A is said to be a Koszul algebra if K has a linear A-resolution. This is equivalent to say that $\mathfrak{m}$ has a linear A-resolution. The bigraded Poincaré series $P_M^A(s,t)$ of M is by definition

$$P_M^A(s,t) = \sum_{i,j} \dim_K \operatorname{Tor}_i^A(M,K)_j s^j t^i.$$

LEMMA 6.3. *Let*

$$\cdots \to M_r \to M_{r-1} \to \cdots \to M_1 \to M_0 \to N \to 0$$

be an exact complex of finitely generated graded A-modules. Then:

(i) *Let* $h \in \mathbb{N}$, *and let* $a \in \mathbb{Z}$ *such that* $t_s(M_r) \leq a + r + s$ *for all* $0 \leq r \leq h$ *and* $0 \leq s \leq h - r$. *Then* $t_h(N) \leq a + h$.

(ii) $\operatorname{reg}_A N \leq \sup\{\operatorname{reg}_A M_r - r\colon\ r \in \mathbb{N}\}$.

Proof. (i) By induction on h. For $h = 0$, one has a surjection $\operatorname{Tor}_0^A(M_0,K)_j \to \operatorname{Tor}_0^A(N,K)_j$ and hence $t_0(N) \leq t_0(M_0) \leq a$. Now let $h > 0$. Let N_1 be the kernel of the map $M_0 \to N$. One has an exact complex

$$\cdots \to M_r \to M_{r-1} \to \cdots \to M_1 \to N_1 \to 0$$

and hence by induction $t_{h-1}(N_1) \leq a + h$. By tensoring the short exact sequence

$$0 \to N_1 \to M_0 \to N \to 0$$

with $\otimes_A K$ we have an exact sequence

$$\operatorname{Tor}_h^A(M_0, K)_j \to \operatorname{Tor}_h^A(N, K)_j \to \operatorname{Tor}_{h-1}^A(N_1, K)_j.$$

We know that $t_{h-1}(N_1) \leq a + h$ and by assumption one has $t_h(M_0) \leq a + h$. It follows that $\operatorname{Tor}_h^A(N, K)_j = 0$ for $j > a + h$, and hence $t_h(N) \leq a + h$.

(ii) If $\sup\{\operatorname{reg}_A M_r - r\colon\ r \in \mathbb{N}\} < \infty$, then set $a = \sup\{\operatorname{reg}_A M_r - r\colon\ r \in \mathbb{N}\}$. For all $s, r \in \mathbb{N}$ one has $t_s(M_r) \leq \operatorname{reg}_A M_r + s \leq a + r + s$. Then by (i) one has $t_h(N) \leq a + h$ for all $h \in \mathbb{N}$. It follows that $\operatorname{reg}_A N \leq a$. □

LEMMA 6.4. *Let A be a Koszul algebra and let M be a graded A-module with a linear A-resolution. Then $\mathfrak{m}^n M$ has a linear A-resolution for all $n \in \mathbb{N}$.*

Proof. Since $\mathfrak{m}(\mathfrak{m}^{n-1}M) = \mathfrak{m}^n M$, it suffices to prove the claim for $n = 1$. Let a be the initial degree of M and μ its minimal number of generators. Tensoring the short exact sequence

$$0 \to \mathfrak{m}M \to M \to M/\mathfrak{m}M \simeq K(-a)^{\mu} \to 0$$

with $\otimes_A K$ one has an exact sequence

$$\operatorname{Tor}_{i+1}^A(K^{\mu}, K)_{j-a} \to \operatorname{Tor}_i^A(\mathfrak{m}M, K)_j \to \operatorname{Tor}_i^A(M, K)_j.$$

For $j > \operatorname{indeg}(\mathfrak{m}M) + i = a + 1 + i$ one has $\operatorname{Tor}_{i+1}^A(K^{\mu}, K)_{j-a} = 0$ because A is Koszul, and $\operatorname{Tor}_i^A(M, K)_j = 0$ because M has a linear A-resolution, by assumption. It follows that $\mathfrak{m}M$ has a linear A-resolution. □

