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KRS and powers of determinantal ideals

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Abstract. The goal of this paper is to determine Gröbner bases of powers of determinantal ideals and to show that the Rees algebras of (products of) determinantal ideals are normal and Cohen-Macaulay if the characteristic of the base field is non-exceptional. Our main combinatorial result is a generalization of Schensted's Theorem on the Knuth–Robinson–Schensted correspondence.

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1. Introduction

Let K be a field and X be an $m \times n$ matrix of indeterminates over K. Denote by $K[X]$ the polynomial ring $K[X_{ij}\colon 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n]$. We equip $K[X]$ with a diagonal term order τ, that is, τ is a term order with the property that the initial term with respect to τ of a minor of X is the product of the elements of its main diagonal. Denote by I_t the ideal of $K[X]$ generated by the t-minors of X. We are interested in the following two questions:

(1) Is the Rees algebra $\mathcal{R}(\mathrm{I}_t) = \oplus_{k \geqslant 0} \mathrm{I}_t^k T^k$ Cohen-Macaulay?
(2) What is a Gröbner basis of I_t^k?

If the characteristic of K is 0 or if $t = \min(m, n)$, then the first question has a positive answer; see Bruns [1] and Eisenbud and Huneke [9]. On the other hand, there exist examples of algebras $\mathcal{R}(\mathrm{I}_t)$ which are not Cohen-Macaulay. They arise when the characteristic is exceptional; see [1, 4.1]. (We say that the characteristic of K is exceptional if $0 < \operatorname{char} K \leqslant \min(t, m \Leftrightarrow t, n \Leftrightarrow t)$.) The best result that one can hope for is a positive answer to (1) in non-exceptional characteristic. Answers to the second question are known when $k = 1$ or $t = \min(m, n)$ (for example see Sturmfels [16] and Conca [5]). The two problems come together in the study of the initial algebra of $\mathcal{R}(\mathrm{I}_t)$. The initial algebra $\mathrm{in}_\tau(\mathcal{R}(\mathrm{I}_t))$ of $\mathcal{R}(\mathrm{I}_t)$ is $\oplus_{k \geqslant 0} \mathrm{in}_\tau(\mathrm{I}_t^k) T^k$.

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It is known that $\mathcal{R}(\mathrm{I}_t)$ is Cohen-Macaulay if $\mathrm{in}_\tau(\mathcal{R}(\mathrm{I}_t))$ is finitely generated and Cohen-Macaulay (Conca, Herzog, and Valla [7, 2.3]).

For the study of $\mathrm{in}_\tau(\mathcal{R}(\mathrm{I}_t))$ we will employ the Knuth–Robinson–Schensted correspondence. It establishes a bijection between standard monomials of X and ordinary monomials of $K[X]$ (Knuth [12]). It is based on the Robinson–Schensted algorithm introduced by Schensted [15] in order to determine the longest increasing subsequence of a sequence of integers. This correspondence was successfully used by Sturmfels [16] in the determinination of a Gröbner basis of I_t. Later variants of the correspondence were applied to the study of Gröbner bases of other classes of determinantal ideals; see Conca [4], [5], and Herzog and Trung [11]. Throughout this paper we will always refer to the version of the Knuth–Robinson–Schensted correspondence described in [11], and we will denote it by KRS.

Our main combinatorial result is a generalization of Schensted's Theorem. It relates the shape of a tableau to the 'shapes' of the decompositions of its KRS sequence into increasing subsequences. It is similar to Greene's generalization [10] of Schensted's Theorem, and its proof also uses the Knuth relations. While Greene's Theorem concerns the maximal length of subsequences s' of a given sequence s of integers such that s' decomposes into t inreasing subsequences s'_i, we are interested in a decomposition of s into increasing subsequences $s''_1, \dots, s''_u$ for which $\Sigma \max(0, |s''_i| \Leftrightarrow t)$ is maximal.

The primary decomposition of the ideals I_t^k (and more generally of the products $\mathrm{I}_{t_1} \dots \mathrm{I}_{t_r}$) in non-exceptional characteristic is well-known; see Eisenbud, De Concini, and Procesi [8] and Bruns and Vetter [3]. They are intersections of symbolic powers of the ideals I_s, $1 \leqslant s \leqslant t$, and the symbolic powers have standard monomial bases. This fact in combination with the generalization of Schensted's Theorem allows us to describe the 'ordinary' monomial basis of $\mathrm{in}_\tau(\mathrm{I}_t^k)$ by combinatorial invariants analogous to those that determine the standard monomial basis of I_t^k. We then show that $\mathrm{in}_\tau(\mathcal{R}(\mathrm{I}_t))$ (and, more generally, $\mathrm{in}_\tau(\mathcal{R}(\mathrm{I}_{t_1} \dots \mathrm{I}_{t_r}))$) is a normal monomial algebra. By Hochster's Theorem $\mathrm{in}_\tau(\mathcal{R}(\mathrm{I}_t))$ is Cohen-Macaulay, so that $\mathcal{R}(\mathrm{I}_t)$ is a normal Cohen-Macaulay ring as desired. Furthermore we will prove that I_t^k has a minimal system of generators that simultaneously is a Gröbner basis. Further applications yield a description of the initial algebra of the subalgebra of $K[X]$ generated by the t-minors and the initial algebra of the symbolic Rees algebra of a ladder determinantal ideal.

We expect that one can investigate the powers of ideals of minors of a symmetric matrix or of the Pfaffian ideals of an alternating one in a similar manner since all the basic results used by us have variants for these cases.

