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Universally Koszul algebras

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Introduction

Let K be a field and let R be a homogeneous K-algebra, i.e. an algebra of the form $R = K[x_1, \dots, x_n]/I$ where I is a homogeneous ideal with respect to the standard grading $\deg x_i = 1$. Recall that R is said to be Koszul if the residue field K has a linear R-free resolution as an R-module. Koszul algebras have been introduced by Priddy [P] in the seventies and since then they have been extensively studied in various contexts. For an updated survey and a rich bibliography on this topic we refer the reader to the recent paper of Fröberg [F].

In this article we introduce and study *universally Koszul algebras*. A homogeneous algebra R is said to be universally Koszul (u-Koszul for short) if every ideal of R generated by linear forms has a linear R-free resolution. This notion can be seen as a strong version of Koszulness. As one may expect u-Koszul algebras are very rare. The goal of the paper is to present some classes of algebras with this property and to classify those u-Koszul algebras which are Cohen-Macaulay domains.

Section 1 contains generalities on u-Koszul algebras. A characterization of them in terms of colon ideals is given and the behaviour of u-Koszul algebras under standard algebra operations (tensor, fiber and Segre products, Veronese subrings, and quotients) is studied.

Section 2 is devoted to Artinian algebras. We show that a quadratic algebra R with Hilbert series $1 + nz + mz^2$ is u-Koszul if either $m \le 1$ or $2m \le n$ and the relations of R are "generic", see 2.1 and 2.4.

As a consequence of a result of Bertini and of Harris' general position theorem one shows that a necessary condition for a Cohen-Macaulay domain to be u-Koszul is to have minimal multiplicity (at least, say, over an algebraically closed

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field of characteristic 0). Cohen-Macaulay domains of minimal multiplicity are classified by a theorem of Bertini and Del Pezzo. They are of essentially of three types: quadric hypersurfaces, coordinate rings of rational normal scrolls, and the coordinate ring of the Veronese embedding $\mathbf{P}^2 \to \mathbf{P}^5$. This opens the way to a classification of the u-Koszul Cohen-Macaulay domains. In Sect. 3 we show that

1) any quadric hypersurface is u-Koszul (3.1),

2) the coordinate ring of the rational normal scroll of type $(a_1, a_2, \dots, a_k)$ is u-Koszul if and only if $k = 1$ (the rational normal curve) or $k = 2$ and $a_1 = a_2$ (3.3),

3) the coordinate ring of the Veronese embedding $\mathbf{P}^2 \to \mathbf{P}^5$ is u-Koszul (3.12).

Some of the results of this paper have been conjectured after (and confirmed by) explicit computations performed with the computer algebra system CoCoA [CNR].

1 Notation, definitions and generalities

Let us first recall the definition of Koszul filtration. This notion, introduced in [CTV], was inspired by the work Herzog, Hibi and Restuccia [HHR] on strongly Koszul algebras.

Definition 1.1 Let R be a homogeneous K-algebra. A family $\mathbf{F}$ of ideals of R is said to be a Koszul filtration of R if:

1) Every ideal $I \in \mathbf{F}$ is generated by linear forms,

2) The ideal 0 and the maximal homogeneous ideal $\mathcal{M}$ of R belong to $\mathbf{F}$,

3) For every $I \in \mathbf{F}$, $I \neq 0$, there exists $J \in \mathbf{F}$ such that $J \subset I$, I/J is cyclic and $J : I \in \mathbf{F}$.

One has (see [CTV]):

Proposition 1.2 *Let $\mathbf{F}$ be a Koszul filtration of R. Then for every $I \in \mathbf{F}$ the quotient R/I has a linear R-free resolution.*

Examples of algebras which admit Koszul filtrations are given in [CTV].

Definition 1.3 A homogeneous algebra R is universally Koszul (u-Koszul for short) if every ideal I generated by linear forms has a linear R-free resolution, that is $\mathrm{Tor}_i^R(R/I, K)_j = 0$ for every $i \neq j$.