Let A and B be positively graded K-algebras. Denote by $A \underline{\otimes} B$ the Segre product

$$A \underline{\otimes} B = \bigoplus_{i \in \mathbb{N}} A_i \otimes_K B_i$$

of A and B. Given graded modules M and N over A and B, one may form the Segre product

$$M \underline{\otimes} N = \bigoplus_{i \in \mathbb{Z}} M_i \otimes_K N_i$$

of M and N. Clearly $M \underline{\otimes} N$ is a graded $A \underline{\otimes} B$-module. It is easy to see that for a given graded A-module M the functor $M \underline{\otimes} -$ from the category of graded B-modules with degree zero maps to the category of graded $A \underline{\otimes} B$-modules with degree zero maps is exact.

LEMMA 6.5. *Let A and B be Koszul K-algebras. Let M be a finitely generated graded A-module, and let N be a finitely generated graded B-module. Assume that*

M and N have linear resolutions over A and B, respectively. Then $M\underline{\otimes}N$ has a linear $A\underline{\otimes}B$-resolution and $\operatorname{reg}_{A\underline{\otimes}B} M\underline{\otimes}N = \max\{\operatorname{reg}_A M, \operatorname{reg}_B N\}$.

Proof. Denote by $\mathfrak{m}_A$ and $\mathfrak{m}_B$ the maximal homogeneous ideals of A and B. Let a and b respectively be the initial degrees of M and N. If $a < b$, then $M\underline{\otimes}N = \mathfrak{m}_A^{b-a}M\underline{\otimes}N$, while if $a > b$ then $M\underline{\otimes}N = M\underline{\otimes}\mathfrak{m}_B^{a-b}N$. Hence by virtue of Lemma 6.4, we may assume $a = b$, and by shifting the degrees, we may also assume that $a = 0$. Consider the minimal free resolution of M

$$\cdots \to F_r \to F_{r-1} \to \cdots \to F_0 \to M \to 0.$$

By assumption F_r is a direct sum of copies of $A(-r)$ for all r. Applying $-\underline{\otimes}N$ to this complex we obtain an exact complex:

$$\cdots \to F_r\underline{\otimes}N \to F_{r-1}\underline{\otimes}N \to \cdots \to F_0\underline{\otimes}N \to M\underline{\otimes}N \to 0.$$

By virtue of Lemma 6.3 it suffices to show that $A(-r)\underline{\otimes}N$ has a linear $A\underline{\otimes}B$-resolution whenever N is a B-module generated in degree 0 and with linear B-resolution. Now taking the minimal free B-resolution of N and applying $A(-r)\underline{\otimes}-$, one sees that it suffices to show that $A(-r)\underline{\otimes}B(-s)$ has a linear resolution over $A\underline{\otimes}B$ for all $r, s \in \mathbb{N}$. Note that $A(-r)\underline{\otimes}B(-s)$ is isomorphic to a shifted copy of $A\underline{\otimes}(\mathfrak{m}_B^{r-s}(r-s))$ or to a shifted copy of $\mathfrak{m}_A^{s-r}(s-r)\underline{\otimes}B$ according to whether $r \geq s$ or $s \geq r$. Hence it is enough to show that $A\underline{\otimes}(\mathfrak{m}_B^k(k))$ and $\mathfrak{m}_A^k(k)\underline{\otimes}B$ have a linear $A\underline{\otimes}B$-resolution for all $k \in \mathbb{N}$. The last statement is equivalent to saying that $t_h(\mathfrak{m}_A^k(k)\underline{\otimes}B) \leq h$ and $t_h(A\underline{\otimes}(\mathfrak{m}_B^k(k)) \leq h$ for all $h, k \in \mathbb{N}$. We argue by induction on h. If $h = 0$, then the claim is trivial. Let $h > 0$. Since A is Koszul, $\mathfrak{m}_A$ has a linear A-resolution and hence, by virtue of Lemma 6.4, $\mathfrak{m}_A^k(k)$ has a linear A-resolution. Applying $-\underline{\otimes}B$ to the minimal free A-resolution of $\mathfrak{m}_A^k(k)$ we have an exact complex

