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Straightening Law and Powers of Determinantal Ideals of Hankel Matrices

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INTRODUCTION

In this paper we establish a standard monomial theory for generic Hankel matrices. By a generic Hankel matrix we mean a matrix $Y=(y_{ij})$ with $y_{ij}=x_{i+j-1}$ where the x_i are indeterminates over a field K. We use this structure to determine the symbolic powers and the primary decomposition of the powers of the determinantal ideals I_t of Y. Further we prove that the symbolic and ordinary Rees algebras of I_t are Cohen–Macaulay normal domains.

The first standard monomial theory was developed by Hodge [H] to study the homogeneous coordinate ring of the Grassmannian variety. Later standard monomial theories were established for generic matrices by Doubilet, Rota and Stein [DRS], and for generic symmetric and generic skew symmetric matrices by De Concini and Procesi [DP]. These are all examples of algebras with straightening law (ASL for short) over a poset or over a doset. The abstract notion of ASL was introduced and developed by Eisenbud [E1], and by De Concini, Eisenbud and Procesi [DEP2], see also [BV]. These structures turned out to be an extremely powerful tool in studying determinantal rings and ideals arising from the above mentioned generic matrices.

Let now $x_1, ..., x_n$ be indeterminates over an arbitrary field K. For $j=1, ..., n$ we denote by X_j the $j\times(n+1-j)$ Hankel matrix with entries $x_1, ..., x_n$, that is,

$$X_j=\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-j+1} \\ x_2 & x_3 & \cdots & \cdots & \cdots \\ x_3 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_j & \cdots & \cdots & \cdots & x_n \end{pmatrix}$$

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For $t=1, ..., \min(j, n-j+1)$ let $I_t(X_j)$ be the ideal of $K[x_1, ..., x_n]$ generated by the t-minors of X_j. It is known that $I_t(X_j)=I_t(X_t)$ for all $t \leqslant j \leqslant n+1-t$, that is, $I_t(X_j)$ does not depend on j but only on t and n, see [GP], [W] or 2.2. Let $I_t=I_t(X_t)$. The ring defined by I_t is known to be a Cohen–Macaulay normal domain, see [E2] and [W]. Further by a result of Valla [V] the ideal I_t is set-theoretic complete intersection. It is well-known that I_2 is the defining ideal of the rational normal curve C of $\mathbf{P}^{n-1}$. The ideal I_t with $t>2$ has also a geometric interpretation, namely it defines the secant variety of order $(t-2)$ of C, see for instance [R] or [E2].

The t-minors of X_t with $t=1, ..., \lfloor (n+1)/2 \rfloor$ play a special role in our theory; we call them maximal minors. In the first section we determine a family of products of maximal minors, called standard monomials, with the following properties: distinct standard monomials have distinct initial monomials and for every ordinary monomial $m \in K[x_1, ..., x_n]$ there exists a standard monomial whose initial monomial is m. Here initial monomials are taken with respect to a diagonal monomial order. It follows then easily that the standard monomials are a K-basis of the polynomial ring $K[x_1, ..., x_n]$, see 1.2. Unfortunately the polynomial ring $K[x_1, ..., x_n]$, equipped with this standard monomial basis is not an ASL in the classical sense. Nevertheless this standard monomial basis has a quadratic "straightening law" which is formally very similar to the straightening laws that one has for generic, generic symmetric and generic skew symmetric matrices, see 2.3 and 2.6. This is proved in Section 2 and it is indeed the core of the paper.

The lack of an ASL structure is very often balanced by the fact that every set of standard monomials is a Gröbner basis for the K-space they generate. This observation is very useful especially for those ideals which have K-bases of standard monomials. In Section 3 we determine a class of ideals $J_{t,k}$ which have K-bases of standard monomials. We show by using generic relations and Gröbner basis arguments that $J_{t,k}$ coincides with the k-symbolic power $I_t^{(k)}$ of I_t, see 3.8. It turns out that, as in the other generic cases, the ideal $I_t^{(k)}$ is generated by the standard monomials μ with $\gamma_t(\mu) \geqslant k$ (see Section 2 for the definition of γ_t). Set $m=\lfloor (n+1)/2 \rfloor$ and $u=\max(1, m-k(m-t))$. Again Gröbner basis arguments allow us to show that the ideal I_t^k has the irredundant primary decomposition:

$$I_t^k = I_t^{(k)} \cap I_{t-1}^{(2k)} \cap \cdots \cap I_u^{((t+1-u)k)},$$

see 3.16. Note that again this result is formally similar to the one that one obtains in the other generic cases. We remark that the primary decomposition of I_t^k and all the results of this paper are characteristic free. This behavior is partially explained by the fact that the ideals I_t are all ideals of "maximal minors" and usually maximal minors are not sensitive to the characteristic.

In the last section we study the symbolic Rees algebra $\mathscr{R}^s(I_t)$ and ordinary Rees algebra $\mathscr{R}(I_t)$. Since I_t^k and $I_t^{(k)}$ have K-bases of standard monomials we are able to control their initial ideals. This allows us to determine the initial algebras of $\mathscr{R}^s(I_t)$ and $\mathscr{R}(I_t)$ and to show that they are normal. Thus we may conclude that $\mathscr{R}^s(I_t)$ and $\mathscr{R}(I_t)$ are Cohen–Macaulay and normal, see 4.2 and 4.5. Furthermore we show that $\mathscr{R}(I_t)$ is defined by a Gröbner basis of quadrics. The normality, the Cohen–Macaulay property and the presentation of the Rees algebra $\mathscr{R}(I_2)$ were determined already in [CHV]. Finally we remark that the knowledge of the primary decomposition of the powers of the ideal I_t can be useful in the study of the zero dimensional schemes lying on rational normal curves or on their secant varieties, see [CEG].

Some of the results of this paper have been conjectured after (and confirmed by) explicit computations performed by the computer algebra system CoCoA[CNR].

1. THE STANDARD MONOMIALS

Let K be a field and $K[X]$ be the polynomial ring $K[x_1, ..., x_n]$. Denote by X the arrangement of indeterminates

$$X = \begin{array}{cccccccc}
x_1 & x_2 & x_3 & \cdots & \cdots & \cdots & x_{n-1} & x_n \\
x_2 & x_3 & \cdots & \cdots & \cdots & \cdots & x_n & \\
x_3 & \cdots & \cdots & \cdots & \cdots & x_n & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & \\
\vdots & \vdots & \vdots & \vdots & & & & \\
\vdots & \vdots & \vdots & & & & & \\
x_{n-1} & x_n & & & & & & \\
x_n & & & & & & &
\end{array}$$

and for $1 \leqslant j \leqslant n$ denote by X_j the submatrix of X

$$X_j = \begin{pmatrix}
x_1 & x_2 & x_3 & \cdots & x_{n-j+1} \\
x_2 & x_3 & \cdots & \cdots & \cdots \\
x_3 & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_j & \cdots & \cdots & \cdots & x_n
\end{pmatrix}$$

Given positive integers $a_1, a_2, ..., a_s, b_1, b_2, ..., b_s$, with $a_i + b_j - 1 \leqslant n$ for all $1 \leqslant i, j \leqslant s$, we denote by $[a_1, a_2, ..., a_s \mid b_1, b_2, ..., b_s]$ the minor of X with row indices $a_1, a_2, ..., a_s$ and column indices $b_1, b_2, ..., b_s$. A minor of the form $[1, 2, ..., s \mid b_1, b_2, ..., b_s]$ will be called *maximal minor* or maximal s-minor, and it will simply be denoted by $[b_1, b_2, ..., b_s]$.

Let $>_L$ be the degree lexicographic monomial order on $K[X]$ induced by the order in the indeterminates $x_1 > x_2 > \cdots > x_n$. Unless otherwise specified $\mathrm{in}(f)$ will always denote the initial monomial of a polynomial f with respect to $>_L$. Similarly $\mathrm{in}(I)$ will always denote the initial ideal of an ideal I with respect to $>_L$. By construction the initial monomial of a minor is the product of the elements of its main diagonal. In $\mathbf{N}$ we introduce the following partial order:

$$i <_1 j \qquad \text{if and only if} \quad i + 1 < j$$

We say that a sequence of integers $a_1, a_2, ..., a_s$ is a $<_1$-chain if $a_1 <_1 a_2 <_1 \cdots <_1 a_s$. Similarly we say that a monomial $x_{a_1} \cdots x_{a_s}$ is a $<_1$-chain if its indices form a $<_1$-chain. Note that a monomial is a $<_1$-chain if and only if it is the initial monomial of a minor of X. Of course, given a $<_1$-chain, say $m = x_{a_1} \cdots x_{a_s}$, there are many s-minors whose initial monomial is equal to m but just one of them is a maximal s-minor, namely $[a_1, a_2 - 1, ..., a_s - s + 1]$. Hence we have a bijective correspondence between the sets:

$$\varphi: \{<_1\text{-chains of } K[X]\} \to \{\text{maximal minors of } X\}.$$

defined by setting $\varphi(x_{a_1} \cdots x_{a_s}) = [a_1, a_2 - 1, ..., a_s - s + 1]$. The inverse of φ is the map in which takes every maximal minor to its initial monomial.

Let m be any monomial of $K[X]$. We describe now a canonical decomposition of m into a product of $<_1$-chains. First let m_1 be the $<_1$-chain which divides m and it is maximal with respect to $>_L$. If $m_1 \neq m$, then let m_2 be the $<_1$-chain which divides m/m_1 and it is maximal with respect to $>_L$, and so on. We end up with a decomposition $m = m_1 m_2 \cdots m_k$ which is uniquely determined by m. Denote by s_i the degree of m_i. The sequence $s_1, s_2, ..., s_k$ is called the shape of m. Note that the shape is a non-increasing sequence. Denote by $a_{j1}, ..., a_{js_i}$ the indices of the j-th factor of the canonical decomposition of m. One represents m by means of the tableau

$$\begin{array}{cccccc} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1s_1} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2s_2} & \\ \vdots & \vdots & \vdots & \vdots & & \\ a_{k1} & \cdots & a_{ks_k} & & & \end{array}$$

It is easy to see that the fact that the decomposition is canonical is equivalent to say that the entries of the tableau are bounded by n, the rows of the tableau are $<_1$-chains and that for all $1 \leqslant h < i \leqslant k$ and $1 \leqslant j \leqslant s_i$ one has:

$$a_{ij} \in \{a_{h1}, a_{h1}+1, ..., a_{hp}, a_{hp}+1, ..., a_{hs_h}, a_{hs_h}+1\}.$$

From these conditions it follows in particular that the columns are non-decreasing from the top to the bottom. Now φ induces a map

$$\Phi\colon \{\text{ordinary monomials of } K[X]\} \to \{\text{products of maximal minors of } X\}.$$

which is defined by $\Phi(m) = \varphi(m_1)\, \varphi(m_2) \cdots \varphi(m_k)$ where $m = m_1 m_2 \cdots m_k$ is the canonical decomposition of m. Note that by construction one has $\mathrm{in}(\Phi(m)) = m$ and hence Φ is injective. We define now the set of the *standard monomials* of X to be the image of Φ. So by construction we have a bijective correspondence:

$$\Phi\colon \{\text{ordinary monomials of } K[X]\} \to \{\text{standard monomials of } X\}$$

whose inverse is given by the map which takes every standard monomial to its initial monomial, i.e. $\mathrm{in}(\Phi(m)) = m$ for all ordinary monomial m and $\Phi(\mathrm{in}(\mu)) = \mu$ for all standard monomial μ.

