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M-Sequences, Graph Ideals, and Ladder Ideals of Linear Type

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INTRODUCTION

In this paper we study monomial ideals and ladder determinantal ideals of linear type and their blow-up algebras. Our main tools are Gröbner bases and Sagbi bases deformations and the notion of M-sequence of monomials.

Let $R = K[X]$ be a polynomial ring over a field K equipped with a monomial order τ. Let I be an ideal of R generated by polynomials $f_1, \dots, f_s$. Consider the presentation

$$\psi\colon R[T] = R[T_1, \dots, T_s] \to \mathscr{R}(I) = R[f_i t, \dots, f_s t]$$

of the Rees algebra $\mathscr{R}(I)$ of I, defined by setting $\psi(T_i) = f_i t$. Let J denote the kernel of ψ. The ideal J is R-homogeneous, and I is said to be of linear type if J is generated by R-homogeneous elements of degree 1. In other words, I is of linear type if and only if the canonical map $S(I) \to \mathscr{R}(I)$ from the symmetric algebra $S(I)$ of the ideal I to the Rees algebra $\mathscr{R}(I)$ is an isomorphism.

It has been shown in [CHV] that I is of linear type provided its initial ideal $\mathrm{in}_\tau(I)$ is of linear type. Moreover, in this situation many good properties (as Cohen–Macaulayness or normality, for instance) are preserved by passing from $\mathscr{R}(\mathrm{in}_\tau(I))$ to $\mathscr{R}(I)$. This approach leads us to look for conditions under which a monomial ideal is of linear type.

The paper is organized in the following way. The first section contains notation and terminology. In the second section we introduce the notion of

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an M-sequence of monomials. A sequence of monomials $m_1, \dots, m_s$ in a set of indeterminates X is said to be an M-sequence if for all i, $1 \leq i \leq s$, there exists a total order on the indeterminates of m_i, say $m_i = x_1^{a_1} \cdots x_n^{a_n}$, such that for all $j > i$ and $1 \leq k \leq n$ with $x_k | m_j$ one has $x_k^{a_k} \cdots x_n^{a_n} | m_j$.

We show that an ideal I generated by an M-sequence $m_1, \dots, m_s$ is of Gröbner linear type, that is, the linear relations form a Gröbner basis of the ideal of presentation of the Rees algebra of I with respect to some monomial order (2.4). Moreover, the m_i's together with their linear relations form a Gröbner basis of the ideal of presentation of the associated graded ring $\mathrm{gr}_I(R)$. The initial ideals of the ideals of presentation of $\mathscr{R}(I)$ and $\mathrm{gr}_I(R)$ turn out to be Cohen–Macaulay (2.3). Furthermore, they are radical ideals provided the m_i's are square-free. It follows that $\mathscr{R}(I)$ and $\mathrm{gr}_I(R)$ are Cohen–Macaulay, and that, if in addition the m_i's are square-free, then $\mathrm{gr}_I(R)$ is reduced and $\mathscr{R}(I)$ is normal. As a consequence we have that the above results hold also for an ideal I whose initial ideal is generated by an M-sequence (2.5, 2.6).

In Section 3 we present a special class of M-sequences, the sequences of interval type. We are able to describe the Hilbert series of the Rees algebra associated with a homogeneous ideal I, whose initial ideal is generated by a sequence $m_1, \dots, m_s$ of interval type of monomials of fixed degree, in terms of the degree of $\gcd(m_i, m_j)$ (3.3).

Given a graph Γ with vertices X and edges E, Villarreal [Vi1] defined the graph ideal $I(G)$ to be the ideal generated by the monomials of the form xy where $(x, y) \in E$. We generalize this notion in Section 4 by considering the ideal $I_t(\Gamma)$ generated by all of the monomials $x_1 \cdots x_t$ such that $x_1, \dots, x_t$ is a path in Γ. If Γ is a tree, then it turns out that the generators of $I_t(\Gamma)$ form an M-sequence. It follows that $I_t(\Gamma)$ is of linear type and that $\mathscr{R}(I_t(\Gamma))$ is normal and Cohen–Macaulay. For ordinary graph ideals of trees (i.e., $t = 2$) these results were already known [SVV, 5.3, 5.9]. Our arguments show that, if Γ is a tree and $I = I_2(\Gamma)$, then the defining ideal of $\mathrm{gr}_I(R)$ has an initial ideal that is Cohen–Macaulay of the form $I_2(\Gamma')$, where Γ' is a graph (which is not a tree in general). Using this we are able to show that the ith component of the h-vector of $\mathrm{gr}_I(R)$ is the number of sets of independent edges of Γ with exactly i elements (4.3).

The ideal I_m of m-minors of an $m \times m + 1$ generic matrix of indeterminates and the ideal P_m of $2m$-pfaffians of a $2m + 1 \times 2m + 1$ skew-symmetric matrix of indeterminates are known to be of linear type, to define normal Cohen–Macaulay Rees algebras, and normal Gorenstein-associated graded rings; see, for instance, [BST], [EH], and [Hu3]. The aim of Section 5 is to present two classes of ladder ideals that are a generalization of the ideals I_m and P_m and to show that they have the above-mentioned properties, too. This is done by showing that the initial ideals of the ideals under consideration are generated by an M-sequence of square-free monomials and by inversion tricks.

1. NOTATION AND GENERALITIES

Let R be a ring (commutative, Noetherian, and with 1) and let I be an ideal of R. The Rees algebra $\mathscr{R}(I)$ of I is defined to be the R-graded algebra $\bigoplus_{i\geq 0} I^i$. It can be identified with the R-subalgebra of $R[t]$ generated by It, where t is an indeterminate over R. Let $f_1, \dots, f_s$ be a system of generators of I and consider the epimorphism of graded R-algebras

$$\psi: R[T] = R[T_1, \dots, T_s] \to \mathscr{R}(I) = R[f_1t, \dots, f_st]$$

defined by setting $\psi(T_i) = f_it$. The ideal $J = \operatorname{Ker} \psi$ of $R[T]$ is R-homogeneous. Denote by J_i the R-homogeneous component of degree i of J. The elements of J_1 are called *linear relations*. The *relation type* $r(I)$ of I is defined to be the minimum of the integers r such that $J = \bigoplus_{i=1}^{r} J_iR[T]$. In other words, $r(I)$ is the highest degree of a minimal homogeneous generator of J. The ideal I is said to be of *linear type* if its relation type is 1, that is, if the linear relations generate J. Several classes of ideals of linear type are known. For instance, ideals generated by d-sequences are of linear type; see [Hu1] or [Va2]. Determinantal ideals of linear type were characterized by Huneke [Hu4] in the generic case and by Kotzev [K] in the symmetric case. The associated graded ring $\operatorname{gr}_I(R)$ of I is by definition the factor ring $\mathscr{R}(I)/I\mathscr{R}(I) = \bigoplus_{i\geq 0} I^i/I^{i+1}$, and it can be identified with the factor ring of $R[T]$ defined by the ideal $J + I$.

Now let R be a standard (or homogeneous) graded K-algebra, that is, $R = \bigoplus_{i\geq 0} R_i$ with $R_0 = K$ a field, and R is generated as a K-algebra by elements $x_1, \dots, x_m$ of degree 1. Let I be a homogeneous ideal and denote by I_h its homogeneous component of degree h. Then $\mathscr{R}(I)$ and $\operatorname{gr}_I(R)$ inherit a natural structure of positively graded K-algebra. Explicitly, the homogeneous components of degree i of $\mathscr{R}(I)$ and $\operatorname{gr}_I(R)$ are

$$\mathscr{R}(I)_i = \bigoplus_{j=0}^{i} I^j_{i-j} \quad \text{and} \quad \operatorname{gr}_I(R)_i = \bigoplus_{j=0}^{i} I^j_{i-j}/I^{j+1}_{i-j}.$$

If I is generated by homogeneous elements $f_1, \dots, f_s$, with $\deg f_i = d_i$, then $\mathscr{R}(I)$ is generated as K-algebra by the x_i's and by the f_it's, and $\operatorname{gr}_I(R)$ is generated by the residue classes of these elements. The degree of x_i in $\mathscr{R}(I)$ and $\operatorname{gr}_I(R)$ is 1, and the degree of f_it is $d_i + 1$. From the presentation point of view, this means that we give degree $d_i + 1$ to the indeterminate T_i, $1 \leq i \leq s$. We will refer to this graded structure as the *nonstandard grading* of $\mathscr{R}(I)$ and $\operatorname{gr}_I(R)$.