$$\cdots \to G_r \to G_{r-1} \to \cdots \to G_0 \to \mathfrak{m}_A^k(k)\underline{\otimes}B \to 0$$

where each G_r is a direct sum of copies of $A(-r)\underline{\otimes}B$. Now we want to apply Lemma 6.3(i) to this exact complex, with $a = 0$. We need to show that $t_s(A(-r)\underline{\otimes}B) \leq r + s$ for all $r = 0, \ldots, h$ and $s = 0, \ldots, h - r$. Since $A(-r)\underline{\otimes}B = (A\underline{\otimes}\mathfrak{m}_B^r(r))(-r)$, one has

$$t_s(A(-r)\underline{\otimes}B) = t_s(A\underline{\otimes}\mathfrak{m}_B^r(r)) + r$$

and by induction $t_s(A\underline{\otimes}\mathfrak{m}_B^r(r)) \leq s$ for all $s < h$. If $s = h$, then $r = 0$ and $t_h(A(-r)\underline{\otimes}B) = t_h(A\underline{\otimes}B) = -\infty$. By virtue of Lemma 6.3 we may now conclude that $t_h(\mathfrak{m}_A^k(k)\underline{\otimes}B) \leq h$. By symmetry one has also $t_h(A\underline{\otimes}(\mathfrak{m}_B^k(k)) \leq h$. ∎

LEMMA 6.6. *Let $S \to R$ be a surjective homomorphism of graded K-algebras. If S is Koszul and* $\operatorname{reg}_S R \leq 1$, *then R is Koszul. Furthermore if* $\operatorname{reg}_S R = 0$, *then* $P^R_K(s,t) \leq P^S_K(s,t)$ *coefficientwise.*

Proof. The standard change of rings spectral sequence

$$\operatorname{Ext}^p_R(M, \operatorname{Ext}^q_S(R,K)) \Rightarrow \operatorname{Ext}^{p+q}_S(M,K)$$

respects the graded structure of the Ext-groups and yields the coefficientwise inequality of formal power series

$$P^R_K(s,t) \leq P^S_K(s,t)(1+t-tP^S_R(s,t))^{-1}.$$

By assumption the term $s^j t^i$ does not appear in the series $P^S_R(s,t)$ for all $j > i+1$. Hence the term $s^j t^i$ does not appear in the series $t - tP^S_R(s,t)$ for all $j > i$. Now $(1+t-tP^S_R(s,t))^{-1} = \sum_{k\in\mathbb{N}}(tP^S_R(s,t)-t)^k$, and hence the term $s^j t^i$ does not appear in the series $(1+t-tP^S_R(s,t))^{-1}$ for all $j > i$. By assumption the term $s^j t^i$ does not appear in the series $P^S_K(s,t)$ for all $j > i$. By virtue of the above inequality one concludes that the term $s^j t^i$ does not appear in the series $P^R_K(s,t)$ for all $j > i$, that is, R is Koszul.

If $\operatorname{reg}_S R$ happens to be 0, then repeating the previous argument one shows that the term $s^j t^i$ does not appear in the series $(1+t-tP^S_R(s,t))^{-1}$ for all $j > i-1 \geq 0$. It follows that $P^R_K(s,t) \leq P^S_K(s,t)$ coefficientwise. □

Remark. The assumption of Lemma 6.6 does not imply that the map $S \to R$ is Golod. This is because the kernel I of $S \to R$ is allowed to contain linear forms.

