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Hilbert function and resolution of the powers of the ideal of the rational normal curve

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Abstract

Let P be the ideal of $R=K[x_0,\dots,x_n]$ generated by the 2-minors of the Hankel matrix

$$X=\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & \cdots & x_{n-1} \\ x_1 & x_2 & x_3 & \cdots & \cdots & x_n \end{pmatrix}.$$

It is well known that P is the defining ideal of the rational normal curve of $\mathbf{P}^n$, that is, the Veronese embedding of $\mathbf{P}^1$ in $\mathbf{P}^n$. The minimal free resolution of R/P is the "generic" one, i.e. the Eagon–Northcott resolution. The resolution of the powers of generic maximal minors has been described by Akin et al. (Adv. Math. 39 (1981) 1–30) and it is linear. It is easy to see that the powers of P do not have the "generic" resolution if $n\geq 5$. The goal of this note is to show that R/P^h has a linear resolution for all h. We determine also the Hilbert function (and hence the Betti numbers) of R/P^h for all h. We compute the Hilbert function of $R/P^{(h)}$ if either $h\leq 3$ or $n\leq 4$. Here $P^{(h)}$ denotes the hth symbolic power of P which in this case coincides with the saturation of P^h. This yields a formula for the Hilbert function of the module of Kähler differentials $\Omega_{A/K}$ of $A=R/P$. Just to avoid trivial cases we will always assume that $n\geq 3$.
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1. Generalities

Let $R=K[x_0,\dots,x_n]$ be the polynomial ring over a field K and let I be a homogeneous ideal. The Hilbert function of R/I will be denoted by $H(R/I,i)$. The a-invariant $a(R/I)$ of R/I is by definition the degree, as a rational function, of the Hilbert series $\sum_i H(R/I,i)z^i$. It is well known that the function $H(R/I,i)$ coincides for large i with a polynomial, the Hilbert polynomial of R/I. It is easy to see that $a(R/I)$ is the maximum

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of all the integers i such that the Hilbert polynomial and the Hilbert function of R/I do not agree at i.

Let us denote by $\beta_{ij}(I)$ the (i,j)th Betti number of I, i.e. the dimension of $Tor_i^R(I,K)$ in degree j. The Castelnuovo–Mumford regularity $\operatorname{reg}(I)$ of I is by definition

$$\operatorname{reg}(I) = \max\{j - i \colon \beta_{ij}(I) \neq 0\}.$$

In general $\operatorname{reg}(I)$ is greater than or equal to the maximum of the degrees of the minimal generators of I. If I is generated by elements all of the same degree, say d, and $\operatorname{reg}(I) = d$ then I is said to have a linear resolution. In other words, I has a linear resolution if its generators have all the same degree and the entries of the matrices of the minimal free resolution of I are linear forms. The saturation I^{sat} of I is the intersection of all the components in a (or equivalently in any) primary decomposition of I whose associated prime is not the maximal homogeneous ideal. One easily sees that $I^{\text{sat}} = \bigcup_i (I : \mathcal{M}^i)$, where $\mathcal{M}$ is the maximal homogeneous ideal of R. It is well known that I^{sat} is the largest ideal which contains I and has the same Hilbert polynomial. Denote by $I_{\geq j}$ the homogeneous ideal generated by the forms of degree $\geq j$ in I. It has been shown in [7, Theorem 1.2] that $\operatorname{reg}(I) \leq j$ if and only if $I_{\geq j}$ has a linear resolution. Let L be a generic linear form for I, i.e. L is not a zero-divisor modulo I^{sat} (L exists for instance if K is infinite). It follows easily from the above characterization of the Castelnuovo–Mumford regularity that

$$\operatorname{reg}(I) \leq \max\{\operatorname{reg}(\bar{I}), \operatorname{sat}(I)\}, \tag{1}$$

where $\operatorname{reg}(\bar{I})$ is the regularity of $\bar{I} = I + L/L$ as an ideal of the polynomial ring R/L and the saturation degree $\operatorname{sat}(I)$ is the least of the integers j such that I and I^{sat} coincide from degree j on. The Hilbert series of R/I can be read off the resolution of R/I and this yields the following inequality:

$$a(R/I) \leq \operatorname{reg}(I) - \operatorname{depth}(R/I) - 1. \tag{2}$$

2. The ideal P^h has a linear resolution

Let P be the ideal of $R = K[x_0, \ldots, x_n]$ generated by the 2-minors of the Hankel matrix

$$X = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 & \cdots & x_{n-1} \\ x_1 & x_2 & x_3 & \cdots & \cdots & x_n \end{pmatrix}.$$

Let τ be the lexicographic term order on R induced by the total order $x_0 > x_1 > \cdots > x_n$. It is shown in [6, Section 3] (and it is easy to see) that the generators of P are a Gröbner basis. It follows that $\operatorname{in}(P)$ is the ideal generated by the monomials $x_i x_j$ with $0 \leq i, j \leq n$ and $j - i > 1$.