In the case in which I is d-equigenerated, that is, $d_i = d$ for all $i = 1, \dots, s$, $\mathscr{R}(I)$ and $\operatorname{gr}_I(R)$ can be viewed as standard graded K-algebras by setting their degree i components to be equal to

$$\mathscr{R}(I)_i = \bigoplus_{j=0}^{i} I^j_{j(d-1)+i} \quad \text{and} \quad \mathrm{gr}_I(R)_i = \bigoplus_{j=0}^{i} I^j_{j(d-1)+i}/I^{j+1}_{j(d-1)+i}.$$

From the presentation point of view, this means that we give degree 1 to the indeterminates T_i. We will refer to this graded structure as the *standard grading* of $\mathscr{R}(I)$ and $\mathrm{gr}_I(R)$. It is easy to see that the Hilbert series $H_{\mathscr{R}(I)}(\lambda)$, $H_{\mathrm{gr}_I(R)}(\lambda)$, and $H_R(\lambda)$ of $\mathscr{R}(I)$, $\mathrm{gr}_I(R)$, and R are related as follows:

$$H_{\mathrm{gr}_I(R)}(\lambda) = \lambda^{d-1}H_R(\lambda) + (1-\lambda^{d-1})H_{\mathscr{R}(I)}(\lambda). \tag{1}$$

The Krull dimension of $\mathrm{gr}_I(R)$ is equal to that of R, and that of $\mathscr{R}(I)$ is equal to $\dim R + 1$, provided I contains a nonzero-divisor; see [Val, 1.6]. Hence under this assumption, the h-polynomials and the multiplicities of $\mathscr{R}(I)$, $\mathrm{gr}_I(R)$, and R satisfy the relations

$$h_{\mathrm{gr}_I(R)}(\lambda) = \lambda^{d-1}h_R(\lambda) + (1+\lambda+\cdots+\lambda^{d-2})h_{\mathscr{R}(I)}(\lambda), \tag{2}$$

$$e(\mathrm{gr}_I(R)) = e(R) + (d-1)e(\mathscr{R}(I)). \tag{3}$$

We assume from now on that R is the polynomial ring $K[X]$ over a field K in a set of indeterminates X. Given a monomial order τ on R, we denote, respectively, by $\mathrm{in}_\tau(f)$ the initial monomial of a polynomial $f \in R$, and by $\mathrm{in}_\tau(I)$ the initial ideal of a homogeneous ideal I of R. Throughout the paper we will use repeatedly the following well-known facts: the Hilbert series of R/I and $R/\mathrm{in}_\tau(I)$ coincide, R/I is Cohen–Macaulay or reduced whenever $R/\mathrm{in}_\tau(I)$ is so, and the polynomials $g_1, \dots, g_m$ are a R/I-regular sequence if their initial monomials are a $R/\mathrm{in}_\tau(I)$-regular sequence.

We say that an ideal I of R is of *Gröbner linear type* if the linear relations form a Gröbner basis of J with respect to some monomial order on $R[T] = K[X, T]$. Of course, ideals of Gröbner linear type are of linear type.

It has been shown in [CHV, 2.2, 2.8] that an ideal I is of (Gröbner) linear type whenever the initial ideal $\mathrm{in}_\tau(I)$ with respect to a monomial order τ has this property. Furthermore, if in addition $\mathscr{R}(\mathrm{in}_\tau(I))$ is Cohen–Macaulay (resp. normal), then $\mathscr{R}(I)$ is Cohen–Macaulay (resp. normal). This fact leads to the following interesting problem: given an ideal I generated by monomials $m_1, \dots, m_s$, find conditions on the m_i's that guarantee that I is of linear type (or of Gröbner linear type).

For instance, it is easy to see that a sequence of three square-free monomials is always of (universal) Gröbner linear type. Of course, d-sequences of monomials are of linear type, but they are rare, see [HSV1, 5.2]. Villarreal characterized explicitly the ideals of linear type generated by square-free monomials of degree 2 (graph ideals) [Vi2, 2.2]. In the next section we present a new class of monomial ideals of Gröbner linear type.

2. *M*-SEQUENCES

In this section we introduce the notion of an *M-sequence* of monomials, and then we show that an ideal generated by an M-sequence is always of Gröbner linear type. Quite surprisingly, it turns out that the Rees algebra associated with an M-sequence is always Cohen−Macaulay. Furthermore, the associated graded ring is reduced and the Rees algebra is normal whenever the monomials are square-free.

DEFINITION 2.1. A sequence of monomials $m_1, \dots, m_s$ in a set of indeterminates X is said to be an M-sequence if for all $1 \le i \le s$ there exists a total order on the set of indeterminates that appear in m_i, say $x_1 < \cdots < x_n$ with $m_i = x_1^{a_1} \cdots x_n^{a_n}$ and $a_1 > 0, \dots, a_n > 0$, such that whenever $x_k | m_j$ with $1 \le k \le n$ and $i < j$, then $x_k^{a_k} \cdots x_n^{a_n} | m_j$.

Note that the total order on the indeterminates of m_i is allowed to depend on i. Obviously the property of being an M-sequence depends on the order in which we list the monomials. For example, xy, yz, zw is an M-sequence, and yz, xy, zw is not. Given a sequence $m_1, \dots, m_s$ of monomials for all $1 \le i, j \le s$, we set

$$m_{ij} = m_i / \gcd(m_i, m_j).$$

We collect some properties of M-sequences in the following:

LEMMA 2.2. *Let $m_1, \dots, m_s$ be an M-sequence in a set of indeterminates X. Then*:

(1) *Every subset of $m_1, \dots, m_s$ is an M-sequence with respect to the induced orders.*

(2) *Let $1 \le b_1 \le \cdots \le b_s$ be integers. Then $m_1^{b_1}, \dots, m_s^{b_s}$ is an M-sequence.*

(3) *Let m be a monomial such that $m | m_1$. Then $m_1/m, m_2, \dots, m_s$ is an M-sequence.*

(4) $\gcd(m_{ij}, \gcd(m_i, m_j)) = 1$ *for all* $1 \le i < j \le s$.

(5) *Let* i, j, k *be integers with* $1 \le i, j, k \le s$, $i < j$, *and* $i < k$. *Then one has either* $\gcd(m_i, m_j) | \gcd(m_i, m_k)$ *or* $\gcd(m_i, m_k) | \gcd(m_i, m_j)$.

(6) *Assume* $\gcd(m_1, m_k) \neq 1$ *for some* $k > 1$. *By virtue of* (5) *there exists* r, $1 < r \le s$, *such that* $\gcd(m_1, m_j) | \gcd(m_1, m_r)$ *for all* $1 < j \le s$ *and* r *is minimal with respect to this property. Then*

(i) $\gcd(m_1, m_j) = \gcd(m_1, m_j, m_r)$ *for all* j, $1 < j \le s$,

(ii) $\gcd(m_1, m_r)/\gcd(m_1, m_r, m_j) | m_{rj}$ *for all* $j > r$.

Proof. Statements (1), (2), and (3) are trivial.

(4) Let $m_i = x_1^{a_1} \cdots x_n^{a_n}$, and let $p = \min\{h: x_h | m_j\}$, with the convention that $p = n + 1$ if $\gcd(m_i, m_j) = 1$. From the definition of M-sequence it follows that $\gcd(m_i, m_j) = x_p^{a_p} \cdots x_n^{a_n}$, and $m_{ij} = x_1^{a_1} \cdots x_{p-1}^{a_{p-1}}$. Hence $\gcd(m_{ij}, \gcd(m_i, m_j)) = 1$.

(5) Let $m_i = x_1^{a_1} \cdots x_n^{a_n}$ and $j > i$. Let $p = \min\{h: x_h | m_j\}$. As above, one has $\gcd(m_i, m_j) = x_p^{a_p} \cdots x_n^{a_n}$. Similarly, $\gcd(m_i, m_k) = x_q^{a_q} \cdots x_n^{a_n}$, where $q = \min\{h: x_h | m_k\}$. Therefore, $\gcd(m_i, m_k) | \gcd(m_i, m_j)$ if $p \le q$, and $\gcd(m_i, m_j) | \gcd(m_i, m_k)$ if $q \le p$.

(6)(i) We fix an indeterminate x and denote by b_1, b_r, b_j the exponents of x in m_1, m_r, m_j. The statement is $\min\{b_1, b_j\} = \min\{b_1, b_j, b_r\}$. If $b_1 = 0$ or $b_j = 0$, then it holds true. If $b_1 \neq 0$ and $b_j \neq 0$, then $b_r \neq 0$ by construction, and $b_1 \le b_r, b_1 \le b_j$, because m is an M-sequence. Hence the statement holds true in this case, too.

(6)(ii) In terms of exponents the statement is $\min\{b_1, b_r\} - \min\{b_1, b_r, b_j\} \le b_r - \min\{b_r, b_j\}$. It is trivial if one among b_1, b_r, and b_j is equal to 0. Assume $b_1 \neq 0$, $b_r \neq 0$, and $b_j \neq 0$. Then $b_1 \le b_r \le b_j$ because m is an M-sequence and hence the statement holds true. ∎

The next proposition deals with the Cohen–Macaulay property of two monomial ideals associated with an M-sequence $m_1, \dots, m_s$. In the sequel we will see that they are the initial ideals of the ideals of definition of the Rees algebra and of the associated graded ring of $m_1, \dots, m_s$.

PROPOSITION 2.3. *Let* $m_1, \dots, m_s$ *be an* M*-sequence of monomials in a set of indeterminates* X, *and let* $T = T_1, \dots, T_s$ *be a set of indeterminates. Let* $I = (m_1, \dots, m_s)$ *and* $Q = (m_{ij}T_j: 1 \le i < j \le s)$. *Then* $K[X, T]/Q$ *and* $K[X, T]/Q + I$ *are Cohen–Macaulay of dimension* $|X| + 1$ *and* $|X|$, *respectively.*

Proof. We argue by induction on s. The claim is trivial for $s = 1$. Hence assume $s > 1$. Set $Q_2 = (m_{ij}T_j: 2 \le i < j \le s)$, and $I_2 = (m_i: 2 \le i \le s)$. By induction $K[X, T_2, \dots, T_s]/Q_2$ and $K[X, T_2, \dots, T_s]/Q_2 + I_2$ are Cohen–Macaulay rings of dimension $|X| + 1$ and $|X|$.