If one represents products of minors as tableaux, then a standard monomial is represented by a single tableau $A = (a_{ij})$ of shape, say, $s_1, ..., s_k$ which is standard in the ordinary sense (i.e. $a_{ij} < a_{ij+1}$ and $a_{ij} \leqslant a_{i+1j}$ whenever these inequalities make sense) and it satisfies the following additional properties: the entries of the jth column of A are bounded by $n+1-j$ and for all $1 \leqslant h < i \leqslant k$ and $1 \leqslant j \leqslant s_i$ one has

$$a_{ij} + j - 1 \in \{a_{h1}, a_{h1}+1 ..., a_{hp}+p-1, a_{hp}+p, ..., a_{hs_h}+s_h-1, a_{hs_h}+s_h\}.$$

EXAMPLE 1.1. Let m be the monomial $x_1 x_2^2 x_3^2 x_4 x_6^2 x_7$. The canonical decomposition of m is $(x_1 x_3 x_6)(x_2 x_4 x_6)(x_2 x_7)(x_3)$ and the shape is 3, 3, 2, 1. Thus m corresponds to the standard monomial $\mu = [1, 2, 4][2, 3, 4][2, 6][3]$. In terms of tableaux

$$m = \begin{matrix} 1 & 3 & 6 \\ 2 & 4 & 6 \\ 2 & 7 & \\ 3 & & \end{matrix} \qquad \mu = \begin{matrix} 1 & 2 & 4 \\ 2 & 3 & 4 \\ 2 & 6 & \\ 3 & & \end{matrix}$$

Now we have the following:

THEOREM 1.2. *The standard monomials form a K-basis of the polynomial ring* $K[X] = K[x_1, ..., x_n]$.

Proof. First note that distinct standard monomials have distinct initial monomials. Hence the standard monomials are linearly independent. Let now let f be any non-zero element of $K[X]$. Let m be the initial monomial of f, and α be the coefficient of m in f. Set $\mu = \Phi(m)$. Since $\mathrm{in}(\mu) = m$, one has $\mathrm{in}(f - \alpha\mu) <_L m$, and by induction we may assume that $f - \alpha\mu$ can be written as a linear combination of standard monomials. Hence f is a linear combination of standard monomials. ∎

Unfortunately $K[X]$ is not an algebra with straightening law in the classical sense. To see this it is enough to observe that if μ_1, μ_2, μ_3 are maximal minors and $\mu_1\mu_2$ and $\mu_2\mu_3$ are standard monomials then the product $\mu_1\mu_2\mu_3$ need not to be a standard monomial. Nevertheless some of the formal properties of an ASL are satisfied in our case too. For instance:

Remark 1.3. (a) A product of maximal minors $\mu_1 \cdots \mu_k$ is a standard monomial if and only if $\mu_i\mu_j$ is a standard monomial for all $1 \leqslant i < j \leqslant k$, (b) A power of a standard monomial is a standard monomial.

Every polynomial f of $K[X]$ can be written in a unique way as linear combination, say $f = \sum_{i=1}^{k} \alpha_i \mu_i$, of standard monomials μ_i with coefficients $\alpha_i \in K \backslash \{0\}$. This expression is called the *standard representation* or *straightening law of f.*

Remark 1.4. Let f be a polynomial and $f = \sum_{i=1}^{k} \alpha_i \mu_i$ its standard representation. Since distinct standard monomials have distinct initial monomials, we may order the μ_i in such a way that $\mathrm{in}(f) = \mathrm{in}(\mu_1) >_L \mathrm{in}(\mu_2) >_L \cdots >_L \mathrm{in}(\mu_k)$.

The proof of 1.2 suggests an algorithm to determine standard representations. For instance:

EXAMPLE 1.5. Suppose we want to determine the standard representation of the product of maximal minors $[1, 2][2, 4]$. We take first the initial monomial of $[1, 2][2, 4]$, that is $x_1x_2x_3x_5$. Then we take its canonical decomposition $x_1x_2x_3x_5 = (x_1x_3x_5)(x_2)$. It corresponds to the standard monomial $[1, 2, 3][2]$. Then we consider the difference $[1, 2][2, 4] - [1, 2, 3][2]$, and take again its initial monomial, that is $x_1x_2x_4^2$. The canonical decomposition of $x_1x_2x_4^2$ is $(x_1x_4)(x_2x_4)$. It corresponds to the standard monomial $[1, 3][2, 3]$. Then we take the difference $[1, 2][2, 4] - [1, 2, 3][2] - [1, 3][2, 3]$ which is 0. Hence

$$[1, 2][2, 4] = [1, 2, 3][2] + [1, 3][2, 3]$$

is the standard representation of $[1, 2][2, 4]$.

2. THE STRAIGHTENING LAW

We have shown that the standard monomials are a K-basis of $K[X]$. As in the classical cases, we would like to show also that all the relevant ideals of minors of X have a K-basis of standard monomials. To this end one has to be able to control somehow the shape of the standard monomials which appear in the straightening law of products of minors. From the algorithm which is implicitly given in the proof of 1.2 it is impossible to control these invariants.

In this section we will show that the straightening law for products of minors can be determined by means of iterated applications of general relations among minors. This will allow us to control the straightening law.

Let δ be a product of minors of X, say $\delta = \delta_1 \cdots \delta_k$. Let s_i denote the size of δ_i, and assume that $s_1 \geqslant s_2 \cdots \geqslant s_k$. The sequence of integers $s_1, ..., s_k$ is the *shape* of δ.

Note first that one has the following elementary relations:

$$[a_1, ..., a_t \mid b_1, ..., b_t] = [b_1, ..., b_t \mid a_1, ..., a_t]$$

and

$$[a_1 + 1, ..., a_t + 1 \mid b_1, ..., b_t] = [a_1, ..., a_t \mid b_1 + 1, ..., b_t + 1].$$

To state other relations we introduce a piece of notation. Let $a = a_1, ..., a_t$ be a sequence of integers and $I \subset \{1, 2, ..., t\}$. We denote by $a^{\wedge I}$ the sequence which is obtained from a by omitting those a_i with $i \in I$, by a_I the sequence of the a_i with $i \in I$. Furthermore we set $e(I) = e_1, ..., e_t$ where $e_i = 1$ if $i \in I$ and $e_i = 0$ if $i \notin I$. If $I = \{i\}$ then $a^{\wedge I}$ will be denoted by $a^{\wedge i}$. One has:

LEMMA 2.1. (a) *Let $\alpha = \alpha_1, ..., \alpha_t$ and $\beta = \beta_1, ..., \beta_t$ be sequences of positive integers. Then for all $k = 1, ..., t$ one has*

$$\sum_{I \subset \{1, ..., t\},\ |I| = k} [\alpha + e(I) \mid \beta] = \sum_{J \subset \{1, ..., t\},\ |J| = k} [\alpha \mid \beta + e(J)]$$

(b) *Let $\alpha = \alpha_1, ..., \alpha_{s+1}$, $\beta = \beta_1, ..., \beta_s$ and $\gamma = \gamma_1, ..., \gamma_{r+1}$, $\delta = \delta_1, ..., \delta_r$ be sequences of integers. Then*:

$$\sum_{i=1}^{s+1} (-1)^i [\alpha^{\wedge i} \mid \beta][\gamma \mid \alpha_i, \delta] = \sum_{j=1}^{r+1} (-1)^j [\gamma^{\wedge j} \mid \delta][\alpha \mid \gamma_j, \beta]$$

Proof. (a) Set $(-1)^I = (-1)^{\sum_{i \in I} i}$. Expanding the minor $[\alpha + e(I) \mid \beta]$ with respect to the rows with indices I and expanding the minor $[\alpha \mid \beta + e(J)]$ with respect to the columns with indices J one has:

$$\sum_I [\alpha + e(I) \mid \beta]$$

$$= \sum_I \sum_J (-1)^I (-1)^J [\alpha_I + 1 \mid \beta_J][\alpha^{\wedge I} \mid \beta^{\wedge J}]$$

$$= \sum_J \sum_I (-1)^J (-1)^I [\alpha_I \mid \beta_J + 1][\alpha^{\wedge I} \mid \beta^{\wedge J}] = \sum_J [\alpha \mid \beta + e(J)].$$

(b) Expanding the minors $[\gamma \mid \alpha_i, \delta]$ and $[\alpha \mid \gamma_j, \beta]$ with respect to the first column one has:

$$\sum_i (-1)^i [\alpha^{\wedge i} \mid \beta][\gamma \mid \alpha_i, \delta]$$

$$= \sum_i (-1)^i [\alpha^{\wedge i} \mid \beta] \sum_j (-1)^{1+j} [\gamma_j \mid \alpha_i][\gamma^{\wedge j} \mid \delta]$$

$$= \sum_{i, j} (-1)^{i+j+1} [\alpha^{\wedge i} \mid \beta][\gamma_j \mid \alpha_i][\gamma^{\wedge j} \mid \delta]$$

$$= \sum_j (-1)^j [\gamma^{\wedge j} \mid \delta] \sum_i (-1)^{1+i} [\alpha_i \mid \gamma_j][\alpha^{\wedge i} \mid \beta]$$

$$= \sum_j (-1)^j [\gamma^{\wedge j} \mid \delta][\alpha \mid \gamma_j, \beta] \quad \blacksquare$$

The first consequence of the lemma is the following well-know result (see [GP, Lemma 2.3] or [W, Proposition 5])

COROLLARY 2.2. (a) *If $j > t$, then every t-minor of X_j is a linear combination of t-minors of X_{j-1},*

(b) *$I_t(X_j) = I_t(X_t)$ for all $t \leqslant j \leqslant n+1-t$,*

(c) *Every t-minor of X is a linear combination of maximal t-minors.*

Proof. (a) Let $[c \mid d] = [c_1, ..., c_t \mid d_1, ..., d_t]$ be a t-minor of X_j. Assume $c_i < c_{i+1}$ and $d_i < d_{i+1}$ for all i. If $c_t < j$ then $[c \mid d]$ is already a t-minor of X_{j-1}. If $c_t = j$, then let h be the smallest integer such that $c_h = j + h - t$. Now applying the equation 2.1(a) to the sequences $\alpha = c_1, ..., c_{h-1}, c_h - 1, ..., c_t - 1$, $\beta = d$ and with $k = t - h + 1$ one writes $[c \mid d]$ as a linear combination of t-minors which are either in X_{j-1} or they are in X_j but with a bigger "h". Arguing by induction on $t - h$ one obtains the desired expression.