Our goal is to show:

Theorem 1. *Set $J = \operatorname{in}(P)$. For all $h \in \mathbf{N}$ one has*

$$\operatorname{reg}(P^h) = \operatorname{reg}(P^{(h)}) = \operatorname{reg}(J^h) = \operatorname{reg}(J^{(h)}) = 2h,$$

where $P^{(h)}$ and $J^{(h)}$ denote the symbolic powers of P and J, respectively. In particular, the ideals P^h and J^h have linear resolutions.

Proof. We recall first some results from [5]. A primary decomposition of the powers P^h of P has been determined in [5, Theorem 3.16]. One has

$$P^h = P^{(h)} \cap \mathcal{M}^{2h}, \tag{3}$$

where $\mathcal{M}$ is the maximal homogeneous ideal $(x_0,\dots,x_n)$. This is an irredundant primary decomposition provided $h > 1$ and $n \geq 4$. The symbolic power $P^{(h)}$ has been explicitly determined in [5, Theorem 3.8]. One has

$$P^{(h)} = \sum I_2^{a_2} I_3^{a_3} \cdots I_m^{a_m}, \tag{4}$$

where the sum is extended to all the sequences of non-negative integers $a_2,\dots,a_m$, with $a_2 + 2a_3 + \cdots + (m-1)a_m = h$. Here m denotes $\lfloor n/2 \rfloor$ and I_k denotes the ideal of the k-minors of the matrix

$$X_k = \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & \cdots & x_{n-k+1} \\ x_1 & x_2 & \cdots & \cdots & \cdots & x_{n-k+2} \\ x_2 & \cdots & \cdots & \cdots & \cdots & x_{n-k+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{k-1} & \cdots & \cdots & \cdots & \cdots & x_n \end{pmatrix}.$$

Note that $I_2 = P$.

The minimal primes of J are the ideals

$$Q_j = (x_i\colon i \neq j \text{ and } i \neq j+1)$$

with $j = 0,1,\dots,n-1$. Hence,

$$J = \bigcap_{j=0}^{n-1} Q_j$$

and

$$J^{(h)} = \bigcap_{j=0}^{n-1} Q_j^h.$$

For our purposes it is also important to recall that the ideal J^k has the following primary decomposition:

$$J^h = J^{(h)} \cap \mathcal{M}^{2h} \tag{5}$$

see [5, Corollary 3.17].

We prove first that $\operatorname{reg}(P^h) = 2h$. By virtue of Eq. (3) the saturation degree of P^h is $2h$. Let L be a generic linear form for P^h, which in this case just means $L \neq 0$. Set $\bar{R}=R/L$, $\bar{P}=P+L/L$ and $\overline{P^h}=P^h+L/L$. One has $\bar{P}^h=\overline{P^h}$ and, since $\dim \bar{R}/\bar{P}=1$, by virtue of [8, Theorem 1.1] we have $\operatorname{reg}(\bar{P}^h) \leq h \operatorname{reg}(\bar{P})$. Since L is a non-zero divisor modulo

P we have $\mathrm{reg}(\bar{P})=\mathrm{reg}(P)$. The ideal P has a linear resolution because R/P is Cohen–Macaulay of minimal multiplicity. More precisely, a minimal free resolution of P is given by the Eagon–Northcott complex which is linear. Anyway we have $\mathrm{reg}(P)=2$. Summing up we have, by virtue of Eq. (1), $\mathrm{reg}(P^h)\le 2h$ and hence $\mathrm{reg}(P^h)=2h$ since P^h is generated in degree $2h$.

The same argument can be applied to show that $\mathrm{reg}(J^h)=2h$. One has just to note that by virtue of Eq. (5) the saturation degree of J^h is $2h$ and that R/J is Cohen–Macaulay with minimal multiplicity. The Cohen–Macaulayness of J can be easily proved by showing that the associated simplicial complex is shellable.