Assume that $\gcd(m_1, m_j) = 1$ for all $j > 1$. Then m_1 is a nonzero-divisor modulo Q_2 and modulo $Q_2 + I_2$. Note that $Q_2 + I_2 + (m_1) = Q + I$. Hence it follows by induction that $K[X,T]/Q + I$ is Cohen–Macaulay of dimension $|X|$. Now note that $Q = Q_2 + (m_1 T_i\colon 1 < i \le s) = (Q_2, m_1) \cap (T_2, \dots, T_s)$, and $(Q_2, m_1) + (T_2, \dots, T_s) = (m_1, T_2, \dots, T_s)$. Hence $K[X,T]/Q$ has dimension $|X| + 1$. Furthermore, from the short exact sequence

$$\begin{aligned} 0 \to K[X,T]/Q \to K[X,T]/(Q_2, m_1) \oplus K[X, T_1] \\ \to K[X, T_1]/(m_1) \to 0, \qquad (4) \end{aligned}$$

by depth chasing it follows that $K[X,T]/Q$ is Cohen–Macaulay.

Now assume that $\gcd(m_1, m_k) \neq 1$ for some $k > 1$, and let r be as in 2.2(6). Let $m_1' = \gcd(m_1, m_r)$, and let m' be the sequence $m_1', m_2, \dots, \hat{m}_r, \dots, m_s$, where $\hat{m}_r$ means that m_r is omitted. Set $m_{1j}' = m_1'/\gcd(m_1', m_j)$, $j > 1$. By virtue of 2.2(1)(3) the sequence m' is an M-sequence. The ideals associated with m' are

$$Q' = \left(m_{1j}' T_j, 1 < j \le s, j \neq r\right) + \left(m_{ij} T_j\colon 2 \le i < j \le s, i \neq r \neq j\right)$$

and

$$I' = \left(m_1', m_2, \dots, \hat{m}_r, \dots, m_s\right).$$

By induction $K[X, T_1, T_2, \dots, \hat{T}_r, \dots, T_s]/Q'$ and $K[X, T_1, T_2, \dots, \hat{T}_r, \dots, T_s]/Q' + I'$ are Cohen–Macaulay rings of dimension $|X| + 1$ and $|X|$, respectively. Set

$$\begin{aligned} J_1 &= (Q', T_r), & J_2 &= (Q_2, m_{1r}), \\ J_3 &= (Q', I', T_r), & J_4 &= (Q_2, I_2, m_{1r}). \end{aligned}$$

We claim that

$$Q = J_1 \cap J_2, \quad Q + I = J_3 \cap J_4, \qquad (5)$$

and

$$J_1 + J_2 = (J_1, m_{1r}), \quad J_3 + J_4 = (J_3, m_{1r}). \qquad (6)$$

Before proving (5) and (6), we conclude the proof of the proposition. By construction the indeterminates of m_{1r}, do not appear in the generators of Q_2 and of I_2, and the indeterminate T_r does not appear in the generators of Q' and of I'. Hence J_1 and J_2 define Cohen–Macaulay factor rings of $K[X,T]$ of dimension $|X| + 1$, and J_3 and J_4 define Cohen–Macaulay factor rings of $K[X,T]$ of dimension $|X|$. Furthermore, (J_1, m_{1r}) and

(J_3, m_{1r}) define Cohen–Macaulay factor rings of $K[X,T]$ of dimension $|X|$ and $|X| - 1$, respectively. As above, it follows from the short exact sequences

$$0 \to K[X,T]/Q \to K[X,T]/J_1 \oplus K[X,T]/J_2 \\ \to K[X,T]/(J_1, m_{1r}) \to 0, \tag{7}$$

and

$$0 \to K[X,T]/Q + I \to K[X,T]/J_3 \oplus K[X,T]/J_4 \\ \to K[X,T]/(J_3, m_{1r}) \to 0 \tag{8}$$

that $K[X,T]/Q$ and $K[X,T]/Q + I$ are Cohen–Macaulay of dimension $|X| + 1$ and $|X|$, respectively.

It remains to prove (5) and (6). One has

$$\begin{aligned} J_1 \cap J_2 = {} & \Big[\big(m'_{1j} T_j, 1 < j \le s, j \ne r \big) \\ & \quad + \big(m_{ij} T_j \colon 2 \le i < j \le s, i \ne r \ne j \big) + (T_r) \Big] \\ & \quad \cap \Big[\big(m_{ij} T_j \colon 2 \le i < j \le s \big) + (m_{1r}) \Big] \\ = {} & \big(m_{ij} T_j \colon 2 \le i < j \le s, i \ne r \big) \\ & \quad + \Big[\big(m'_{1j} T_j, 1 < j \le s, j \ne r \big) + (T_r) \Big] \\ & \quad \cap \Big[\big(m_{rj} T_j \colon r < j \le s \big) + (m_{1r}) \Big]. \end{aligned}$$

By 2.2(6)(ii) $m'_{1j} \mid m_{rj}$ for all $j > r$. It follows that

$$\begin{aligned} J_1 \cap J_2 = {} & \big(m_{ij} T_j \colon 2 \le i < j \le s \big) \\ & \quad + \Big[\big(m'_{1j} T_j, 1 < j \le s, j \ne r \big) + (T_r) \Big] \cap (m_{1r}). \end{aligned}$$

Now note that by virtue of 2.2(4) the indeterminates of m_{1r} do not appear in m'_{1j} for all $j > 1$ and that by 2.2(6)(i) $m_{1r} m'_{1j} = m_{1j}$. It follows that $J_1 \cap J_2 = (m_{ij} T_j \colon 1 \le i < j \le s) = Q$. By means of similar arguments one shows the other equalities. ∎

Let I be an ideal generated by an M-sequence $m_1, \dots, m_s$ in a set of indeterminates X. Consider the presentation

$$\psi \colon K[X][T_1, \dots, T_s] \to \mathcal{R}(I)$$

defined by setting $\psi(T_i) = m_i t$ and denote by J the kernel of ψ. For $1 \le i < j \le s$ set

$$\ell_{ij} = m_{ij} T_j - m_{ji} T_i.$$

Let τ be a monomial order on $K[X,T]$ such that $\mathrm{in}_\tau(\ell_{ij}) = m_{ij}T_j$ (for instance, one can take the lexicographic order induced by the total order $T_s > \cdots > T_1$). The main result of this section is:

THEOREM 2.4. (i) *The linear relations* ℓ_{ij} *form a Gröbner basis of J with respect to* τ, *that is,* $\mathrm{in}_\tau(J) = (m_{ij}T_j\colon 1 \le i < j \le s)$ *and I is of Gröbner linear type.*

(ii) *The linear relations* ℓ_{ij} *and the* m_i *form a Gröbner basis of the ideal of presentation of* $\mathrm{gr}_I(R)$ *with respect to* τ, *that is,* $\mathrm{in}_\tau(J+I) = (m_{ij}T_j\colon 1 \le i < j \le s) + (m_i\colon 1 \le i \le s)$.

Proof. (i) Denote by Q the ideal $(m_{ij}T_j\colon 1 \le i < j \le s)$. To show that the linear relations ℓ_{ij} are a Gröbner basis of J with respect to τ, we argue by contradiction. Suppose the claim is false. Since the binomial relations are known to be a universal Gröbner basis of J [St, 2.2], there exists a binomial relation $aT^\alpha - bT^\beta \in J$, with a and b monomials in the x's, whose initial monomial is not in the ideal Q. We may assume that T^α, T^β have no common factors. Furthermore, we also may assume that both aT^α and bT^β are not in Q. This is because one of the two monomials (the initial) is not in Q by assumptions, and if the other is in Q, then we may reduce it by means of the relations ℓ_{ij} and repeat this procedure until we obtain a binomial relation with the desired properties.

Now let i be the smallest index such that T_i appears in T^α or in T^β. By symmetry, we may assume that T_i appears in T^α. Since $aT^\alpha - bT^\beta \in J$, m_i divides $b\psi(T^\beta)$.

If $m_i|b$, then let T_j be any of the indeterminates in T^β. One has $m_{ij}T_j|m_iT_j|bT^\beta$ and $i<j$. This is a contradiction.

Thus m_i does not divide b. Let $x_1 < \cdots < x_n$ be the total order of the indeterminates of m_i, and let $m_i = x_1^{a_1}\cdots x_n^{a_n}$. Let k be the minimum of the indices such that $x_k^{a_k}$ does not divide b. Then $x_1^{a_1}\cdots x_{k-1}^{a_{k-1}}|b$. Since $x_k^{a_k}|b\psi(T^\beta)$, there exists j such that T_j appears in T^β and $x_k|m_j$. Since m is an M-sequence, one has $x_k^{a_k}\cdots x_n^{a_n}|m_j$ and $m_{ij}|x_1^{a_1}\cdots x_{k-1}^{a_{k-1}}|b$. It follows that $m_{ij}T_j|bT^\beta$, which again is a contradiction.

