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Hilbert function of powers of ideals of low codimension

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Abstract. We study the Hilbert function of the powers of homogeneous ideals which are either Cohen-Macaulay of codimension 2 or Gorenstein of codimension 3. We show that that if I is an ideal in one of these classes and it is of linear type then for all k the Hilbert function of I^k depends only on the Hilbert function of I. In other words, if I and J are ideals in one of the above mentioned classes which are both of linear type and they have the same Hilbert function then also I^k and J^k have the same Hilbert function for all k.

Introduction

Let I be a homogeneous ideal in a polynomial ring $R = K[X_1, \dots, X_n]$ and let $P_{R/I}(z)$ denote the Hilbert series of R/I. Consider the class $\mathcal{CI}$ of the ideals of R which are homogeneous complete intersection. If $I \in \mathcal{CI}$, then the degrees of a minimal set of generators of I are uniquely determined by $P_{R/I}(z)$ and they determine the Hilbert series of R/I^k for all $k \in \mathbf{N}$. In other words, if $I, J \in \mathcal{CI}$ and $P_{R/I}(z) = P_{R/J}(z)$, then $P_{R/I^k}(z) = P_{R/J^k}(z)$ for all $k \in \mathbf{N}$.

One may ask whether other classes of ideals have this property. Let $\mathcal{C}$ be a class of homogeneous ideals of R. We say that $\mathcal{C}$ has *rigid powers* (with respect to Hilbert functions) if for all $I, J \in \mathcal{C}$ with $P_{R/I}(z) = P_{R/J}(z)$ then $P_{R/I^k}(z) = P_{R/J^k}(z)$ for all $k \in \mathbf{N}$. More generally, we say that $\mathcal{C}$ has *rigid t-powers* if for all $I, J \in \mathcal{C}$ with $P_{R/I}(z) = P_{R/J}(z)$ then $P_{R/I^k}(z) = P_{R/J^k}(z)$ for all $1 \leq k \leq t$.

Let us denote by $\mathcal{CM}(2)$ the class of the homogeneous ideals of R which are Cohen-Macaulay of codimension 2 and by $\mathcal{G}(3)$ the class of the homogeneous ideals of R which are Gorenstein of codimension 3. It is easy to show that $\mathcal{CM}(2)$ and $\mathcal{G}(3)$ have not rigid powers. Our goal is to determine (big) subclasses of $\mathcal{CM}(2)$ and $\mathcal{G}(3)$ which have rigid powers or rigid t-powers for some $t \in \mathbf{N}$. We denote by $S_k(I)$ the k-th symmetric power of the ideal I. Recall that an ideal I is said to be t-syzygetic if $S_k(I) \simeq I^k$ for all $k = 1, \dots, t$. Furthermore I is said to be of linear type if $S_k(I) \simeq I^k$ for all $k \in \mathbf{N}$. Our main results are:

$i)$ The class $\{I \in \mathcal{CM}(2) : I$ is of linear type$\}$ has rigid powers, [see 2.3].

$ii)$ The class $\{I \in \mathcal{CM}(2) : I$ is t-syzygetic $\}$ has rigid t-powers, [see 2.4].

$iii)$ The class $\{I \in \mathcal{G}(3) : I$ is of linear type$\}$ has rigid powers, [see 4.5].

$iv)$ The class $\{I \in \mathcal{G}(3) : I$ is t-syzygetic $\}$ has rigid t-powers, [see 4.6].

$v)$ Let $\mathcal{C}$ be the class of ideals $I \in \mathcal{CM}(2)$ such that the h-vector of R/I is $1 + 2z + \cdots + mz^{m-1}$ for some $m \in \mathbf{N}$ and $\mu(I_P) \leq \operatorname{height} P$ for all the prime ideal P with $\operatorname{height} P < \dim R$. Then $\mathcal{C}$ has rigid powers, [see 3.2].

Since every ideal $I \in \mathcal{G}(3)$ is 2-syzygetic, from $iv)$ it follows that:

$vi)$ $\mathcal{G}(3)$ has rigid 2-powers, [see 4.2].

This result was conjectured by Geramita, Pucci and Shin [GPS, 4.11]. Another proof of it is given independently by J.Kleppe in [Kl, Prop.2.5] by different methods.

Actually $vi)$ is a consequence of a stronger result; we show that for every ideal I in $\mathcal{G}(3)$ the h-polynomial of R/I^2 is given by $(1-z^c)h(z) + (1+z)^3 h(z^2)/2 - (1-z)^3 h(z)^2/2$ where $h(z)$ is the h-polynomial of R/I and $c = \deg h(z) + 3$, see 4.1. Similarly we show that there exist polynomial formulas for the h-polynomial of the t-power of I in terms of the h-polynomial of I which work for any ideal I either in $\mathcal{CM}(2)$ or $\mathcal{G}(3)$ which is t-syzygetic.

The results of $ii)$ and $v)$ apply, for instance, to ideals of points of $\mathbf{P}^2$. One has that the class of the defining ideals of sets of distinct points in $\mathbf{P}^2$ has rigid 2-powers. This result was conjectured by Geramita, Pucci and Shin [GPS, 4.12]. Further the class of the defining ideals of $\binom{m}{2}$ distinct points in $\mathbf{P}^2$ with generic Hilbert function has rigid powers.

Conjecture [GPS, 4.11] arose in the study of the scheme $\operatorname{Gor}(H)$ of all the Gorenstein ideals I with a given h-polynomial H. By results of Iarrobino and Kanev (in the 0-dimensional case) and of Kleppe (for higher dimension) the dimension of the tangent space of $\operatorname{Gor}(H)$ at I can be expressed in terms of the Hilbert function of R/I and of the Hilbert function of the conormal module I/I^2. Our explicit formula relating the Hilbert series of R/I^2 to that of R/I for any ideal $I \in \mathcal{G}(3)$ yields explicit formulas for the dimension of $\mathrm{Gor}(H)$ only in terms of H, see Sect. 5. One should compare these results with the dimension computations in [Kl] where one has always to introduce Betti numbers even if one knows that the final results depend only on the Hilbert function.

Our approach to the study of the Hilbert function of the powers of an ideal I is based on the study of the bigraded structure of the Rees algebra of I. Roughly speaking, the Hilbert series of the powers of I are the coefficients of the bigraded Hilbert series of the Rees algebra of I. Hence to prove that a certain class $\mathcal{C}$ of ideals has rigid powers it suffices to show that for any $I \in \mathcal{C}$ the bigraded Hilbert series of the Rees algebra of I is determined by the Hilbert series of R/I.

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1 Notation and generalities

Let $R = K[X_1, \dots, X_n]$ be a polynomial ring over a field K and let M be a finitely generated graded R-module. We denote by $H_M(t)$ the Hilbert function and by $P_M(z)$ the Hilbert series of M, that is $H_M(t) = \dim_K M_t$ and $P_M(z) = \sum_t H_M(t) z^t$. For large t the Hilbert function of M coincides with a polynomial which is uniquely determined and it is called the Hilbert polynomial of M.

The Hilbert series can be computed from a free resolution of M. Consider a free resolution (minimal or not) of M

$$\begin{aligned} 0 \to \bigoplus_j R(-j)^{\beta_{rj}} \to \cdots \to \bigoplus_j R(-j)^{\beta_{ij}} \\ \to \cdots \to \bigoplus_j R(-j)^{\beta_{0j}} \to M \to 0. \end{aligned}$$

Then

$$P_M(z) = \sum_{i,j} (-1)^i \beta_{ij} z^j / (1-z)^n.$$

Let I be a homogeneous ideal of R. It is well known that the Hilbert series of R/I can be expressed as

$$P_{R/I}(z) = h(z)/(1-z)^d$$

where $h(z) \in \mathbf{Z}[z]$ and d is the Krull dimension of R/I. The polynomial $h(z)$ is called the h-polynomial of R/I (sometime improperly the h-polynomial of I). The a-invariant $a(R/I)$ of R/I is, by definition, the degree of the rational function $P_{R/I}(z)$, that is, $\deg h(z) - d$. It is well known that $a(R/I)$ is the maximum of the integers k for which the Hilbert function and the Hilbert polynomial of R/I do not agree at k.

Let $B = \bigoplus_{(a,b)\in\mathbf{N}^2} B_{(a,b)}$ be a finitely generated $\mathbf{N}^2$-graded K-algebra such that $B_{(0,0)} = K$. Denote by $(a_1, b_1), \dots, (a_r, b_r)$ the degrees of the generators of B.

The bigraded Hilbert series $P_B(z,u)$ of B is defined as follows:

$$P_B(z,u) = \sum_{(s,t)\in\mathbf{N}^2} \dim_K B_{(s,t)} z^s u^t.$$

The series $P_B(z,u)$ can be expressed as a rational function

$$P_B(z,u) = F(z,u) / \prod_i (1 - z^{a_i} u^{b_i})$$

where $F(z,u) \in \mathbf{Z}[z,u]$. The polynomial $F(z,u)$ is determined by any bigraded free resolution of B over $K[X_1, \dots, X_r]$ where the bigraded structure on $K[X_1, \dots, X_r]$ is given by the weights $\deg X_i = (a_i, b_i)$. The bigraded algebra B is said to be *standard* (or homogeneous) if all the K-algebra generators of B have degree either $(1,0)$ or $(0,1)$.

1.1 The bigraded structure of the Rees algebra

In order to study the powers of the ideal I, it is natural to consider the Symmetric and the Rees algebra of I. The *Symmetric algebra* $S(I)$ of the R-module I has a natural structure of $\mathbf{N}$-graded algebra, namely

$$S(I) = \bigoplus_{j\in\mathbf{N}} S_j(I)$$

where $S_j(I)$ is the j-th symmetric power of I. Since I is homogeneous and $S_j(I)$ is a quotient of the tensor product $I^{\otimes j}$, $S_j(I)$ has a natural structure of graded R-module. Hence $S(I)$ is a $\mathbf{N}^2$-graded algebra

$$S(I) = \bigoplus_{(i,j)\in\mathbf{N}^2} S_j(I)_i.$$

The *Rees algebra* $\mathcal{R}(I)$ of I is the subalgebra of $R[T]$ consisting of the polynomials $\sum a_j T^j$ where $a_j \in I^j$. As a subalgebra of the bigraded algebra $R[T] = \oplus R_i T^j$, the Rees algebra of I has a natural structure of bigraded algebra

$$\mathcal{R}(I) = \bigoplus_{(i,j)\in \mathbf{N}^2} (I^j)_i T^j.$$

The canonical epimorphism from $S(I)$ to $\mathcal{R}(I)$ is a homogeneous homomorphism of $\mathbf{N}^2$-graded algebras. An ideal I is said to be *syzygetic* (or 2-syzygetic) if $S_2(I) \simeq I^2$. Furthermore I is said to be m*-syzygetic* if $S_j(I) \simeq I^j$ for $1 \leq j \leq m$. Finally I is of *linear type* if the Rees algebra and the Symmetric algebra of I are isomorphic, i.e. $S_j(I) \simeq I^j$ for all j.