First of all note that u-Koszul algebras exist. For instance polynomial rings and rings of the type $K[x_1, \dots, x_n]/(x_1, \dots, x_n)^2$ are simple examples of u-Koszul algebras. Set

$$\mathcal{L}(R) = \{I \subset R : I \text{ is an ideal of } R \text{ generated by linear forms}\}$$

We have the following characterization of u-Koszul algebras:

Proposition 1.4 *Let R be a homogeneous K-algebra. The following conditions are equivalent:*

1) R is u-Koszul,

2) For every $I \in \mathcal{L}(R)$ one has $\operatorname{Tor}_2^R(R/I, K)_j = 0$ for every $j > 2$,

3) For every $I \in \mathcal{L}(R)$ and $x \in R_1 \setminus I$ one has $I : (x) \in \mathcal{L}(R)$,

4) $\mathcal{L}(R)$ is a Koszul filtration of R.

Proof. 1) $\Rightarrow$ 2) is obvious, 3) $\Rightarrow$ 4) is easy, and 4) $\Rightarrow$ 1) follows from 1.2. For 2) $\Rightarrow$ 3), let $y_1, \dots, y_k$ be a system of generators of I and consider a presentation $\phi : R \oplus R^k \to (x)+I$ of $(x)+I$. The projection of the syzygy module $\Omega = \operatorname{Ker}\phi$ on the first coordinate is $I : (x)$ and it is generated by linear forms because by assumption Ω is generated by elements of degree 1. □

The following simple lemma will be useful for inductive arguments:

Lemma 1.5 *Let R be a homogeneous K-algebra. Assume that for every non-zero $z \in R_1$ one has $0 : (z) \in \mathcal{L}(R)$ and $R/(z)$ is u-Koszul. Then R is u-Koszul.*

Proof. We show that $I : (x) \in \mathcal{L}(R)$ for every $I \in \mathcal{L}(R)$ and $x \in R_1 \setminus I$. If $I = 0$ then this is true by assumption. If $I \neq 0$, then take a non-zero linear form $z \in I$. By assumption $R/(z)$ is u-Koszul, and hence the ideal $I : (x)/(z)$ is generated by linear forms. It follows that $I : (x) \in \mathcal{L}(R)$. □

It is known that the Koszul property behaves well under standard operations on algebras, see [BF]. As far as u-Koszul algebras are concerned we have:

Lemma 1.6 *Let R, S be u-Koszul K-algebras. One has:*

1) Let $I \in \mathcal{L}(R)$. Then the algebra R/I is u-Koszul,

2) $R[x_1, \dots, x_n]$ is u-Koszul,

3) The fiber product $R \circ S$ is u-Koszul.

Proof. 1) Set $A = R/I$. Let $J \in \mathcal{L}(A)$ and $y \in A_1 \setminus J$. We have $J = Q/I$ with $Q \in \mathcal{L}(R)$ and $y = x + I$ with $x \in R_1 \setminus Q$. One has $J : (y) = Q : (x)/I$. It follows that $J : (y) \in \mathcal{L}(A)$ since $Q : (x) \in \mathcal{L}(R)$.

As for 2), we may assume $n = 1$. Let $J \in \mathcal{L}(R[x])$. If the generators of J belong to R, then $J = IR[x]$ where $I \in \mathcal{L}(R)$. The ideal I has a linear R-free resolution F. Then $F \otimes_R R[x]$ is a linear $R[x]$-free resolution of J. Now assume that some of the generators of J do not belong to R. We may decompose J as $I + (x + L)$ where $I \in \mathcal{L}(R)$ and L is a linear form in R. We have seen already that I has a linear $R[x]$-free resolution. Note that $x + L$ is a non-zerodivisor on $R[x]/I$. Then it follows from the short exact sequence

$$0 \to R[x]/I[-1] \to R[x]/I \to R[x]/J \to 0$$

that J has a linear $R[x]$-free resolution.