2. Invariants of the Knuth–Robinson–Schensted correspondence

The polynomial ring $K[X]$ has two distinguished K-bases, the standard monomial basis and the ordinary monomial basis. Hence KRS induces a degree preserving K-linear map KRS: $K[X] \to K[X]$ which is our bridge from standard monomial theory to ordinary monomial theory. To be precise, one has the following lemma, whose part (a) is essentially due to Sturmfels [16].

LEMMA 2.1. (a) *Let I be an ideal of* $K[X]$ *which has a* K*-basis, say* B*, of standard monomials, and let* S *be a subset of I. Assume that for all* $b \in B$ *there exists* $s \in S$ *such that* $\mathrm{in}_\tau(s) \mid \mathrm{KRS}(b)$. *Then* S *is a Gröbner basis of I and* $\mathrm{in}_\tau(I) = \mathrm{KRS}(I)$.

(b) *Let* I *and* J *be homogeneous ideals such that* $\mathrm{in}_\tau(I) = \mathrm{KRS}(I)$ *and* $\mathrm{in}_\tau(J) = \mathrm{KRS}(J)$. *Then* $\mathrm{in}_\tau(I) + \mathrm{in}_\tau(J) = \mathrm{in}_\tau(I + J) = \mathrm{KRS}(I + J)$ *and* $\mathrm{in}_\tau(I) \cap \mathrm{in}_\tau(J) = \mathrm{in}_\tau(I \cap J) = \mathrm{KRS}(I \cap J)$.

Proof. (a) Let Q denote the ideal generated by the initial monomials of the elements of S. By hypothesis one has $\mathrm{KRS}(I) \subseteq Q \subseteq \mathrm{in}_\tau(I)$. But $\mathrm{KRS}(I)$ and I have the same Hilbert function because KRS is degree preserving, and $\mathrm{in}_\tau(I)$ and I have the same Hilbert function because of the general properties of the initial ideal. It follows that $\mathrm{KRS}(I) = Q = \mathrm{in}_\tau(I)$.

(b) One has $\mathrm{KRS}(I+J) = \mathrm{KRS}(I)+\mathrm{KRS}(J) = \mathrm{in}_\tau(I)+\mathrm{in}_\tau(J) \subseteq \mathrm{in}_\tau(I+J)$, and $\mathrm{KRS}(I\cap J) = \mathrm{KRS}(I)\cap\mathrm{KRS}(J) = \mathrm{in}_\tau(I)\cap\mathrm{in}_\tau(J) \supseteq \mathrm{in}_\tau(I\cap J)$. The Hilbert function argument concludes the proof. □

The ordinary and symbolic powers of determinantal ideals have K-bases of standard monomials (in non-exceptional characteristic). We may use 2.1 to study the initial ideals of these ideals provided we have some control on the KRS image of their basis elements. The elements of the standard monomial K-bases of $\mathrm{I}_t^{(k)}$ and of I_t^k are described in terms of certains functions γ_t (we will recall the precise results in Section 3). To this end we introduce some notation and define the functions γ_t.

Let $S = s_1, \ldots, s_r$ be a sequence of positive integers. For $t \in N$ we define

$$\gamma_t(S) = \sum_{i=1}^{r} (s_i \Leftrightarrow t + 1)_+,$$

where $(a)_+ = \max\{a, 0\}$, and

$$\Gamma_t(S) = \max\{k \in \mathbb{N}\colon \gamma_j(S) \geqslant k(t + 1 \Leftrightarrow j) \text{ for all } j = 1, \ldots, t\}.$$

For instance if $S = 4, 3, 3, 1$ then $\gamma_4(S) = 1$, $\gamma_3(S) = 4$, $\gamma_2(S) = 7$, $\gamma_1(S) = 11$, $\Gamma_4(S) = 1$, $\Gamma_3(S) = 3$, $\Gamma_2(S) = 5$, $\Gamma_1(S) = 11$.

Let δ be a product of minors of X, say $\delta = \delta_1 \ldots \delta_r$, where δ_i is a s_i-minor. The *shape* of δ is the sequence of integers $S(\delta) = s_1, \ldots, s_r$. Then we set

$$\gamma_t(\delta) = \gamma_t(S(\delta)), \quad \text{and} \quad \Gamma_t(\delta) = \Gamma_t(S(\delta)).$$

Now let M be a monomial of $K[X]$. We define

$$\gamma_t(M) = \max\{\gamma_t(\delta)\colon \delta \text{ is a product of minors and in } i_\tau(\delta) = M\},$$

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We warn the reader that in general there does not exist a product of minors δ with $\mathrm{in}_\tau(\delta) = M$ and $\gamma_t(M) = \gamma_t(\delta)$ for all t; for instance see the monomial given in 3.9.

Schensted's result [15] on the longest incresing subsequence of a sequence of integers can be expressed in terms of the function γ_t by saying that for a standard monomial δ one has

$$\gamma_t(\delta) \neq 0 \Leftrightarrow \gamma_t(\mathrm{KRS}(\delta)) \neq 0.$$

For the application we have in mind we need a much stronger result.