(ii) The ideal of presentation of $\mathrm{gr}_I(R)$ is $J+I$. We know that $J = (\ell_{ij}\colon 1 \le i < j \le s)$ and that the ℓ_{ij}'s form a Gröbner basis with respect to a monomial order τ such that $\mathrm{in}_\tau(\ell_{ij}) = m_{ij}T_j$. To show that the ℓ_{ij}'s and the m_i's form a Gröbner basis of $J+I$ with respect to τ, we apply the Buchberger criterion. It suffices to analyze the S-pairs $S(\ell_{ij}, m_k) = m_{ji}T_im_k/\mathrm{gcd}(m_k, m_{ij})$ for all $1 \le i < j \le s$ and $1 \le k \le s$. We have to distinguish two cases.

Assume first $i \le k$. By virtue of 2.2(5) we have either $\gcd(m_i, m_j) | \gcd(m_i, m_k)$ or $\gcd(m_i, m_k) | \gcd(m_i, m_j)$. If $\gcd(m_i, m_k) | \gcd(m_i, m_j)$, then $\gcd(m_k, m_{ij}) = 1$ and $S(\ell_{ij}, m_k) = m_{ji} T_i m_k$. Hence the polynomial $S(\ell_{ij}, m_k)$ reduces to 0 via m_k. If $\gcd(m_i, m_j) | \gcd(m_i, m_k)$, then we claim that $m_j | m_{ji} m_k / \gcd(m_k, m_{ij})$. From the claim it follows that $S(\ell_{ij}, m_k)$ reduces to 0 via m_j. To prove the claim we fix an indeterminate x and denote by b_i, b_k, b_j the exponents of x in m_i, m_k, m_j. In terms of the exponents the assertion is

$$b_j \le b_j - \min\{b_i, b_j\} + b_k - \min\Big\{b_k, b_i - \min\{b_i, b_j\}\Big\}. \tag{9}$$

If $b_j = 0$, then (9) clearly holds. If $b_j > 0$, then $b_i \le b_j$ because m is an M-sequence. Since $\gcd(m_i, m_j) | \gcd(m_i, m_k)$, $b_i = \min\{b_i, b_j\} \le \min\{b_i, b_k\}$. Hence $b_i \le b_k$. Now (9) reads $b_j \le b_j - b_i + b_k$ and holds true.

If $k < i$, then arguing as above one shows that $m_{ki} | m_{ji} m_k / \gcd(m_k, m_{ij})$. Then $m_{ji} T_i m_k / \gcd(m_k, m_{ij})$ reduces to $S_1 = m_i m_{ji} T_k / \gcd(m_k, m_{ij})$ via ℓ_{ki}. By means of the same argument one shows that $m_j | S_1$ and then S_1 reduces to 0 via m_j. This concludes the proof of (ii). ∎

An important consequence of 2.4 is

THEOREM 2.5. *Let I be a homogeneous ideal of a polynomial ring $R = K[X]$ over a field K. Suppose that there exists a monomial order, say δ, such that $\operatorname{in}_\delta(I)$ is generated by an M-sequence $m_1, \dots, m_s$. Then let $f_1, \dots, f_s$ be a Gröbner basis of I with $\operatorname{in}_\delta(f_i) = m_i$, and consider the presentation ψ: $R[T] \to \mathscr{R}(I)$ defined by $\psi(T_i) = f_i t$. Denote by J the kernel of ψ. Then there exists a monomial order σ on $K[X, T]$ such that*

(i) $\operatorname{in}_\sigma(J) = (m_{ij} T_j\colon 1 \le i < j \le s)$. *In particular, I is of Gröbner linear type.*

(ii) $\operatorname{in}_\delta(I^i) = \operatorname{in}_\delta(I)^i$ *for all $i \in \mathbb{N}$.*

(iii) $\operatorname{in}_\sigma(J + I) = \operatorname{in}_\sigma(J) + \operatorname{in}_\delta(I) = (m_{ij} T_j\colon 1 \le i < j \le s) + (m_1, \dots, m_s)$.

(iv) *$\mathscr{R}(I)$ and $\operatorname{gr}_I(R)$ are Cohen–Macaulay.*

(v) *If the m_i's are square-free, then $\operatorname{gr}_I(R)$ is reduced.*

Proof. Set $Q = (m_{ij} T_j\colon 1 \le i < j \le s)$. By assumption, $\operatorname{in}_\delta(I) = (m_1, \dots, m_s)$. By virtue of 2.4(i), $\operatorname{in}_\delta(I)$ is of Gröbner linear type, and there exists a monomial order τ on $K[X, T]$ such that the initial ideal of the ideal of presentation of the Rees algebra of $\operatorname{in}_\delta(I)$ is Q. It follows from [CHV, 2.2, 2.7, 2.8] that I is of Gröbner linear type, that $\operatorname{in}_\delta(I^i) = \operatorname{in}_\delta(I)^i$ for all $i \in \mathbb{N}$, and that there exists a monomial order σ on $K[X, T]$ (which is defined by combining τ and δ) and linear relations $L_{ij} \in J$ such that $\operatorname{in}_\sigma(L_{ij}) = m_{ij} T_j$ and $\operatorname{in}_\sigma(J) = Q$. This proves (i) and (ii).

(iii) To prove that $\mathrm{in}_\sigma(J + I) = Q + \mathrm{in}_\delta(I)$, we note that by the very definition of σ (see [CHV, 2.2]) one has $\mathrm{in}_\sigma(f_i) = \mathrm{in}_\delta(f_i) = m_i$. Hence we have $Q + \mathrm{in}_\delta(I) \subseteq \mathrm{in}_\sigma(J + I)$. For the other inclusion, it is enough to show that the Hilbert function (with respect to the nonstandard grading) of $A = K[X, T]/\mathrm{in}_\sigma(J + I)$ and that of $B = K[X, T]/Q + \mathrm{in}_\delta(I)$ coincide. Denote by G and by G_1 the rings $\mathrm{gr}_I(R)$ and $\mathrm{gr}_{\mathrm{in}_\delta(I)}(R)$. Note that G and G_1 have the same Hilbert function (with respect to the nonstandard grading) because by (ii) $\mathrm{in}_\delta(I^i) = \mathrm{in}_\delta(I)^i$ for all i. Since A and G have the same Hilbert function, it suffices to show that G_1 and B have the same Hilbert function. But this follows from 2.4(ii), since one knows that $Q + \mathrm{in}_\delta(I)$ is the initial ideal of the ideal of presentation of G_1.

(iv) We have shown in (i) and in 2.3 that Q is the initial ideal of J with respect to τ and that it defines a Cohen–Macaulay ring. This implies that $\mathscr{R}(I) \simeq K[X, T]/J$ is Cohen–Macaulay. The Cohen–Macaulayness of $\mathrm{gr}_I(R)$ follows from that of $\mathscr{R}(I)$ [Hu2, 1.1], but also from (iii) and 2.3.

(v) Since the m_i's are square-free, it follows from (iii) that $\mathrm{in}_\sigma(J + I)$ is generated by square-free monomials and hence it is radical. But this forces $J + I$ to be a radical ideal. Hence $\mathrm{gr}_I(R)$ is reduced. ∎

With the assumption and notation of 2.4 one has

COROLLARY 2.6. (1) *Assume that the m_i's are square-free; then*:

(i) *$\mathscr{R}(I)$ is normal, it is F-injective and F-rational in positive characteristic, and it has rational singularities in characteristic* 0.

(ii) $\mathrm{gr}_I(R)$ *is F-injective. Furthermore,* $\mathrm{gr}_I(R)$ *is Gorenstein if I is unmixed and it is a domain if I is prime.*

(2) *If the m_i's have degree* 2, *then* $\mathrm{gr}_I(R)$ *is a Koszul algebra with respect to the standard grading.*

Proof. (1) By 2.5(v) $\mathrm{gr}_I(R)$ is reduced and hence $\mathscr{R}(I)$ is normal; see [Ba, 5]. Furthermore, $\mathscr{R}(I)$ and $\mathrm{gr}_I(R)$ are F-injective because by Theorem 2.5 and Proposition 2.3 their ideals of presentation have initial ideals that are square-free and Cohen–Macaulay; see [CH, 2.1]. The F-rationality and the rational singularities of $\mathscr{R}(I)$ follow from [CHV, 2.3], because the initial algebra of $\mathscr{R}(I)$ is $\mathscr{R}(\mathrm{in}_\delta(I))$ and it is normal. If I is unmixed, it follows from [HSV2, 4.2.3] that $\mathrm{gr}_I(R)$ is Gorenstein. Finally, if I is prime, then it follows from [HuSV, 1.1] that $\mathrm{gr}_I(R)$ is a domain.

(2) If the m_i's have degree 2, then by virtue of 2.5(iii) the ideal $J + I$ is generated by a Gröbner basis of quadrics. It follows from [BHV, 2.2] that $\mathrm{gr}_I(R)$ is a Koszul algebra with respect to the standard grading. ∎

For the next applications we record the following:

LEMMA 2.7. *With the assumptions and the notation of* 2.5, *denote by* t_i *the residue class of* f_i *in* $I/I^2 \subset \mathrm{gr}_I(R)$. *Let* h *be an integer such that* $\gcd(m_i, m_j) = 1$ *for all* $1 \leq i < j \leq h$. *Then*

(1) *The indeterminates* $T_1, \dots, T_h$ *do not appear in the generators of* $\mathrm{in}_\sigma(J + I)$.

(2) $t_1, \dots, t_h$ *is a* $\mathrm{gr}_I(R)$*-regular sequence*.

(3) *If in addition the monomials* $m_1, \dots, m_s$ *are square-free, then* $K[X, T]/\mathrm{in}_\sigma(J + I + T_1 \cdots T_i)$ *is reduced and Cohen–Macaulay for all* $1 \leq i \leq h$.