Let now $F_1, \ldots, F_r$ be homogeneous generators of the ideal I. We may present the Rees algebra $\mathcal{R}(I)$ as a quotient of the polynomial ring $A = R[T_1, \ldots, T_r]$ via the epimorphism of R-algebras

$$\phi : A \to \mathcal{R}(I) = R[F_1T, \ldots, F_rT]$$

sending T_j to F_jT. Since the degree of F_jT in $\mathcal{R}(I)$ is $(\deg(F_i), 1)$, we give to A the $\mathbf{N}^2$-graded structure induced by the weights $\deg X_i = (1, 0)$ and $\deg T_j = (\deg(F_j), 1)$ so that ϕ is a $\mathbf{N}^2$-graded homomorphism.

It is well known (and easy to see) that $S(I)$ has a presentation $S(I) \simeq A/J$ where J is the ideal generated by the polynomials $\sum D_iT_i$ such that $\sum D_iF_i = 0$.

We will make frequently use of the following formulas. Since $\mathcal{R}(I)_{(s,t)} = (I^t)_sT^t$ we have

$$P_{I^t}(z) = \sum_{s\geq 0} \dim_K (I^t)_s z^s = \sum_{s\geq 0} \dim_K \mathcal{R}(I)_{(s,t)} z^s \tag{1}$$

and

$$P_{\mathcal{R}(I)}(z, u) = \sum_{t\geq 0} P_{I^t}(z)u^t. \tag{2}$$

From this we also get

$$P_{I^t}(z) = \frac{1}{t!} \left[\frac{\partial^t}{\partial u^t} P_{\mathcal{R}(I)}(z, u)\right]_{u=0} \tag{3}$$

Note that a class $\mathcal{C}$ of ideals has rigid powers if and only if for all $I, J \in \mathcal{C}$ with $P_{R/I}(z) = P_{R/J}(z)$ one has $P_{\mathcal{R}(I)}(z, u) = P_{\mathcal{R}(J)}(z, u)$.

The associated graded ring $G(I) = \bigoplus_{t\geq 0} I^t/I^{t+1}$ inherits form $\mathcal{R}(I)$ the bigraded structure. One has exact sequences of bigraded modules and homomorphisms

$$0 \to I\mathcal{R}(I) \to \mathcal{R}(I) \to G(I) \to 0$$

and

$$0 \to I\mathcal{R}(I)(0,-1) \to \mathcal{R}(I) \to R \to 0$$

where in the second sequence the first map is the multiplication by T which has bidegree $(0,1)$. From this one can deduce the following relation between the bigraded Hilbert series of $\mathcal{R}(I)$ and that of $G(I)$:

$$(1-u)P_{\mathcal{R}(I)}(z,u) + uP_{G(I)}(z,u) = P_R(z) \tag{4}$$

Let $a_1,\dots,a_k$ be the degrees of the minimal generators of I and assume $a_1 \le a_2 \le \cdots \le a_k$. We have seen that

$$P_{\mathcal{R}(I)}(z,u) = F(z,u)/(1-z)^n \prod_i (1-z^{a_i}u)$$

where $F(z,u) \in \mathbf{Z}[z,u]$. We may write $F(z,u) = \sum_{i=0}^h g_i(z)u^i$ where $g_i(z) \in \mathbf{Z}[z]$. Then one has

$$P_{I^p}(z) = \sum_{i=0}^h g_i(z) \sum_{|\alpha|=p-i} z^{\alpha a}/(1-z)^n$$

where $\alpha \in \mathbf{N}^k$, $|\alpha|$ denotes the sum of the components of α and αa denotes the scalar product. It follows then easily that

$$pa_1 - 1 - n + \operatorname{height} I \le a(R/I^p) \le pa_k + \max_j(\deg g_j - ja_k) - n \tag{5}$$

If the ideal I is generated by polynomials $F_1,\dots,F_r$ all of the same degree, say d, then the Rees algebra can be given a bigraded standard structure by setting $\deg X_i = (1,0)$ and $\deg F_iT = (0,1)$. As above $G(I)$ inherits form $\mathcal{R}(I)$ the bigraded standard structure. In this case T has degree $(-d,1)$ and hence the relation between the Hilbert series is:

$$(1-z^{-d}u)P_{\mathcal{R}(I)}(z,u) + z^{-d}uP_{G(I)}(z,u) = P_R(z). \tag{6}$$

In this case the Rees algebra has also a $\mathbf{N}$-graded standard structure which is obtained by setting $\deg X_i = 1$ and $\deg F_iT = 1$. The relation between the Hilbert series is:

$$(1-z^{-d+1})P_{\mathcal{R}(I)}(z) + z^{-d+1}P_{G(I)}(z) = P_R(z). \tag{7}$$

2 Cohen-Macaulay codimension 2

Let us denote by $\mathcal{CM}(2)$ the class of homogeneous ideals of the polynomial ring $R = K[X_1, \ldots, X_n]$ which are Cohen-Macaulay of codimension two. It is easy to see that the class $\mathcal{CM}(2)$ does not have rigid powers. Take for instance the ideals $I = (X^2, Y^2)$ and $J = (X^2, XY, Y^3)$ of $R = K[X, Y]$. They are 0-dimensional and hence Cohen-Macaulay of codimension two. They have the same Hilbert series, namely $P_{R/I}(z) = P_{R/J}(z) = 1 + 2z + z^2$, but $P_{R/I^2}(z) \neq P_{R/J^2}(z)$. In this section we will determine a subclass of $\mathcal{CM}(2)$ with rigid powers.

Let $I \in \mathcal{CM}(2)$. By the Hilbert-Burch theorem, R/I has a minimal free resolution

$$0 \to \oplus_{i=1}^{r} R(-b_i) \xrightarrow{f} \oplus_{i=1}^{r+1} R(-a_i) \to R \to R/I \to 0$$

and it is generated by the maximal minors, say $D_1, \ldots, D_{r+1}$, of the $r \times (r+1)$ matrix $H = (F_{ij})$ associated with the map f. The degree of D_i is a_i, while the degree of F_{ij} is $b_i - a_j$ whenever $F_{ij} \neq 0$. The Symmetric algebra of I has the presentation

$$S(I) = A/(G_1, \ldots, G_r)$$

where $A = R[T_1, \ldots, T_{r+1}]$ and $G_i = \sum F_{ij} T_j$. We observe that $F_{ij} T_j$ in the bigraded algebra A has bidegree

$$(b_i - a_j, 1) + (a_j, 1) = (b_i, 1)$$

so that G_i is homogeneous of bidegree $(b_i, 1)$.

A result of Avramov (see [A, Prop.1]) gives a necessary and sufficent condition under which the elements $G_1, \ldots, G_r$ form a regular sequence in A . The condition is that the ideal $I_t(H)$ generated by the $t \times t$ minors of H has $\mathrm{height}\, I_t(H) \geq r - t + 1$ for every $t = 1, \ldots, r$.

We show that for any $I \in \mathcal{CM}(2)$ with a complete intersection Symmetric algebra, the bigraded Hilbert series of $S(I)$ can be expressed in terms of the h-polynomial of R/I.

Proposition 2.1 *Let $I \in \mathcal{CM}(2)$ such that $S(I)$ is a complete intersection. Let $h(z)$ be the h-polynomial of R/I and set $\sum q_i z^i = h(z)(1-z)^2$. Then we have*

$$P_{S(I)}(z, u) = \frac{\prod_{i>0} (1 - z^i u)^{q_i}}{(1-z)^n}.$$

Proof. Let

$$0 \to \oplus_{i=1}^{r} R(-b_i) \to \oplus_{i=1}^{r+1} R(-a_i) \to R \to R/I \to 0$$

be a minimal free resolution of R/I. We are assuming that

$$S(I) = A/(G_1, \dots, G_r)$$

is a complete intersection. Since $\deg(X_i) = (1,0)$, $\deg(T_i) = (a_i, 1)$, $\deg(G_i) = (b_i, 1)$, we have

$$P_{S(I)}(z,u) = \frac{\prod_{i=1}^{r}(1 - z^{b_i}u)}{(1-z)^n \prod_{i=1}^{r+1}(1 - z^{a_i}u)}$$

Note that

$$\sum q_i z^i = h(z)(1-z)^2 = 1 - \sum z^{a_j} + \sum z^{b_j}.$$

Hence we clearly have

$$q_i = \sharp\{m \mid b_m = i\} - \sharp\{m \mid a_m = i\}$$

for every $i > 0$. We may conclude that

$$P_{S(I)}(z,u) = \frac{\prod_{i>0}(1 - z^i u)^{q_i}}{(1-z)^n}.$$

We will apply 2.1 to the study of the rigidity of the subclass of $\mathcal{CM}(2)$ consisting of the ideals of linear type. The characterization of the Cohen-Macaulay codimension two ideals which are of linear type is due to Huneke [Hu, Thm1.1] and [Hu1, Thm.1.16]:

Theorem 2.2 *Let $I \in \mathcal{CM}(2)$ with Hilbert-Burch matrix H of size $r \times (r+1)$. Then the following are equivalent:*
1*) I is of linear type,*
2*) $\operatorname{height} I_t(H) \geq r - t + 2$ for $1 \leq t \leq r$,*
3*) $\mu(I_P) \leq \operatorname{height} P$ for every prime ideal P with $P \supseteq I$.*

Here $I_t(H)$ denotes the ideal generated by the t-minors of H and $\mu(I_P)$ denotes the minimal number of generators of I_P.

From 2.1 it follows:

Theorem 2.3 *Let $I \in \mathcal{CM}(2)$. Assume that I is of linear type. Let $h(z)$ be the h-polynomial of R/I and set $\sum q_i z^i = h(z)(1-z)^2$. Then we have*

$$P_{\mathcal{R}(I)}(z,u) = \frac{\prod_{i>0}(1 - z^i u)^{q_i}}{(1-z)^n}$$

and

$$P_{R/I^t}(z) = \frac{1}{(1-z)^n} - \frac{1}{t!(1-z)^n}\left[\frac{\partial^t}{\partial u^t}\prod_{i>0}(1 - z^i u)^{q_i}\right]_{u=0}.$$

In particular the class $\{I \in \mathcal{CM}(2) : I$ is of linear type$\}$ has rigid powers.

Proof. Since I is of linear type then $S(I) = \mathcal{R}(I)$. One knows that $\dim \mathcal{R}(I) = \dim R + 1$ and hence $S(I) = \mathcal{R}(I)$ is a complete intersection. Then the first assertion follows immediately from 2.1. As for the second formula, it follows immediately from the first and from the formula (3) of Sect. 1.