THEOREM 2.2. *Let δ be a standard monomial. Then one has for all t:*

$$\text{(i)}\ \ \gamma_t(\delta) = \gamma_t(\mathrm{KRS}(\delta)), \qquad \text{(ii)}\ \ \Gamma_t(\delta) = \Gamma_t(\mathrm{KRS}(\delta)).$$

As we shall see, 2.2 can be reduced to a theorem on the decomposition of sequences of integers into increasing ones. To this end we extend the functions γ_t to this context. Let $b = b_1, \dots, b_s$ be a sequence of integers. A subsequence $b_{i_1}, \dots, b_{i_k}$, with $i_1 < \cdots < i_k$, is *increasing* if $b_{i_1} < \cdots < b_{i_k}$. A decomposition g of b into increasing subsequences, an *inc-decomposition* for short, is said to have shape $S = s_1, \dots, s_r$ if the ith subsequence has s_i elements. We set

$$\gamma_t(g) = \gamma_t(S), \quad \Gamma_t(g) = \Gamma_t(S),$$

$$\gamma_t(b) = \max\{\gamma_t(g) \colon g \text{ is an inc} \Leftrightarrow\text{decomposition of } b\},$$

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Suppose that $M = \mathrm{KRS}(\delta) = X_{a_1b_1} \dots X_{a_sb_s}$, where the factors $X_{a_ib_i}$ are ordered such that $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_s$ and $b_i \geqslant b_{i+1}$ whenever $a_i = a_{i+1}$. A representation of M as $\mathrm{in}_\tau(\delta)$ with δ a product of minors of shape $s_1, \dots, s_r$ is equivalent to an inc-decomposition of $b_1, \dots, b_s$ of shape $s_1, \dots, s_r$. Therefore $\gamma_t(M) = \gamma_t(b)$ and $\Gamma_t(M) = \Gamma_t(b)$. Since the shape of the standard monomial δ and, hence, $\gamma_t(\delta)$ and $\Gamma_t(\delta)$ are determined only by $b_1, \dots, b_s$, we may assume that $a_i = i$ for $i = 1, \dots, s$. Furthermore, exchanging the roles of rows and columns and using [11, 1.1(b)], we may also assume that $\{b_1, \dots, b_s\} = \{1, \dots, s\}$. After this reduction, 2.2 follows from:

THEOREM 2.3. *Let $b = b_1, \dots, b_s$ be a sequence of distinct integers and let P be the tableau obtained from b by the Robinson–Schensted algorithm. Denote the shape of P by $S = s_1, \dots, s_r$. Then*

$$\text{(i)}\ \ \gamma_t(b) = \gamma_t(S), \quad \text{(ii)}\ \ \Gamma_t(b) = \Gamma_t(S).$$

Proof. We first recall the definition of Knuth relation (Sagan [14, 3.6.2]). Let a and b be two sequences of integers. One says that a and b *differ by a Knuth relation* if one of the following conditions holds with $z < y < w$ or $w < y < z$:

(1) $a = x_1, \dots, x_i, y, z, w, x_{i+4}, \dots, x_s$ and

$b = x_1, \dots, x_i, y, w, z, x_{i+4}, \dots, x_s$,

(2) $a = x_1, \dots, x_i, z, w, y, x_{i+4}, \dots, x_s$ and

$b = x_1, \dots, x_i, w, z, y, x_{i+4}, \dots, x_s$.

Two sequences are said to be *Knuth equivalent* if one can pass from one to the other via a chain of Knuth relations.

Denote the entries of the ith row of the tableau P by $P_{i1}, \dots, P_{is_i}$. The sequence $P_{r1}, \dots, P_{rs_r}, P_{r-11}, \dots, P_{r-1s_{r-1}}, \dots, P_{11}, \dots, P_{1s_1}$ is said to be the *P-canonical sequence*. It is known that the sequence b is Knuth equivalent to the P-canonical one [14, 3.6.3]. So, to prove the theorem it suffices to show that

(iii) $\gamma_t(b) = \gamma_t(S)$ and $\Gamma_t(b) = \Gamma_t(S)$ if b is the P-canonical sequence, and
(iv) γ_t and Γ_t have the same values on sequences that differ by a Knuth relation.

For the proof of (iii) let b be the P-canonical sequence. First of all one notes that b has an inc-decomposition into the **i** subsequences $P_{i1}, \dots, P_{is_i}$, $i = 1, \dots, r$. The shape of this decomposition is $s_1, \dots, s_r$. Hence $\gamma_t(S) \leqslant \gamma_t(b)$ and $\Gamma_t(S) \leqslant \Gamma_t(b)$. Now it is enough to show that $\gamma_t(Q) \leqslant \gamma_t(S)$ for all t if $Q = q_1, \dots, q_v$ is the shape of an inc-decomposition of b. Since the longest increasing subsequence of b has s_1 elements, one has $q_i < t$ for all i if $t > s_1$, and thus $\gamma_t(Q) = 0$. So we may assume $t \leqslant s_1$.

Let k be the integer for which $s_1 = s_k$ and $s_1 > s_{k+1}$. Let b' be the sequence which is obtained from b by deleting the element P_{ks_k}. Then b' is the P'-canonical sequence where the tableau P' is obtained from P by deleting the last element of the kth row. The shape of P' is $S' = s_1, \dots, s_{k-1}, s_k \Leftrightarrow 1, s_{k+1}, \dots, s_r$. Every inc-decomposition of b gives, by restriction, an inc-decomposition of b'. So we obtain a decomposition of b' with shape $Q' = q'_1, \dots, q'_v$, and there exists j such that $q'_i = q_i$ for all $i \neq j$ and $q'_j = q_j \Leftrightarrow 1$. By induction on the number of entries of P we may assume that $\gamma_t(Q') \leqslant \gamma_t(S')$. Then $\gamma_t(Q) \leqslant \gamma_t(Q') + 1 \leqslant \gamma_t(S') + 1 = \gamma_t(S)$.