Proof. The initial ideal of $J + I$ is $(m_1, \dots, m_s) + (m_{ij}T_j: 1 \leq i < j \leq s)$. By assumption $m_{ij} = m_i$ for all $1 \leq i < j \leq h$. It follows that the indeterminates $T_1, \dots, T_h$ do not appear in the generators of $\mathrm{in}_\sigma(J + I)$. Statements (2) and (3) follow directly from (1) and Proposition 2.3. ∎

Note that the assumption of Lemma 2.7 is trivially satisfied if one takes $h = 1$.

3. SEQUENCES OF INTERVAL TYPE

The aim of this section is to present a class of M-sequences, the sequences of interval type. We show how the Hilbert series of the Rees algebra and of the associated graded ring of an ideal I generated by a sequence of monomials $m_1, \dots, m_s$ of interval type can be expressed in terms of the degree of the monomials m_{ij}.

Given a monomial n and an indeterminate x, denote by $O_x(n)$ the exponent of x in n. We start with the definition:

DEFINITION 3.1. A sequence of monomials $m_1, \dots, m_s$ in the set of indeterminates X is said to be of interval type if for all $1 \leq i < j \leq s$ and $x | \gcd(m_i, m_j)$, one has $O_x(m_i) \leq O_x(m_k)$ for all $i \leq k \leq j$. In other words, any indeterminate appears in a "subinterval" of $m_1, \dots, m_s$ and with nondecreasing exponents from the left to the right.

Let $m_1, \dots, m_s$ be a sequence of interval type. For all $x \in X$ set

$$i(x) = \min\{k: x | m_k\} \quad \text{and} \quad j(x) = \max\{k: x | m_k\}.$$

One has

PROPOSITION 3.2. *A sequence of interval type is an M-sequence.*

Proof. Let $m = m_1, \dots, m_s$ be a sequence of interval type. First we order the indeterminates according to the function $j(..)$: $x \leq y$ if and only if $j(x) < j(y)$ or $x = y$. Then we extend this partial order to a total order. We claim that, with this order on the indeterminates, m is an M-sequence. To show that this is the case, we write $m_i = x_1^{a_1} \cdots x_n^{a_n}$ with $a_h > 0$ for all $h = 1, \dots, n$, and $j(x_1) \leq \cdots \leq j(x_n)$, and assume $x_k | m_j$ with $j > i$. Note that for all $h = k, \dots, n$ one has $i(x_h) \leq i < j \leq j(x_k) \leq j(x_h)$, and hence $x_h | m_j$. It follows that $x_h^{a_h} | m_j$, for all $h = k, \dots, n$. ∎

Let I be a homogeneous ideal whose initial ideal is generated by a sequence $m_1, \dots, m_s$ of interval type, and denote by Q the ideal $(m_{ij}T_j\colon 1 \leq i < j \leq s)$. Note that the integer r defined in 2.2(6), for a sequence of interval type, is always equal to 2. As a by-product, from the short exact sequences (4) and (7) it follows that the Hilbert series of $K[X, T]/Q$ can be expressed in terms of $\deg m_{12}$ and of the Hilbert series of rings of the same kind, but associated with the shorter sequences $m_2, \dots, m_s$, and $\gcd(m_1, m_2), m_3, \dots, m_s$, which are again of interval type. To be more explicit, let us denote by $h(m_1, \dots, m_s)(\lambda)$ the h-polynomial of $K[X, T]/Q$. Then one has

$$h(m_1, \dots, m_s)(\lambda) = \frac{(1 - \lambda^{\deg m_{12}})}{(1 - \lambda)} h(m_2, \dots, m_s)(\lambda) + \lambda^{\deg m_{12}} h(\gcd(m_1, m_2), m_3, \dots, m_s)(\lambda). \quad (10)$$

This recursive formula allows one to describe the h-vector of $K[X, T]/Q$ in terms of numerical invariants of the sequence. For all $1 \leq i < j \leq s$ set

$$[i, j] = \deg m_{ij}, \quad (i, j) = \deg m_{ij} - \deg m_{ij-1},$$

and for systematic reasons put $[i, i] = 0$. Note that by virtue of Lemma 2.2(5),

$$\begin{aligned}(i, j) &= \deg m_{ij}/m_{ij-1} = \deg \gcd(m_i, m_{j-1})/\gcd(m_i, m_j) \\ &= \deg \gcd(m_i, m_{i+1}, \dots, m_{j-1})/\gcd(m_i, m_{i+1}, \dots, m_j).\end{aligned}$$

Now from (10), by induction on s, it follows that

$$h(m_1, \dots, m_s)(\lambda) = \sum_{k \geq 0} \sum_{1 = i_0 < i_1 < \cdots < i_k \leq s} \left(\prod_{j=1}^{k} \lambda^{[i_{j-1}, i_j - 1]} \frac{1 - \lambda^{(i_{j-1}, i_j)}}{1 - \lambda} \right) \lambda^{[i_k, s]}.$$

If the m_i's all have the same degree, then by virtue of Lemma 2.5(i) the Hilbert function of $\mathscr{R}(I)$, equipped with the standard degree, coincides with that of $K[X,T]/Q$. Hence we have shown

COROLLARY 3.3. *Let I be a homogeneous ideal whose initial ideal is generated by a sequence $m_1,\dots,m_s$ of interval type of elements of fixed degree d. Then the h-polynomial of $\mathscr{R}(I)$ is*

$$h(\mathscr{R}(I))(\lambda) = \sum_{k\geq 0} \sum_{1=i_0<i_1<\cdots<i_k\leq s} \left(\prod_{j=1}^{k} \lambda^{[i_{j-1},i_j-1]} \frac{1-\lambda^{(i_{j-1},i_j)}}{1-\lambda} \right) \lambda^{[i_k,s]},$$

and its multiplicity is

$$e(\mathscr{R}(I)) = \sum_{k\geq 0} \sum_{1=i_0<i_1<\cdots<i_k\leq s} \left(\prod_{j=1}^{k} (i_{j-1},i_j) \right).$$

EXAMPLE 3.4. Given two positive integers r and s, consider the sequence of monomials $m_1,\dots,m_s$ defined as follows: $m_i = x_i x_{i+1} \cdots x_{i+r-1}$ for all $i=1,\dots,s$. This sequence is clearly a sequence of interval type. It is easy to see that

$$(i,j) = \begin{cases} 1 & \text{if } j-i\leq r \\ 0 & \text{if } j-i>r. \end{cases}$$

By virtue of 3.3, $e(\mathscr{R}(I))$ coincides with the cardinality of the set

$$A = \Big\{ \{i_0,\dots,i_k\} \subset \mathbb{N} \colon k\geq 0, \quad 1=i_0<\cdots<i_k\leq s, i_j - i_{j-1}\leq r \quad \text{for all } 1\leq j\leq k \Big\}.$$

The cardinality of A can be computed by means of the "inclusion-exclusion principle." One obtains

$$e(\mathscr{R}(I)) = \sum_{j\geq 0} (-1)^j \binom{s-1-jr}{j} 2^{s-1-j(r+1)}.$$

Given an M-sequence $m_1,\dots,m_s$, we would like to know when the sequence $m_i, m_1,\dots,\hat{m}_i,\dots,m_s$ is again an M-sequence. This is important since, by virtue of Lemma 2.7, it would then follow that the residue class of T_i in $\mathrm{gr}_I(R)$ is a nonzero-divisor. To this end we introduce the following definition: let $m_1,\dots,m_s$ be a sequence of interval type of square-free monomials. We say that m_i satisfies the *last-in–first-out* property if for every pair of indeterminates x, y in m_i with $j(x) \le j(y)$, then $i(y) \le i(x)$. One has

LEMMA 3.5. *Let $m_1, \dots, m_s$ be a sequence of interval type of square-free monomials in the set of indeterminates X. Assume that $m_{i_1}, \dots, m_{i_k}$ satisfy the last-in–first-out property. Then $m_{i_1}, \dots, m_{i_k}$, $m_1, \dots, \hat{m}_{i_1}, \dots, \hat{m}_{i_k}, \dots, m_s$ is an M-sequence.*

Proof. It is clear that one may assume that $k = 1$. Set $i_1 = i$. We order the indeterminates according to the function $j(..)$. Since $m_1, \dots, \hat{m}_i, \dots, m_s$ is an M-sequence, it suffices to show that for any indeterminate x that divides $\gcd(m_i, m_j)$ for some $1 \le j \le s$, $j \ne i$, then $y | m_j$ whenever y is an indeterminate such that $y | m_i$ and $j(x) \le j(y)$. By assumption, $i(y) \le i(x) \le j \le j(x) \le j(y)$, and hence $y | m_j$. ∎

4. GENERALIZED GRAPH IDEALS OF TREES

Ordinary graph ideals have been studied by Villarreal in [Vi1], [Vi2], and subsequently by Simis, Vasconcelos, and Villarreal [SVV]. The aim of this section is to generalize the notion of graph ideals and to show that generalized graph ideals of trees are generated by M-sequences.

A *graph* Γ is a pair (X, E), where X is a finite set and $E \subset X \times X \setminus \{(x, x): x \in X\}$. The elements of X are called *vertices*, and the elements of E *edges*. A *path* from x to y is a sequence of vertices $x = x_1, \dots, x_t = y$ of Γ such that $(x_i, x_{i+1}) \in E$ for all $i = 1, \dots, t-1$. A *path of length* t is a path with t vertices.