As an example of application of the corollary, let us consider the following

Example. Let I be defining ideal of a set of four distinct points in $\mathbf{P}^2$ which are the complete intersection of two quadrics. In this case I is of linear type with h-polynomial $h(z) = 1 + 2z + z^2$. Let now J be the defining ideal of four distinct points in $\mathbf{P}^2$, three of them lying on a line. Then J is generated by two quadrics and a cubic, hence it is not a complete inetersection but it is of linear type (because of part 3 of 2.2). We may hence conclude that the Hilbert functions of the powers of I and J coincide. We have

$$q(z) = (1-z)^2(1+2z+z^2) = 1 - 2z^2 + z^4$$

hence

$$P_{\mathcal{R}(I)}(z,u) = P_{\mathcal{R}(J)}(z,u) = \frac{(1-z^4u)}{(1-z)^3(1-z^2u)^2}.$$

This implies that

$$P_{I^t}(z) = P_{J^t}(z) = \frac{(t+1)z^{2t} - tz^{2t+2}}{(1-z)^3}$$

and

$$P_{R/I^t}(z) = P_{R/J^t}(z) = \frac{1-(t+1)z^{2t} + tz^{2t+2}}{(1-z)^3} =$$

$$= \frac{1 + 2z + 3z^2 + \cdots + (2t)z^{2t-1} + tz^{2t}}{1-z}.$$

$\odot$

Remark. Let $X = (X_{ij})$ be a $r \times r+1$ matrix of distinct indeterminates over a field K. Let $R = K[X_{ij}]$ and I be the ideal generated by the r-minors of X. Since I is of linear type [EH], one has:

$$P_{\mathcal{R}(I)}(z,u) = \frac{(1-z^{r+1}u)^r}{(1-z^ru)^{r+1}(1-z)^{r(r+1)}}$$

and from this one can deduce the formula for the Hilbert series of R/I^k which is given in [C, Thm.3.3]. $\odot$

As an extension of 2.3 we have:

Proposition 2.4 *Let $I \in \mathcal{CM}(2)$ and assume that I is t-syzygetic. Let $h(z)$ be the h-polynomial of R/I and set $\sum q_i z^i = h(z)(1-z)^2$. Then we have*

$$P_{R/I^t}(z) = \frac{1}{(1-z)^n} - \frac{1}{t!(1-z)^n}\left[\frac{\partial^t}{\partial u^t}\prod_{i>0}(1-z^iu)^{q_i}\right]_{u=0}$$

In particular the class $\{I \in \mathcal{CM}(2) : I$ is t-syzygetic$\}$ has rigid t-powers.

Proof. Let

$$0 \to \oplus_{i=1}^{r} R(-b_i) \to \oplus_{i=1}^{r+1} R(-a_i) \to R \to R/I \to 0,$$

be the minimal free resolution of R/I, so that

$$P_{R/I}(z) = \frac{h(z)}{(1-z)^{n-2}} = \frac{1-\sum_{i=1}^{r+1} z^{a_i} + \sum_{i=1}^{r} z^{b_i}}{(1-z)^n}.$$

For every $j = 1, \ldots, t$, we have that $I^j = S_j(I)$ and hence $S_j(I)$ is torsion-free. Under this assumption Tchernev [Tc, 5.4] has shown that the symmetric power complex $\mathcal{G}^t F$ of the complex F

$$F : 0 \to \oplus_{i=1}^{r} R(-b_i) \to \oplus_{i=1}^{r+1} R(-a_i) \to 0$$

is a minimal free resolution of I^t.

We claim that there exists a Cohen-Macaulay codimension 2 ideal J in a polynomial ring, say T, such that J is of linear type and I and J have the same graded Betti numbers. Note that since I and J have the same graded Betti numbers they have also the same h-polynomial. Since $J^j = S_j(J)$ for all $j \in \mathbf{N}$, as above we may conclude that $\mathcal{G}^t F_1$ is a minimal free resolution of J^t where F_1 is the complex

$$0 \to \oplus_{i=1}^{r} T(-b_i) \to \oplus_{i=1}^{r+1} T(-a_i) \to 0.$$

Given a complex C the complex $\mathcal{G}^t C$ is built by canonical combinations of symmetric and exterior powers of the modules of C. It follows that I^t and J^t have the same graded Betti numbers. This implies that

$$P_{J^t}(z) = g(z)/(1-z)^m \quad \text{and} \quad P_{I^t}(z) = g(z)/(1-z)^n$$

where $m = \dim T$. Then the desired result follows since by 2.3 we know that

$$g(z) = \frac{1}{t!}\left[\frac{\partial^t}{\partial u^t}\prod_{i>0}(1-z^iu)^{q_i}\right]_{u=0}.$$

It remains to prove the claim. We may assume $a_1 \le a_2 \le \cdots \le a_{r+1}$ and $b_1 \le b_2 \le \cdots \le b_r$. Denote by $H = (F_{ij})$ the Hilbert-Burch matrix in the resolution of I and set $u_{ij} = b_i - a_j$. One has $\deg F_{ij} = u_{ij}$ if

$u_{ij} > 0$ and $F_{ij} = 0$ if $u_{ij} \le 0$. In the degree matrix (u_{ij}) the entries do not decrease if one moves down or left. Since the resolution of R/I is minimal, the maximal minors of H do not vanish and hence $u_{ij} > 0$ for every i, j with $j \le i+1$, see for instance [HTV, Sect. 2]. Set $\alpha_i = u_{ii}$ and $\beta_i = u_{ii+1}$ for $i = 1, \dots, r$, and consider the matrix

$$Z = \begin{pmatrix} X_1^{\alpha_1} & Y_1^{\beta_1} & 0 & \dots & 0 & 0 \\ 0 & X_2^{\alpha_2} & Y_2^{\beta_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & X_r^{\alpha_r} & Y_r^{\beta_r} \end{pmatrix}.$$

Now we define J to be the ideal of $T = K[X_i, Y_i : i = 1, \dots, r]$ generated by maximal minors of Z . The ideal J has codimension two so that it is Cohen-Macaulay. Further, since I and J have the same degree matrix, they have the same Betti numbers. In order to show that J is of linear type one may use 2.2. Alternatively, one may note that J is generated by monomials which form an M-sequence of interval type in the sense of [CD, Def.3.1]. Then it follows from [CD, Thm.2.4,Prop.3.2] that J is of linear type.

It is well known that for a Cohen-Macaulay ideal I of codimension two, being syzygetic is equivalent to be generically a complete intersection, i.e. I_P is a complete intersection for all $P \in Min(I)$, see [SV, Th.2.2]. Thus for instance any radical ideal of $\mathcal{CM}(2)$ is syzygetic. If $I \in \mathcal{CM}(2)$ is syzygetic, then Hilbert series of R/I^2 and of R/I are related in a nice and somehow unexpected way.

Proposition 2.5 *Let $I \in \mathcal{CM}(2)$ and assume that I is syzygetic (i.e. generically complete intersection). Let $h(z)$ be the h-polynomial of R/I and set $q(z) = h(z)(1-z)^2$. Then we have*

$$\begin{aligned} P_{R/I^2}(z) - P_{R/I}(z) = P_{I/I^2}(z) &= \frac{q(z^2) - q(z)^2}{2(1-z)^n} \\ &= \frac{(1+z)^2 h(z^2) - (1-z)^2 h(z)^2}{2(1-z)^{n-2}}. \end{aligned}$$

In particular the h-polynomial of R/I^2 is:

$$h(z) + \frac{(1+z)^2 h(z^2) - (1-z)^2 h(z)^2}{2}.$$

Proof. Let us denote by $Q(z,u)$ the rational function $\prod_{i>0}(1 - z^i u)^{q_i}$. One easily sees that

$$\frac{\partial}{\partial u} Q(z,u) = -Q(z,u) \sum_{i>0} q_i z^i/(1 - z^i u)$$

Set $F_1(z,u) = \sum_{i>0} q_i z^i/(1-z^i u)$. Then we have

$$\frac{\partial^2}{\partial u^2} Q(z,u) = \left[F_1(z,u)^2 - F_2(z,u)\right] Q(z,u)$$

where

$$F_2(z,u) = \frac{\partial}{\partial u} F_1(z,u) = \sum_{i>0} q_i z^{2i}/(1-z^i u)^2.$$

Hence

$$\left[\frac{\partial^2}{\partial u^2} Q(z,u)\right]_{u=0} = (F_1(z,0)^2 - F_2(z,0))Q(z,0) =$$

$$= (q(z)-1)^2 - (q(z^2)-1) = q(z)^2 - 2q(z) - q(z^2) + 2.$$

From this we get

$$\begin{aligned} P_{R/I^2}(z) &= \frac{1}{(1-z)^n} - \frac{q(z)^2 - q(z^2) - 2q(z) + 2}{2(1-z)^n} \\ &= \frac{q(z)}{(1-z)^n} + \frac{q(z^2) - q(z)^2}{2(1-z)^n}. \end{aligned}$$

Since $P_{R/I}(z) = q(z)/(1-z)^n$, the first equality follows. The second and the third one are consequences of the first because $q(z) = h(z)(1-z)^2$.

Remark. One easily see by induction that the t-th partial derivative of the rational function $Q(z,u) = \prod_{i>0}(1-z^i u)^{q_i}$ can be written as $Q(z,u)W_t(z,u)$ where $W_t(z,u)$ is a polynomial in $F_1(z,u), \dots, F_t(z,u)$ with integer coefficents and where

$$F_k(z,u) = \frac{\partial}{\partial u} F_{k-1}(z,u) = (k-1)! \sum_{i>0} q_i z^{ki}/(1-z^i u)^k.$$

Since $F_k(z,0) = (k-1)! \sum_{i>0} q_i z^{ki} = (k-1)!(q(z^k)-1)$, it follows that there exists a polynomial

$$C_t(z, x_1, \dots, x_t) \in \mathbf{Q}[z, x_1, \dots, x_t]$$

such that for every ideal $I \in \mathcal{CM}(2)$ which is t-syzygetic the h-polynomial of R/I^t is given by $C_t(z, h(z), h(z^2), \dots, h(z^t))$ where $h(z)$ is the h-polynomial of R/I. The polynomial $C_t(z, x_1, \dots, x_t)$ can be explicitely determined by carrying out all the computations. We have already seen that

$$C_2(z, x_1, x_2) = x_1 + (1/2)x_2(1+z)^2 - (1/2)x_1^2(1-z)^2.$$

For instance, for $t = 3$ one gets:

$$C_3(z,x_1,x_2,x_3) = x_1 + (1/2)x_2(1+z)^2 + (1/3)x_3(1+z+z^2)^2$$
$$-(1/2)x_1x_2(1-z^2)^2 - (1/2)x_1^2(1-z)^2$$
$$+(1/6)x_1^3(1-z)^4.$$

$\odot$

We single out a special case of 2.5:

Corollary 2.6 *i) Let I be the defining ideal of a set of distinct points in $\mathbf{P}^2$ and let $h(z)/(1-z)$ be the Hilbert series of $K[X_0,X_1,X_2]/I$. Then the Hilbert series of I/I^2 is*

$$P_{I/I^2}(z) = \frac{(1+z)^2h(z^2)-(1-z)^2h(z)^2}{2(1-z)}.$$

ii) Let I and J be the defining ideals of two sets of distinct points in $\mathbf{P}^2$. If R/I and R/J have the same Hilbert function then the same is true for R/I^2 and R/J^2. In other words, the class of the defining ideals of sets of distinct points in $\mathbf{P}^2$ has rigid second powers.