We are now ready for the proof of Theorem 6.2:

Proof. Let $c, e \in \mathbb{N} - \{0\}$ and denote by Δ the diagonal $\{(sc, se) \in \mathbb{N}^2\colon\ s \in \mathbb{N}\}$. From the free resolution of R over S

$$0 \to F_p \to F_{p-1} \to \cdots \to F_1 \to S \to R \to 0$$

one obtains an the exact complex

$$0 \to (F_p)_\Delta \to (F_{p-1})_\Delta \to \cdots \to (F_1)_\Delta \to S_\Delta \to R_\Delta \to 0$$

of S_Δ-modules. One has $S_\Delta = A\underline{\otimes}B$, where A denotes the cth Veronese subring of $K[X]$, and B denotes the eth Veronese subring of $K[Y]$. The ring $A\underline{\otimes}B$ is known to be Koszul [3, Theorem 4], and by virtue of Lemma 6.3 one has

$$\operatorname{reg}_{A\underline{\otimes}B} R_\Delta \leq \sup\{\operatorname{reg}_{A\underline{\otimes}B}(F_i)_\Delta - i\colon\ i = 1, \ldots, p\}.$$

It follows from Lemma 6.6 that R_Δ is Koszul whenever

$$\operatorname{reg}_{A\underline{\otimes}B}(F_i)_\Delta - i \leq 1 \text{ for all } i = 1, \ldots, p.$$

Since

$$(F_i)_\Delta = \bigoplus_{(a,b)\in\mathbb{N}^2} S(-a,-b)_\Delta^{\beta_{i,a,b}},$$

one has

$$\operatorname{reg}_{A\underline{\otimes}B}(F_i)_\Delta = \max\{\operatorname{reg}_{A\underline{\otimes}B} S(-a,-b))_\Delta\colon\ \beta_{i,a,b} \neq 0\}.$$

We now have to evaluate $\operatorname{reg}_{A\underline{\otimes}B} S(-a,-b)_\Delta$. To this end denote by $M_0, \ldots, M_{c-1}$ the relative Veronese submodules of $K[X]$, that is, $M_j = \oplus_{k\in\mathbb{N}} K[X]_{kc+j}$ for $j = 0, \ldots, c-1$. Similarly denote $N_0, \ldots, N_{e-1}$ the relative Veronese submodules of $K[Y]$.

One has

$$S(-a,-b)_\Delta = \bigoplus_s K[X]_{sc-a} \otimes K[Y]_{se-b} = M_i(-\lceil a/c\rceil)\underline{\otimes} N_j(-\lceil b/e\rceil)$$

where $i = -a \bmod c$, $0 \leq i \leq c-1$, and $j = -b \bmod e$, $0 \leq j \leq e-1$.

The relative Veronese submodules of a polynomial ring are known to have a linear resolution over the Veronese ring [1, 2.1]. Hence by virtue of Lemma 6.5 one has:

$$\operatorname{reg}_{A\underline{\otimes}B} S(-a,-b)_\Delta = \max\{\lceil a/c\rceil, \lceil b/e\rceil\}.$$

Summing up we see that R_Δ is Koszul if

$$\max\{\lceil a/c\rceil, \lceil b/e\rceil\colon\ \beta_{i,a,b} \neq 0\} \leq i+1$$

for all $i = 1, \ldots, p$. This concludes the proof of the theorem. □

As a corollary to the proof of the theorem we have

COROLLARY 6.7. *Let c, e be positive integers such that*

$$\max\{a/c, b/e\colon\ \beta_{i,a,b} \neq 0\} \leq i$$

for all $i = 1, \ldots, p$. Then $P_K^{R_\Delta}(s,t) \leq P_K^{A\underline{\otimes}B}(s,t)$ coefficientwise, where A and B are the cth and the eth Veronese subrings of $K[X]$ and $K[Y]$ respectively.