(b) By (a) we have $I_t(X_{n+1-t}) \subseteq I_t(X_{n-t}) \subseteq \cdots \subseteq I_t(X_t)$ and $I_t(X_t) = I_t(X_{n+1-t})$ because X_{n+1-t} is the transpose on X_t. Finally, statement (c) follows immediately from (b). $\blacksquare$

The crucial case in determining the straightening law for products of minors is the following:

PROPOSITION 2.3. *Let δ_1 and δ_2 be minors of X of size s and r. Let μ be a standard monomial which appears in the standard representation of $\delta_1\delta_2$. Then μ has at most two factors and one of its factors has size bigger than or equal to* $\max\{s, r\}$.

Proof. We may assume that $s \geqslant r$. We argue by induction on r and in $(\delta_1\,\delta_2)$. In the case $r=0$ the claim follows from 2.2(c). Now let $r>0$. Let $\delta_1=[a_1, ..., a_s \mid b_1, ..., b_s]$ and $\delta_2=[c_1, ..., c_r \mid d_1, ..., d_r]$. Let $\delta_1=\mu+\sum_i \alpha_i\mu_i$ and $\delta_2=\mu'+\sum_i \alpha'_i\mu'_i$ be the standard representations of δ_1 and δ_2, with $\mathrm{in}(\delta_1)=\mathrm{in}(\mu)>_L \mathrm{in}(\mu_i)$ and $\mathrm{in}(\delta_2)=\mathrm{in}(\mu')>_L \mathrm{in}(\mu'_i)$. By construction $\mu=[a_1+b_1-1, a_2+b_2-2, ..., a_s+b_s-s]$ and $\mu'=[c_1+d_1-1, c_2+d_2-2, ..., c_r+d_r-r]$. We have that

$$\delta_1\,\delta_2=\mu\mu'+\begin{array}{l}\text{linear combination of products of minors of shape } s, r\\ \text{and initial monomial smaller than } \mathrm{in}(\delta_1\,\delta_2).\end{array}$$

If $\mu\mu'$ is standard then the desired results follows by induction. If $\mu\mu'$ is not standard, then there exists $1 \leqslant k \leqslant r$ such that

$$a_h+b_h<_1 c_k+d_k<a_{h+1}+b_{h+1}$$

for some $0 \leqslant h \leqslant r$, where for systematic reason we have put

$$a_0=b_0=0 \qquad \text{and} \qquad a_{s+1}=b_{s+1}=\infty.$$

We may assume that k is the minimum of the integers with these properties. Note that h is uniquely determined by k. We will argue also by induction on (k, h). Now let

$$a'_i=\lfloor (a_i+b_i)/2 \rfloor, \qquad b'_i=\lceil (a_i+b_i)/2 \rceil,$$

$$c'_i=\lfloor (c_i+d_i)/2 \rfloor, \qquad d'_i=\lceil (c_i+d_i)/2 \rceil,$$

where $\lfloor x \rfloor=\max\{n \in \mathbf{N} : n \leqslant x\}$ and $\lceil x \rceil=\min\{n \in \mathbf{N} : n \geqslant x\}$. Let $\delta_3=[a'_1, ..., a'_s \mid b'_1, ..., b'_s]$ and $\delta_4=[c'_1, ..., c'_s \mid d'_1, ..., d'_s]$. By construction $a_i+b_i=a'_i+b'_i$ and $c_i+d_i=c'_i+d'_i$. Hence we have as above that

$$\delta_3\,\delta_4=\mu\mu'+\begin{array}{l}\text{linear combination of products of minors of shape } s, r\\ \text{and initial monomial smaller than } \mathrm{in}(\delta_1\,\delta_2),\end{array}$$

and thus we have

$$\delta_1\,\delta_2=\delta_3\,\delta_4+\begin{array}{l}\text{linear combination of products of minors of shape } s, r\\ \text{and initial monomial smaller than } \mathrm{in}(\delta_1\,\delta_2).\end{array}$$

Therefore by induction it suffices to prove the claim for $\delta_3\,\delta_4$, that is, we may assume that

$$a_i \leqslant b_i \leqslant a_i + 1 \qquad \text{and} \qquad c_j \leqslant d_j \leqslant c_j + 1$$

for $i = 1, ..., s$ and $j = 1, ..., r$.

Now from $a_h + b_h <_1 c_k + d_k < a_{h+1} + b_{h+1}$ it follows easily that either:

$$(1) \quad a_h < c_k < a_{h+1} \qquad \text{and} \qquad b_h < d_k \leqslant b_{h+1}$$

or

$$(2) \quad a_h < c_k \leqslant a_{h+1} \qquad \text{and} \qquad b_h < d_k < b_{h+1}$$

In the case (1) we apply the equation 2.1(b) to the sequences $\alpha = (c_k, a)$, $\beta = b$, $\gamma = d$, $\delta = c\,{}^{\wedge k}$, and we obtain the expression:

$$\delta_1\,\delta_2 = \sum_{i=1}^{s} \pm [c_k, a\,{}^{\wedge i} \mid b][a_i, c\,{}^{\wedge k} \mid d] + \begin{array}{l}\text{linear combination of products of}\\ \text{minors of shape } s+1, r-1\end{array}$$

By virtue 2.4 the desired conclusion follows by induction. In case (2) we apply the equation 2.1(b) to the sequences $\alpha = (d_k, b)$, $\beta = a$, $\gamma = c$, $\delta = d\,{}^{\wedge k}$, and as above the desired conclusion follows from 2.5. ▮

LEMMA 2.4. *Assume* (1). *Then for all i such that* $[c_k, a\,{}^{\wedge i} \mid b][a_i, c\,{}^{\wedge k} \mid d] \neq 0$ *one has either*

$(*)$ $\quad \mathrm{in}([c_k, a\,{}^{\wedge i} \mid b][a_i, c\,{}^{\wedge k} \mid d]) <_L \mathrm{in}(\delta_1\,\delta_2)$,

or

$(**)$ $\quad \mathrm{in}([c_k, a\,{}^{\wedge i} \mid b][a_i, c\,{}^{\wedge k} \mid d]) = \mathrm{in}(\delta_1\,\delta_2)$ *and* $[c_k, a\,{}^{\wedge i} \mid b][a_i, c\,{}^{\wedge k} \mid d]$ *has a bigger "*(k, h)*" than* $\delta_1\,\delta_2$ *with respect to the lexicographical order* (*which includes also the case in which it is standard*)

LEMMA 2.5. *Assume* (2). *For all i such that* $[a \mid d_k, b\,{}^{\wedge i}][c \mid b_i, d\,{}^{\wedge k}] \neq 0$ *one has either*

$(*)$ $\quad \mathrm{in}([a \mid d_k, b\,{}^{\wedge i}][c \mid b_i, d\,{}^{\wedge k}]) <_L \mathrm{in}(\delta_1\,\delta_2)$,

or

$(**)$ $\quad \mathrm{in}([a \mid d_k, b\,{}^{\wedge i}][c \mid b_i, d\,{}^{\wedge k}]) = \mathrm{in}(\delta_1\,\delta_2)$ *and* $[a \mid d_k, b\,{}^{\wedge i}][c \mid b_i, d\,{}^{\wedge k}]$ *has a bigger "*(k, h)*" than* $\delta_1\,\delta_2$ *with respect to the lexicographical order* (*which includes also the case in which it is standard*).

Now to complete the proof of 2.3 it remains to prove 2.4 and 2.5. Since the nature of the two cases is essentially the same we prove only 2.4.

Proof of 2.4. By assumption we have

$$a_h < c_k < a_{h+1} \qquad \text{and} \qquad b_h < d_k \leqslant b_{h+1}$$

Let $1 \leqslant i \leqslant s$ such that $[c_k, a^{\wedge i} \mid b][a_i, c^{\wedge k} \mid d] \neq 0$. In order to compare the initial terms, we have to rewrite the sequences $c_k, a^{\wedge i}$ and $a_i, c^{\wedge k}$ in ascending order. There exists j, $1 \leqslant j \leqslant r+1$, such that

$$c_{j-1} < a_i < c_j$$

(by systematic reason we have put $c_0 = 0$ and $c_{s+1} = \infty$). It is easy to see that:

$$a_i < c_k \Leftrightarrow j \leqslant k \Leftrightarrow i \leqslant h.$$

We discuss first the:

Case (1). $a_i < c_k$, $j \leqslant k$, $i \leqslant h$. After reordering the sequences one has:

$$\begin{array}{rcccccccccc} c_k, a^{\wedge i}: & a_1 & \cdots & a_{i-1} & a_{i+1} & \cdots & a_h & c_k & a_{h+1} & \cdots & a_s \\ b: & b_1 & \cdots & b_{i-1} & b_i & \cdots & b_{h-1} & b_h & b_{h+1} & \cdots & b_s \end{array}$$

and

$$\begin{array}{rcccccccccc} a_i, c^{\wedge k}: & c_1 & \cdots & c_{j-1} & a_i & c_j & \cdots & c_{k-1} & c_{k+1} & \cdots & c_r \\ d: & d_1 & \cdots & d_{j-1} & d_j & d_{j+1} & \cdots & d_k & d_{k+1} & \cdots & d_r \end{array}$$

The first $i-1$ and last $s-h$ row and column indices of $[c_k, a^{\wedge i} \mid b]$ and $[a \mid b]$ coincide and the same is also true for the first $j-1$ and last $r-k$ indices of $[a_i, c^{\wedge k} \mid d]$ and $[c \mid d]$. Hence to compare initial monomials we may restrict our attention to the following subsequences that we denote for simplicity again by c_k, $a^{\wedge i}$, b, ... etc.:

$$\begin{array}{rcccc} c_k, a^{\wedge i}: & a_i + 1 & \cdots & a_h & c_k \\ b: & b_i & \cdots & b_{h-1} & b_h \end{array} \qquad \begin{array}{rcccc} a_i, c^{\wedge k}: & a_i & c_j & \cdots & c_{k-1} \\ d: & d_j & d_{j+1} & \cdots & d_k \end{array}$$

$$\begin{array}{rcccc} a: & a_i & \cdots & a_{h-1} & a_h \\ b: & b_i & \cdots & b_{h-1} & b_h \end{array} \qquad \begin{array}{rcccc} c: & c_j & c_{j+1} & \cdots & c_k \\ d: & d_j & d_{j+1} & \cdots & d_k \end{array}$$

There are four subcases to be discussed:

(1.1). $i=h$ and $j=k$. One has

$$\begin{array}{ll} c_k, a^{\wedge h} : c_k & a_h, c^{\wedge k} : a_h \\ b : b_h & d : d_k \\ \\ a : a_h & c : c_k \\ b : b_h & d : d_k \end{array}$$

and $a_h + b_h < \min(a_h + d_k, c_k + b_h)$. Hence $(*)$ holds.

(1.2). $i=h$ and $j<k$. One has:

$$\begin{array}{llllll} c_k, a^{\wedge h} : c_k & a_h, c^{\wedge k} : a_h & c_j & & \cdots & c_{k-1} \\ b : b_h & & d : d_j & d_{j+1} & \cdots & d_k \\ a : a_h & & c : c_j & c_{j+1} & \cdots & c_k \\ b : b_h & & d : d_j & d_{j+1} & \cdots & d_k \end{array}$$

If $b_h < d_j$ then $a_h + b_h < \min(c_k + b_h, a_h + d_j)$ and $(*)$ holds. Otherwise $d_j \leqslant b_h \leqslant a_h + 1 \leqslant c_j \leqslant d_j$ and hence $d_j = b_h = a_h + 1 = c_j$, and $a_h + b_h = a_h + d_j$. But then $c_j + d_j < \min(c_k + b_h, c_j + d_{j+1})$ and thus $(*)$ holds.