Part ii) of 2.6 has been conjectured in [GPS, 4.12]. The following example shows that the assumptions we made in 2.4 and 2.5 are necessary.

Example. Let I and J be the ideals of $R = K[X_0,X_1,X_2]$ generated by the maximal minors of the following matrices:

$$\begin{pmatrix} 0 & X_1-2X_2 & -X_0 \\ X_1(X_1-X_2) & (X_0-X_2)(X_0-2X_2) & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & X_1-X_2 & X_2-X_0 & 0 \\ X_1 & 2X_2-X_0 & 0 & 0 \\ 0 & 0 & (X_1-2X_2)(X_1-3X_2) & -X_0 \end{pmatrix}$$

Both I and J are the defining ideals of a set of 7 points in $\mathbf{P}^2$. It is easy to prove that

$$P_{R/I}(z) = P_{R/J}(z) = \frac{1+2z+3z^2+z^3}{1-z}.$$

Further I is of linear type because it is an almost complete intersection while J is syzygetic but not of linear type. By virtue of 2.5 we have

$$P_{R/I^2}(z) = P_{R/J^2}(z) = \frac{1+2z+3z^2+4z^3+5z^4+6z^5+z^6-z^7}{1-z}.$$

By virtue of 2.3

$$P_{R/I^3}(z) = \frac{1+2z+3z^2+4z^3+5z^4+6z^5+7z^6+8z^7+9z^8-3z^{10}}{1-z},$$

and finally by direct (computer) computation

$$P_{R/J^3}(z) = \frac{1+2z+3z^2+4z^3+5z^4+6z^5+7z^6+8z^7+9z^8-2z^{10}-z^{11}}{1-z}.$$

$\odot$

3 Cohen-Macaulay codimension 2 with linear presentation

We now describe another class of Cohen-Macaulay codimension 2 ideals with rigid powers. We consider the class $\mathcal{CM}(2)^*$ of the homogeneous ideals I of the polynomial ring $R = K[X_1, \ldots, X_n]$ which are Cohen-Macaulay of codimension two and such that the following conditions are satisfied:

1) I has a linear presentation, which means that the entries of the Hilbert-Burch matrix H of I are linear forms in R.

2) I verifies the property G_n, which means $\mu(I_P) \leq \operatorname{height} P$ for every prime ideal $P \supseteq I$ with $\operatorname{height} P \leq n-1$.

Remark. Note that condition 1) (for an ideal $I \in \mathcal{CM}(2)$) is equivalent to say that the h-vector of R/I is of the form $1 + 2z + \cdots + mz^{m-1}$. If this is the case, the size of the Hilbert-Burch matrix H is $m \times (m+1)$ and I is minimally generated by $m+1$ forms of degree m. By 2.2, condition $2)$ is equivalent to say that I is of linear type on the punctured spectrum. $\odot$

If $m+1 \leq n$, then, by virtue of 2.2, I is of linear type. We know already how the Hilbert function of the powers I^k of an ideal of linear type I is related to that of I. So we restrict for the moment our attention to the case $m+1 > n$.

The presentation of Rees algebra of the ideals of the class $\mathcal{CM}(2)^*$ (with $m+1 > n$) has been determined by Morey and Ulrich in [MU]. Let us recall their result. Let $F_1, \ldots, F_{m+1}$ be the maximal minors of H and let

$$\phi : S = R[T_1, \ldots, T_{m+1}] \to \mathcal{R}(I)$$

be the presentation of $\mathcal{R}(I)$ obtained by sending T_j to F_jT. Set $J = \operatorname{Ker} \phi$. Let $G_1, \ldots, G_m$ be the defining equations of the Symmetric algebra $S(I)$. As a vector, $(G_1 \ldots G_m)$ is given by

$$(G_1 \ldots G_m) = (T_1 \ldots T_{m+1})\, {}^tH.$$

Let B be the (unique) matrix of linear forms in $K[T_1, \ldots, T_{m+1}]$ such that one has the matrix factorization

$$(T_1 \ldots T_{m+1})\, {}^tH = (X_1 \ldots X_n)B.$$

Note that B has size $n \times m$. Now let $I_n(B)$ be the ideal generated by the maximal minors of B. Then J is equal to $(G_1, \ldots, G_m) + I_n(B)$ [MU, Thm.1.3]. Bruns, Kustin and Miller have studied in [BKM] a class of ideals which includes J. Since $\dim \mathcal{R}(I) = n + 1$, and $\dim S = n + m + 1$, the ideal J has height m, the generic height for an ideal of this type. Therefore Corollary 4.3 in [BKM] applies and we may describe a minimal bigraded free resolution of $S/J = \mathcal{R}(I)$. This is given by

$$\begin{aligned} 0 &\to L_m \to L_{m-1} \to \cdots \to L_n \to L_{n-1} \oplus Q_{n-1} \to \ldots \\ &\to L_1 \oplus Q_1 \to Q_0 \to \mathcal{R}(I) \to 0 \end{aligned}$$

where

$$Q_i = S(-i(m+1), -i)^{\binom{m}{i}}$$

and

$$L_i = \oplus_{a,b} S(-b - m(n+a), -n-a)^{\binom{n+a-b-1}{a}\binom{n+a}{b}\binom{m}{n+a}}$$

where the sum is taken over all $a \geq 0$ and $b \geq 0$ such that $a + b = i - 1$ (note that out of the range $0 \leq a \leq m - n$ and $0 \leq b \leq n - 1$ the binomial exponent vanishes). From this we may deduce the following expression for the bigraded Hilbert series of the Rees algebra of I:

Theorem 3.1 *Let $I \in \mathcal{CM}(2)^*$ and assume $m \geq n$. Then one has:*

$$P_{\mathcal{R}(I)}(z, u) =$$

$$\frac{\sum_{i=0}^{n-1}(-1)^i\binom{m}{i}z^{i(m+1)}u^i + \sum_{a,b\geq 0}(-1)^{a+b+1}\binom{n+a-b-1}{a}\binom{n+a}{b}\binom{m}{n+a}z^{b+m(n+a)}u^{n+a}}{(1-z)^n(1-z^m u)^{m+1}}.$$

As a corollary we have:

Corollary 3.2 *The class $\mathcal{CM}(2)^*$ has rigid powers.*

Proof. Let $I, J \in \mathcal{CM}(2)^*$ and assume they have the same Hilbert series. We have to show that R/I^k and R/J^k have the same Hilbert series for all k. The ideals I and J have the same h-polynomial, say $h(z) = 1 + 2z + \cdots + mz^{m-1}$. If $m < n$, then I, J are of linear type and by virtue of 2.3 R/I^k and R/J^k have the same Hilbert series for all k. If $m \geq n$, then the desired conclusion follows immediately from 3.1.

It is clear that 3.2 cannot be deduced neither from 2.3 since I is not of linear type, nor from 2.4, where only a finite number of powers of I are considered.

Another important property of the ideals in $\mathcal{CM}(2)^*$ is :

Corollary 3.3 *Let $I \in \mathcal{CM}(2)^*$. Then for every integer t we have:*

$$a(R/I^t) \leq mt - 1.$$

Proof. If $n \leq m$ then it follows from 3.1 that the "numerator" of $P_{\mathcal{R}(I)}(z,u)$ can be written as $\sum_{j=0}^{m} g_j(z)u^j$ where $\deg g_j(z) \leq jm + n - 1$ for $j = 0,\ldots,m$. If $n > m$ then I is of linear type and by 2.3 the "numerator" of $P_{\mathcal{R}(I)}(z,u)$ is $(1 - z^{m+1}u)^m$. This polynomial can be written as $\sum_{j=0}^{m} g_j(z)u^j$ where $\deg g_j(z) = j(m+1) \leq jm+n-1$ for $j = 0,\ldots,m$.

Then in both cases, by virtue of Eq.5 of Sect. 1, we have

$$a(R/I^t) \leq tm + \max_j(\deg g_j(z) - jm) - n$$

and since $\deg g_j(z) - jm \leq n-1$ one concludes that $a(R/I^t) \leq mt - 1$.

In some special case the formula of 3.1 simplifies:

Example. In the case $n = 2$, the ideals of the class $\mathcal{CM}(2)^*$ are the powers of (X_1, X_2). If $I = (X_1, X_2)^m$ then obviously

$$P_{R/I^t}(z) = \sum_{j=0}^{mt-1} (j+1)z^j.$$

It is also easy to see that

$$P_{\mathcal{R}(I)}(z,u) = \frac{1 + (m-1-mz)z^m u}{(1-z)^2(1-z^m u)^2}.$$

$\odot$

Example. If $m = n$ then $a = 0$ and $0 \leq b \leq n-1$ so that:

$$P_{\mathcal{R}(I)}(z,u) = \frac{\sum_{i=0}^{n-1}(-1)^i \binom{n}{i} z^{i(n+1)} \left[z^{i(n+1)}u^i - z^{n^2+i}u^n\right]}{(1-z)^n(1-z^n u)^{n+1}} =$$

$$= \frac{(1-z^{n+1}u)^n - z^{n^2}u^n(1-z)^n}{(1-z)^n(1-z^n u)^{n+1}}.$$

$\odot$

We have also:

Proposition 3.4 *Let I be a homogeneous ideal in $K[X_1, X_2, X_3]$ which is perfect of codimension 2. Assume that I verifies G_3 (i.e. I is generically a complete intersection) and has h-polynomial $1 + 2z + \cdots + mz^{m-1}$. Then*

$$P_{R/I^t}(z) = \frac{1 + 2z + \cdots + (mt)z^{mt-1} - \binom{t}{2}\binom{m}{2}z^{mt}}{(1-z)}$$

and

$$\begin{aligned} &P_{\mathcal{R}(I)}(z,u) \\ &= \frac{1 + (m-2-mz)z^m u + \left[\binom{m-1}{2} - m(m-2)z + \binom{m}{2}z^2\right](z^m u)^2}{(1-z)^3(1-z^m u)^3}. \end{aligned}$$

Proof. Denote by $h(z)$ the h-polynomial of R/I^t. By 3.3 $a(R/I^t) \le mt-1$. Since $\dim R/I^t = 1$ we have $\deg h(z) \le mt$ for all m. The initial degree of I^t is mt. This implies that $h(z) = 1+2z+\cdots+(mt)z^{mt-1}+cz^{mt}$ for some $c \in \mathbf{Z}$. By assumption I is generically a complete intersection. Therefore it follows from the multiplicity formula that $e(R/I^t) = \binom{t+1}{2}e(R/I) = \binom{t+1}{2}\binom{m+1}{2}$. Then $c = \binom{t+1}{2}\binom{m+1}{2} - \binom{mt+1}{2} = -\binom{t}{2}\binom{m}{2}$ and this proves the first formula. The second formula follows easily from the first because $P_{\mathcal{R}(I)}(z,u) = \sum_{t\ge 0}[1/(1-z)^3 - P_{R/I^t}(z)]u^t]$.