Since $P^{(h)}$ coincides with P^h from degree $2h$ on and $\mathrm{reg}(P^h)=2h$, it follows from [7, Theorem 1.2] that $\mathrm{reg}(P^{(h)})\le 2h$. To prove equality it suffices to show that $P^{(h)}$ has a minimal generator in degree $2h$. By contradiction, if $P^{(h)}$ is generated in degree $<2h$ then by Eq. (4) we have that $P^{(h)}\subset I_3$ and hence $P^h\subset I_3$. But this is a contradiction because I_3 is prime and does not contain P. Similarly, to show that $\mathrm{reg}(J^{(h)})=2h$, it suffices to prove that $J^{(h)}$ has a minimal generator in degree $2h$. This can be seen by using the description of $J^{(h)}$ given in [5, 3.4, 3.5]. □

Remark 2. Let X be an $m\times n$ matrix of linear forms over a field K, and let $I_m(X)$ be the ideal of the maximal minors of X. In [1, Theorem 5.4] Akin, Buchsbaum and Weymann have shown that for all $h\ge 1$ the ideal $I_m(X)^h$ has a linear resolution provided height $I_j(X)\ge (m+1-j)(n-m)+1$ for all $j=1,2,\ldots,m$.

Applying this result to the ideal P one gets $\mathrm{reg}(P^h)=2h$ if $n\le 4$. Note that for $n>4$ there are quadric relations among the 2-minors of the Hankel matrix of size $2\times n$ which are not of Plücker type, see [6, Section 3]. Hence the ideal P^h for $n>4$ cannot have the "generic" resolution because already the minimal number of generators of P^h is strictly smaller than the generic one.

Remark 3. In general, the powers of an ideal with linear resolution need not have linear resolutions. For examples, let Δ be a triangulation of the real projective plane and let I be the monomial ideal associated with the simplicial complex Δ. The ideal I is generated by the 10 monomials

$$I=(x_2x_5x_6,\, x_1x_5x_6,\, x_3x_4x_6,\, x_1x_4x_6,\, x_2x_3x_6,\, x_3x_4x_5,\, x_2x_4x_5,\, x_1x_3x_5,\, x_1x_2x_4,\, x_1x_2x_3).$$

The resolutions of I and I^2 (computed with CoCoA [4]) in characteristic 0 are:

$$0\to R^6(-5)\to R^{15}(-4)\to R^{10}(-3)\to I\to 0,$$

$$0\to R(-12)\to R^{21}(-10)\to R^{80}(-9)\to R^{150}(-8)\to R^{144}(-7)\to R^{55}(-6),$$

$$\to I^2\to 0.$$

Hence I has a linear resolution and the resolution of I^2 is not linear.

It is natural to ask whether the powers of ideals I_t of t-minors of a generic Hankel matrix also have linear resolution. We believe that this is indeed the case but we are not able to prove it. Let us explain why our proof works only for $t=2$. By looking at the primary decomposition of I_t^h one has that the saturation degree of I_t^h is th. But by taking a general hyperplane section of R/I_t^h, one gets a $(2t-3)$-dimensional ring and so one cannot use [8, Theorem 1.1] unless $t=2$.

3. The Hilbert function and Betti numbers of R/P^h

The Hilbert function of R/P is known to be

$$H(R/P,i)=n(i+1)-(n-1)\quad \text{for } i\geq 0.$$

The Krull dimension of the ring R/P^h is 2 and hence the Hilbert polynomial of R/P^h has degree 1. Therefore there exist integers $e_0(h)$ and $e_1(h)$ such that the Hilbert polynomial of R/P^h is

$$e_0(h)(x+1)-e_1(h).$$

Hence, we have

$$H(R/P^h,i)=e_0(h)(i+1)-e_1(h)\quad \text{for } i>a(R/P^h).$$

From the multiplicity formula [3, Corollary 4.6.8] one knows that

$$e_0(h)=e_0(1)\,\text{length}(R_P/P^hR_P)=n\binom{n+h-2}{n-1}. \tag{6}$$

We have:

Lemma 4. *For all h one has $a(R/P^h)\leq 2h-1$ and $a(R/P^{(h)})\leq 2h-2$.*

Proof. By Eq. (2) and Theorem 1 we have

$$a(R/P^h)\leq \text{reg}(P^h)-1-\text{depth}\,R/P^h\leq 2h-1$$

and

$$a(R/P^{(h)})\leq \text{reg}(P^{(h)})-1-\text{depth}\,R/P^{(h)}\leq 2h-1-1=2h-2. \qquad \square$$