(1.3). $i<h$ and $j=k$. One has:

$$\begin{array}{llllll} c_k, a^{\wedge i} : a_{i+1} & \cdots & a_h & c_k & & a_i, c^{\wedge k} : a_i \\ b : \ b_i & \cdots & b_{h-1} & b_h & & d : d_k \\ \\ a : \ a_i & \cdots & a_{h-1} & a_h & & c : c_k \\ b : \ b_i & \cdots & b_{h-1} & b_h & & d : d_k \end{array}$$

If $b_i < d_k$ then $(*)$ holds. In the opposite case one has $d_k \leqslant b_i \leqslant a_i + 1 \leqslant a_h < c_k \leqslant d_k$ which is a contradiction.

(1.4). $i<h$ and $j<k$. Since $a_i + b_i < a_{i+1} + b_i$, if $a_i + b_i < a_i + d_j$ then $(*)$ holds. If $d_j \leqslant b_i$ then one has $d_j \leqslant b_i \leqslant a_i + 1 \leqslant c_j \leqslant d_j$, that is to say $d_j = b_i = a_i + 1 = c_j$. Set $x = d_j$. Now let v be the maximum of the integers with the properties $0 \leqslant v \leqslant \min(h-i, k-j)$, and

$$b_{i+u} = c_{j+u} = d_{j+u} = x + u, \quad \text{and} \quad a_{i+u} = x + u - 1 \quad \text{for} \quad u = 0, ..., v.$$

Then the sequences have the form:

$$
\begin{array}{lcccccccccc}
c_k, a^{\wedge i}: & x & x+1 & \cdots & x+v-1 & a_{i+v+1} & \cdots & \cdots & a_h & c_k \\
b: & x & x+1 & \cdots & x+v-1 & x+v & b_{i+v+1} & \cdots & b_{h-1} & b_h
\end{array}
$$

$$
\begin{array}{lccccccccc}
a: & x-1 & x & \cdots & \cdots & x+v-1 & a_{i+v+1} & \cdots & \cdots & a_h \\
b: & x & x+1 & \cdots & \cdots & x+v & b_{i+v+1} & \cdots & \cdots & b_h
\end{array}
$$

$$
\begin{array}{lccccccccc}
a_i, c^{\wedge k}: & x-1 & x & \cdots & x+v-1 & x+v & c_{j+v+1} & \cdots & c_{k-1} \\
d: & x & x+1 & \cdots & x+v & d_{j+v+1} & \cdots & \cdots & d_k
\end{array}
$$

$$
\begin{array}{lcccccccc}
c: & x & x+1 & \cdots & x+v & c_{j+v+1} & \cdots & \cdots & c_k \\
d: & x & x+1 & \cdots & x+v & d_{j+v+1} & \cdots & \cdots & d_k
\end{array}
$$

As above we may skip from the sequences those indices whose contribution to the initial monomials cancel one against the other. Hence we may restrict our attention to the following subsequences:

$$
\begin{array}{lccccc}
c_k, a^{\wedge i}: & a_{i+v+1} & \cdots & \cdots & a_h & c_k \\
b: & x+v & b_{i+v+1} & \cdots & b_{h-1} & b_h
\end{array}
$$

$$
\begin{array}{lcccc}
a: & a_{i+v+1} & \cdots & \cdots & a_h \\
b: & b_{i+v+1} & \cdots & \cdots & b_h
\end{array}
$$

$$
\begin{array}{lcccc}
a_i, c^{\wedge k}: & x+v & c_{j+v+1} & \cdots & c_{k-1} \\
d: & d_{j+v+1} & \cdots & \cdots & d_k
\end{array}
$$

$$
\begin{array}{lccccc}
c: & x+v & c_{j+v+1} & \cdots & \cdots & c_k \\
d: & x+v & d_{j+v+1} & \cdots & \cdots & d_k
\end{array}
$$

We have three subcases

(1.4.1). $v=k-j\leqslant h-i$. We have $c_k=d_k=b_{i+k-j}=x+k-j$ and $a_{i+k-j}=x+k-j-1$. Since $h\geqslant i+k-j$ we have $a_h+b_h\geqslant a_{i+k-j}+b_{i+k-j}=c_k+d_k-1$ which contradicts the assumption $a_h<c_k$ and $b_h<d_k$.

(1.4.2): $v = h - i < k - j$. The sequences have the form:

$$\begin{array}{llllll} c_k, a^{\wedge i}: & c_k & a_i, c^{\wedge k}: x+h-i & c_{j+h-i+1} & \cdots & c_{k-1} \\ & b: x+h-i & d: d_{j+h-i+1} & \cdots & \cdots & d_k \\ & a: & c: x+h-i & c_{j+h-i+1} & \cdots \ \cdots & c_k \\ & b: & d: x+h-i & d_{j+h-i+1} & \cdots \ \cdots & d_k \end{array}$$

Since $x+h-i = d_{j+h-i} < d_{j+h-i+1}$ and $x+h-i = c_{j+h-i} < c_k$, we have $2(x+h-1) < \min(x+h-i+d_{j+h-i+1}, x+h-i+c_k)$ and hence $(*)$ holds.

(1.4.3): $v < \min(h-i, k-j)$. If $a_{i+v+1} > x+v$, then $(*)$ holds. If $a_{i+v+1} \leqslant x+v$, then $a_{i+v+1} = x+v$ because $a_{i+v} = x+v-1$. In this case, if $b_{i+v+1} < d_{j+v+1}$, then $(*)$ holds. If instead one has $b_{i+v+1} \geqslant d_{j+v+1}$, then $b_{i+v+1} \leqslant a_{i+v+1} + 1 = x+v+1 = c_{j+v} + 1 \leqslant c_{j+v+1} \leqslant d_{j+v+1} \leqslant b_{i+v+1}$. It follows that $c_{j+v+1} = d_{j+v+1} = b_{i+v+1} = x+v+1$ which is a contradiction because of the definition of v.

Case (2). $a_i > c_k, j > k, i > h$: After reordering the sequences and skipping those indices whose contribution to the initial monomials cancel one against the other, one has:

$$\begin{array}{lllll} c_k, a^{\wedge i}: c_k & a_{h+1} & \cdots & a_{i-1} & \qquad a_i, c^{\wedge k}: c_{k+1} \ \cdots \ c_{j-1} \ \ a_i \\ b: b_{h+1} & b_{h+2} & \cdots & b_i & \qquad d: d_k \ \cdots \ d_{j-2} \ d_{j-1} \\ a: a_{h+1} & \cdots & \cdots & a_i & \qquad c: c_k \ \cdots \ \cdots \ c_{j-1} \\ b: b_{h+1} & \cdots & \cdots & b_i & \qquad d: d_k \ \cdots \ \cdots \ d_{j-1} \end{array}$$

There are four subcases:

(2.1). $i = h+1$ and $j = k+1$. The sequences have the form:

$$\begin{array}{ll} c_k, a^{\wedge h+1}: \ c_k & a_{h+1}, c^{\wedge k}: a_{h+1} \\ b: b_{h+1} & d: \ d_k \\ a: a_{h+1} & c: \ c_k \\ b: b_{h+1} & d: \ d_k \end{array}$$

If $d_k < b_{h+1}$ then $(*)$ holds. If $d_k \geqslant b_{h+1}$ then $d_k = b_{h+1}$. In this case the initial monomials of $[a \mid b][c \mid d]$ and $[c_k, a^{\wedge h+1} \mid b][a_{h+1}, c^{\wedge k} \mid b]$ coincide. We show that in this case $(**)$ holds. Given a minor (or better a bivector) $[\alpha \mid \beta] = [\alpha_1, ..., \alpha_p \mid \beta_1, ..., \beta_p]$, we put $[\alpha \mid \beta]_u = \alpha_u + \beta_u$. Now assume that $[c_k, a^{\wedge h+1} \mid b]_u <_1 [a_{h+1}, c^{\wedge k} \mid b]_v < [c_k, a^{\wedge h+1} \mid b]_{u+1}$. We have to show that (v, u) is bigger than (k, h) in the lexicographical order. Assume by contradiction that (v, u) is smaller than or equal to (k, h) in the lexicographical order. Note that $[c_k, a^{\wedge h+1} \mid b]_p = [a \mid b]_p$ for all $p \neq h+1$ and $[a_{h+1}, c^{\wedge k} \mid b]_q = [c \mid d]_q$ for all $q \neq k$. If $v < k$ and $u < h$, then one has a contradiction because k was chosen minimally. If $v < k$ and $u = h$, then one has a contradiction as in the previous case because $[c_k, a^{\wedge h+1} \mid b]_{h+1} = c_k + b_{h+1} < a_{h+1} + b_{h+1}$. If $v < k$ and $u > h$, then

$$\begin{aligned} c_k + d_k = c_k + b_{h+1} &= [c_k, a^{\wedge h+1} \mid b]_{h+1} \\ &\leqslant [c_k, a^{\wedge h+1} \mid b]_u <_1 [a_{h+1}, c^{\wedge k} \mid b]_v = c_v + d_v \end{aligned}$$

which is a contradiction. If $v = k$ and $u \leqslant h$ then

$$\begin{aligned} a_{h+1} + d_k = [a_{h+1}, c^{\wedge k} \mid b]_k &< [c_k, a^{\wedge h+1} \mid b]_{u+1} \leqslant [c_k, a^{\wedge h+1} \mid b]_{h+1} \\ &= c_k + b_{h+1} = c_k + d_k < a_{h+1} + d_k \end{aligned}$$

which is again a contradiction. This conclude the discussion of (2.1).

(2.2). $i = h+1$ and $j > k+1$. The sequences have the form:

$$\begin{array}{llllllll} c_k, a^{\wedge h+1}: & c_k & \qquad a_{h+1}, c^{\wedge k}: & c_{k+1} & \cdots & c_{j-1} & a_{h+1} \\ b: & b_{h+1} & \qquad d: & d_k & \cdots & d_{j-2} & d_{j-1} \\ & & & & & & \\ a: & a_{h+1} & \qquad c: & c_k & \cdots & \cdots & c_{j-1} \\ b: & b_{h+1} & \qquad d: & d_k & \cdots & \cdots & d_{j-1} \end{array}$$

If $d_k < b_{h+1}$ then $(*)$ holds. As above in the opposite case one has $d_k = b_{h+1}$. If $a_{h+1} < c_{k+1}$ then $(*)$ holds. If $a_{h+1} \geqslant c_{k+1}$ then $c_{k+1} \leqslant a_{h+1} \leqslant b_{h+1} = d_k \leqslant c_k + 1 \leqslant c_{k+1}$ and hence $c_{k+1} = a_{h+1} = b_{h+1} = d_k = c_k + 1$. It follows that $(*)$ holds unless $j = k+2$. In this last case the initial monomials of $[a \mid b][c \mid d]$ and $[c_k, a^{\wedge h+1} \mid b][a_{h+1}, c^{\wedge k} \mid d]$ coincide. As in (2.1) one shows that $(**)$ holds.