We remark that the defining ideal I of a set of distinct points in $\mathbf{P}^2$ is generically a complete intersection so that it verifies G_3. On the other hand, the ideal of $\binom{m+1}{2}$ points which are not on a curve of degree $m-1$ has h-polynomial $h(z) = 1 + 2z + \cdots + mz^{m-1}$. Hence for such an ideal I the above proposition applies and one has:

Corollary 3.5 *Let I be the defining ideal of a set of $\binom{m+1}{2}$ distinct points in $\mathbf{P}^2$ which are not on a curve of degree $m-1$. Then*

$$P_{R/I^t}(z) = \frac{1 + 2z + \cdots + (mt)z^{mt-1} - \binom{t}{2}\binom{m}{2}z^{mt}}{(1-z)}.$$

In particular the class of these ideals has rigid powers.

4 Gorenstein codimension 3

We turn now to the study of the Hilbert function of the powers of ideals which are Gorenstein of codimension three. Let us denote by $\mathcal{G}(3)$ the class of the homogeneous ideals of the polynomial ring $R = K[X_1, \ldots, X_n]$ which are Gorenstein of codimension three. It is easy to see that the class

$\mathcal{G}(3)$ has not rigid powers. Take for instance the ideals $I = (X^3, X^2Y, Y^2 - XZ, YZ, Z^2)$ and $J = (X^2, Y^2, Z^2)$ of $R = K[X, Y, Z]$. They are both Gorenstein codimension three ideal and they have the same Hilbert series, namely $P_{R/I}(z) = P_{R/J}(z) = 1+3z+3z^2+z^3$. One has $(X, Y, Z)^8 \subset J^3$ and $X^8 \notin I^3$. It follows that $P_{R/I^3}(z) \neq P_{R/J^3}(z)$. The goal of this section is to determine rigid subclasses of $\mathcal{G}(3)$.

Let $I \in \mathcal{G}(3)$. By virtue of Buchsbaum-Eisenbud structure theorem [BE], one knows that there exists a skew-symmetric matrix A of size $(2g+1) \times (2g+1)$ with homogeneous entries such that I is minimally generated by the $2g$-pfaffians of A and R/I has a minimal free resolution

$$\begin{aligned} 0 \to R(-c) \to F^* = \oplus_{i=1}^{2g+1} R(-b_i) \to F \\ = \oplus_{i=1}^{2g+1} R(-a_i) \to R \to R/I \to 0 \end{aligned}$$

In [KU] Kustin and Ulrich consider the free complex $\mathcal{D}^q$ which is built by canonical combinations of symmetric and exterior powers of the free modules which appear in the minimal free resolution of I. They prove that $\mathcal{D}^q$ is a minimal free resolution of I^q provided the ideals of pfaffians of the skew-symmetric matrix A verify the "sliding grade" condition SPC$_r$ for a certain r depending on q and g, see [KU, Def.5.9,Thm.6.2,6.17] for the precise statements. They also show that the sliding grade condition SPC$_r$ implies that $S_q(I) \simeq I^q$. Boffi and Sanchez also determined the resolution of I^q in the generic case [BS].

For the second power I^2 the corrisponding sliding grade condition SPC$_r$ is always verified. In particular every $I \in \mathcal{G}(3)$ is 2-syzygetic, see also [HSV, Prop.2.8]. The minimal free resolution of I^2 is given by:

$$0 \to \wedge^2 F^* \to F \otimes F^*/\eta \to S_2(F) \to I^2 \to 0$$

where $e_1, \ldots, e_{2g+1}$ is the basis of F, $e_1^*, \ldots, e_{2g+1}^*$ is the dual basis and $\eta = e_1 \otimes e_1^* + \cdots + e_{2g+1} \otimes e_{2g+1}^*$. Since the degree of $e_i \otimes e_i^*$ is $a_i + b_i = c$, it follows that the shifts in the resolutions are:

$$\begin{aligned} 0 \to \bigoplus_{1 \le i < j \le 2g+1} R(-b_i - b_j) \to \bigoplus_{\substack{1 \le i,j \le 2g+1 \\ (i,j) \neq (1,1)}} R(-a_i - b_j) \to \\ \to \bigoplus_{1 \le i \le j \le 2g+1} R(-a_i - a_j) \to R \to R/I^2 \to 0. \end{aligned}$$

Using this resolution we show that the Hilbert series of R/I and R/I^2 are releted by an expression which is similar to the one of 2.5.

Proposition 4.1 *Let $I \in \mathcal{G}(3)$ and let $h(z)$ be the h-polynomial of R/I. Set $p(z) = h(z)(1-z)^3$ and $c = \deg p(z) = \deg h(z) + 3$. Then we have*

$$\begin{aligned} P_{R/I^2}(z) - (1-z^c)P_{R/I}(z) &= \frac{p(z^2) - p(z)^2}{2(1-z)^n} \\ &= \frac{(1+z)^3 h(z^2) - (1-z)^3 h(z)^2}{2(1-z)^{n-3}}, \end{aligned}$$

and the h-polynomial of R/I^2 is

$$(1-z^c)h(z) + (1+z)^3 h(z^2)/2 - (1-z)^3 h(z)^2/2.$$

Proof. The Hilbert series of R/I and R/I^2 can be computed from the the minimal free resolutions. One has $P_{R/I}(z) = p(z)/(1-z)^n$ and $P_{R/I^2}(z) = g(z)/(1-z)^n$ where

$$p(z) = 1 - \sum_{j=1}^{2g+1} z^{a_j} + \sum_{j=1}^{2g+1} z^{b_j} - z^c,$$

and

$$g(z) = 1 - \sum_{1 \le i \le j \le 2g+1} z^{a_i + a_j} + \sum_{\substack{1 \le i,j \le 2g+1 \\ (i,j) \ne (1,1)}} z^{a_i + b_j} - \sum_{1 \le i < j \le 2g+1} z^{b_i + b_j}.$$

One has:

$$2 \left(\sum_{1 \le i \le j \le 2g+1} z^{a_i + a_j} \right) = \sum_{1 \le i,j \le 2g+1} z^{a_i + a_j} + \sum_{i=1}^{2g+1} z^{2a_i},$$

$$\begin{aligned} 2 \left(\sum_{\substack{1 \le i,j \le 2g+1 \\ (i,j) \ne (1,1)}} z^{a_i + b_j} \right) &= 2 \left(\sum_{1 \le i,j \le 2g+1} z^{a_i + b_j} \right) - 2z^c \\ &= \sum_{1 \le i,j \le 2g+1} \left(z^{a_i + b_j} + z^{a_j + b_i} \right) - 2z^c, \end{aligned}$$

$$2 \left(\sum_{1 \le i < j \le 2g+1} z^{b_i + b_j} \right) = \sum_{1 \le i,j \le 2g+1} z^{b_i + b_j} - \sum_{j=1}^{2g+1} z^{2b_j}.$$

Hence

$$
\begin{aligned}
g(z) = 1 - \frac{1}{2}&\left(\sum_{1\le i\le j\le 2g+1} z^{a_i+a_j} + \sum_{i=1}^{2g+1} z^{2a_i}\right)\\
&+\frac{1}{2}\left(\sum_{1\le i,j\le 2g+1}(z^{a_i+b_j}+z^{a_j+b_i})\right) - z^c\\
&-\frac{1}{2}\left(\sum_{1\le i,j\le 2g+1} z^{b_i+b_j} - \sum_{j=1}^{2g+1} z^{2b_j}\right)\\
&= 1 - z^c - \frac{1}{2}\sum_{1\le i,j\le 2g+1}\left(z^{a_i+a_j}+z^{b_i+b_j}-z^{a_i+b_j}-z^{a_j+b_i}\right)\\
&+\frac{1}{2}\sum_{j+1}^{2g+1}(z^{2b_j}-z^{2a_j})\\
&= 1 - z^c - \frac{1}{2}\left(\sum_{j=1}^{2g+1} z^{b_j} - \sum_{j=1}^{2g+1} z^{a_j}\right)^2 + \frac{1}{2}\left(\sum_{j=1}^{2g+1} z^{2b_j} - \sum_{j=1}^{2g+1} z^{2a_j}\right).
\end{aligned}
$$

We finally get

$$
g(z) = 1 - z^c - \frac{1}{2}\left(p(z) - (1-z^c)\right)^2 + \frac{1}{2}\left(p(z^2) - (1-z^{2c})\right) =
$$

$$
= (1-z^c)p(z) + \frac{p(z^2)-p(z)^2}{2}.
$$

The first equality follows, while the second one is an easy consequence of the first.

As a consequence of the proposition we have:

Theorem 4.2 *The class $\mathcal{G}(3)$ of the ideals which are Gorenstein of codimension 3 has rigid 2-powers.*

This result was conjectured (and proved for some subclasses) by Geramita, Pucci and Shin [GPS, 4.11] and it was the starting point of our analysis. A proof of 4.2 is also given independently by J.Kleppe in [Kl, Prop.2.5] by different methods.

From the formula of 4.1 it follows immediately that for any ideal $I \in \mathcal{G}(3)$ the multiplicity $e(R/I^2)$ of R/I^2 is equal to $4e(R/I)$, a result which is due to Herzog, see [Her].

We prove now a result which is the analogous of 2.3. The Gorenstein case is more complicated than the Cohen-Macaulay one because the Rees algebra of an ideal $I \in \mathcal{G}(3)$ of linear type is not a complete intersection.

The characterization of the ideals $I \in \mathcal{G}(3)$ which are of linear type is due to Eisenbud and Huneke [EH, Thm.3.4]

Theorem 4.3 *Let $I \in \mathcal{G}(3)$. The following are equivalent:*
1*) I is of linear type,*
2*) $\mu(I_P) \leq \operatorname{height} P$ for every prime ideal P with $P \supseteq I$.*

The equivalent conditions of 4.3 are also equivalent to the sliding grade condition SPC_1, see [KU, Obs.6.23].