Proof. The assumption implies that $\operatorname{reg}_{A\underline{\otimes}B} R_\Delta = 0$. Then the claim follows from Lemma 6.6. □

If the algebra R happens to be Cohen-Macaulay, then the shifts in the resolution of R over S can be bounded in term of the a-invariant $a(R)$ of R. Indeed, if $\beta_{i,a,b} \neq 0$, then $a + b \leq a(R) + \dim R + i$. Thus we get

PROPOSITION 6.8. *Assume that R is Cohen-Macaulay. If $c, e \geq (a(R) + \dim R + 1)/2$, then R_Δ is Koszul.*

Proof. If $\beta_{i,a,b} \neq 0$, we have $a/c \leq (a + b)/c \leq (a(R) + \dim R + i)/c$, and similarly $b/e \leq (a(R) + \dim R + i)/e$. By virtue of Theorem 6.2, we have that R_Δ is Koszul if $(a(R) + \dim R + i)/c$ and $(a(R) + \dim R + i)/e$ are less than or equal to $i + 1$ for all $i = 1, \dots, \operatorname{codim} R$, that is to say $c, e \geq (a(R) + \dim R + i)/i + 1$ for all $i = 1, \dots, \operatorname{codim} R$. Since $a(R) + \dim R \geq 1$, the last statement is equivalent to $c, e \geq (a(R) + \dim R + 1)/2$. □

COROLLARY 6.9. *Let I be a homogeneous ideal of a polynomial ring $R = K[X_1, \dots, X_n]$. Denote by d the highest degree of a generator of I. Then there exist integers a, b such the K-algebra $K[(I^e)_{ed+c}]$ is Koszul for all $c \geq a$ and $e \geq b$.*

Proof. By replacing I with the ideal generated by I_d we may assume that I is generated by forms of degree d. The Rees algebra $R[It]$ is a standard bigraded algebra by setting $\deg X_i = (1, 0)$ and $\deg ft = (0, 1)$ for all $f \in I_d$. The claim follows now from Theorem 6.2 since $R[It]_\Delta = K[(I^e)_{ed+c}]$. □

The integers a, b of the corollary can be explicitly computed whenever one knows the shifts in the bigraded resolution of $R[It]$ over the polynomial ring. For instance in the complete intersection case one has:

COROLLARY 6.10. *Let I be an ideal of the polynomial ring $K[X_1, \dots, X_n]$ generated by a regular sequence $f_1, \dots, f_r$ of polynomials of degree d. Then one has:*

(1) *If $c \geq d/2$, and $e > 0$, then the ideal of definition of $K[I^e_{ed+c}]$ as a quotient of the polynomial ring $K[T_{(\alpha,\beta)}$: $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^r$, $|\alpha| = c$, $|\beta| = e]$ is generated by forms of degree less than or equal to 2.*

(2) *If $c \geq d(r - 1)/r$ and $e > 0$, then $K[I^e_{ed+c}]$ is Koszul.*

Proof. The resolution of $R[It]$ over $S = K[X_1, \dots, X_n, T_1, \dots, T_r]$, as observed in Section 4, is given by the Eagon-Northott complex. It follows that

$$0 \to D_{r-1} \to \cdots \to D_i \to \cdots \to D_1 \to S \to \mathcal{R}(I) \to 0$$

where

$$D_i = \bigoplus_{j=1}^{i} S(-jd, -i - 1 + j)^{\binom{r}{i+1}}.$$

Hence the claim follows from Propositiion 6.1 and Theorem 6.2. □

Perhaps the bound that we obtained in the complete intersection case can be improved. It could even be true that for a complete intersection ideal I generated by elements of degree d the algebra $K[I^e_{ed+c}]$ is Koszul as soon as it is defined by quadrics, that is, if $c \geq d/2$.

It was proved by Backelin ([4]) that the Veronese subrings of a Koszul algebra are all Koszul. Furthermore it is known ([9, Theorem 2]) that large Veronese subrings of a standard graded K-algebra are defined by a Gröbner basis of quadrics. One may ask whether the same properties hold for diagonal algebras too, that is:

Question 1. Suppose that a bigraded standard algebra R is Koszul. Are all the diagonal algebras of R Koszul?

Question 2. Let R be a bigraded standard K-algebra. Do there exist integers a, b such that for all $c \geq a$ and $e \geq b$ the (c, e)-diagonal R_Δ of R can be presented as a quotient of a polynomial ring by a Gröbner bases of quadrics?

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