(2.3). $i > h+1$ and $j = k+1$. The sequences have the form:

$$\begin{array}{llllllll} c_k, a^{\wedge i}: & c_k & a_{h+1} & \cdots & a_{i-1} & \qquad a_i, c^{\wedge k}: & a_i \\ b: & b_{h+1} & b_{h+2} & \cdots & b_i & \qquad d: & d_k \\ & & & & & & \\ a: & a_{h+1} & \cdots & \cdots & a_i & \qquad c: & c_k \\ b: & b_{h+1} & \cdots & \cdots & b_i & \qquad d: & d_k \end{array}$$

If $d_k < b_{h+1}$ then $(*)$ holds. Otherwise $d_k = b_{h+1}$ and then $a_{h+1} + b_{h+1} < \min(a_{h+1} + b_{h+2}, a_i + d_k)$. Hence $(*)$ holds.

(2.4). $i>h+1$, $j>k+1$. If $d_k<b_{h+1}$ then $(*)$ holds. So we may assume $d_k=b_{h+1}$. If $a_{h+1}<c_{k+1}$ then $(*)$ holds. On the other hand if $a_{h+1}\geqslant c_{k+1}$, then it follows as above that $c_{k+1}=a_{h+1}=b_{h+1}=d_k=d_{k+1}-1$. Therefore we may define v to be the maximum of the integers with the properties: $1\leqslant v\leqslant \min(j-k-1, i-h)$ and $a_{h+u}=b_{h+u}=c_{k+u}=d_{k+u}-1=d_k-1+u$ for all $1\leqslant u\leqslant v$. Set $x=d_k-1$. Again we may consider only the relevant part of the sequences. They are:

$$\begin{array}{rcccc} c_k, a^{\wedge i}: & x+v & a_{h+v+1} & \cdots & a_{i-1} \\ b: & b_{h+v+1} & \cdots & \cdots & b_i \end{array}$$

$$\begin{array}{rcccc} a: & a_{h+v+1} & \cdots & \cdots & a_i \\ b: & b_{h+v+1} & \cdots & \cdots & b_i \end{array}$$

$$\begin{array}{rcccc} a_i, c^{\wedge k}: & c_{k+v+1} & \cdots & c_{j-1} & a_i \\ d: & x+v+1 & \cdots & d_{j-2} & d_{j-1} \end{array}$$

$$\begin{array}{rcccc} c: & x+v & c_{k+v+1} & \cdots & c_{j-1} \\ d: & x+v+1 & d_{k+v+1} & \cdots & d_{j-1} \end{array}$$

We have three subcases:

(2.4.1). $v=i-h\leqslant j-k-1$. One has $a_i=a_{h+i-h}=x+i-h+1$ and $c_{j-1}\geqslant c_{k+i-h}=x+i-h+1$. This is a contradiction because by assumption $c_{j-1}<a_i$.

(2.4.2). $v=j-k-1<i-h$. The sequences are:

$$\begin{array}{rcccc} c_k, a^{\wedge i}: & x+v & a_{h+v+1} & \cdots & a_{i-1} \\ b: & b_{h+v+1} & \cdots & \cdots & b_i \end{array}$$

$$\begin{array}{rcccc} a: & a_{h+v+1} & \cdots & \cdots & a_i \\ b: & b_{h+v+1} & \cdots & \cdots & b_i \end{array}$$

$$\begin{array}{rc} a_i, c^{\wedge k}: & a_i \\ d: & x+v+1 \end{array}$$

$$\begin{array}{rc} c: & x+v \\ d: & x+v+1 \end{array}$$

One observes that $(*)$ holds unless $b_{h+v+1}=x+v+1$ and $v+1=i-h$. In the exceptional case the initial monomials of $[c_k, a^{\wedge i} \mid b][a_i, c^{\wedge k} \mid d]$ and $[a \mid b][c \mid d]$ are equal and one verifies that $(**)$ holds.

(2.4.3). $v<\min(j-k-1, i-h)$. If $x+v+1<b_{k+v+1}$ then $(*)$ holds. The same is true if $x+v+1=b_{k+v+1}$ and $a_{h+v+1}<c_{k+v+1}$. Finally if $x+v+1=b_{k+v+1}$ and $a_{h+v+1}\geqslant c_{k+v+1}$ then $c_{k+v+1}\leqslant a_{h+v+1}\leqslant b_{h+v+1}=x+v+1=d_{k+v}\leqslant d_{k+v+1}-1\leqslant c_{k+v+1}$, and hence $c_{k+v+1}=a_{h+v+1}=b_{h+v+1}=d_{k+v+1}-1=x+v+1$ which contradicts the definition of v. ▌

In order to state the straightening law for products of minors, we need to recall some notation. Given a product of minors δ of shape $s=s_1, ..., s_k$ and $t\in\mathbf{N}$ one defines

$$\gamma_t(\delta)=\sum_{i=1}^{k} \max\{s_i+1-t, 0\}$$

These functions were introduced in [DEP1] to describe the symbolic powers of generic determinantal ideals. We are now in the position to state the following important:

THEOREM 2.6 (Straightening law). *Let δ be a product of minors of X and let μ be a standard monomial which appears in the standard representation of δ. Then for all $t\in\mathbf{N}$ one has $\gamma_t(\delta)\leqslant\gamma_t(\mu)$.*

Proof. We introduce an order in the set of the products of minors. Let $\delta=\delta_1\cdots\delta_k$ and $\mu=\mu_1\cdots\mu_h$ be products of minors and assume that in$(\delta_1)\geqslant_L\cdots\geqslant_L$ in(δ_k) and in$(\mu_1)\geqslant_L\cdots\geqslant_L$ in(μ_h). We set $\delta\succ\mu$ if $(k, \text{in}(\delta), \text{in}(\delta_k), ..., \text{in}(\delta_1))$ is bigger than $(h, \text{in}(\mu), \text{in}(\mu_h), ..., \text{in}(\mu_1))$ in the lexicographical order, i.e., $k>h$ or $k=h$ and $\text{in}(\delta)>_L\text{in}(\mu)$ or $k=h$, $\text{in}(\delta)=\text{in}(\mu)$ and there exists p such that $\text{in}(\delta_p)>_L\text{in}(\mu_p)$ and $\text{in}(\delta_i)=\text{in}(\mu_i)$ for $i=p+1, ..., k$.

We argue by induction on $\succ$. Let $\delta=\delta_1\cdots\delta_k$ be a product of minors with the factors ordered as above, and let s_i be the size of δ_i. For $k=1, 2$ the claim follows from 2.3. Assume $k>2$. By virtue of 2.2(c) we may replace each δ_i by a linear combination of maximal minors of size s_i. By induction we may hence assume that the δ_i are maximal minors. If δ is a standard monomial then there is nothing to prove. If δ is not a standard monomial, then there exist i, j with $1\leqslant i<j\leqslant k$ such that $\delta_i\delta_j$ is not a standard monomial. Let $\delta_i\delta_j=\mu_1+\sum_{h=2}^{r}\alpha_h\mu_h$ be the standard representation of $\delta_i\delta_j$ and assume that $\text{in}(\delta_i\delta_j)=\text{in}(\mu_1)>_L\text{in}(\mu_h)$ for all $h>1$. By 2.3 we have that $\gamma_t(\delta_i\delta_j)\leqslant\gamma_t(\mu_h)$ for all h.

Replacing $\delta_i\delta_j$ with its standard representation we may write δ as a linear combinations of $\delta^{(h)}=\delta_1\cdots\delta_{i-1}\delta_{i+1}\cdots\delta_{j-1}\delta_{j+1}\cdots\delta_k\mu_h$, $h=1, ..., r$, and $\gamma_t(\delta)\leqslant\gamma_t(\delta^{(h)})$ for all h. It suffices now to show that $\delta\succ\delta^{(h)}$ for all h.

If $h>1$, then $\delta^{(h)}$ has at most k factors and $\operatorname{in}(\delta^{(h)}) <_L \operatorname{in}(\delta)$ and hence $\delta \succ \delta^{(h)}$.

For $h=1$, one notes that $\mu = n_1 n_2$ where n_1, n_2 are maximal minors with, say, $\operatorname{in}(n_1) >_L \operatorname{in}(n_2)$ and $\operatorname{in}(n_2) <_L \operatorname{in}(\mu_i)$, $\operatorname{in}(\mu_j)$. It follows that $\delta \succ \delta^{(1)}$. ∎

3. SYMBOLIC POWERS AND PRIMARY DECOMPOSITION OF POWERS OF IDEALS OF MINORS

We have seen in 2.2 that the ideal $I_t(X_j)$ of $K[X]$ generated by the t-minors of X_j does not depend on j but only on t and n. Hence we will denote it simply by I_t. The highest order of a minor in X is $\lfloor (n+1)/2 \rfloor$. Thus we consider only I_t with $1 \leqslant t \leqslant \lfloor (n+1)/2 \rfloor$. We set

$$m = \lfloor (n+1)/2 \rfloor.$$

It is well-known that I_2 is the defining ideal of the rational normal curve C of $\mathbf{P}^{n-1}$, while I_t defines the $(t-2)$-secant variety of C, see for instance [R, pag. 91, pag. 229] or [E2, Proposition 4.3]. The quotient ring $K[X]/I_t$ is known to be a Cohen–Macaulay normal domain of dimension $2(t-1)$, see [E2] and [W]. Further by a result of Valla [V] the ideal I_t is known to be a set-theoretic complete intersection. This section is devoted to determine the symbolic powers and the primary decomposition of the powers of the ideals I_t.

We say that an ideal I of $K[X]$ is an *ideal of standard monomials* if I has a basis as a K-vector space which consists of standard monomials. The class of ideals of standard monomials is obviously closed under sum and intersection.

Since distinct standard monomials have distinct initial monomials it follows immediately:

LEMMA 3.1. *Let I be an ideal of standard monomials. Let B denote the standard monomial K-basis of I. Then B is a Gröbner basis of I with respect to $>_L$. Furthermore, the monomials* $\operatorname{in}(\mu)$ *with $\mu \in B$ form a K-basis of* $\operatorname{in}(I)$.

DEFINITION 3.2. Let t and k be positive integers. We define $J_{t,k}$ to be the K-vector space generated by the standard monomials μ with $\gamma_t(\mu) \geqslant k$.