Theorem 4.4 *Let $I \in \mathcal{G}(3)$. Assume that I is of linear type and denote by $h(z)$ the h-polynomial of R/I. Set $p(z) = \sum p_i z^i = h(z)(1-z)^3$ and $c = \deg h(z) + 3 = \deg p(z)$. Then we have*

$$P_{\mathcal{R}(I)}(z,u) = \frac{\prod_{0<i<c}(1-z^i u)^{p_i} + z^c u}{(1-z)^n(1-z^c u^2)}$$

Proof. By virtue of Buchsbaum and Eisenbud structure theorem there exists a skew-symmetric matrix A of size $(2g+1)\times(2g+1)$, with homogeneous entries such that I is minimally generated by the pfaffians, say $D_1, \dots, D_{2g+1}$, of order $2g$ of A.

Furthermore R/I has a minimal free resolution

$$0 \to R(-c) \to \oplus_{i=1}^{2g+1} R(-b_i) \to \oplus_{i=1}^{2g+1} R(-a_i) \to R \to R/I \to 0$$

where the matrix of the map $\oplus_{i=1}^{2g+1} R(-b_i) \to \oplus_{i=1}^{2g+1} R(-a_i)$ is A. By duality one has $a_i + b_i = c$ for all i. The Hilbert series of R/I is

$$P_{R/I}(z) = (1 - \sum_i z^{a_i} + \sum_i z^{b_i} - z^c)/(1-z)^n$$

and hence

$$p(z) = (1 - \sum_i z^{a_i} + \sum_i z^{b_i} - z^c).$$

Let $T_1, \dots, T_{2g+1}$ be indeterminates and let

$$(G_1 \dots G_{2g+1}) = (T_1 \dots T_{2g+1})A.$$

By assumption the ideal $J = (G_1, \dots, G_{2g+1})$ is the defining ideal of the Rees algebra of I as a quotient of $S = K[X_1, \dots, X_n, T_1, \dots, T_{2g+1}]$. The bigraded structure of S is given by the assignment $\deg(X_i) = (1,0)$, $\deg(T_i) = (a_i, 1)$. Since the (i,j)-entry of A has degree $b_i - a_j$, we have $\deg(G_i) = (b_i, 1)$. Note that $\mathcal{R}(I)$ is a domain of Krull dimension $n+1$, J

is a prime ideal of height $2g$, and hence J is an almost complete intersection. We claim that

$$\sum_{i=1}^{2g+1} T_iG_i = 0 = \sum_{i=1}^{2g+1} D_iG_i.$$

In the following, if M is a matrix, we denote by ${}^tM$ the transpose of M. We have

$$\begin{aligned}\sum_{i=1}^{2g+1} T_iG_i &= (T_1 \dots T_{2g+1})\, {}^t(G_1 \dots G_{2g+1}) \\ &= (T_1 \dots T_{2g+1})\, {}^tA\, {}^t(T_1 \dots T_{2g+1}) = 0\end{aligned}$$

because A is skew-symmetric. This proves the first equality. As for the second one, we have

$$0 = (D_1 \dots D_{2g+1})\, {}^tA$$

hence

$$\begin{aligned}(D_1 \dots D_{2g+1})\, {}^tA\, {}^t(T_1 \dots T_{2g+1}) &= (D_1 \dots D_{2g+1})\, {}^t(G_1 \dots G_{2g+1}) \\ &= \sum_{i=1}^{2g+1} D_iG_i = 0.\end{aligned}$$

Let now $E = (G_1, \dots, G_{2g})$. It follows from the claim that

$$E + (T_{2g+1}, D_{2g+1}) \subseteq E : G_{2g+1}.$$

Since J has height 2g, we may assume that any $2g$ of the generators of J form a regular sequence. Further, by virtue of [HuU, 2.12] we know that links specialize; thus we may assume, as in the generic case, that the matrix A is chosen so that

$$G_1, \dots, G_{2g} \quad \text{form a regular sequence in S} \tag{8}$$

and

$$E + (T_{2g+1}, D_{2g+1}) = E : G_{2g+1}. \tag{9}$$

We claim now that the following equalities hold:

$$E : T_{2g+1} = J \tag{10}$$

$$[E + (T_{2g+1})] : D_{2g+1} = (T_1, \dots, T_{2g+1}). \tag{11}$$

Since $\sum_{i=1}^{2g+1} T_iG_i = 0$, we have $E : T_{2g+1} \supseteq J$. The other inclusion follows since J is a prime ideal, $E \subseteq J$ and $T_{2g+1} \notin J$.

Since $(T_1, \ldots, T_{2g+1})$ is a prime ideal in S and $D_{2g+1} \notin (T_1, \ldots, T_{2g+1})$ while $E \subseteq (T_1, \ldots, T_{2g+1})$, we have $[E + (T_{2g+1})] : D_{2g+1} \subseteq (T_1, \ldots, T_{2g+1})$. The other inclusion follows immediately from :

Claim: Let X be a skew-symmetric matrix of size $n \times n$, n even, and let F be its pfaffian. Set $(G_1 \ldots G_n) = (T_1 \ldots T_n)X$. Then $T_i F \in (G_1, \ldots, G_n)$ for every $i = 1, \ldots, n$.

To prove the claim one notes that there exists a matrix X^* such that

$$X\ X^* = F\ I_n$$

where I_n is the identity matrix. Then one has

$$(G_1 \ldots G_n)\ X^* = (T_1 \ldots T_n)\ X\ X^* = (T_1 \ldots T_n)\ F\ I_n$$

which proves the claim.

For simpicity let us set $D = D_{2g+1}, G = G_{2g+1}, T = T_{2g+1}, a = a_{2g+1}$ and $b = b_{2g+1}$. We have a short exact sequence

$$0 \to S/(E : G)(-b, -1) \to S/E \to \mathcal{R}(I) \to 0 \qquad (12)$$

where the first map is the multiplication by G. By Eq.(9) we know that $S/(E : G) = S/(E + (T, D))$, and hence we have

$$0 \to S/\left[(E + (T)) : D\right](-a, 0) \to S/(E + (T)) \to S/(E : G) \to 0 \qquad (13)$$

where the first map is the multiplication by D. Finally we have another short exact sequence

$$0 \to S/(E : T)(-a, -1) \to S/E \to S/(E + (T)) \to 0 \qquad (14)$$

where the first map is the multiplication by T. By Eq.(11) we have also

$$S/\left[(E + (T)) : D\right] = S/(T_1, \ldots, T_{2g+1}) = R,$$

while by Eq.(10)

$$S/(E : T)\ =\ S/J = \mathcal{R}(I).$$

Since $c = a + b$, we get

$$P_{\mathcal{R}(I)}(z, u) \overset{(12)}{=} P_{S/E}(z, u) - z^b u P_{S/(E:G)}(z, u) =$$

$$\overset{(13)}{=} P_{S/E}(z, u) - z^b u \left[P_{S/(E+(T))}(z, u) - z^a P_R(z, u)\right] =$$

$$\overset{(14)}{=} P_{S/E}(z, u) + z^c u P_R(z, u) - z^b u \left[P_{S/E}(z, u) - z^a u P_{\mathcal{R}(I)}(z, u)\right].$$

It follows that

$$P_{\mathcal{R}(I)}(z,u)(1-z^c u^2) = (1-z^b u)P_{S/E}(z,u) + z^c u P_R(z,u) =$$

$$\overset{(8)}{=} \frac{\prod_{i=1}^{2g+1}(1-z^{b_i}u)}{(1-z)^n \prod_{i=1}^{2g+1}(1-z^{a_i}u)} + \frac{z^c u}{(1-z)^n}.$$

Since

$$\sum p_i z^i = h(z)(1-z)^3 = 1 - \sum z^{a_j} + \sum z^{b_j} - z^c,$$

we clearly have for every $0 < i < c$

$$p_i = \sharp\{m \mid b_m = i\} - \sharp\{m \mid a_m = i\}.$$

Hence

$$P_{\mathcal{R}(I)}(z,u)(1-z^c u^2) = \frac{\prod_{0<i<c}(1-z^i u)^{p_i} + z^c u}{(1-z)^n}$$

and this concludes the proof of the theorem.

As a corollary of the theorem we have:

Corollary 4.5 *Let $I \in \mathcal{G}(3)$ be an ideal of linear type. Let $h(z)$ be the h-polynomial of R/I. Set $c = s+3$ and $p(z) = \sum p_i z^i = h(z)(1-z)^3$. Then*

$$P_{R/I^t}(z) = \frac{1}{(1-z)^n} - \frac{1}{t!(1-z)^n}\left[\frac{\partial^t}{\partial u^t}\frac{\prod_{0<i<c}(1-z^i u)^{p_i} + z^c u}{(1-z^c u^2)}\right]_{u=0}.$$

In particular the class $\{I \in \mathcal{G}(3) : I$ is of linear type $\}$ has rigid powers.

Remark. Let $X = (X_{ij})$ be a $2g+1 \times 2g+1$ skew-symmetric matrix of indeterminates over K. Denote by R the polynomial ring $K[X_{ij}]$, by $n = \dim R = (2g+1)g$ and by I the ideal generated by the pfaffians of order $2g$ of I. The ideal I is of linear type [EH]. Then one has

$$P_{\mathcal{R}(I)}(z,u) = \frac{1}{(1-z)^n(1-z^{2g+1}u^2)}\left[\frac{(1-z^{g+1}u)^{2g+1}}{(1-z^g u)^{2g+1}} + z^{2g+1}u\right]$$

Since the generators of the ideal I have all degree g, the Rees algebra has also a bigraded and **N**-graded standard structure. The Hilbert series of $\mathcal{R}(I)$ with respect to the standard bigraded structure is obtained from the above expression by replacing $z^g u$ with u. Hence it is

$$P_{\mathcal{R}(I)}(z,u) = \frac{1}{(1-z)^n(1-zu^2)}\left[\frac{(1-zu)^{2g+1}}{(1-u)^{2g+1}} + z^{g+1}u\right].$$

The Hilbert series of $\mathcal{R}(I)$ with respect to the standard $\mathbf{N}$-graded structure is obtained from the last expression by replacing u with z. Hence it is

$$\begin{aligned} P_{\mathcal{R}(I)}(z) &= \frac{1}{(1-z)^n(1-z^3)}\left[\frac{(1-z^2)^{2g+1}}{(1-z)^{2g+1}} + z^{g+2}\right] \\ &= \frac{(1+z)^{2g+1} + z^{g+2}}{(1-z)^n(1-z^3)} \end{aligned}$$

Hence the h-polynomial of $\mathcal{R}(I)$ (with respect to the standard $\mathbf{N}$-graded structure) is:

$$\frac{(1+z)^{2g+1} + z^{g+2}}{1+z+z^2}$$

from this it follows that the multiplicity of $\mathcal{R}(I)$ (with respect to the standard $\mathbf{N}$-graded structure) is $e(\mathcal{R}(I)) = (2^{2g+1}+1)/3$, see [HTU, Ex.3.7] $\odot$

As in the Cohen-Macaulay codimension 2 case we have:

Theorem 4.6 *Let $I \in \mathcal{G}(3)$ and let $t \in \mathbf{N}$ such that $I^j = S_j(I)$ for every $j = 1, \dots, t$. Let $h(z)$ be the h-polynomial of R/I and set $\sum p_i z^i = h(z)(1-z)^3$. Then we have*

$$P_{R/I^t}(z) = \frac{1}{(1-z)^n} - \frac{1}{t!(1-z)^n}\left[\frac{\partial^t}{\partial u^t}\frac{\prod_{0<i<c}(1-z^i u)^{p_i} + z^c u}{(1-z^c u^2)}\right]_{u=0}.$$

In particular the class $\{I \in \mathcal{G}(3) : I \text{ is } t\text{-syzygetic } \}$ has rigid t-powers.