For the proof of (iv) we consider sequences a and b of integers that differ by a Knuth relation. In order to prove that $\gamma_t(a) = \gamma_t(b)$ (respectively $\Gamma_t(a) = \Gamma_t(b)$) it suffices to show that for every inc-decomposition g of a there exists an inc-decomposition h of b such that $\gamma_t(g) \leqslant \gamma_t(h)$ (respectively $\Gamma_t(g) \leqslant \Gamma_t(h)$). So let g be an inc-decomposition of a. If z and w belong to distinct subsequences of the decomposition g, then the increasing subsequences of g are not affected by the Knuth relation, and we may take h equal to g.

It remains the case in which z and w belong to the same increasing subsequence. In particular $z < w$ and hence we are in the case $z < y < w$. Denote by $u = p_1, z, w, p_2$ and $v = p_3, y, p_4$ the subsequences of g which contain z, w and y.

Here the p_i are increasing subsequences of a. Now we have to distinguish two cases. Assume first the Knuth relation is of type (1). We have three ways of rearranging the elements of the sequences u and v into increasing subsequences of b:

$$\text{(I)}\begin{cases} u' = p_1, z, p_4, \\ v' = p_3, y, w, p_2, \end{cases} \qquad \text{(II)}\begin{cases} u' = p_1, w, p_2, \\ v' = p_3, y, p_4, \\ z, \end{cases} \qquad \text{(III)}\begin{cases} u' = p_1, y, w, p_2, \\ v' = p_3, p_4, \\ z. \end{cases}$$

Now we show how to define h if one wants to control the function γ_t for a given t. If $|u| < t$ (respectively $|v| < t$), then the sequence u (respectively v) does not contribute to $\gamma_t(g)$. Hence we may take h to be the decomposition of b that is obtained from g by replacing u and v by the sequences in (II) (respectively (III)), and we have $\gamma_t(g) = \gamma_t(h)$. If $|u|, |v| \geqslant t$, then we take h to be the decomposition of b which is obtained from g by replacing u and v by the sequences in (I). Then it is easy to see that

$$\gamma_t(h) \Leftrightarrow \gamma_t(g) = (|u'| \Leftrightarrow t + 1)_+ + (|v'| \Leftrightarrow t + 1)_+ \Leftrightarrow |u| \Leftrightarrow |v| + 2t \Leftrightarrow 2 \geqslant 0.$$

Next we explain how to choose h in order to control Γ_t for a given t. Let $k = \Gamma_t(g)$, and let $e = \min(|u|, |v|)$. If $\gamma_i(g) > k(t + 1 \Leftrightarrow i)$ for all $i = 2, \dots, e$, then we may use (II) or (III) (depending on whether $|u| \leqslant |v|$ or $|u| \geqslant |v|$) to define h. In fact, in this case we have $\gamma_i(h) = \gamma_i(g) \Leftrightarrow 1$ for $i = 2, \dots, e$ and $\gamma_i(h) = \gamma_i(g)$ for $i > e$; hence $\Gamma_t(h) \geqslant k$ as desired. Next assume $\gamma_i(g) = k(t + 1 \Leftrightarrow i)$ for some i, $2 \leqslant i \leqslant e$. We claim that in this last case one may define h by means of (I). Note that for $i \leqslant e$ the above argument shows that $\gamma_i(g) \leqslant \gamma_i(h)$. We have to control what happens to $\gamma_j(h)$ for $e < j \leqslant t$. To this end we set $\alpha_j = \gamma_j(g) \Leftrightarrow k(t + 1 \Leftrightarrow j)$ and denote by c_i the number of the subsequences in the decomposition g with at least i elements. By assumption one has $\alpha_j \geqslant 0$ for $1 \leqslant j \leqslant t$ and $\alpha_i = 0$ for some $2 \leqslant i \leqslant e$. In order to conclude that $\Gamma_t(h) \geqslant k$ it suffices to show that $\gamma_j(g) \Leftrightarrow \gamma_j(h) \leqslant \alpha_j$ for $e < j \leqslant t$. It is easy to see that $\gamma_j(g) = \gamma_{j+1}(g) + c_j$. Therefore $\alpha_j \Leftrightarrow \alpha_{j+1} = c_j \Leftrightarrow k$. We claim that $(*)$ $c_j < k$ for all $j > e$. If $c_j \geqslant k$ for some $j > e$, then, since $c_e > c_j$, one has $c_i > k$ for all $i \leqslant e$. But then $\alpha_i \geqslant c_i \Leftrightarrow k > 0$ for all $i \leqslant e$, which contradicts the assumption. It follows from the inequality $(*)$ that $\alpha_{j+1} > \alpha_j$ for $e < j < t$, and hence $\alpha_j \geqslant j \Leftrightarrow e \Leftrightarrow 1$ for $e < j \leqslant t$. This is exactly what it is needed to show that $\gamma_j(g) \Leftrightarrow \gamma_j(h) \leqslant \alpha_j$ for $e < j \leqslant t$. The proof of (iv) in the case of a Knuth relation of type (1) is complete, and the dual argument works in the case of a Knuth relation of type (2). □

In the next section we shall need that the function γ_t behaves well with respect to taking powers.

LEMMA 2.4. *Let M be a monomial of $K[X]$ and $h \in \mathbb{N}$. Then $\gamma_t(M^h) = h\gamma_t(M)$.*

Proof. Let δ be the standard monomial such that $\mathrm{KRS}(\delta) = M$. Any power of δ is a standard monomial and it is easy to see that $\mathrm{KRS}(\delta^h) = M^h$. From 2.2 it follows that $\gamma_t(M) = \gamma_t(\delta)$ and $\gamma_t(M^h) = \gamma_t(\delta^h)$. Clearly one has $\gamma_t(\delta^h) = h\gamma_t(\delta)$, and hence $\gamma_t(M^h) = h\gamma_t(M)$. □

Note that in general $\Gamma_t(M^h) \neq h\Gamma_t(M)$. One only has $[\Gamma_t(M^h)/h] = \Gamma_t(M)$. Furthermore note that the functions γ_t are not additive: in general, $\gamma_t(M_1M_2) > \gamma_t(M_1) + \gamma_t(M_2)$.