A *tree* is a graph Γ with a vertex x, called a *root*, such that for any vertex y of Γ there exists a unique path from x to y. It is easy to see that a tree has a unique root. Furthermore, given a tree Γ with root x, any other vertex y is the end point of a unique edge. Figure 1 illustrates a tree (the edges are oriented from the top to the bottom).

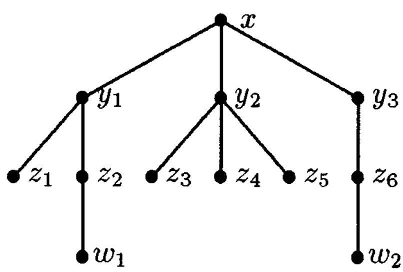

FIG. 1. A tree.

Let Γ be a graph and let K be a field. We consider the polynomial ring $K[X]$ over K whose indeterminates are the vertices X of Γ. We define $I_t(\Gamma)$ to be the ideal of $K[X]$ generated by the monomials of the form $x_1 \cdots x_t$, where $x_1, \dots, x_t$ is a path of Γ. In the sequel we will identify the paths with the corresponding monomials. For instance, if Γ is the tree of Fig. 1, the ideal $I_3(\Gamma)$ is generated by the monomials $w_1 z_2 y_1$, $w_2 z_6 y_3$, $z_1 y_1 x$, $z_2 y_1 x$, $z_3 y_2 x$, $z_4 y_2 x$, $z_5 y_2 x$, $z_6 y_3 x$. Note that $I_2(\Gamma)$ is the ordinary graph ideal as introduced by Villarreal [Vi1].

PROPOSITION 4.1. *Let Γ be a tree. Then for all t the ideal $I_t(\Gamma)$ is generated by an M-sequence.*

Proof. Let us denote by x the root of Γ. Given $y \in X$, the rank rk y is by definition the length of the path from x to y. We introduce a partial order on X by setting $y \leq z$ if $y = z$ or $\operatorname{rk} y > \operatorname{rk} z$. Then we extend this partial order to a total order, and we denote it by $<$. Let $y_1, \dots, y_t$ be a path in Γ. By construction one has $\operatorname{rk} y_i = \operatorname{rk} y_t + i - t$ and hence $y_t < \cdots < y_1$. The total order on X gives rise to a total order on the set of the paths of length t of Γ by setting $y_1, \dots, y_t \leq z_1, \dots, z_t$ if and only if $y_t \leq z_t$. We want to show that the set of paths of length t, equipped with this total order, is an M-sequence if the indeterminates in the monomials are ordered according to $<$. To this end, consider a path $y = y_1, \dots, y_t$ and a path $z = z_1, \dots, z_t$, and assume that $y < z$ and that y and z have a common vertex, say $y_i = z_j$. Then $\operatorname{rk} y_t + i - t = \operatorname{rk} y_i = \operatorname{rk} z_j = \operatorname{rk} z_t + j - t$, and $j - i = \operatorname{rk} y_t - \operatorname{rk} z_t \geq 0$. Hence $y_k = z_{k+j-i}$ for all $k = 1, \dots, i$. This shows that $y_i y_{i-1} \cdots y_1 | z$ and concludes the proof. ∎

Now by virtue of Proposition 4.1, Theorem 2.5, and Corollary 2.6 we have

COROLLARY 4.2. *Let Γ be a tree and set $I = I_t(\Gamma)$. Then*

(1) *I is of Gröbner linear type.*

(2) *The Rees algebra $\mathscr{R}(I)$ is normal and Cohen–Macaulay, and* $\operatorname{gr}_I(R)$ *is reduced.*

(3) *If $t = 2$, then* $\operatorname{gr}_I(R)$ *is a Koszul algebra with respect to the standard grading.*

Consider now the ordinary graph ideal $I = I_2(\Gamma)$ associated with a tree $\Gamma = (X, E)$. As in the proof of Proposition 4.1, we fix a total order $<$ on X that refines the rank function. The total order $<$ induces a total order on E in such a way that the monomials associated with the edges form an M-sequence. Then the initial ideal $\operatorname{in}_\tau(J + I)$ of the ideal $J + I$ of definition of $\operatorname{gr}_I(R)$ with respect to the term order τ is generated by square-free monomials of degree 2. Hence $\operatorname{in}_\tau(J + I)$ is again a graph ideal associated

FIGURE 2

with a graph Γ' (which is not a tree in general) and, by Proposition 2.3, it is Cohen–Macaulay. The vertices of Γ' are the elements of $X \cup E$, while its edges are the elements of E and the pairs $(y, (z, w)) \in X \times E$ such that $(w, y) \in E$ (see Fig. 2) or $(z, y) \in E$ and $y < w$ in the total order of the elements of X (see Fig. 3).

Note that the graph Γ' depends strongly on the order $<$ on X. Figure 4 illustrates the two graphs Γ' obtained by the same graph Γ with two different orders.

The Hilbert series of $\mathrm{gr}_I(R)$ is equal to that of the ring $K[X, E]/I_2(\Gamma')$. Next we want to give a combinatorial interpretation. To this end we introduce the following definition: given a graph $\Gamma = (X, E)$, a subset $F \subset E$ is said to be a *set of independent edges* of Γ if for all $x \in X$, x belongs to at most one edge in F. We have:

THEOREM 4.3. *Let $\Gamma = (X, E)$ be a tree, and set $I = I_2(\Gamma)$. Denote by $h_0, h_1, \ldots, h_s$ the h-vector of $\mathrm{gr}_I(R)$. Then h_i is equal to the number of sets of independent edges of Γ with i elements.*

Proof. Let us denote by g_i the number of sets of independent edges of Γ with i elements. Then set $h_\Gamma(\lambda) = \sum_{i \geq 0} h_i \lambda^i$ and $g_\Gamma(\lambda) = \sum_{i \geq 0} g_i \lambda^i$. Consider a total order $<$ on the vertices X of Γ defined as in the proof of Proposition 4.1. Let w_1 be the smallest element of X. Let z be the vertex such that $e_1 = (z, w) \in E$, and denote by $w_2, \ldots, w_n$ the vertices of $X \setminus \{w_1\}$ such that $e_i = (z, w_i) \in E$. Finally, denote (if it exists) by y the vertex of Γ such that $f = (y, z) \in E$. Now set

$$\Gamma_1 = (X_1, E_1) = \left(X \setminus \{w_1\}, E \setminus \{e_1\}\right) \quad \text{and}$$
$$\Gamma_2 = (X_2, E_2) = \left(X \setminus \{w_1, \ldots, w_n, z\}, E \setminus \{e_1, \ldots, e_n, f\}\right).$$

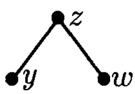

FIGURE 3

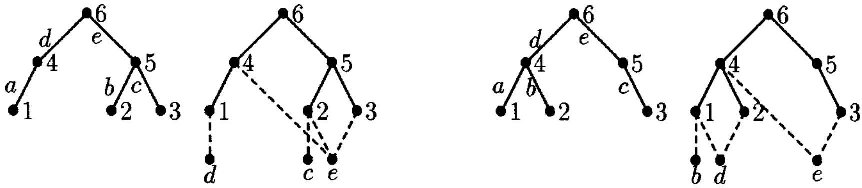

FIGURE 4

By construction, Γ_1 and Γ_2 are trees. The sets of independent edges of Γ that do not contain e_1 are exactly the sets of independent edges of Γ_1. The sets of independent edges of Γ that contain e_1 are of the form $\{e_1\} \cup F$, where F is a set of independent edges of Γ_2. These facts yield the following equation: $g_\Gamma(\lambda) = g_{\Gamma_1}(\lambda) + \lambda g_{\Gamma_2}(\lambda)$. To show that $g_\Gamma(\lambda) = h_\Gamma(\lambda)$, it suffices to show that

$$h_\Gamma(\lambda) = h_{\Gamma_1}(\lambda) + \lambda h_{\Gamma_2}(\lambda), \tag{13}$$

because the functions h and g agree for trees with few vertices. Consider the following short exact sequence:

$$0 \to K[X, E]/(I_2(\Gamma'): w_1)[-1] \xrightarrow{w_1} K[X, E]/I_2(\Gamma') \\ \to K[X, E]/I_2(\Gamma') + (w_1) \to 0 \tag{14}$$

and note that $(I_2(\Gamma'): w_1) = I_2(\Gamma'_2) + (e_2, \dots, e_n, f, z)$ and $I_2(\Gamma') + (w_1) = I_2(\Gamma'_1) + (w_1)$. Hence

$$K[X, E]/(I_2(\Gamma'): w_1) = K[X_2, E_2][w_1, \dots, w_n, e_1]/I_2(\Gamma'_2)$$

and

$$K[X, E]/I_2(\Gamma') + (w_1) = K[X, E][e_1]/I_2(\Gamma'_1).$$

Then, taking into account that the Hilbert series of $K[X, E]/I_2(\Gamma')$ is equal to that of $\mathrm{gr}_I(R)$ and that the dimension of $\mathrm{gr}_I(R)$ is $|X|$, (13) follows from (14). ∎

Theorem 4.3 allows us to determine the h-polynomial of $\mathrm{gr}_I(R)$ for certain trees:

EXAMPLE 4.4. Let S_n be the star with n edges, that is, the edges of S_n are $(x, y_1), (x, y_2), \dots, (x, y_n)$. Denote by $h_n(\lambda)$ the h-polynomial of $\mathrm{gr}_I(R)$, $I = I_2(S_n) = (xy_1, xy_2, \dots, xy_n)$. The only sets of independent edges of S_n are the empty set and the sets $\{(x, y_i)\}$. Hence $h_n(\lambda) = 1 + n\lambda$.