Proof. As in the proof of 2.4, it is enough to show that there exists a Gorenstein ideal J of codimension three in a suitable polynomial ring T such that J is of linear type and I and J, I^t and J^t have the same graded Betti numbers.

Let

$$0 \to R(-c) \to \oplus_{i=1}^{2g+1} R(-b_i) \to \oplus_{i=1}^{2g+1} R(-a_i) \to R \to R/I \to 0$$

be a minimal free resolution of R/I and assume that $a_1 \le a_2 \cdots \le a_{2g+1}$ and $b_1 \ge b_2 \cdots \ge b_{2g+1}$. Let $U = (u_{ij})$ be the degree matrix of I, that is $u_{ij} = b_i - a_j$. It is known that $u_{ij} > 0$ for $i + j \le 2g+3$, see for instance [HTV, Sect. 5]. Then consider the skew-symmetric matrix B with entries

$$B_{ij} = \begin{cases} 0 & \text{if } i+j < 2g+1 \text{ or } i+j > 2g+3 \\ X_{ij}^{u_{ij}} & \text{if } 2g+1 \le i+j \le 2g+3 \text{ and } 1 \le i < j \le 2g+1 \end{cases}$$

where the X_{ij} with $2g+1 \le i+j \le 2g+3$ and $1 \le i < j \le 2g+1$ form a set of $3g$ distinct indeterminates. Let J be the ideal of $T = K[X_{ij}]$ generated the pfaffians of order $2g$ of B. It is easy to see that J has codimension 3.

It follows that $J \in \mathcal{G}(3)$ (of T) and that J and I have the same graded Betti numbers because, by construction, they have the same degree matrix. In order to prove that J is of linear type one can apply 4.3. Alternatively one may also note that by virtue of [HT, 6.6], the pfaffians of B form a Gröbner basis of J with respect to a lexicographic term order and the initial terms of the pfaffians are their anti-diagonal terms. It is easy to see that the anti-diagonal terms form a M-sequence of interval type in the sense of [CD, Def.3.1]. It follows from [CD, Thm.2.5] that J is of linear type. It remains to show that I^t and J^t have the same h-vector. Actually we will see that I^t and J^t have the same graded Betti numbers. This will be a consequence of the following result of Tchernev [Tc1]:

Let $I \in \mathcal{G}(3)$ and assume that $I^j = S_j(I)$ for every $j = 1, \dots, t$. Then the Kustin-Ulrich complex $\mathcal{D}^t$ is a (minimal) free resolution of I^t.

Since I and J have the same graded Betti numbers, it follows that the Kustin-Ulrich complexes $\mathcal{D}^t$ of I and J are numerically the same. By the above result the complexes $\mathcal{D}^t$ for I and J are free resolution of I^t and J^t, hence I^t and J^t have the same graded Betti numbers.

The formula of 4.2 for the Hilbert series of R/I^2 can be easily recovered from the formula of 4.5 by computing explicitely the second partial derivative. In details, set $Q(z,u) = \prod_{0<i<c}(1-z^iu)^{p_i}/(1-z^cu^2)$ and $P(z,u) = z^cu/(1-z^cu^2)$. Then note that:

$$\frac{\partial}{\partial u}Q(z,u) = -Q(z,u)(\sum_{1<i<c}(p_iz^i/(1-z^iu)) - 2z^cu/(1-z^cu^2))$$

Set $F_1(z,u) = \sum_{1<i<c} p_iz^i/(1-z^iu) - 2z^cu/(1-z^cu^2)$. Then we have

$$\frac{\partial^2}{\partial u^2}Q(z,u) = \left[F_1(z,u)^2 - F_2(z,u)\right]Q(z,u)$$

where

$$\begin{aligned} F_2(z,u) = \frac{\partial}{\partial u}F_1(z,u) &= \sum_{1<i<c}(p_iz^{2i}/(1-z^iu)^2) \\ &+(-2z^c - 2z^{2c}u^2)/(1-z^cu^2)^2. \end{aligned}$$

Further

$$\left[\frac{\partial^t}{t!\partial u^t}P(z,u)\right]_0 = \begin{cases} 0 & \text{if } t \text{ is even} \\ z^{(k+1)c} & \text{if } t \text{ is odd, } t = 2k+1 \end{cases}$$

Hence

$$\left[\frac{\partial^2}{\partial u^2}\left(Q(z,u)+P(z,u)\right)\right]_{u=0} = (F_1(z,0)^2 - F_2(z,0))Q(z,0) =$$

$$\left(\sum_{1<i<c} p_i z^i\right)^2 - \left(\sum_{1<i<c} p_i z^{2i}\right) + 2z^c$$

Since we have

$$p(z) = 1 + \sum_{1<i<c} p_i z^i - z^c$$

and

$$p(z^2) = 1 + \sum_{1<i<c} p_i z^{2i} - z^{2c}$$

we obtain

$$\begin{aligned}\left[\frac{\partial^2}{\partial u^2}\left(Q(z,u)+P(z,u)\right)\right]_{u=0} &= (p(z) - 1 + z^c)^2 - p(z^2) \\ &\quad +1 - z^{2c} + 2z^c = p(z)^2 \\ &\quad -p(z^2) - 2p(z)(1-z^c) + 2.\end{aligned}$$

It follows that

$$P_{R/I^2}(z) = \frac{1}{(1-z)^n} - \frac{p(z)^2 - p(z^2) - 2p(z)(1-z^c) + 2}{2(1-z)^n} =$$

$$= (1-z^c)P_{R/I}(z) + \frac{p(z^2) - p(z)^2}{2(1-z)^n}$$

Remark. As in the Cohen-Macaulay case, by computing the higher partial derivatives one proves by induction that for all $t \in \mathbf{N}$ there exists a polynomial

$$G_t(z, z_1, x_1, \ldots, x_t) \in \mathbf{Q}[z, z_1, x_1, \ldots, x_t]$$

such that for every ideal $I \in \mathcal{G}(3)$ which is t-syzygetic the h-polynomial of R/I^t is given by $G_t(z, z^c, h(z), h(z^2), \ldots, h(z^t))$ where $h(z)$ is the h-polynomial of R/I and $c = \deg h(z) + 3$. $\odot$

5 The dimension of some schemes

In this section, to illustrate the formulas found in the preceding sections, we perform some computations which simplify and extend several results that appeared recently in different papers.

Example. By using the formula of corollary 2.6 we can easily prove Proposition 3.6.1 in [IK] concerning the Hilbert function of the square of the defining ideal of a set of generic points in $\mathbf{P}^2$*. The* h*-polynomial of the defining ideal of a set of generic points in* $\mathbf{P}^2$ *is*

$$h(z) = 1 + 2z + 3z^2 + \cdots + tz^{t-1} + az^t$$

for some $a, 0 \leq a < t+1$*. Accordingly to corollary 2.6 the* h*-polynomial of* R/I^2 *is*

$$g(z) = h(z) + \frac{(1+z)^2 h(z^2) - h(z)q(z)}{2}$$

where $q(z) = h(z)(1-z)^2$ *is the second difference of* $h(z)$*.*

We can visualize $h(z)$ *and its first two differences as follows:*

$h(z)$	1	2	3	$\dots$	t	a	0	0
	1	1	1	$\dots$	1	$a-t$	$-a$	0
$q(z)$	1	0	0	$\dots$	0	$a-t-1$	$t-2a$	a

From this we get

$$\begin{aligned} g(z) &= h(z) \\ &+ \frac{(1+z)^2 h(z^2) - h(z)(1 + (a-t-1)z^t + (t-2a)z^{t+1} + az^{t+2})}{2} \\ &= \sum_{i=0}^{2t-1} (i+1)z^i + \left(2t + 1 - \binom{t+2-a}{2}\right) z^{2t} \\ &\quad + (a-t+1)az^{2t+1} - \binom{a}{2} z^{2t+2}. \end{aligned}$$

We remark that Proposition 3.6.1 in [IK] has been proved for points which have generic Hilbert Function and minimum number of generators possible for that Hilbert function. Here we prove that the formula holds for every set of points with generic (maximal) Hilbert function. $\odot$

Next we use the formula of Proposition 4.1 to compute the dimensions of the scheme $\mathrm{Gor}(H)$ which parametrize homogeneous codimension 3 Gorenstein ideals I in the polynomial ring $R = k[X_1, \dots, X_n]$ with h-polynomial $H = \sum_{i=0}^{s} h_i z^i$ with $h_i = h_{s-i}$, $h_0 = 1$ and $h_1 = 3$.