3) Set $A = R \circ S$. Recall that A is by definition the quotient of $R \otimes_K S$ by the ideal generated by the elements of the form $x \otimes y$ with $x \in R_1$ and $y \in S_1$. Let $I \in \mathcal{L}(A)$, say $I = (a_1 + b_1, \dots, a_k + b_k)$ with $a_i \in R_1$ and $b_i \in S_1$, and $x \in A_1 \setminus I$, say $x = y + z$ with $y \in R_1$ and $z \in S_1$. We have to show that $I : (x) \in \mathcal{L}(A)$. Denote by Q and W respectively the projection of I on R and S, i.e. $Q = (a_1, \dots, a_n)$ and $W = (b_1, \dots, b_n)$. Set $J = QA + WA$. We have $I \subseteq J$ and $I_k = J_k$ for every $k \geq 2$. It follows that $I : (x) = (J : (x)) \cap \mathcal{M}_A$, where $\mathcal{M}_A$ denotes the homogeneous maximal ideal of A. If $x \in J$ then $I : (x) = \mathcal{M}_A \in \mathcal{L}(A)$. If $x \notin J$, then $I : (x) = J : (x)$. Now we distinguish three cases:

i) $y \notin Q$ and $z \notin W$. Then $J : (x) = (Q :_R (y))A + (W :_S (z))A$

ii) $y \notin Q$ and $z \in W$. Then $J : (x) = J : (y) = (Q :_R (y))A + (S_1)A$.

iii) $y \in Q$ and $z \notin W$. Then $J : (x) = J : (z) = (R_1)A + (W : (z))A$.

In any case $J : (x)$ is generated by linear forms because $Q :_R (y)$ and $W :_S (z)$ are generated by linear forms. □

As one may expect, u-Koszul algebras are quite rare. For instance one has:

Lemma 1.7 *Let K be an algebraically closed field of characteristic 0 and let R be a homogeneous Cohen-Macaulay domain K-algebra. If R is u-Koszul then R has minimal multiplicity.*

Recall that if R is a Cohen-Macaulay homogeneous K-algebra, then one has $e(R) \geq 1 + \operatorname{codim}(R)$. Here $e(R)$ denotes the multiplicity (or degree) of R and $\operatorname{codim}(R) = \dim_K R_1 - \dim R$. The algebra R is said to have minimal multiplicity if $e(R) = 1 + \operatorname{codim}(R)$, see [EG]. In order to prove 1.7 we need the following:

Lemma 1.8 *Let K be an infinite field and let R be the coordinate ring of a set of points $\{P_1, \dots, P_s\}$ in the projective space $\mathbf{P}^n_K$. Assume that $s > n + 1$ and that $P_1, \dots, P_{s-1}$ are not contained in a hyperplane of $\mathbf{P}^n_K$. Then R is not u-Koszul.*

Proof. Let I be the defining ideal of the given set of points. Then $I = \cap_{i=1}^{s} \mathcal{P}_i$ where $\mathcal{P}_i$ is the defining ideal of P_i. Let x be a linear form in $\mathcal{P}_s \setminus \cup_{i=1}^{s-1} \mathcal{P}_i$ (it exists because K is infinite). Then $0 : (x) = \cap_{i=1}^{s-1} \mathcal{P}_i / I$ is non-zero. By assumption $\cap_{i=1}^{s-1} \mathcal{P}_i$ does not contain linear forms and hence $0 : (x)$ cannot be generated by linear forms. □

Proof of 1.7: Set $d = \dim R$ and $n = \operatorname{codim} R$ and assume by contradiction that $e(R) > 1 + n$. Let $x_1, \dots, x_{d-2}$ be general linear forms. Then $S = R/(x_1, \dots, x_{d-2})$ is a 2-dimensional Cohen-Macaulay K-algebra with $e(S) = e(R)$ and $\operatorname{codim} S = \operatorname{codim} R$. Furthermore, by virtue of Bertini's theorem [Z, p.68] (for an algebraic form of the result see also [S, Prop.3.2]) the ring S is a domain. Now let x be a general linear form on S. By virtue of Harris' "general position theorem" [ACGH, pg.109] $S/(x)$ is the coordinate ring of a set of $e(R)$ points of $\mathbf{P}^n_K$ in general position and therefore the assumption of 1.8 is satisfied by $S/(x)$. Hence $S/(x)$ is not u-Koszul. It follows from 1.6 that R is not u-Koszul. □

Remark 1.9 In general the tensor products, the Segre products and the Veronese subrings of u-Koszul algebras are not u-Koszul. As far as the tensor product is concerned, one can consider for instance the algebra $R = K[x, y]/(x, y)^2$ which is u-Koszul while the algebra

$$S = R \otimes_K R = K[x, y, z, t]/(x, y)^2 + (z, t)^2$$

is not u-Koszul since in S one has $0 : (x + z) = (x - z, yt)$ and $yt \notin (x - z)$.