One has:

PROPOSITION 3.3. (a) *$J_{t,k}$ is an ideal of standard monomials,*

(b) *If δ is a product of minors and $\gamma_t(\delta) \geqslant k$, then $\delta \in J_{t,k}$,*

(c) *$I_t^k \subseteq J_{t,k}$, and $I_t = J_{t,1}$.*

Proof. (a) and (b) follow from the straightening law 2.6. The first part of (c) follows from (b). It remains to prove that $I_t \subseteq J_{t,1}$. To this end one notes that if μ is a standard monomial with $\gamma_t(\mu) \geqslant 1$ then μ has a factor of size $\geqslant t$. ∎

By virtue of 3.1 and 3.3 we have that the standard monomials in $J_{t,k}$ are a infinite Gröbner basis of $J_{t,k}$. A finite Gröbner basis is:

PROPOSITION 3.4. *The set $G_{t,k}$ of the standard monomials μ which have all the factors of size $\geqslant t$ and $\gamma_t(\mu) = k$ is a finite Gröbner basis of $J_{t,k}$ with respect to $<_L$.*

Proof. It is easy to see that whenever μ is a standard monomial with $\gamma(\mu) \geqslant k$ there exists $\mu_1 \in G_{t,k}$ with $\mathrm{in}(\mu_1) \mid \mathrm{in}(\mu)$. This proves that $G_{t,k}$ is a Gröbner basis of $J_{t,k}$. Note that the degree of the elements of $G_{t,k}$ is bounded by kt. Hence $G_{t,k}$ is finite. ∎

In particular $G_{t,1}$ is the set of the maximal t-minors of X and it is a Gröbner basis of I_t. It follows that the ideal $\mathrm{in}(I_t)$ is generated the $<_1$-chains of length t and hence it is a square-free monomial ideal associated with a simplicial complex that we denote by Δ_t. If $j = j_1, ..., j_{t-1}$ is a $<_1$-chain with $j_{t-1} \leqslant n-1$ then the set $F_j = \{j_1, j_1+1, ..., j_{t-1}, j_{t-1}+1\}$ is clearly a facet of Δ_t. Furthermore it is easy to see that any facets of Δ_t is of the form F_j for some $<_1$-chain j of length $t-1$ and bounded by $n-1$. Denote A_{t-1} the set of the $<_1$-chain of length $t-1$ bounded by $n-1$, and for $j \in A_{t-1}$ denote by P_j the ideal $(x_i \colon i \notin F_j)$. We have:

$$\mathrm{in}(I_t) = \bigcap_{j \in A_{t-1}} P_j$$

More generally:

LEMMA 3.5. *For all $t = 1, ..., m$ and $k \in \mathbf{N}$ one has:*

$$\mathrm{in}(J_{t,k}) = \bigcap_{j \in A_{t-1}} P_j^k$$

Proof. Let $\mu = \mu_1 \cdots \mu_p$ be a standard monomial, and let s_i denote the size of μ_i. The initial monomial $\mathrm{in}(\mu_j)$ is a $<_1$-chain of length s_i. Each facet of Δ_t contains at most $t-1$ points of the support of $\mathrm{in}(\mu_i)$. It follows that $\mathrm{in}(\mu_i) \in \bigcap_{j \in A_{t-1}} P_j^{\gamma_t(\mu_i)}$ and thus $\mathrm{in}(\mu) \in \bigcap_{j \in A_{t-1}} P_j^{\gamma_t(\mu)}$. Since $\mathrm{in}(J_{t,k})$ is generated by the initial monomials $\mathrm{in}(\mu)$ of the standard monomials μ with $\gamma_t(\mu) \geqslant k$, we have $\mathrm{in}(J_{t,k}) \subseteq \bigcap_{j \in A_{t-1}} P_j^k$.

Now let m be a monomial in $\bigcap_{j \in A_{t-1}} P_j^k$. Let μ the standard monomial which corresponds to m. We have to show that $\gamma_t(\mu) \geqslant k$. Let m_1 be the first factor in the canonical decomposition of m, that is, m_1 is the maximum (w.r.t. $>_L$) of the $<_1$-chains which divide m. Say $m_1 = x_{i_1} \cdots x_{i_s}$. It follows that $m = \prod_{h=1}^{s} x_{i_h}^{a_h} x_{i_h+1}^{b_h}$ with $a_h > 0$ (we have put $x_{n+1} = 1$ for systematic reasons). If $s < t$ then there exists $j \in A_{t-1}$ such that $\{i_1, i_1+1, ..., i_s, i_s+1\} \subset F_j$, and since $m \in P_j^k$, it follows that $k = 0$ which is a trivial case. So we may assume that $s \geqslant t$. Let $n = m/m_1 = \prod_{h=1}^{s} x_{i_h}^{a_h - 1} x_{i_h+1}^{b_h}$. By induction it suffices to show that $n \in \bigcap_{j \in A_{t-1}} P_j^{k-s-1+t}$, that is,

$$\sum_{i_h \notin F_j} (a_h - 1) + \sum_{i_h + 1 \notin F_j} b_h \geqslant k - s - 1 + t \tag{1}$$

for all $j \in A_{t-1}$. Denote by $O_P(m) = \max\{z : m \in P^z\}$. Note that the left hand side of (1) is equal to

$$O_{P_j}(m) - |\{h : i_h \notin F_j\}|$$

which is equal to

$$O_{P_j}(m) - s + |\{h : i_h \in F_j\}|$$

Set $r(j) = |\{h : i_h \in F_j\}|$. Thus (1) is equivalent to

$$O_{P_j}(m) + r(j) \geqslant k + t - 1 \tag{2}$$

for all $j \in A_{t-1}$. We prove (2) by decreasing induction on $r(j)$. In general $r(j) \leqslant t-1$. If $r(j) = t-1$ then (2) is trivially true because $O_{P_j}(m) \geqslant k$ by assumption. Assume that $r(j) < t-1$. Set $G = \{i_1, i_1+1, ..., i_s, i_s+1\}$. It suffices to prove the following:

Claim. Let $j \in A_{t-1}$ such that $r(j) < t-1$. Then there exists $z \in A_{t-1}$ such that $F_j \cap G \subset F_z$ and $r(z) > r(j)$.

From the claim it follows by straightforward computations that $O_{P_j}(m) + r(j) \geqslant O_{P_z}(m) + r(z)$, and, since $r(z) > r(j)$, (2) follows by induction.

We prove now the claim: Let $j = j_1, ..., j_{t-1}$.

If for some k one has $\{j_k, j_k+1\} \cap G = \emptyset$, then F_z is defined to be a facet which contains $(F_j \setminus \{j_k, j_k+1\}) \cup \{i_h\}$ where $i_h \notin F_j$.

Assume now that for all $k = 1, ..., t-1$ one has $\{j_k, j_k+1\} \cap G \neq \emptyset$. Since $r(j) < t-1$, there exists k such that $\{j_k, j_k+1\} \cap \{i_1, ..., i_s\} = \emptyset$. It follows that $j_k = i_q + 1$ for some q and $j_k + 1 \notin G$. Set $p = \max\{u : u < j_k$ and $u \notin F_j \cap G\}$. By construction one has either $p \in G \setminus F_j$ or $p \in F_j \setminus G$.

In the first case $p = i_h$ for some h, and F_z is defined to be equal to $(F_j \setminus \{j_k+1\}) \cup \{i_h\}$.

In the second case $q = j_d$ for some d and F_z is defined to be a facet of Δ_t which contains $(F_j \setminus \{j_d, j_k+1\}) \cup \{i_h\}$ where $i_h \notin F_j$. ∎

For later applications we record the following result which is implicit in [STV]:

LEMMA 3.6. *Let J be an ideal of a polynomial ring R and let τ be any monomial order. Then*

$$\max\{\text{height}(P)\colon P \in \text{Ass}(R/J)\} \leqslant \max\{\text{height}(P)\colon P \in \text{Ass}(R/\text{in}_\tau(J))\}$$

Proof. For $k \in \mathbf{N}$ denote by $J_{\leqslant k} = \{f \in R : \text{height}(J : f) > k\}$. One knows that $J_{\leqslant k}$ is the intersection of the primary components of height $\leqslant k$ of any primary decomposition of J, and that $\text{in}_\tau(J_{\leqslant k}) \subseteq \text{in}_\tau(J)_{\leqslant k}$, see [STV, pag. 420]. Put $h = \max\{\text{height}(P)\colon P \in \text{Ass}(R/\text{in}_\tau(J))\}$. Then one has

$$\text{in}_\tau(J) \subseteq \text{in}_\tau(J_{\leqslant h}) \subseteq \text{in}_\tau(J)_{\leqslant h} = \text{in}_\tau(J).$$

It follows that $\text{in}_\tau(J) = \text{in}_\tau(J_{\leqslant h})$ and therefore $J = J_{\leqslant h}$. Thus the associated prime ideals of J have height bounded by h. ∎

LEMMA 3.7. *Let r, s be integers with $1 \leqslant r, s \leqslant m$ and $s > r+1$. Then one has*

$$I_r I_s \subset I_{r+1} I_{s-1}.$$

Proof. Since $I_s(X_s)$ is the ideal of maximal minors of the matrix X_s, by virtue of [BV, Lemma 10.10] one has $I_r(X_s)\, I_s(X_s) \subseteq I_{r+1}(X_s)\, I_{s-1}(X_s)$. It follows that $I_r I_s = I_r(X_s)\, I_s(X_s) \subseteq I_{r+1}(X_s)\, I_{s-1}(X_s) = I_{r+1} I_{s-1}$. ∎

Recall that the k-symbolic power $P^{(k)}$ of a prime ideal P of a Noetherian ring R is defined to be the P-primary component of P^k. In other words,

$$P^{(k)} = R \cap P^k R_P = \{x \in R : \text{there exists } f \in R \backslash P \text{ such that } fx \in P^k\}.$$

The symbolic powers of a prime ideal form a filtration, that is, $P^{(k+1)} \subseteq P^{(k)}$ and $P^{(k)} P^{(h)} \subseteq P^{(k+h)}$.

THEOREM 3.8. *For all $t = 1, ..., m$ and $k \in \mathbf{N}$ one has*

$$I_t^{(k)} = J_{t,k}.$$

In particular:

$$I_t^{(k)} = \sum I_t^{a_t} I_{t+1}^{a_{t+1}} \cdots I_m^{a_m}$$

the sum being extended over all the sequences of non-negative integers $a_t, a_{t+1}, ..., a_m$, with $a_t + 2a_{t+1} + \cdots + (m-t+1)\, a_m = k$.

Proof. For $t=1$ the claim is trivial. Hence we assume that $1<t\leqslant m$. Let $s\geqslant t$. By virtue of 3.7

$$I_{t-1}^{s-t}I_s\subseteq I_{t-1}^{s-t-1}I_tI_{s-1}\subseteq\cdots\subseteq I_{t-1}^{s-t-j}I_t^jI_{s-j}\subseteq\cdots\subseteq I_t^{s-t}I_t=I_t^{s-t+1}.$$

This implies that $I_s\subseteq I_t^{(s-t+1)}$. Since the symbolic powers form a filtration for any product of minors μ one has $\mu\in I_t^{(\gamma_t(\mu))}$. Therefore $J_{t,k}\subseteq I_t^{(k)}$, and by 3.3, we know also that $I_t^k\subseteq J_{t,k}$. By virtue of 3.5 the ideal $\mathrm{in}(J_{t,k})$ is the intersection of the ideals P_j^k with $j\in A_{t-1}$. The ideals P_j are prime and complete intersections, and hence their powers are primary. It follows that $\mathrm{in}(J_{t,k})$ has no embedded prime ideals, and further all its minimal primes have the same height. By virtue of 3.6 we have that $J_{t,k}$ has no embedded prime ideals. Since $I_t^k\subseteq J_{t,k}\subseteq I_t^{(k)}$, it follows that $J_{t,k}=I_t^{(k)}$. ∎

Remark 3.9. Since the ideal I_2 defines a ring with isolated singularities (i.e. the rational normal curve is smooth), the only possible embedded prime ideal of I_2^k is the homogeneous maximal ideal. Thus $I_2^{(k)}$ is the saturation of I_2^k.