3. Gröbner bases of powers of determinantal ideals

We keep the notation of the previous section and assume for simplicity that $m \leqslant n$. Denote by I_t the ideal of $K[X]$ generated by the t-minors of X. The set of all the standard monomials δ of $K[X]$ with $\gamma_t(\delta) \geqslant k$ is a K-basis of the kth symbolic power $\mathrm{I}_t^{(k)}$ (see [8] or or [3]). Furthermore, a product of minors δ is in $\mathrm{I}_t^{(k)}$ if and only if $\gamma_t(\delta) \geqslant k$. It follows that the symbolic Rees algebra $\mathcal{R}^s(\mathrm{I}_t) = \oplus_{k\geqslant 0}\mathrm{I}_t^{(k)}T^k$ is given by

$$\mathcal{R}^s(\mathrm{I}_t) = K[X][\mathrm{I}_tT, \mathrm{I}_{t+1}T^2, \ldots, \mathrm{I}_mT^{m-t+1}].$$

PROPOSITION 3.1. *The set of products δ of minors with $\gamma_t(\delta) = k$ is a Gröbner basis of $\mathrm{I}_t^{(k)}$. Further $\mathrm{in}_\tau(\mathrm{I}^{(k)})$ is generated by the monomials M with $\gamma_t(M) = k$.*

Proof. Let S be the set of the products of minors δ with $\gamma_t(\delta) = k$. One has $S \subset \mathrm{I}_t^{(k)}$. By virtue of 2.2(i) we know that for all standard monomials δ with $\gamma_t(\delta) \geqslant k$ there exists ω in S with $\mathrm{in}_\tau(\omega) \mid \mathrm{KRS}(\delta)$. Thus the claim follows from 2.1(a). □

Rewriting 3.1 in terms of the symbolic Rees algebra and its initial algebra yields

LEMMA 3.2. *The initial algebra $\mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t))$ of the symbolic Rees algebra $\mathcal{R}^s(\mathrm{I}_t)$ is equal to $K[X][\mathrm{in}_\tau(\mathrm{I}_t)T, \mathrm{in}_\tau(\mathrm{I}_{t+1})T^2, \ldots, \mathrm{in}_\tau(\mathrm{I}_m)T^{m-t+1}]$. In particular, a monomial MT^k is in $\mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t))$ if and only if $\gamma_t(M) \geqslant k$.*

An important consequence is

COROLLARY 3.3. *The monomial algebra $\mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t))$ is finitely generated and normal. In particular $\mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t))$ and $\mathcal{R}^s(\mathrm{I}_t)$ are Cohen-Macaulay.*

Proof. It follows from 3.2 that the algebra $\mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t))$ is finitely generated. Let NT^k be a monomial of $K[X][T]$. For the normality of $\mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t))$ it suffices that $NT^k \in \mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t))$ whenever a power of NT^k belongs to $\mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t))$. So let us assume $(NT^k)^h \in \mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t))$. Then $\gamma_t(N^h) \geqslant hk$ in view of 3.2. But by 2.4 we have $\gamma_t(N^h) = h\gamma_t(N)$, and hence $\gamma_t(N) \geqslant k$. Using 3.2 once more, one gets $NT^k \in \mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t))$. The rest follows from 7, 2.3]. □

Application of 2.1(b) yields

LEMMA 3.4. *Let* $k_1, \ldots, k_m \in N$. *Then*

$$\mathrm{in}_\tau\left(\bigcap_{1\leqslant j\leqslant m} \mathrm{I}_j^{(k_j)}\right) = \bigcap_{1\leqslant j\leqslant m} \mathrm{in}_\tau(\mathrm{I}_j^{(k_j)}).$$

Let $t_1, \ldots, t_r$ be positive integers and set $g_j = \gamma_j(t_1, \ldots, t_r)$. If $\operatorname{char} K = 0$ or $\operatorname{char} K > \max_i(\min(t_i, m \Leftrightarrow t_i))$, one has

$$\mathrm{I}_{t_1} \ldots \mathrm{I}_{t_r} = \bigcap_{1\leqslant j\leqslant m} \mathrm{I}_j^{(g_j)};$$

see [3, 10.9] and [8, 8.1]. Thus 3.4 implies

THEOREM 3.5. *Suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \max_i(\min(t_i, m \Leftrightarrow t_i))$. *Then*

$$\mathrm{in}_\tau(\mathrm{I}_{t_1} \ldots \mathrm{I}_{t_r}) = \bigcap_{1\leqslant j\leqslant m} \mathrm{in}_\tau(\mathrm{I}_j^{(g_j)}).$$

In particular, a monomial M *belongs to* $\mathrm{in}_\tau(\mathrm{I}_{t_1} \ldots \mathrm{I}_{t_r})$ *if and only if* $\gamma_j(M) \geqslant g_j$ *for all* $j = 1, \ldots, m$.