Let R and S be polynomial rings with $1 < \dim R \leq \dim S$. Among the Segre products and Veronese subrings of polynomial rings those with minimal multiplicity are:

i) $R \circ S$ with $\dim R = 2$,

ii) $R^{(d)}$ with $\dim R = 2$ and $d \geq 1$,

iii) $R^{(2)}$ with $\dim R = 3$.

Here $R \circ S$ denotes the Segre product of R and S and $R^{(d)}$ denotes the d-th Veronese subring of R. Segre products and Veronese subrings of polynomial rings are Cohen-Macaulay domains. Hence it follows from 1.7 that $R \circ S$ is not u-Koszul if $\dim R, \dim S \geq 3$ (at least, say, over an algebraically closed field of characteristic 0). Analogously, the d-Veronese subring $R^{(d)}$ is not u-Koszul if either $\dim R \geq 4$ and $d \geq 2$ or $\dim R = 3$ and $d \geq 3$.

A more explicit argument (not using Bertini's theorem) to show that, for instance, the third Veronese subring of $R = K[x, y, z]$ is not u-Koszul is the following (assume char $K \neq 2, 3$): Let I be the ideal of $R = K[x, y, z]$ generated by $x(x - z)(x - 2z)$ and $y(y - z)(y - 2z)$. The ideal I is a complete intersection and defines the set X of the 9 points in $\mathbf{P}^2$ with coordinates $(a, b, 1)$ where $a = 0, 1, 2$ and $b = 0, 1, 2$. The point $(0, 0, 1)$ is in X, hence the ideal $I : (x, y)$ is the ideal of the points in $X \setminus \{(0, 0, 1)\}$. By [BH, Cor.2.3.10] one knows that $I : (x, y) = I + (h)$ where $h = (x - z)(x - 2z)(y - z)(y - 2z)$. Now if one takes $f \in R$ to be a cubic form such that $\{f = 0\} \cap X = \{(0, 0, 1)\}$ (take for instance $f = x^3 + y^3$) then one has that $I : f = I : (x, y) = I + (h)$. Intersecting with $A = R^{(3)}$ one has $I :_A f = I + (hR_2)$. Since (hR_2) it is not contained in I, we have that $I :_A f$ is not generated by linear forms of A and hence A is not u-Koszul.

The question of whether the Segre and Veronese rings with minimal multiplicity are u-Koszul will be answered in Sect. 3.

It is known that an algebra defined by monomials $m_1, \dots, m_k$ is Koszul if and only if $\deg m_i = 2$ for every i. The following example shows that the u-Koszul property of an algebra defined by monomials depends on the characteristic of the base field.

Example 1.10 Let $R = K[x, y, z]/(x^2, y^2, z^2)$ and char $K \neq 2$. Then the ideal $0 : (x + y + z)$ is generated by quadrics and hence R is not u-Koszul. Note that the algebra $K[x_1, x_2, \dots, x_n]/(x_1^2, x_2^2, \dots, x_n^2)$ is u-Koszul if char $K = 2$.

It would be interesting to characterize the quadratic monomial ideals defining u-Koszul algebras at least, say, over a field of characteristic 0.

2 Artinian algebras

In this section we describe some families of Artinian algebras which are u-Koszul. We mentioned already that algebras with Hilbert series $1 + nz$ are examples of u-Koszul algebras. A less trivial class is given by the following:

Proposition 2.1 *Let n be an integer with $n \geq 2$. Let R be a homogeneous algebra defined by quadrics and with Hilbert series $1 + nz + z^2$. Then R is u-Koszul.*

To prove 2.1 we use the following criterion for an algebra to be u-Koszul which is similar to Fitzgerald's result [Fi, Thm.3.6]:

Proposition 2.2 *Let $R = \oplus_{i \geq 0} R_i$ be a homogeneous algebra. Assume that for every $x \in R_1 \setminus \{0\}$ one has $(0 : (x))R_1 \supseteq R_2$. Then R is u-Koszul.*

Proof. Let $I \in \mathcal{L}(R)$ and $x \in R_1 \setminus I$. By 1.4 it suffices to show $I : (x) \in \mathcal{L}(R)$. Note that $0 : (x) \subseteq I : (x)$. Since the linear forms in $0 : (x)$ generate an ideal containing R_2, it follows that the same is true also for the linear forms in $I : (x)$. Then $I : (x) \in \mathcal{L}(R)$. □