Let F be the K-subalgebra of R generated by the 2-minors of X. The algebra F is the special fiber of the Rees algebra of P. It is $\mathbf{N}$-graded and its component of degree h is $(P^h)_{2h}$, that is the component of degree $2h$ of P^h. The Hilbert series $H_F(z)$ of F has been determined in [6, Theorem 4.7]. For $n\geq 4$ one has

$$H_F(z)=\sum_{i\geq 0}\binom{n+1}{2i}z^i+\alpha z+\beta z^2/(1-z)^{n+1},$$

where

$$\alpha=-(2n+1)\quad\text{and}\quad \beta=-(n^2-3n+1),$$

while for $n=3$ the algebra F is a polynomial ring and hence $H_F(z)=1/(1-z)^3$.

From the Hilbert series $H_F(z)$ one can read-off the Hilbert function of F. For $n \geq 4$ one obtains

$$\dim_K(P^h)_{2h} = \binom{n+2h}{n} + \alpha\binom{n+h-1}{n} + \beta\binom{n+h-2}{n}, \tag{7}$$

while for $n=3$ one has $\dim_K(P^h)_{2h} = \binom{2+h}{2}$.

Now we are ready to prove:

Theorem 5. *For all $h \in \mathbf{N}$ one has*

$$H(R/P^h, i) = \begin{cases} e_0(h)(i+1) - e_1(h) & \text{for } i \geq 2h, \\ \binom{n+i}{n} & \text{for } 0 \leq i < 2h, \end{cases}$$

where

$$e_0(h) = n\binom{n+h-2}{n-1}$$

and

$$e_1(h) = (n-1)(n+2)\binom{n+h-2}{n} + (n-1)\binom{n+h-2}{n-1}$$

Proof. We have seen in Lemma 4 that $a(R/P^h) \leq 2h-1$. Hence the Hilbert function of R/P^h coincides with the Hilbert polynomial in degree greater than or equal to $2h$. Since the initial degree of P^h is $2h$, one has

$$H(R/P^h, i) = \binom{n+i}{n} \quad \text{for } 0 \leq i < 2h.$$

It remains to determine the Hilbert polynomial of R/P^h, that is, $e_0(h)$ and $e_1(h)$. We already know that $e_0(h) = n\binom{n+h-2}{n-1}$, see Eq. (6). Now, since we know that $H(R/P^h, 2h) = e_0(h)(2h+1) - e_1(h)$ and $H(R/P^h, 2h) = \dim_K R_{2h} - \dim_K(P^h)_{2h}$ we have

$$e_1(h) = e_0(h)(2h+1) - \binom{n+2h}{n} + \dim_K(P^h)_{2h}.$$

For $n \geq 4$, by virtue of Eq. (7) we obtain

$$e_1(h) = e_0(h)(2h+1) + \alpha\binom{n+h-1}{n} + \beta\binom{n+h-2}{n}$$

while for $n=3$ we obtain

$$e_1(h) = e_0(h)(2h+1) - \binom{3+2h}{3} + \binom{2+h}{2}.$$

An easy computation shows now that in both cases one has

$$e_1(h) = (n-1)(n+2)\binom{n+h-2}{n} + (n-1)\binom{n+h-2}{n-1}. \qquad \square$$

We know already that P^h has a linear resolution:

$$0 \to R(-2h-n)^{\beta_n} \to \cdots \to R(-2h-1)^{\beta_1} \to R(-2h)^{\beta_0} \to P^h \to 0.$$

The Betti numbers β_i of P^h can be read off the Hilbert function of R/P^h. One has

$$\sum_{i=0}^{n}(-1)^i\beta_i z^i = (1 - H_h(z)(1-z)^{n-1})/z^{2h}, \tag{8}$$

where

$$H_h(z) = \sum_{i=0}^{2h-1}\binom{n+i}{n} z^i(1-z)^2 + z^{2h}(2he_0(h) - e_1(h))(1-z) + z^{2h}e_0(h)$$

is the h-vector of P^h. Therefore Eq. (8) determines (at least implicitly) the Betti numbers of P^h. The Betti numbers of $J^h = \mathrm{in}(P)^h = \mathrm{in}(P^h)$ are equal to those of P^h because J^h has also a linear resolution and its Hilbert function equals that of P^h.