THEOREM 3.6. *Suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \max_i(\min(t_i, m \Leftrightarrow t_i))$. *Then*

(a) $\mathrm{in}_\tau(\mathcal{R}(\mathrm{I}_{t_1} \ldots \mathrm{I}_{t_r}))$ *is finitely generated and normal,*

(b) $\mathcal{R}(\mathrm{I}_{t_1} \ldots \mathrm{I}_{t_r})$ *is Cohen-Macaulay and normal, and*

(c) *the associated graded ring of* $K[X]$ *with respect to* $\mathrm{I}_{t_1} \ldots \mathrm{I}_{t_r}$ *is Cohen-Macaulay.*

Proof. Set $J = \mathrm{I}_{t_1} \ldots \mathrm{I}_{t_r}$. One has $\mathrm{in}_\tau(\mathcal{R}(J)) = \oplus_{k\geqslant 0}\mathrm{in}_\tau(J^k)T^k$. By 3.5 $\mathrm{in}_\tau(J^k) = \cap_{1\leqslant j\leqslant m}\mathrm{in}_\tau(\mathrm{I}_j^{(kg_j)})$. Hence

$$\mathrm{in}_\tau(\mathcal{R}(J)) = \cap_{1\leqslant j\leqslant m} \oplus_{k\geqslant 0} \mathrm{in}_\tau(\mathrm{I}_j^{(kg_j)})T^k.$$

The monomial algebra $\oplus_{k\geqslant 0}\mathrm{in}_\tau(\mathrm{I}_j^{(kg_j)})T^k$ is isomorphic to the g_jth Veronese subalgebra of the monomial algebra $\mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_j))$ (in the relevant case $g_j > 0$ and equal to $K[X,T]$ otherwise). By 3.3 the latter is normal and finitely generated, and therefore $\oplus_{k\geqslant 0}\mathrm{in}_\tau\mathrm{I}_j^{(kg_j)})T^k$ is a normal, finitely generated monomial algebra. Thus $\mathrm{in}_\tau(\mathcal{R}(J))$ is finitely generated and normal. In fact, the intersection of a finite number of finitely generated normal monomial algebras is finitely generated and normal. (This follows easily from standard results about normal affine semigroup rings; see Bruns and Herzog [2, 6.1.2 and 6.1.4].)

For (b) one again applies [7, 2.3], and (c) is a standard consequence of (b): $K[X]$ and the associated graded ring are residue class rings of the Rees algebra modulo the isomorphic ideals $JT\mathcal{R}(J)$ and $J\mathcal{R}(J)$ respectively. □

We single out the most important case.

THEOREM 3.7. *Suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(t, m \Leftrightarrow t)$. *Then* $\mathcal{R}(\mathrm{I}_t)$ *is Cohen-Macaulay and normal.*

Remark 3.8. In order to obtain a version of 3.7 that is valid in arbitrary characteristic one must replace the Rees algebra by its integral closure. The integral closure is always equal to the intersection of symbolic Rees algebras that in non-exceptional characteristic gives the Rees algebra itself [1].

Theorem 3.5 is satisfactory if one only wants to determine the initial ideal of the product $\mathrm{I}_{t_1} \ldots \mathrm{I}_{t_r}$, but it does not tell us how to find a Gröbner basis. A natural guess is that a Gröbner basis of $\mathrm{I}_{t_1} \ldots \mathrm{I}_{t_r}$ is given by the products of minors (standard or not) which are in $\mathrm{I}_{t_1} \ldots \mathrm{I}_{t_r}$. Unfortunately this is wrong in general.

EXAMPLE 3.9. Suppose that $m \geqslant 4$ and $\operatorname{char} K = 0$ or > 3, and consider the ideal $\mathrm{I}_2\mathrm{I}_4$. The monomial $M = X_{11}X_{13}X_{22}X_{34}X_{43}X_{45}$ has $\gamma_4(M) = 1, \gamma_3(M) = 2, \gamma_2(M) = 4, \gamma_1(M) = 6$. Hence, by virtue of 3.4, we know that $M \in \mathrm{in}_\tau(\mathrm{I}_2\mathrm{I}_4)$. The products of minors of degree 6 in $\mathrm{I}_2\mathrm{I}_4$ have the shapes 6 or 5, 1, or 4, 2. Clearly M is not the initial monomial of a product of minors of shape 6 or of shape 5, 1. The only initial monomial of a 4-minor that divides M is $X_{11}X_{22}X_{34}X_{45}$ but the remaining factor $X_{13}X_{43}$ is not the initial monomial of a 2-minor. Hence M is not the initial monomial of a product of minors that belongs to $\mathrm{I}_2\mathrm{I}_4$.

Nevertheless, if we confine our attention to powers of determinantal ideals, the result is optimal.

THEOREM 3.10. *Suppose that* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > \min(t, m \Leftrightarrow t)$. *Then a Gröbner basis of* I_t^k *is given by the products of minors* δ *such that* δ *has at most* k *factors,* $\Gamma_t(\delta) = k$, *and* $\deg \delta = kt$. *Therefore* I_t^k *has a minimal system of generators which is a Gröbner basis.*

Proof. Since $\mathrm{I}_t^k = \cap_{1 \leqslant j \leqslant t} \mathrm{I}_j^{((t+1-j)k)}$, a K-basis of the ideal I_t^k is given by the standard monomials δ with $\gamma_j(\delta) \geqslant (t+1 \Leftrightarrow j)k$ for $1 \leqslant j \leqslant t$, that is, $\Gamma_t(\delta) \geqslant k$. Further a product of minors δ is in $\mathrm{I}_t^{(k)}$ if and only if $\Gamma_t(\delta) \geqslant k$. Consider the set S of the products of minors δ with $\Gamma_t(\delta) \geqslant k$. We have $S \subset \mathrm{I}_t^k$, and by virtue of

2.2(ii) the monomial $\mathrm{KRS}(\delta)$ is divisible by the initial term of an element of S for all standard monomials δ with $\Gamma_t(\delta) \geqslant k$. By 2.1(a) S is a Gröbner basis of I_t^k.