Proof of 2.1: Let $x \in R_1 \setminus \{0\}$. Denote by V the space of linear forms in $0 : (x)$. We have to show that $V R_1 = R_2$. Since R_2 is one-dimensional we have $n - 1 \leq \dim V \leq n$. If $\dim V = n$, that is $V = R_1$, then $V R_1 = R_2$ clearly holds. Now assume that $\dim V = n - 1$. Assume by contradiction that $V R_1 \neq R_2$. Then $V R_1 = 0$. Let $R = S/I$ be a presentation of R where $S = K[x_1, \dots, x_n]$ and identify R_1 with S_1. It follows that $V S_1 \subseteq I_2$ and then they are equal because they have both codimension 1 in S_2. Since I is generated by quadrics we have then $I = (V R_1)$ and this contradicts the fact that $I_3 = S_3$

More generally, we consider now quadratic algebras with Hilbert series $1 + nz + mz^2$, with say m small relatively to n. Not all of them are u-Koszul. For instance

Example 2.3 The algebra $A = K[x_1, x_2, x_3]/(x_1x_2, x_1x_3, x_2x_3, x_1^2 + x_2^2 + x_3^2)$ has Hilbert series $1 + 3z + 2z^2$. Let $y = a_1x_1 + a_2x_2 + a_3x_3$ be a non-zero linear form of A. Set $\rho(y) = |\{i : a_i \neq 0\}|$. It is easy to see that $yA_1 = A_2$ if and only if $\rho(y) > 1$. Using this one easily shows that $0 : (x_1 + x_2) = (x_3) + A_2 = (x_3, x_1^2)$ and that the generator of degree 2 is needed. Hence A is not u-Koszul. The same argument works for any principal ideal (y) generated by a linear form y with $\rho(y) = 2$. On the other hand A is "almost u-Koszul". It is not difficult to prove that the set of the ideals $I \in \mathcal{L}(A)$ which are not of the type (y) with $\rho(y) = 2$ is a Koszul filtration of A.

Now take $B = K[x_4, \dots, x_n]/(x_4, \dots, x_n)^2$ and set $C = A \circ B$. The Hilbert series of C is $1 + nz + 2z^2$. Furthermore, C is not u-Koszul. This follows from 1.6 because the quotient $C/(x_4, \dots, x_n)$ is A which is not u-Koszul.

For generic algebras we have:

Theorem 2.4 *Let m, n be positive integers with $2m \leq n$ and let R be a quadratic algebra with Hilbert series $1 + nz + mz^2$. If R is generic then R is u-Koszul. More precisely, if the variety of the linear forms x such that $xR_1 \neq R_2$ has the expected codimension (which is $n - m + 1$) then R is u-Koszul.*

Let us first explain what is the meaning of "generic" in the statement of the theorem. Let $S = K[x_1, \dots, x_n]$ be polynomial ring and let m be an integer. For every space of quadrics $V \subseteq S_2$ of codimension m we get a quadratic algebra $R = S/(V)$; here (V) denotes the ideal generated by the elements in V. The set of the spaces of quadrics of codimension m is a projective variety, the Grassmannian $\mathrm{Grass}(m, S_2)$, which is embedded in the projective space $\mathbf{P}^N$ via the Plücker map. Here $N = \binom{\dim S_2}{m} - 1$. Therefore the family of the quadratic algebras with $\dim R_1 = n$ and $\dim R_2 = m$ gets identified with the Grassmannian $\mathrm{Grass}(m, S_2)$ via the correspondence $V \to S/(V)$. We then say that a property P holds for a generic algebra if there exists a non-empty Zariski open subset U of $\mathrm{Grass}(m, S_2)$ such that for every $V \in U$ the algebra $R = S/(V)$ has property P.