Example 6. The polynomials $\sum_{i\geq 0}^{n}(-1)^i\beta_i z^i$ for $n = 5, 6$ and $1 \leq h \leq 7$ are:

n	h	$\sum_{i=0}^{n}(-1)^i\beta_i z_i$
5	1	$-4z^3 + 15z^2 - 20z + 10$
5	2	$-4z^5 + 30z^4 - 100z^3 + 170z^2 - 144z + 49$
5	3	$-30z^5 + 201z^4 - 560z^3 + 795z^2 - 570z + 165$
5	4	$-120z^5 + 755z^4 - 1940z^3 + 2526z^2 - 1660z + 440$
5	5	$-350z^5 + 2115z^4 - 5180z^3 + 6405z^2 - 3990z + 1001$
5	6	$-840z^5 + 4935z^4 - 11740z^3 + 13980z^2 - 8400z + 2030$
5	7	$-1764z^5 + 10150z^4 - 23520z^3 + 27405z^2 - 16044z + 3744$
6	1	$5z^4 - 24z^3 + 45z^2 - 40z + 15$
6	2	$10z^6 - 80z^5 + 285z^4 - 560z^3 + 630z^2 - 384z + 100$
6	3	$91z^6 - 672z^5 + 2121z^4 - 3640z^3 + 3570z^2 - 1896z + 427$
6	4	$428z^6 - 3024z^5 + 9030z^4 - 14560z^3 + 13356z^2 - 6608z + 1379$
6	5	$1435z^6 - 9856z^5 + 28470z^4 - 44240z^3 + 38990z^2 - 18480z + 3682$
6	6	$3892z^6 - 26208z^5 + 74025z^4 - 112224z^3 + 96300z^2 - 44352z + 8568$
6	7	$9114z^6 - 60480z^5 + 168070z^4 - 250320z^3 + 210735z^2 - 95088z + 17970$

4. The Hilbert function of $P^{(h)}$ for $h \leq 3$ or $n \leq 4$

First of all note that as a consequence of Lemma 4 and Theorem 5 one has:

Theorem 7. *The Hilbert polynomial of $R/P^{(h)}$ is $e_0(h)(x+1) - e_1(h)$ where $e_0(h)$ and $e_1(h)$ are those of Theorem 5. Furthermore, $H(R/P^{(h)}, i) = e_0(h)(i+1) - e_1(h)$ for $i \geq 2h-1$.*

Since the symbolic square of P is $P^{(2)} = P^2 + I_3$, the initial degree of $P^{(2)}$ is ≥ 3 (it is equal to 3 if $n \geq 4$). By virtue of Theorem 7 the Hilbert function of $R/P^{(2)}$ coincides with the Hilbert polynomial from degree 3 on. It follows that

Proposition 8.

$$H(R/P^{(2)}, i) = \begin{cases} e_0(2)(i+1) - e_1(2) & \text{for } i \geq 3, \\ \binom{n+i}{n} & \text{for } 0 \leq i < 3, \end{cases}$$

where $e_0(2) = n^2$ *and* $e_1(2) = 2(n^2 - 1)$.

The symbolic cube of P is

$$P^{(3)} = I_2^3 + I_2 I_3 + I_4.$$

In degree ≥ 5 the Hilbert function of $R/P^{(3)}$ is equal to the Hilbert polynomial. The degree 4 component of $P^{(3)}$ coincides with the degree 4 component of I_4. The dimension of this space is the number of the 4-minors of X_4 because the minors are linearly independent (to see this one has just to note that they have distinct leading terms). It follows that the dimension of $P^{(3)}$ in degree 4 is $\binom{n-2}{4}$. Then one has:

Proposition 9.

$$H(R/P^{(3)}, i) = \begin{cases} e_0(3)(i+1) - e_1(3) & \text{for } i \geq 5, \\ \binom{n+4}{n} - \binom{n-2}{4} & \text{for } \quad i = 4, \\ \binom{n+i}{n} & \text{for } i < 4, \end{cases}$$

where $e_0(3) = n^2(n+1)/2$ *and* $e_1(3) = 2(n^2 - 1)(n+1)$.