Let now S_1 be the set of all the products of minors δ with at most k factors and $\Gamma_t(\delta) = k$. In order to show that S_1 is a Gröbner basis of I_t^k, it suffices to show that for every $\omega \in S$ with more than k factors or $\Gamma_t(\omega) > k$ there exists $\delta \in S$ such that $\mathrm{in}_\tau(\delta)$ is a proper divisor of $\mathrm{in}_\tau(\omega)$. If ω has more than k factors, then δ is taken to be the product of minors which is obtained from ω by skipping the shortest minor. If $\Gamma_t(\omega) > k$, then δ is taken to be the product of minors which is obtained from ω by replacing the longest minor, say $[a_1, \ldots, a_h|b_1, \ldots, b_h]$, with $[a_1, \ldots, a_{h-1}|b_1, \ldots, b_{h-1}]$. It is easy to see that $\delta \in S$ in both cases.

Next we choose S_2 as the set of all $\delta \in S_1$ with $\deg \delta = kt$. Pick $\omega \in S_1$. If $\deg \omega > kt$, then we define ω' by omitting one position from the shortest minor in ω. It is easily verified that $\omega' \in S_1$, and proceeding iteratively we eventually reach $\delta \in S_2$ such that $\mathrm{in}_\tau(\delta)$ divides $\mathrm{in}_\tau(\omega)$.

In order to find a minimal set of generators we select a minimal subset S_3 of S_2 with the property that its set of initial terms still generates $\mathrm{in}_\tau(\mathrm{I}_t^k)$. Since all the elements of S_3 have degree kt, $|S_3| = \dim_K(\mathrm{I}_t^k)_{kt}$. □

We conclude this section by describing the initial algebra of the subalgebra A_t of $K[X]$ that is generated by the minors of size t.

THEOREM 3.11. *Suppose that* $\mathrm{char}\, K = 0$ *or* $\mathrm{char}\, K > \min(t, m - t)$. *Then the products* δ *of minors with* $\deg \delta = kt$ *and* $\Gamma_t(\delta) = k$ *for some* $k \in \mathbb{N}$ *form a Sagbi basis of* A_t. *The initial algebra* $\mathrm{in}_\tau(A_t)$ *is finitely generated and normal. Hence* A_t *is a normal Cohen-Macaulay ring.*

Proof. The degree kt component of $\mathrm{in}_\tau(A_t)$ is the degree kt component of $\mathrm{in}_\tau(\mathrm{I}_t)$. Therefore the first part of the theorem follows from 3.10.

Let B_t be the subalgebra of $\mathrm{in}_\tau(\mathcal{R}(\mathrm{I}_t))$ generated by all monomials MT^k such that $\deg M = kt$. Then B_t is obviously isomorphic to $\mathrm{in}_\tau(A_t)$. Furthermore the semigroup H of monomials belonging to B_t is the intersection of the semigroup of monomials of $\mathrm{in}_\tau(\mathcal{R}(\mathrm{I}_t))$ with the subset of monomials of $K[X, T]$ whose exponent vector satisfies a homogeneous linear equation. Therefore H is finitely generated and normal. □

4. Symbolic powers of ladder determinantal ideals

In this section we apply the results of the previous one to the study of the powers of ladder determiantal ideals. For generalities about ladders and ladder determinantal ideals we refer the reader [11] and [13]. Let Y be a ladder of X, and let $\mathrm{I}_t(Y)$ denote the ideal of $K[Y]$ generated by the t-minors of Y. It is known that $\mathrm{I}_t(Y)$ is a prime ideal, that $\mathrm{I}_t(Y) = \mathrm{I}_t \cap K[Y]$ and that the t-minors of Y form a Gröbner basis of $\mathrm{I}_t(Y)$. Our goal is to gain some information about the symbolic powers of ladder determinantal ideals.

THEOREM 4.1. *Let Y be a ladder and $t > 1$.*

(a) *The set of products of minors δ of Y with $\gamma_t(\delta) = k$ is a Gröbner basis of* $\mathrm{I}_t(Y)^{(k)}$ *and* $\mathrm{I}_t(Y)^{(k)} = \mathrm{I}_t^{(k)} \cap K[Y]$,

(b) $\mathrm{I}_t(Y)$ *has primary powers if and only if Y does not contain $(t+1)$-minors,*

(c) $\mathcal{R}^s(\mathrm{I}_t(Y)) = K[Y][\mathrm{I}_t(Y)T, \mathrm{I}_{t+1}(Y)T^2, \ldots, \mathrm{I}_m(Y)T^{m-t+1}]$, *and*

$$\begin{aligned}&\mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t(Y)))\\ &\quad = K[X][\mathrm{in}_\tau(\mathrm{I}_t(Y))T, \mathrm{in}_\tau(\mathrm{I}_{t+1}(Y))T^2, \ldots, \mathrm{in}_\tau(\mathrm{I}_m(Y))T^{m-t+1}],\end{aligned}$$

(d) $\mathcal{R}^s(\mathrm{I}_t(Y))$ *is normal and Cohen-Macaulay.*

Proof. First we claim that a Gröbner basis of the ideal $\mathrm{I}_t^{(k)} \cap K[Y]$ is given by the set S of the products of minors δ of Y with $\gamma_t(\delta) = k$. To show this one notes that $S \subset \mathrm{I}_t^{(k)} \cap K[Y]$. Further, if $f \in \mathrm{I}_t^{(k)} \cap K[Y]$, then we know by virtue of 3.1 that there exists a product of minors δ of X with $\gamma_t(\delta) = k$ and $\mathrm{in}_\tau(\delta) \mid \mathrm{in}_\tau(f)$. Finally $\delta \in S$ because Y is a ladder and $\mathrm{in}_\tau(\delta) \in K[Y]$.