Proof of 2.4 It is easy to see that for a generic $V \in \mathrm{Grass}(m, S_2)$ one has $VS_1 = S_3$, that is, the Hilbert series of the corresponding algebra is $1+nz+mz^2$. Let $V \in \mathrm{Grass}(m, S_2)$, $R = S/(V)$ and consider the set

$$X_V = \{x \in \mathbf{P}(S_1) : xS_1 + V \neq S_2\}$$

Fix a set $u_1, \dots, u_m$ of monomials of degree 2 in the x_i's and let us restrict our attention to the open affine subset of the Grassmannian of the elements V such that $V + \langle u_1, \dots, u_m \rangle = S_2$. In other words, we are assuming that $u_1, \dots, u_m$ is a basis of R_2. Note that $x = \sum_i a_i x_i$ is in X_V if and only if the multiplication map from $R_1 \xrightarrow{x} R_2$ is not surjective. The matrix $M(x)$ of the map $R_1 \xrightarrow{x} R_2$ has size

$n \times m$ and its entries are linear forms in the a_i whose coefficients are polynomial functions of the Plücker coordinates of V. Hence X_V is the subvariety of $\mathbf{P}(S_1)$ defined by the maximal minors of $M(x)$. Therefore the codimension of X_V is bounded by $n-m+1$. We claim that for a generic V the codimension of X_V is equal to $n-m+1$. By the semicontinuity of the fiber dimension (see for instance [E, Cor.14.19]) it is enough to exhibit a space $V \in \mathrm{Grass}(m, S_2)$ such that the codimension of X_V is $n-m+1$. We will first complete the proof of the theorem and then present the example. By 2.2 it suffices to show that for every $x \in R_1 \setminus \{0\}$ one has $WR_1 = R_2$ where W denotes the space of linear forms in $0:(x)$. The dimension of the vector space W is greater than or equal to $n-m$. Hence W cannot be contained in X_V, otherwise by comparing the dimensions one will have $n \leq 2m-1$, a contradiction with the assumption. Therefore there exists a linear form $y \in W \setminus X_V$ and so $WR_1 = R_2$.

It remains to present the example of a space $V \in \mathrm{Grass}(m, S_2)$ such that X_V has codimension $n-m+1$. Set $s = n-m$ and $B = K[y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_s]$ where $y_1, \ldots, y_m$ and $z_1, z_2, \ldots, z_s$ are indeterminates. Then consider the algebra $R = B/(V)$ where V is the space generated by the following quadrics:

(1)	$z_i^2 - \sum_{j=1}^m c_{ij} y_j y_m$	with $1 \leq i \leq s$
(2)	$z_i z_j$	with $1 \leq i < j \leq s$
(3)	$z_i y_j$	with $1 \leq i \leq s$ and $1 \leq j \leq m$
(4)	$y_i y_j$	with $1 \leq i \leq j \leq m-1$ and $i+j \leq m$
(5)	$y_i y_j - y_{i+j-m} y_m$	with $1 \leq i \leq j \leq m-1$ and $i+j > m$

where $c_{ij} \in K$. It is easy to see that the Hilbert series of R is $1+nz+mz^2$ provided $c_{1m} \neq 0$. By construction the monomials $y_1y_m, y_2y_m, \ldots, y_m^2$ form a basis of R_2. Now let x be a linear form of R, say $x = \sum_{i=1}^s a_i z_i + \sum_{i=1}^m b_i y_i$. In order to describe the matrix $M(x)$ we express the elements $z_1x, z_2x, \ldots, z_sx, y_1x, \ldots, y_mx$ in terms of the basis $y_1y_m, y_2y_m, \ldots, y_m^2$. One has

$$\begin{array}{ll} z_i x = a_i z_i^2 = \sum_{j=1}^m a_i c_{ij} y_j y_m & \text{with } 1 \leq i \leq s \\ y_j x = \sum_{i=1}^m b_i y_i y_j = \sum_{k=1}^j b_{m-j+k} y_k y_m & \text{with } 1 \leq j \leq m \end{array}$$

Hence the matrix $M(x)$ is

$$M(x) = \begin{pmatrix}
a_1c_{11} & a_1c_{12} & \dots & \dots & a_1c_{1j} & \dots & \dots & a_1c_{1m} \\
a_2c_{21} & a_2c_{22} & \dots & \dots & a_2c_{2j} & \dots & \dots & a_2c_{2m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_sc_{s1} & a_sc_{s2} & \dots & \dots & a_sc_{sj} & \dots & \dots & a_sc_{sm} \\
b_m & 0 & \dots & \dots & 0 & \dots & \dots & 0 \\
b_{m-1} & b_m & 0 & \dots & 0 & \dots & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_{m-j} & b_{m-j+1} & \dots & \dots & b_m & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b_2 & b_3 & \dots & \dots & \dots & \dots & b_m & 0 \\
b_1 & b_2 & \dots & \dots & \dots & \dots & \dots & b_m
\end{pmatrix}$$