The theorem has the following corollaries:

COROLLARY 3.10. *The ideal $I_t^{(k)}$ is an ideal of standard monomials. In particular, the set of the standard monomials μ with $\gamma_t(\mu)\geqslant k$ is a K-basis of $I_t^{(k)}$. Further $G_{t,k}$ is a Gröbner basis of $I_t^{(k)}$.*

Proof. It follows from 3.3, 3.4 and 3.8. ∎

COROLLARY 3.11. *The ideal I_t has primary powers if and only if $t=1$ or $t=m$.*

Let $\tau=t_1,t_2,\ldots,t_k$ be a sequence of integers with $m\geqslant t_1\geqslant t_2\geqslant\cdots\geqslant t_k\geqslant 1$. Define

$$\gamma_j(\tau)=\sum_{i=1}^{k}\max\{t_i+1-j,0\}$$

For all the products of minors $\mu=\mu_1\cdots\mu_k$ of shape $\tau=t_1,t_2,\ldots,t_k$ and for all $j\in\mathbf{N}$ one has $\mu\in I_j^{(\gamma_j(\tau))}$ and thus

$$I_{t_1}\cdots I_{t_k}\subseteq\bigcap_{j=1}^{t_1}I_j^{(\gamma_j(\tau))}.$$

We want to show that:

THEOREM 3.12. *Let* $\tau = t_1, t_2, ..., t_k$ *be a sequence of integers with* $m \geqslant t_1 \geqslant t_2 \geqslant \cdots \geqslant t_k \geqslant 1$. *Then*

$$I_{t_1} \cdots I_{t_k} = \bigcap_{j=1}^{t_1} I_j^{(\gamma_j(\tau))}$$

is a possibly redundant primary decomposition of $I_{t_1} \cdots I_{t_k}$.

To prove the theorem we need some preliminary results. Note that $\bigcap_{j=1}^{t_1} I^{(\gamma_j(\tau))}$, being the intersection of ideals of standard monomials, is an ideal of standard monomials. Its K-basis is the set of the standard monomials μ with $\gamma_j(\mu) \geqslant \gamma_j(\tau)$ for all $j = 1, ..., t_1$.

LEMMA 3.13. *Let* n_1 *and* n_2 *be* $<_1$*-chains of* $K[X]$ *of length* s *and* r, *respectively, with* $s > r+1$. *Then there exist two* $<_1$*-chains* n_3, n_4 *of length* $s-1$ *and* $r+1$, *respectively, such that* $n_1 n_2 = n_3 n_4$.

Proof. Let $n_1 = x_{i_1} \cdots x_{i_s}$ and $n_2 = x_{j_1} \cdots x_{j_r}$. For $h = 1, ..., r$ we set $i'_h = \min(i_h, j_h)$ and $j'_h = \max(i_h, j_h)$. The sequences $i'_1, ..., i'_r, i_{r+1}, ..., i_s$ and $j'_1, ..., j'_r$ are $<_1$-chains, and hence we may assume that $i_h \leqslant j_h$ for all $h = 1, ..., r$. We have to distinguish two cases:

If $i_k <_1 j_k$ for some k, then take k be the minimum of the integers with this property. Then $j_{k-1} \leqslant i_{k-1} + 1 < i_k <_1 i_{k+1}$. Thus $j_1, ..., j_{k-1}, i_{k+1}, ..., i_s$ and $i_1, ..., i_k, j_k, ..., j_r$ are $<_1$-chains and one takes n_3 and n_4 to be the associated monomials.

If $i_k \not<_1 j_k$ for all k, then in particular $j_r \leqslant i_r + 1 < i_{r+1} <_1 i_s$. Thus $i_1, ..., i_{s-1}$ and $j_1, ..., j_r, i_s$ are $<_1$-chains and one takes n_3 and n_4 to be the associated monomials. ∎

LEMMA 3.14. *Let* $\tau = t_1, t_2, ..., t_k$ *be a sequence of integers with* $m \geqslant t_1 \geqslant t_2 \geqslant \cdots \geqslant t_k \geqslant 1$. *Let* $\mu = \mu_1 \cdots \mu_q$ *be a product of minors such that* $\gamma_j(\mu) \geqslant \gamma_j(\tau)$ *for all* $j = 1, ..., t_1$. *Then there exists a product of minors* $\delta_1, ..., \delta_k$ *of shape* τ *such that* $\mathrm{in}(\delta_1 \cdots \delta_k) \mid \mathrm{in}(\mu)$.

Proof. If one of the μ_i is a t_1-minor, then one concludes by induction on q because $\gamma_j(\mu_1 \cdots \mu_{i-1} \mu_{i+1} \cdots \mu_q) = \gamma_j(\mu) - (t_1 + 1 - j) \geqslant \gamma_j(\tau) - (t_1 + 1 - j) = \gamma_j(t_2, ..., t_k)$ for all $j = 1, ..., t_1$. Otherwise we may arrange the factors μ_i in ascending order according to their size and assume that $\mu_1, ..., \mu_p$ have size $< t_1$ and $\mu_{p+1}, ..., \mu_q$ have size $> t_1$. Let r be the size of μ_p and s be the size of μ_{p+1}. By virtue of 3.13 we may find two minors ρ_1 and ρ_2 of size $r+1$ and $s-1$ respectively such that $\mathrm{in}(\rho_1 \rho_2) = \mathrm{in}(\mu_p \mu_{p+1})$. Set $\mu' = \mu_1 \cdots \mu_{p-1} \rho_1 \rho_2 \mu_{p+2} \cdots \mu_q$. We claim that $\gamma_j(\mu') \geqslant \gamma_j(\tau)$ for $j = 1, ..., t_1$. Since $\mathrm{in}(\mu) = \mathrm{in}(\mu')$ we may then conclude by induction because μ' has either a factor of size t_1 or a smaller "$s - r$". To prove the claim one first notes that $\gamma_j(\mu')=\gamma_j(\mu)$ for all $j=1, ..., r+1$ and that $\gamma_j(\mu')=\gamma_j(\mu)-1$ for all $j=r+2, ..., t_1$. Arguing by contradiction we may assume that $\gamma_i(\mu)=\gamma_i(\tau)$ for some i, $r+2\leqslant i\leqslant t_1$. Since μ has no factors which has size bigger than r and smaller than or equal to t_1, one has $\gamma_{i-1}(\mu)=\gamma_i(\mu)+q-p$. It follows that $q-p+\gamma_i(\tau)\geqslant\gamma_{i-1}(\tau)$ and then

$$q-p\geqslant\gamma_{i-1}(\tau)-\gamma_i(\tau)=|\{h\colon t_h\geqslant i-1\}|.$$

One obtains

$$\begin{aligned}\gamma_i(\mu)=\gamma_i(\mu_{p+1}\cdots\mu_q)&\geqslant(q-p)(t_1+2-i)\geqslant|\{h\colon t_h\geqslant i-1\}|\,(t_1+2-i)\\&\geqslant|\{h\colon t_h\geqslant i\}|\,(t_1+2-i)>|\{h\colon t_h\geqslant i\}|\,(t_h+1-i)=\gamma_i(\tau)=\gamma_i(\mu)\end{aligned}$$

which is a contradiction. ∎

Proof of Theorem 3.12 Let J denote the ideal generated by the initial monomials of the products of minors of shape τ. Since $\mathrm{in}(\bigcap_{j=1}^{t_1} I^{(\gamma_j(\tau))})$ is generated by the initial monomials of the standard monomials μ with $\gamma_j(\mu)\geqslant\gamma_j(\tau)$ for all $j=1, ..., t_1$, by virtue of 3.14 one has $\mathrm{in}(\bigcap_{j=1}^{t_1} I_j^{(\gamma_j(\tau))})\subseteq J$. Then

$$J\subseteq\mathrm{in}(I_{t_1})\cdots\mathrm{in}(I_{t_k})\subseteq\mathrm{in}(I_{t_1}, ..., I_{t_k})\subseteq\mathrm{in}\left(\bigcap_{j=1}^{t_1} I_j^{(\gamma_j(\tau))}\right)\subseteq J$$

It follows that $I_{t_1}\cdots I_{t_k}=\bigcap_{j=1}^{t_1} I_j^{(\gamma_j(\tau))}$. ∎

The proof of the theorem as the following important:

COROLLARY 3.15. *Let $\tau=t_1, t_2, ..., t_k$ be a sequence of integers with $m\geqslant t_1\geqslant t_2\geqslant\cdots\geqslant t_k\geqslant 1$. Then the product of minors of shape τ form a Gröbner basis of the ideal $I_{t_1}\cdots I_{t_k}$. In particular one has*:

$$\mathrm{in}(I_{t_1}\cdots I_{t_k})=\mathrm{in}(I_{t_1})\cdots\mathrm{in}(I_{t_k})$$

We single out the most important case:

THEOREM 3.16. (a) *Let $1\leqslant t\leqslant m$ and $k\in\mathbf{N}$. Set $u=\max(1, m-k(m-t))$. Then*:

$$I_t^k=\bigcap_{j=u}^{t} I_j^{(k(t+1-j))}$$

is an irredundant primary decomposition of I_t^k.

(b) $\mathrm{in}(I_t^k)=\mathrm{in}(I_t)^k$ *for all k.*

Proof. (a) To determine an irredundant primary decomposition of I_t^k one has only to detect those components which are superflous in the decomposition $I_t^k = \bigcap_{j=1}^t I_j^{(k(t+1-j))}$ which is given in 3.12. Hence the question is whether one of the $I_j^{(k(t+1-j))}$ already contains the intersection of the others or not. This question can be refrased completely in terms of whether there exists an admissible shape with given γ-functions and it is exactly the same question that one has to answer in the generic case, see [DEP1, Corollary 7.3] and [BV, Corollary 10.13]. Hence the answer is the same as in the generic case.

(b) It is just a special case of 3.15. ∎

As a by-product of the proof of Theorem 3.12 we obtain a primary decomposition of the ideal $\mathrm{in}(I_{t_1}) \cdots \mathrm{in}(I_{t_k})$:

Corollary 3.17. *Let $\tau = t_1, t_2, ..., t_k$ be a sequence of integers with $m \geqslant t_1 \geqslant t_2 \geqslant \cdots \geqslant t_k \geqslant 1$. Then*

$$\mathrm{in}(I_{t_1}) \cdots \mathrm{in}(I_{t_k}) = \bigcap_{j=1}^{t_1} \bigcap_{z \in A_{j-1}} P_z^{\gamma_j(\tau)}$$

is a possibly redundant primary decomposition of $\mathrm{in}(I_{t_1}) \cdots \mathrm{in}(I_{t_k})$.