Now, since $\mathrm{I}_t(Y)^{(k)} \subseteq \mathrm{I}_t^{(k)} \cap K[Y]$, it suffices for (a) to verify that $S \subset \mathrm{I}_t(Y)^{(k)}$. Let ω be a minor of size $t+j \Leftrightarrow 1$ of Y, and let Z be a submatrix of Y which contains ω. Then by [3, Prop. 10.2] we have $\omega \in \mathrm{I}_t(Z)^{(j)}$ and since $\mathrm{I}_t(Z)^{(j)} \subset \mathrm{I}_t(Y)^{(j)}$, it follows that $\omega \in \mathrm{I}_t(Y)^{(j)}$. The symbolic powers form a filtration, and hence $S \subset \mathrm{I}_t(Y)^{(k)}$.

(b) If Y does not contain $(t+1)$-minors, then (a) implies $\mathrm{I}_t(Y)^{(k)}$ is equal to $\mathrm{I}_t(Y)^k$. Conversely, assume Y contains a $(t+1)$-minor, say ω. Then pick a t-minor α of Y. Since $\gamma_t(\omega\alpha^{k-2}) = k$, it follows from (a) that $\omega\alpha^{k-2} \in \mathrm{I}_t(Y)^{(k)}$ for all $k > 1$. But $\deg \omega\alpha^{k-2} = t + 1 + t(k \Leftrightarrow 2) = tk + 1 \Leftrightarrow t < tk$, and hence $\omega\alpha^{k-2} \notin \mathrm{I}_t(Y)^k$.

Statement (c) is just (a) rewritten in terms of the symbolic Rees algebra and its initial algebra. Finally, to prove (d) one notes that $\mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t(Y))) = \mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t)) \cap K[Y][T]$, and hence $\mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t(Y)))$ inherits normality from $\mathrm{in}_\tau(\mathcal{R}^s(\mathrm{I}_t))$. Then the claim follows from [7, 2.3]. □

Unfortunately we are not able to determine the primary decomposition of $\mathrm{I}_t(Y)^k$. Of course one has $\mathrm{I}_t(Y)^k \subseteq \mathrm{I}_t(Y)^{(k)} \cap \mathrm{I}_{t-1}(Y)^{(2k)} \cap \cdots \cap \mathrm{I}_1(Y)^{(tk)}$, but equality does not hold in general.

EXAMPLE 4.2. One can check with a computer algebra system that $\mathrm{I}_1(Y)\mathrm{I}_3(Y) \not\subset \mathrm{I}_2(Y)^2$, where Y is the ladder obtained from a 5×5 matrix by skipping X_{45}, X_{54}, X_{55}.

Neither can we expect to descend properties of $\mathcal{R}(\mathrm{I}_t(Y))$ directly from those of $\mathcal{R}(\mathrm{I}_t)$, because the previous example tells us that in general $\mathrm{I}_t(Y)^k$ is strictly smaller than $\mathrm{I}_t^k \cap K[Y]$.

References

1. Bruns, W.: Algebras defined by powers of determinantal ideals, *J. of Algebra* 142 (1991) 150–163.
2. Bruns, W. and Herzog, J.: *Cohen-Macaulay Rings*, Cambridge University Press, 1993.
3. Bruns, W. and U. Vetter, U.: *Determinantal Rings*, Lect. Notes Math. 1327, Springer, 1988.
4. Conca, A.: Gröbner bases of ideals of minors of a symmetric matrix, *J. Algebra* 166 (1994) 406–421.
5. Conca, A.: Gröbner bases of powers of ideals of maximal minors, *J. Pure Appl. Alg.* (in press).
6. Conca, A. and Herzog, A.: Ladder determinantal rings have rational singularities, *Advances in Math.* (in press).
7. Conca, A., Herzog, J. and Valla, G.: Sagbi bases and application to blow-up algebras, *J. Reine Angew. Math*. 474 (1996) 113–138.
8. De Concini, C., Eisenbud, D. and Procesi, C.: Young diagrams and determinantal varieties, *Invent. Math*. 56 (1980) 129–165.
9. Eisenbud, D. and Huneke, C.: Cohen-Macaulay Rees algebras and their specializations, *J. Algebra* 81 (1983) 202–224.
10. Greene, C.: An extension of Schensted's theorem, *Advances in Math*. 14 (1974) 254–265.
11. Herzog, J. and Trung, N. V.: Gröbner bases and multiplicity of determinantal and pfaffian ideals, *Advances in Math*. 96 (1992) 1–37.
12. Knuth, D.E.: Permutations, matrices, and generalized Young tableaux, *Pacific J. Math*. 34 (1970) 709–727.
13. Narasimhan, H.: The irreducibility of ladder determinantal varieties, *J. of Algebra* 102 (1986) 162–185.
14. Sagan, B.: *The Symmetric Group*, Wadsworth and Brooks/Cole 1991.
15. Schensted, C.: Longest increasing and decreasing subsequences, *Can. J. Math*. 13 (1961) 179–191.
16. Sturmfels, B.: Gröbner bases and Stanley decompositions of determinantal rings, *Math. Z*. 205 (1990) 137–144.