Let Q be the ideal of $K[a_1, \dots, a_s, b_1, \dots, b_m]$ generated by the minors of size m of $M(x)$. We will show that the codimension of Q is $n-m+1$ provided every minor of the matrix $C = (c_{ij})$ is non-zero. This will complete the proof because a matrix C with that property always exists if K is infinite. Hence assume that every minor of the matrix $C = (c_{ij})$ is non-zero. Let J_i be the ideal generated by the square-free monomials of degree i in the indeterminates $a_1, \dots, a_s$. Set

$$W = (b_m) + (b_{m-1})J_1 + (b_{m-2})J_2 + \cdots + (b_1)J_{m-1} + J_m$$

We claim that W is the radical $\mathrm{Rad}(Q)$ of Q. From this it follows then easily that the codimension of Q is $n-m+1$. The inclusion $Q \subseteq W$ is clear. Hence it is enough to show that $W \subseteq \mathrm{Rad}(Q)$. Since $b_m^m \in Q$ one has $b_m \in \mathrm{Rad}(Q)$. Therefore we may assume by induction that

$$(b_m) + (b_{m-1})J_1 + (b_{m-2})J_2 + \cdots + (b_{j+1})J_{m-j-1} \subseteq \mathrm{Rad}(Q) \qquad (*)$$

and show that $(b_j)J_{m-j} \in \mathrm{Rad}(Q)$. Take a minor of size m of $M(x)$ which involves $m-j$ rows among the first s, say the rows with indices $1 \le i_1 < \cdots < i_{m-j} \le s$ and the last j rows of $M(x)$. Expanding the minor and taking into consideration $(*)$ one gets that

$$\delta b_j^j a_{i_1} \dots a_{i_{m-j}} \in \mathrm{Rad}(Q)$$

where δ is the minor of C which involves the rows $i_1, i_2, \dots, i_{m-j}$ and the last $m-j$ columns. By assumption $\delta \neq 0$ and hence $b_j a_{i_1} \dots a_{i_{m-j}} \in \mathrm{Rad}(Q)$. It follows that $(b_j)J_{m-j} \in \mathrm{Rad}(Q)$. This concludes the proof of Theorem 2.4. □

Remark 2.5 In 2.3 we have seen a quadratic algebra with Hilbert series $1+3z+2z^2$ which is not u-Koszul. One can prove something more: an algebra R with

Hilbert series $1 + 3z + 2z^2$ and with generic quadratic relations is not u-Koszul. With the notation of the proof of 2.4, the set X_V is defined by the of 2-minors of a 2×3 matrix. Therefore, for a generic V, the codimension of X_V in $\mathbf{P}^2 = \mathbf{P}(R_1)$ is 2, that is, X_V is a finite set. Take $x \in X_V$, then $0 :_{R_1} (x)$ is a subspace of R_1 of dimension > 1 and it cannot be contained in X_V. Therefore we may take, $y \in R_1 \setminus X_V$ such that $yx = 0$. By construction $0 : (y) = (x) + (R_2)$ and it is not generated by linear forms.

On the other hand, there are (non-generic) quadratic algebras with Hilbert series $1 + 3z + 2z^2$ which are u-Koszul. For instance:

Example 2.6 The algebra $R = K[x, y, z]/(x^2, xy, y^2, z^2)$ is u-Koszul. It is enough to show that $R/(f)$ is u-Koszul and $0 : (f) \in \mathcal{L}(R)$ for every non-zero linear form f of R. Since no linear form annihilates R_1, we have that $S = R/(f)$ has always $\dim S_2 \leq 1$ and hence is u-Koszul by 2.1. Now write $f = ax + by + cz$. If $c = 0$ then obviously $0 : (f) = (x, y)$ and if $c \neq 0$ then $0 : (f) = (ax + by - cz)$.