A consequence of 3.16 is the following

Corollary 3.18. *The set of associated primes of I_t^k is*

$$\mathrm{Ass}(K[X]/I_t^k) = \{I_u, I_{u+1}, ..., I_t\}.$$

In particular, if $t < m$ and $k \geqslant (m-1)/(m-t)$ then $u = 1$ and hence

$$\mathrm{Ass}(K[X]/I_t^k) = \{I_1, I_2, ..., I_t\} \text{ and depth } K[X]/I_t^k = 0.$$

4. SYMBOLIC AND ORDINARY REES ALGEBRAS

We next turn to the study of the symbolic and ordinary Rees algebras associated to the ideals I_t.

Let I be an ideal of a ring R. The Rees algebra $\mathscr{R}(I)$ of I is the R-graded algebra $\bigoplus_{k=0}^{\infty} I^k T^k$, where T is an indeterminate over R. In other words, $\mathscr{R}(I)$ can be identified with the R-subalgebra of $R[T]$ generated by IT. If I happens to be prime we may also consider the symbolic Rees algebra $\mathscr{R}^s(I)$, that is, $\mathscr{R}^s(I) = \bigoplus_{k=0}^{\infty} I^{(k)} T^k$. If R is a polynomial ring and τ a monomial order, then the initial algebra $\mathrm{in}(\mathscr{R}(I))$ of $\mathscr{R}(I)$ is $\mathrm{in}_\tau(\mathscr{R}(I)) = \bigoplus_{k=0}^{\infty} \mathrm{in}_\tau(I^k)\, T^k$. Similarly the initial algebra of $\mathrm{in}(\mathscr{R}^s(I))$ of $\mathscr{R}^s(I)$ is $\mathrm{in}_\tau(\mathscr{R}^s(I)) = \bigoplus_{k=0}^{\infty} \mathrm{in}_\tau(I^{(k)})\, T^k$.

It follows from 3.8 and 3.3 that:

PROPOSITION 4.1. *One has*:

$$\mathcal{R}^s(I_t) = K[X][I_t T, I_{t+1} T^2, \ldots, I_m T^{m-t+1}]$$
$$in(\mathcal{R}^s(I_t)) = K[X][\mathrm{in}(I_t)\, T, \mathrm{in}(I_{t+1})\, T^2, \ldots, \mathrm{in}(I_m)\, T^{m-t+1}].$$

In particular $\mathcal{R}^s(I_t)$ *and* $\mathrm{in}(\mathcal{R}^s(I_t))$ *are Noetherian.*

By virtue of 3.5 we have:

$$\mathrm{in}(\mathcal{R}^s(I_t)) = \bigoplus_{k=0}^{\infty} \left(\bigcap_{j \in A_{t-1}} P_j^k \right) T^k = \bigcap_{j \in A_{t-1}} \left(\bigoplus_{k=0}^{\infty} P_j^k T^k \right) = \bigcap_{j \in A_{t-1}} \mathcal{R}(P_j)$$

Since the ideal P_j is generated by indeterminates, $\mathcal{R}(P_j)$ is normal. It follows that $\mathrm{in}(\mathcal{R}^s(I_t))$ is normal, too. By virtue of [CHV, Corollary 2.3], this suffices to conclude:

THEOREM 4.2. *The symbolic Rees algebra* $\mathcal{R}^s(I_t)$ *and its initial algebra are Cohen–Macaulay normal domains.*

Let $\tau = t_1, t_2, \ldots, t_k$ be a sequence of integers with $m \geqslant t_1 \geqslant t_2 \geqslant \cdots \geqslant t_k \geqslant 1$. Set $J = I_{t_1} \cdots I_{t_k}$ and $g_j = \gamma_j(\tau)$. Note that from 3.15 it follows that:

PROPOSITION 4.3. $\mathrm{in}(\mathcal{R}(J)) = \mathcal{R}(\mathrm{in}(J)) = \mathcal{R}(\mathrm{in}(I_{t_1}) \cdots \mathrm{in}(I_{t_k}))$.

Now 3.17 yields:

$$\mathrm{in}(\mathcal{R}(J)) = \bigoplus_{i=0}^{\infty} \left(\bigcap_{j=1}^{t_1} \bigcap_{z \in A_{j-1}} P_z^{g_j i} \right) T^i = \bigcap_{j=1}^{t_1} \bigcap_{z \in A_{j-1}} \mathcal{R}(P_z^{g_j})$$

Since $\mathcal{R}(P_z^{g_j})$ is a direct summand of $\mathcal{R}(P_z)$, it is normal. It follows that $\mathrm{in}(\mathcal{R}(J))$ is normal. As above one has:

THEOREM 4.4. *Let* $J = I_{t_1} \cdots I_{t_k}$. *Then the Rees algebra* $\mathcal{R}(J)$ *and its initial algebra are Cohen–Macaulay normal domains. Further the same conclusion holds for the special fibre* $\mathcal{R}(J)/(X)\,\mathcal{R}(J)$ *of* $\mathcal{R}(J)$.

Proof. Set $\tau = t_1, \ldots, t_k$. The only statement which still needs an argument is the one concerning the special fibre of $\mathcal{R}(J)$. Let us denote it by $A(\tau)$. Since the generators of J have all the same degree (i.e. $t_1 \cdots t_k$) the algebra $A(\tau)$ can be identified with the K-algebra generated by the products of minors of shape τ. The initial algebra $\mathrm{in}(A(\tau))$ of $A(\tau)$ is the algebra generated by the initial monomials of the product of minors of shape τ. Thus $\mathrm{in}(A(\tau))$ is a direct summand of $\mathrm{in}(\mathcal{R}(J))$. It follows that $\mathrm{in}(A(\tau))$ is normal. The desired conclusion follows from [CHV, Corollary 2.3]. ∎

Again we single out the most important case:

THEOREM 4.5. *The Rees algebra $\mathcal{R}(I_t)$, its initial algebra and its special fibre are Cohen–Macaulay normal domains.*

For the special case $t=2$ (and for more general matrices) Theorem 4.4 has been proved in [CHV, Theorem 3.8].

Remark 4.6. Similarly one can show that the multi-homogeneous Rees algebra $\mathcal{R}(I_{t_1}, ..., I_{t_k})$ is normal and Cohen–Macaulay.

We turn now to the study of the presentation of the Rees algebras. We content ourself to treat the case of the Rees algebra of I_t. We have:

THEOREM 4.7. *The Rees algebra $\mathcal{R}(I_t)$ is defined by a Gröbner basis of quadrics.*

Proof. By virtue of [CHV, Corollary 2.2], it suffices to show that the initial algebra of $\mathcal{R}(I_t)$ is defined by a Gröbner basis of quadrics. In this case the initial algebra is $\mathcal{R}(\mathrm{in}(I_t))$. Let $A=\{(a_1, ..., a_t) : a_1 <_1 \cdots <_1 a_t\}$ and take a family of indeterminates $Y=(Y_a)_{a\in A}$. Consider the (minimal) presentation of $\mathrm{in}(\mathcal{R}(I_t))$

$$g\colon K[X][Y] \to \mathcal{R}(\mathrm{in}(I_t))$$

defined by sending Y_a to $x_a T = x_{a_1} x_{a_2} \cdots x_{a_t} T$. In the kernel of g there are three types of polynomials.

(1) Linear relations:

$$x_k Y_a - x_{a_h} Y_b$$

with $a_{h-1} <_1 k < a_h$ for some h, $1 \leqslant h \leqslant t$ and where b is the sequence $(a_1, ..., a_{h-1}, k, a_{h+1}, ..., a_t)$.

(2) Plücker-type relations:

$$Y_a Y_b - Y_{a\wedge b} Y_{a\vee b}$$

where

$$a \wedge b = (\min(a_1, b_1), ..., \min(a_t, b_t)),$$
$$a \vee b = (\max(a_1, b_1), ..., \max(a_t, b_t))$$

and $a_h < b_h$, $a_k > b_k$ for some h and k.

(3) Relation of type:

$$Y_a Y_b - Y_c Y_d$$

with $a_i \leqslant b_i$ for all i, and there exist $1 \leqslant h \leqslant k < t$ with

$$b_{h-1} \leqslant a_h,$$

$$b_h > a_{h+1}, b_{h+1} > a_{h+2}, ..., b_k > a_{k+1},$$

$$b_{k+1} \leqslant a_{k+2},$$

where

$$c = (a_1, ..., a_h, b_h, b_{h+1}, ..., b_k, a_{k+2}, ..., a_t)$$

and

$$d = (b_1, ..., b_{h-1}, a_{h+1}, a_{h+2}, ..., a_{k+1}, b_{k+1}, ..., b_t).$$

These relations form a Gröbner basis of *Kerg*. To prove this one uses an argument which is standard, see for instance [S, Chapter 14] or [CHV, Lemma 3.1, Proposition 3.2]. We give just the main details.

We put $(a_1, ..., a_t) \prec (b_1, ..., b_t)$ if $a_1 \leqslant b_1 \leqslant a_2 \leqslant \cdots \leqslant a_t \leqslant b_t$. Applying the above relations one shows that every monomial m in $\mathscr{R}(\mathrm{in}(I_t))$ has a representation $m = n x_{a_1} T x_{a_2} T \cdots x_{a_k} T$ (here the a_i are elements of A, say $a_i = (a_{i1}, ..., a_{it})$) with the properties

(i) $a_i \prec a_j$ for all $1 \leqslant i \leqslant j \leqslant k$

(ii) for all the x_h in n and $i = 1, ..., k$ one has either $0 \leqslant k - a_{ij} \leqslant 1$ for some j or $k \geqslant a_{it}$.

One shows that this representation is unique. Then one determines a term order in $K[X][Y]$ such that the initial monomials of the polynomials (1) (2) and (3) are those on the left hand side, and the desired conclusion follows. ∎

The analytic spread $\ell(I)$ of an homogeneous ideal I in a polynomial ring R is defined to be the dimension of the special fiber $\mathscr{R}(I)/m_R\mathscr{R}(I)$ of the Rees algebra $\mathscr{R}(I)$. Here m_R denotes the homogeneous maximal ideal of R. In general $\ell(I) \leqslant \min(\mu(I), \dim R)$ where $\mu(I)$ denotes the minimum number of generators of I. We have:

PROPOSITION 4.8. *Let $I = I_t$. Then one has:*

$$\ell(I) = \min(\mu(I), n) = \begin{cases} n & \text{if } \; 1 \leqslant t < m \\ 1 & \text{if } \; t = m \; \text{ and } \; n \text{ is odd} \\ m+1 & \text{if } \; t = m \; \text{ and } \; n \text{ is even} \end{cases}$$

Proof. If $1 \leqslant t < m$ we know by 3.18 that depth $K[X]/I^k = 0$ for $k \gg 0$. The associated graded ring $\mathrm{gr}_I(K[X])$ is Cohen–Macaulay because $\mathscr{R}(I)$ is. Then it follows from [BV, Corollary 9.24] that $\ell(I) = \dim \mathrm{gr}_I(K[X]) = \dim K[X] = n$. Alternatively one may observe that the t-minors are non-maximal minors of the matrix X_m and hence, as in the proof of [BV, Proposition 10.16], one may show that field of fractions of $K[X]$ is algebraic over the field of fractions of the special fiber of $\mathscr{R}(I)$.

If $t = m$ then it follows form the proof of 4.7 (and can be easily seen directly) that the m-minors are algebraically independent. It follows that $\ell(I) = \mu(I)$. ∎

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