For $n = 3$ one has $P^{(h)} = P^h$. For $n = 4$ one has

$$P^{(h)} = I_2^h + I_2^{h-2} I_3 + I_2^{h-4} I_3^2 + \cdots + I_2^{h-2j} I_3^j + \cdots .$$

Since $\mathscr{M} I_3 \subseteq I_2^2$ (see [5, Lemma 3.7]), we have

$$I_2^{h-2j} I_3^j \mathscr{M} \subset I_2^{h-2(j-1)} I_3^{j-1}.$$

Then the component of degree $2h - j$ of $P^{(h)}$ coincides with the component of degree $2h - j$ of $I_2^{h-2j} I_3^j$. Since I_3 is principal the dimension of $P^{(h)}$ in degree $2h - j$ is equal to the dimension of I_2^{h-2j} in degree $2h - 4j$. By virtue of Eq. (7), the last is

$$\binom{4 + 2h - 4j}{4} - 9\binom{3 + h - 2j}{4} - 5\binom{2 + h - 2j}{4}$$

which is equal to

$$2\binom{h - 2j + 4}{4} - \binom{h - 2j + 3}{3}.$$

So we have shown that:

Proposition 10. *For* $n = 4$ *one has*

$$H(R/P^{(h)}, i) = \begin{cases} e_0(h)(i+1) - e_1(h) & \text{for } i \geq 2h - 1, \\ \binom{4+i}{4} - 2\binom{2i-3h+4}{4} + \binom{2i - 3h + 3}{3} & \text{for } i < 2h - 1. \end{cases}$$

In order to determine the Hilbert function of $R/P^{(h)}$ in general it would be enough to compute the minimal number of generators of $I_t^a I_{t+1}^b$ for all a, b, t. But we do not know how to solve this problem in general.

5. The module of differentials of R/P

Set $A = R/P$ and denote by $\Omega_{A/K}$ the module of Kähler differentials of A over K. One has short exact sequence of A-modules:

$$0 \to P/P^{(2)} \to A^{n+1} \to \Omega_{A/K} \to 0. \qquad (9)$$

The map $\phi : P/P^{(2)} \to A^{n+1}$ is given by

$$\phi(f) = \sum_{i=0}^{n} (\partial f/\partial x_i)\,\mathrm{d}x_i,$$

where $\mathrm{d}x_i$ are the basis elements of A^{n+1}. Hence ϕ has degree -1. One obtains the following expressions for the Hilbert function of $\Omega_{A/K}$:

$$\begin{aligned} H(\Omega_{A/K}, i) &= (n+1)H(A,i) - H(P, i+1) + H(P^{(2)}, i+1) \\ &= (n+1)H(A,i) + H(A, i+1) - H(R/P^{(2)}, i+1). \end{aligned}$$

A simple calculation using Proposition 8 now yields:

Proposition 11.

$$H(\Omega_{A/K}, i) = \begin{cases} 2n(i+1) & \text{if } i \geq 2, \\ (n+1)^2 - \binom{n}{2} & \text{if } i = 1, \\ n+1 & \text{if } i = 0. \end{cases}$$

Note that $H(\Omega_{A/K}, 2) = 6n$ is smaller than $H(\Omega_{A/K}, 1) = (n+1)^2 - \binom{n}{2}$ for $n \geq 7$. It follows that depth $\Omega_{A/K} = 0$ for $n \geq 7$. Bruns has shown in his Ph.D. thesis [2] that depth $\Omega_{A/K} = 0$ for all $n \geq 3$. The following is an alternative proof of Bruns' result;

Theorem 12. *Let $A = R/P$. Then*

$$\operatorname{depth} \Omega_{A/K} = 0$$

for all $n \geq 3$.

Proof. By virtue of (9) and since A is Cohen–Macaulay, it suffices to show that

$$\operatorname{depth} P/P^{(2)} = 1.$$

We have the short exact sequence

$$0 \to P/P^{(2)} \to R/P^{(2)} \to A \to 0.$$

Therefore, it is enough to show that depth $R/P^{(2)}=1$. We have shown that $\operatorname{reg}(P^{(2)})=4$ and hence by Eq. (2) we have

$$\operatorname{depth} R/P^{(2)} \leq 3 - a(R/P^{(2)}).$$

The difference between the Hilbert polynomial and the Hilbert function of $R/P^{(2)}$ at $i=2$ is

$$(3n^2 - 2(n^2-1)) - \binom{n+2}{2} = 1/2(n-2)(n-1)$$

and it does not vanish for $n \geq 3$. It follows that $a(R/P^{(2)}) = 2$ and hence depth $R/P^{(2)}=1$. □

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