EXAMPLE 4.5. Let L_n be the line with n points, that is, the edges of the tree L_n are $(x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n)$. Denote by $h_n(\lambda)$ the h-polynomial of $\mathrm{gr}_I(R)$, $I = I_2(L_n) = (x_1x_2, x_2x_3, \dots, x_{n-1}x_n)$. Note that (13) yields the following equation: $h_n(\lambda) = h_{n-1}(\lambda) + \lambda h_{n-2}(\lambda)$ for all $n > 2$. Since $h_1(\lambda) = 1$ and $h_2(\lambda) = 1 + \lambda$, one has

$$h_n(\lambda) = \sum_{i \geq 0} \binom{n-i}{i} \lambda^i.$$

EXAMPLE 4.6. Let $\Gamma = (X, E)$ be a tree, with vertex set $X = \{x_1, \dots, x_n\}$. By definition the suspension S of Γ is the tree with vertices $X \cup \{y_1, \dots, y_n\}$ and edges $E \cup \{(x_1, y_1), \dots, (x_n, y_n)\}$. Denote by I and J the ideals $I_2(\Gamma)$ and $I_2(S)$, respectively. A set of independent edges F of Γ with i elements involves exactly $2i$ distinct vertices, say $x_{j_1}, \dots, x_{j_{2i}}$. Adding to F any subset of $\{(x_1, y_1), \dots, (x_n, y_n)\} \setminus \{(x_{j_1}, y_{j_1}), \dots, (x_{j_{2i}}, y_{j_{2i}})\}$, one obtains a set of independent edges of S, and any set of independent edges of S has exactly one such "presentation." It follows that, if the h-polynomial of $\mathrm{gr}_I(R)$ is $\sum_{i \geq 0} h_i \lambda^i$, then the h-polynomial of $\mathrm{gr}_J(R[Y])$ is

$$\sum_{i \geq 0} \left(\sum_{j \geq 0} h_j \binom{n-2j}{i-j} \right) \lambda^i, \tag{15}$$

and in particular the multiplicity of $\mathrm{gr}_J(R[Y])$ is $\sum_{i \geq 0} h_i 2^{n-2i}$. The polynomial (15) has degree n, and it is symmetric. This is not surprising, because it was proved by Villarreal [Vi1, 2.4, 2.5] that J is unmixed, and hence by 2.6 $\mathrm{gr}_J(R[Y])$ is a Gorenstein ring.

5. LADDER IDEALS OF LINEAR TYPE

In this section we show that certain ladder ideals of minors and of pfaffians are of Gröbner linear type, that their Rees algebras are Cohen–Macaulay and normal, and their associated graded rings are Gorenstein normal domains.

Ladder determinantal ideals of minors have been introduced and studied by Abhyankar [A], and subsequently by other authors; see [N], [AK], [M], [HT], [CH], and [C]. They are defined as the ideals generated by the minors of certain subregions, called ladders, of a generic matrix of indeterminates. Likewise, one defines ladder ideals of pfaffians as the ideals generated by pfaffians of certain subregions of a skew-symmetric matrix of indeterminates; see [D].

To avoid confusion, we call *generic ladders* the ladders of a generic matrix, and *ladders of pfaffians* those of a skew-symmetric matrix. We introduce now the class of generic ladders Y_{mn}, and the class of ladders of pfaffians L_{mn} that we want to investigate.

We start with the definition of Y_{mn}. Let $X = (x_{ij})$ be a matrix of indeterminates over a field K, and fix two positive integer m and n. Denote by $[a, b] \times [c, d]$ the submatrix $(x_{ij})_{a \le i \le b, c \le j \le d}$ of X. For $k \ge 0$ we set

$$X_{m,2k+1} = [k+1, k+m] \times [k+1, k+1+m], \quad \text{and}$$
$$X_{m,2k+2} = [k+1, k+1+m] \times [k+2, k+1+m].$$

Note that $X_{m,2k+1}$ and $X_{m,2k+2}$ are matrices of size $m \times m+1$ and $m+1 \times m$, respectively, and that they intersect in a submatrix of size $m \times m$.

Then we set

$$Y_{mn} = \bigcup_{i=1}^{n} X_{m,i}.$$

For example,

$$Y_{34} = \begin{array}{ccccc} x_{11} & x_{12} & x_{13} & x_{14} & \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} \\ & x_{42} & x_{43} & x_{44} & x_{45} \\ & & x_{53} & x_{54} & x_{55} \end{array}.$$

Now let $Z = (z_{ij})$ be a skew-symmetric matrix of indeterminates over a field K, and fix two positive integers m and n. For $k \ge 1$, we set

$$Z_{m,k} = [k, k+2m] \times [k, k+2m].$$

Note that $Z_{m,k}$ is a skew-symmetric matrix of size $2m+1 \times 2m+1$. Then we set

$$L_{mn} = \bigcup_{k=1}^{n} Z_{m,k}.$$

For example,

$$L_{23} = \begin{matrix} 0 & z_{12} & z_{13} & z_{14} & z_{15} & & \\ -z_{12} & 0 & z_{23} & z_{24} & z_{25} & z_{26} & \\ -z_{13} & -z_{23} & 0 & z_{34} & z_{35} & z_{36} & z_{37} \\ -z_{14} & -z_{24} & -z_{34} & 0 & z_{45} & z_{46} & z_{47} \\ -z_{15} & -z_{25} & -z_{35} & -z_{45} & 0 & z_{56} & z_{57} \\ & -z_{26} & -z_{36} & -z_{46} & -z_{56} & 0 & z_{67} \\ & & -z_{37} & -z_{47} & -z_{57} & -z_{67} & 0 \end{matrix}.$$

The *main antidiagonal* of a minor of X (resp. of a pfaffian of Z) with row indices $a_1 < \cdots < a_t$ and column indices $b_1 < \cdots < b_t$ (resp. with row and column indices $a_1 < \cdots < a_{2t}$) is the monomial $\prod_{i=1}^{t} x_{a_i b_{t-i+1}}$ (resp. $\prod_{i=1}^{t} z_{a_i a_{2t-i+1}}$).

The set Y_{mn} is a generic ladder, that is, if the main antidiagonal of a minor is in Y_{mn}, then the minor is in Y_{mn}. Analogously, the set L_{mn} is a ladder of pfaffians, that is, if the main antidiagonal of a pfaffian is in L_{mn}, then the pfaffian is in L_{mn}.

DEFINITION 5.1. Let $K[Y_{mn}]$ and $K[L_{mn}]$ be the polynomial rings over the field K in the set of indeterminates of Y_{mn} and L_{mn}, respectively. We define I_{mn} to be the ideal of $K[Y_{mn}]$ generated by the m-minors of Y_{mn}, and P_{mn} to be the ideal of $K[L_{mn}]$ generated by the $2m$-pfaffians of L_{mn}.

A monomial order τ on the polynomial ring $K[X]$ (resp. $K[Z]$) is said to be *antidiagonal* if the leading monomial of any minor of X (resp. pfaffian of Z) is its main antidiagonal. It is known that the t-minors (resp. $2t$-pfaffians) of a generic ladder (resp. ladder of pfaffians) form a Gröbner basis of the ideal they generate with respect to an antidiagonal monomial order; see [N, 3.4] and [D, 1.4].

In general, initial ideals of ideals of pfaffians of ladders are also initial ideals of ideals of minors of ladders. For instance, in the case under investigation, the initial ideal of P_{mn} and the initial ideal of $I_{m,2n}$ are the same.

To apply Theorem 2.5 and Corollary 2.6 to the ideals I_{mn} and P_{mn}, it is enough to check that the main antidiagonals of the m-minors of Y_{mn} and of the $2m$-pfaffians of L_{mn} are M-sequences. It is easy to see that they even form a sequence of interval type. To this end, it suffices to note that listing the main antidiagonals as they appear in the ladder from the left to the right, one passes from one antidiagonal to the next just by replacing an indeterminate with a new one. This is enough to conclude that the main antidiagonals are a sequence of interval type.

For instance, the main antidiagonals of the 3-minors of Y_{34} are

$$x_{31}x_{22}x_{13}, \quad x_{31}x_{22}x_{14}, \quad x_{31}x_{23}x_{14}, \quad x_{32}x_{23}x_{14}, \quad x_{42}x_{23}x_{14},$$
$$x_{42}x_{33}x_{14}, \quad x_{42}x_{33}x_{24}, \quad x_{42}x_{33}x_{25}, \quad x_{42}x_{34}x_{25},$$
$$x_{43}x_{34}x_{25}, \quad x_{53}x_{34}x_{25}, \quad x_{53}x_{44}x_{25}, \quad x_{53}x_{44}x_{35}.$$

Combining the above discussion with Theorem 2.5, Corollary 2.6, and Proposition 3.2 and the fact that ladder ideals are prime ([N, 4.2] and [D, 1.3]), one has

THEOREM 5.2. *Let* $I = I_{mn}$ *and* $R = K[Y_{mn}]$ *or* $I = P_{mn}$ *and* $R = K[L_{mn}]$. *Then*

(i) *I is of Gröbner linear type.*

(ii) *If τ is an antidiagonal monomial order, then* $\mathrm{in}_\tau(I^i) = \mathrm{in}_\tau(I)^i$ *for all $i \in \mathbb{N}$.*

(iii) *If $m = 2$, then* $\mathrm{gr}_I(R)$ *is a Koszul algebra with respect to the standard grading.*

(iv) *The Rees algebra $\mathscr{R}(I)$ is a Cohen–Macaulay normal domain, it is F-injective and F-rational in positive characteristic, and it has rational singularities in characteristic* 0.