Example. Let us first consider the case when $n = 3$. By results of Iarrobino and Kanev [IK], Diesel [D, Thm.2.7] and J.Kleppe [Kl, Cor.1.5] one knows that $\mathrm{Gor}(H)$ is a smooth irreducible scheme (if it is non-empty) of dimension

$$\dim \mathrm{Gor}(H) = H_{I/I^2}(s)$$

where I is any ideal of $\mathrm{Gor}(H)$. By 4.1 we have

$$P_{I/I^2}(z) = (1+z)^3 h(z^2)/2 - (1-z)^3 h(z)^2/2 - z^c h(z).$$

Hence $\dim \mathrm{Gor}(H)$ is equal to the coefficient of z^s in $[(1+z)^3 H(z^2) - (1-z)^3 H(z)^2]/2$. The coefficient of z^s in $(1+z)^3 H(z^2)/2$ is $(h_r + 3h_{r-1})/2$ if $s = 2r$ or $(3h_r + h_{r-1})/2$ if $s = 2r+1$. The coefficient of z^s in $(1-z)^3 H(z)^2/2$ is $\sum_{i=0}^{s} h_i p_{s-i}/2$ where

$$p_i = (h_i - 3h_{i-1} + 3h_{i-2} - h_{i-3})$$

are the coefficients of the "third difference" $(1-z)^3 H(z)$ of $H(z)$. Summing up and taking into account the symmetry of the polynomial $H(z)$ we have:

$$\dim \mathrm{Gor}(H) = \begin{cases} (h_r + 3h_{r-1} - \sum_{i=0}^{s} h_i p_i)/2 & \text{if } s \text{ is even, } s = 2r \\ (3h_r + h_{r-1} - \sum_{i=0}^{s} h_i p_i)/2 & \text{if } s \text{ is odd, } s = 2r+1. \end{cases}$$

For instance, for $s = 4, 5, \ldots, 11$ we have:

s	$\dim \mathrm{Gor}(H)$
4	$-1/2h_2^2 + 13/2h_2 - 7$
5	$1/2h_2^2 - 1/2h_2 + 5$
6	$-5/2h_2^2 + 3h_2h_3 - 1/2h_3^2 + 21/2h_2 - 15/2h_3 - 1$
7	$-1/2h_2^2 + 1/2h_3^2 + 13/2h_2 - 7/2h_3 - 1$
8	$-h_2^2 + 4h_2h_3 - 3h_2h_4 + 6h_2 - 5/2h_3^2 + 3h_3h_4$ $-13/2h_3 - 1/2h_4^2 + 7/2h_4 - 1$
9	$-h_2^2 + 3h_2h_3 - 2h_2h_4 + 6h_2 - 1/2h_3^2 - 15/2h_3$ $+1/2h_4^2 + 9/2h_4 - 1$
10	$-h_2^2 + 3h_2h_3 - 3h_2h_4 + h_2h_5 + 6h_2 - h_3^2 + 4h_3h_4$ $-3h_3h_5 - 8h_3 - 5/2h_4^2 + 3h_4h_5 + 9/2h_4 - 1/2h_5^2 + 1/2h_5 - 1$
11	$-h_2^2 + 3h_2h_3 - 3h_2h_4 + h_2h_5 + 6h_2 - h_3^2 + 3h_3h_4$ $-2h_3h_5 - 8h_3 - 1/2h_4^2 + +7/2h_4 + 1/2h_5^2 + 3/2h_5 - 1$

Notice that the two tables in [IK, 3.5.1] are particular cases of the lines $s = 9$ and $s = 11$ in our table. $\odot$

Example. Here we consider the following polynomial which was extensively studied in [GPS]. Let t and j be positive integers such that $\binom{t+2}{2} \leq j < \binom{t+3}{2}$. For every $m \geq 4$ set

$$H(z) = \sum_{i=0}^{t} \binom{i+2}{2} z^i + \sum_{i=t+1}^{t+m-1} j z^i + \sum_{i=t+m}^{2t+m} \binom{2t+m-i+2}{2} z^i.$$

Here we have $s = 2t + m$ and, no matter whether m is even or odd, the coefficient of z^s in $(1+z)^3 H(z^2)/2$ is $2j$. In this case we better write the coefficient of z^s in $(1-z)^3 H(z)^2/2$ as $\sum_{i=0}^{s} r_i q_{s-i}/2$ where r_i and q_i are the coefficients of the "first" and "second " difference $r(z) = (1-z)H(z)$ and $q(z) = (1-z)^2 H(z)$ of $H(z)$ respectively. We can visualize $H(z)$ and its first two differences as follows:

1	3	...	$\binom{t+1}{2}$	$\binom{t+2}{2}$	j	j	j	...	j	$\binom{t+2}{2}$	$\binom{t+1}{2}$	...	3	1
1	2	...	t	$t+1$	a	0	0	...	0	$-a$	$-(t+1)$	...	-3	-2
1	1	...	1	1	b	$-a$	0	...	0	$-a$	b	...	1	1

where $a = j - \binom{t+2}{2}$ and $b = a - (t+1)$. From this table we easily get

$$\sum_{i=0}^{s} r_i q_{s-i} = 1 + 2 + \cdots + (t-1) + t(a-t-1) - a(t+1)$$
$$-a - (t+1) - t - (t-1) - \cdots - 2 = -t^2 - 3t - 2a.$$

Summing up we get

$$\dim \operatorname{Gor}(H) = 2j + \frac{t^2 + 3t + 2a}{2} = 2j + \frac{2j-2}{2} = 3j - 1.$$

$\odot$

Example. We consider here the case $n = 4$. Hence we are dealing with zero-dimensional schemes in $\mathbf{P}^3$ which are arithmetically Gorenstein. By [Kl, Prop. 3.1] we have

$$\dim \operatorname{Gor}(H) = 3d - H_{I/I^2}(s-1)$$

where I is any ideal of $\operatorname{Gor}(H)$ and d is the degree of R/I. Hence $d = \sum_{i=0}^{s} h_i$ and as before one can explicitely write down the a formula for $\dim \operatorname{Gor}(H)$ only in terms of the h_i. For instance for $s = 4, 5, \ldots, 9$ one obtains:

s	$\dim \operatorname{Gor}(H)$
4	$4h_2 + 11$
5	$1/2h_2^2 + 3/2h_2 + 17$
6	$-h_2^2 + h_2h_3 + 8h_2 - 2h_3 + 11$
7	$1/2h_2^2 - h_2h_3 + 7/2h_2 + 1/2h_3^2 + 7/2h_3 + 11$
8	$2h_2h_3 - 2h_2h_4 + 3h_2 - h_3^2 + h_3h_4 - h_3 + 6h_4 + 11$
9	$h_2h_3 - h_2h_4 + 3h_2 + 1/2h_3^2 - h_3h_4 - 5/2h_3 + 1/2h_4^2 + 17/2h_4 + 11$

$\odot$

Example. Here we consider arithmetically Gorenstein smooth curves in $\mathbf{P}^4$ and for every such curve C, we determine the dimension of the Hilbert scheme of $\mathbf{P}^4$ at C in terms of the h-polynomial $H(z) = \sum_{i=0}^{s} h_i z^i$ of C. By [KM, Example 2.9] we know that

$$\dim Hilb_{[C]}\mathbf{P}^4 = 5d + 1 - g + H_{I/I^2}(s-2)$$

where d and g are the degree and the genus of the curve respectively. Since $d = \sum_{i=0}^{s} h_i$ and $g = \sum_{i=1}^{s}(i-1)h_i$ again one can explicitely write down a formula for $\dim \mathrm{Gor}(H)$ only in terms of the h_i. For instance for $s = 4, 5, 6, 7$ one obtains:

s	$\dim \mathrm{Gor}(H)$
4	$3h_2 + 37$
5	$5h_2 + 43$
6	$-1/2h_2^2 + 21/2h_2 + 2h_3 + 33$
7	$h_2^2 - h_2h_3 + 3h_2 + 9h_3 + 35$

$\odot$

References

[A] L.Avramov, Complete intersection and Symmetric algebras, J. of Alg. **73** (1981), 248-263.

[BS] G.Boffi, R.Sanchez, On the resolution of the powers of the pfaffian ideal, J. of Alg. **152** (1992), 463-491.

[BKM] W.Bruns, A.Kustin, M.Miller, The resolution of the generic residual intersection of a complete intersection, J. of Alg. **128** (1990), 214-239.

[BE] D.Buchsbaum, D.Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. of Math. **99** (1977) 447-485.

[CNR] A.Capani, G.Niesi, L.Robbiano, CoCoA, a system for doing Computations in Commutative Algebra, (1995). Available via anonymous ftp from `lancelot.dima.unige.it`

[C] A.Conca, Gröbner bases of powers of ideals of maximal minors, Preprimt 1995, (to appear in J. of Pure and Appl. Alg.).

[CD] A.Conca, E.De Negri, M-sequences, graph ideals and ladder ideals of linear type, Preprint 1995 (to appear in J. of Alg.).

[D] S.Diesel, Irreducibility and dimension theorems for families of height 3 Gorenstein algebras, Pacific J. of Math. **172** (1996), 363-395.

[EH] D.Eisenbud, C.Huneke, Cohen-Macaulay Rees algebras and their specialization, J. of Alg. **81** (1983), 202-224.

[GPS] A.Geramita, M.Pucci, Y. Shin, Smooth points of $Gor(T)$, Queen's papers in Pure and Appl. Math., The Curves seminar at Queen's, Vol.X No. 102 (1996), 256- 297.

[Her] J. Herzog, Ein Cohen-Macaulay Kriterium mit Anwendungen auf den Konormlenmodul und Differentialmodul, Math. Zeit. **163** (1978), 149-162.

[HSV] J. Herzog, A.Simis, W.Vasconcelos, Koszul homology and blowing-up rings, in Proc. of the Conference "Commutative Algebra", Ed. S.Greco, G.Valla, Lect. Notes in Pure and Appl. Math. Vol. 84 (1983), 79-169.

[HT] J. Herzog, N. V. Trung, Gröbner bases and multiplicity of determinantal and pfaffian ideals, Adv. Math. **96** (1992), 1-37.

[HTU] J. Herzog, N. V. Trung, B.Ulrich, On the multiplicity of blow-up rings of ideals generated by d-sequences, J. of Pure and Appl. Alg. **80** (1992), 273-297.

[HTV] J. Herzog, N. V. Trung, G.Valla, On hyperplane sections of reduced irriducible varieties of low codimension, J. of Math. of Kyoto Univ. **34**, (1994), 47-72.

[Hu] C.Huneke, On the Symmetric algebra of a module, J. of Alg. **69** (1981), 113-119.

[Hu1] C. Huneke, Linkage and the Koszul homology of ideals, Amer. J. of Math. **104** (1982), 1043-1062.

[HuU] C.Huneke, B.Ulrich, Divisor class group and deformations, Amer. J. of Math. **107** (1985), 1265-1303.

[IK] A.Iarrobino, V.Kanev, The length of a homogeneous form, determinantal loci of catalecticants and Gorenstein algebras, Preprint 1996.

[Kl] J.Kleppe, The smoothness and the dimension of PGOR(H) and of other strata of the punctual Hilbert scheme, Preprint 1997.

[KM] J.Kleppe, R.M.Miró-Roig, The dimension of the Hilbert scheme of Gorenstein codimension 3 subschemes, Preprint 1997.

[KU] A.Kustin, B.Ulrich, A family of complexes associated to an almost alternating map, with applications to residual intersection, Memoirs of the A.M.S., Vol.95 N.461, (1992).

[MU] S.Morey, B.Ulrich, Rees algebras of ideals with low codimension, Proc. A.M.S. **124** (1996), 3653-3661.

[SV] A.Simis, W.Vasconcelos, The syzygies of the conormlal module, Amer. J. of Math. **103** (1981), 203-224.

[Tc] A.Tchernev, Acyclicity of symmetric and exterior powers of complexes, J. of Alg. **184** (1996), 1113-1135.

[Tc1] A.Tchernev, On the acyclicity of Kustin-Ulrich complexes, in preparation 1997.