(v) *The associated graded ring* $\mathrm{gr}_I(R)$ *is a normal Gorenstein domain and $I^{(i)} = I^i$ for all $i \in \mathbb{N}$. Furthermore,* $\mathrm{gr}_I(R)$ *is F-pure and F-regular if K is perfect of positive characteristic, and it has rational singularities in characteristic* 0.

Proof. The only statements that still need to be proved are the normality, the F-purity, the F-regularity, and the rational singularities of $\mathrm{gr}_I(R)$. First of all, note that $\mathrm{gr}_I(R)$ is Gorenstein and F-injective and hence is F-pure; see [F, 1.5]. F-regularity for Gorenstein rings is equivalent to F-rationality [HH, 4.7]. Furthermore, F-rationality in positive characteristic together with the fact that the ring $\mathrm{gr}_I(R)$ defined over $\mathbb{Z}$ is $\mathbb{Z}$-free (which is easy to see) implies rational singularities in characteristic 0 [Sm, 4.3]. So it is enough to show that $\mathrm{gr}_I(R)$ is normal (in arbitrary characteristic) and that it is F-rational over a perfect field of positive characteristic. We make use of the Fedder–Watanabe F-rationality criterion [FW, 2.13], which says that if the base field is perfect and there exists a nonzero element c in $\mathrm{gr}_I(R)$ such that $\mathrm{gr}_I(R)[c^{-1}]$ is regular and $\mathrm{gr}_I(R)/c\,\mathrm{gr}_I(R)$ is F-injective, then $\mathrm{gr}_I(R)$ is F-rational.

We distinguish the determinantal from the pfaffian case. So consider first $I = I_{mn}$ and $R = K[Y_{mn}]$, and denote by f_i the ith minor of Y_{mn} (counting from the left to the right), by t_i the residue class in $I/I^2 \subset \mathrm{gr}_I(R)$ of f_i, and by A_i the main antidiagonal of f_i. For systematic reasons, denote by I_{m0} the ideal generated by the determinant of the $m \times m$ matrix Y_{m0} of the first m rows and columns of X. It is easy to see that A_{km+1} has the last-in–first-out property for $k = 0, \dots, n$. Furthermore, $\gcd(A_{km+1}, A_{hm+1}) = 1$ for all $k \neq h$. By virtue of Lemma 3.5, Lemma 2.7, and Theorem 2.5 there exists a monomial order σ on $K[Y_{mn}, T]$ such that $T_1, T_{m+1}, \dots, T_{nm+1}$ do not appear in the generators of $\text{in}_\sigma(J + I) = \text{in}_\sigma(J) + \text{in}_\tau(I)$. Let g be the $(m-1)$-minor of X of the first $m-1$ rows and columns. The elements of the main antidiagonal of g do not appear in the generators of $\text{in}_\sigma(J + I)$. We set $C = T_1 \cdots T_{(n-1)m+1} g$, and $c = t_1 \cdots t_{(n-1)m+1} g$. Note that $\text{in}_\sigma(J + I + (C)) = \text{in}_\sigma(J + I) + \text{in}_\sigma(C) = \text{in}_\sigma(J + I) + (T_1 \cdots T_{nm+1} \text{in}_\tau(g))$, and this implies that $\text{in}_\sigma(J + I + (C))$ is square-free and Cohen–Macaulay because $\text{in}_\sigma(J + I)$ is so. By virtue of [CH, 2.1], $R[T]/J + I + (C) \simeq \text{gr}_I(R)/c\, \text{gr}_I(R)$ is F-injective. To show that $\text{gr}_I(R)[c^{-1}]$ is regular, we argue by induction on n. If $n = 0$, then $\text{gr}_I(R)$ is a polynomial extension of $K[Y_{m0}]/\det Y_{m0}$, and after inversion of g, it becomes a localization of a polynomial ring. Now assume $n > 0$. We claim that

$$\text{gr}_{I_{mn}}(R)\left[t_{(n-1)m+1}^{-1}\right] \simeq \text{gr}_{I_{mn-1}}(R)\left[t_{(n-1)m+1}^{-1}\right]\left[T_{(n-1)m+2}, \dots, T_{mn+1}\right]. \tag{11}$$

From (11) it follows by induction that $\text{gr}_I(R)[c^{-1}]$ is a localization of a polynomial ring. So it remains to show (11). The ladder that is obtained from Y_{mn} by deleting the last column or the last row (depending on whether n is odd or even) is Y_{mn-1}. Denote by B the K-subalgebra of $\text{gr}_{I_{mn}}(R)$ generated by the residue classes of the elements of Y_{mn-1} and by the t_i's. By using the linear relations among $f_{(n-1)m+1}, \dots, f_{mn+1}$ that arise from the matrix X_{mn}, one shows that $\text{gr}_{I_{mn}}(R)[t_{(n-1)m+1}^{-1}]$ coincides with $B[t_{(n-1)m+1}^{-1}]$. By comparing the dimensions, one sees immediately that the only relations in B are those that define $\text{gr}_{I_{mn-1}}(R)$. This proves (11).

The pfaffian case can be treated similarly. We just indicate the main steps. Set $C = T_1 T_{2m+1} \cdots T_{2(n-1)m+1} g$, where g is the pfaffian of the first $2m-2$ rows and columns of L_{m1}, and $c = t_1 t_{2m+1} \cdots t_{2(n-1)m+1} g$. The F-injectivity of $\text{gr}_I(R)/c\, \text{gr}_I(R)$ follows from [CH, 2.1] and from the fact that, by virtue of Lemma 3.5, Lemma 2.7, and Theorem 2.5, one knows that $\text{in}_\sigma(J + I + (C))$ is square-free and Cohen–Macaulay for a suitable monomial order σ. By induction, and using the isomorphism

$$\text{gr}_{P_{mn}}(R)\left[t_{2(n-1)m+1}^{-1}\right] \simeq \text{gr}_{P_{mn-1}}(R)\left[t_{2(n-1)m+1}^{-1}\right]\left[t_{2(n-1)m+1}, \dots, t_{2nm+1}\right], \tag{12}$$

one shows that $\text{gr}_I(R)[c^{-1}]$ is a localization of a polynomial ring.

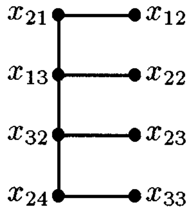

FIGURE 5

Finally the normality of $\mathrm{gr}_I(R)$ follows from (11) and (12) (which hold for any field K) because $(c)\mathrm{gr}_I(R)$ is a radical ideal. ∎

The ideals $I_m = I_{m1}$ and $P_m = P_{m1}$ are the ideal of the m-minors of an $m \times m + 1$ generic matrix and the ideal of the $2m$-pfaffians of a $2m + 1 \times 2m + 1$ skew-symmetric matrix. In addition to the properties listed in Theorem 5.2, they are known to be generated by d-sequences and to be strongly Cohen–Macaulay. We do not know whether this is true for the ideals I_{mn} and P_{mn}, $n > 1$.

Note that Corollary 3.3 gives us the possibility of determining the h-vector of the Rees algebras associated with the ideals I_{mn} and P_{mn}. Unfortunately, we are not able to derive a compact expression unless $m = 2$. But in this case one can also argue as follows. One observes that the initial ideal of I_{2n} (resp. P_{2n}) can be interpreted as the ordinary graph ideal associated with the suspension of a line with $n + 1$ points (resp. $2n + 1$ points). For example, the main antidiagonals of Y_{23} are $x_{21}x_{12}$, $x_{21}x_{13}$, $x_{22}x_{13}$, $x_{32}x_{13}$, $x_{32}x_{23}$, $x_{32}x_{24}$, $x_{33}x_{24}$, and they can be arranged as in Fig. 5. The h-polynomial of the associated graded ring of the graph ideal of the suspension of a line is determined by Examples 4.5 and 4.6. It follows that the ith component of the h-vector of $\mathrm{gr}_I(R)$ is

$$h_i(\mathrm{gr}_I(R)) = \begin{cases} \displaystyle\sum_{j \geq 0} \binom{n+1-j}{j}\binom{n+1-2j}{i-j} & \text{if } I = I_{2n}, \\ \displaystyle\sum_{j \geq 0} \binom{2n+1-j}{j}\binom{2n+1-2j}{i-j} & \text{if } I = P_{2n}, \end{cases}$$

and its multiplicity is

$$e(\mathrm{gr}_I(R)) = \begin{cases} \displaystyle\sum_{j \geq 0} \binom{n+1-j}{j} 2^{n+1-2j} & \text{if } I = I_{2n}, \\ \displaystyle\sum_{j \geq 0} \binom{2n+1-j}{j} 2^{2n+1-2j} & \text{if } I = P_{2n}. \end{cases}$$

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