Let us also record the following:

Remark 2.7 Let I be an ideal in a polynomial ring S. One knows that the Koszul property behaves well under Gröbner deformations. One may wonder whether the same is true also for the u-Koszul property, that is, whether S/I is u-Koszul provided $S/\operatorname{in}(I)$ is u-Koszul. The answer is no.

To see this, we consider an algebraically closed field K of characteristic $\neq 2$ and a quadratic K-algebra $R = K[x, y, z]/I$ with Hilbert series $1 + 3z + 2z$. It is know that any such an algebra in defined by a Gröbner basis of quadrics (in a suitable system of coordinates), see [C]. It follows that an initial ideal of I is $\operatorname{in}(I) = (x^2, y^2, z^2, xy) \subset K[x, y, z]$. This is because that is the only quadratic monomial ideal with the correct Hilbert function (up to permutations). By virtue of 2.6 the algebra $K[x, y, z]/\operatorname{in}(I)$ is u-Koszul, but by 2.5 we know that if we take R generic then it is not u-Koszul. Explicitly, one can take $R = K[x, y, z]/(x^2 - yz, xy, y^2, z^2)$. The initial ideal of $(x^2 - yz, xy, y^2, z^2)$ is of (x^2, xy, y^2, z^2) and in R one has $0 : x = (y, xz)$.

3 Universally Koszul algebras of minimal multiplicity

Let K be an algebraically closed field of characteristic 0. We have seen in 1.7 that a necessary condition for a Cohen-Macaulay domain R to be u-Koszul is to have minimal multiplicity. The homogeneous domains with minimal multiplicity are classified by a theorem of Del Pezzo and Bertini, see for instance [EH]. There are essentially (i.e. up to polynomial extensions) three classes of such algebras:

1) Quadric hypersurfaces, i.e. rings of the form $K[x_1, \ldots, x_n]/(f)$ with f an irreducible polynomial of degree 2,

2) The coordinate rings of rational normal scrolls, i.e. rings of the form $R_a = K[x^{(1)}, \dots, x^{(k)}]/I_2(X)$ where the $x^{(i)}$ are pairwise disjoint sets of variables, say $x^{(i)} = \{x_0^{(i)}, x_1^{(i)}, \dots, x_{a_i}^{(i)}\}$, with $a_i > 0$ and $I_2(X)$ is the ideal of the 2-minors of the $2 \times (a_1 + \cdots + a_k)$ matrix

$$X = \left(X^{(1)} \mid X^{(2)} \mid \dots \mid X^{(k)} \right)$$

where each $X^{(i)}$ is the $2 \times a_i$ Hankel block:

$$\begin{pmatrix} x_0^{(i)} & x_1^{(i)} & x_2^{(i)} & \dots & x_{a_i-1}^{(i)} \\ x_1^{(i)} & x_2^{(i)} & \dots & \dots & x_{a_i}^{(i)} \end{pmatrix}.$$

The sequence $a = (a_1, a_2, a_3, \dots, a_k)$ is said to be the type of the corresponding scroll. These algebras have also a presentation as semigroup algebras. Let us recall how. Consider the polynomial ring $S = K[x, y][s_1, s_2, \dots, s_k]$ equipped with the bigraded structure given by $\deg(x) = \deg(y) = (1, 0)$ and $\deg(s_i) = (0, a_i)$. Denote by $S_{(i,j)}$ the homogeneous component of bidegree (i, j) of S. The subalgebra $S_a = \oplus_u S_{(u,u)}$ of S is (isomorphic to) the coordinate ring of the rational normal scroll of type a. For instance, if $a = (3, 2)$ then $S_a = K[x^3 s_1, x^2 y s_1, x y^2 s_1, y^3 s_1, x^2 s_2, x y s_2, y^2 s_2]$ and it is isomorphic to

$$R_a = K[x_0^{(1)}, x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_0^{(2)}, x_1^{(2)}, x_2^{(2)}]/I_2(X)$$

where

$$X = \begin{pmatrix} x_0^{(1)} & x_1^{(1)} & x_2^{(1)} & \mid & x_0^{(2)} & x_1^{(2)} \\ x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & \mid & x_1^{(2)} & x_2^{(2)} \end{pmatrix}.$$

3) The coordinate ring of the Veronese embedding of $\mathbf{P}^2 \to \mathbf{P}^5$, that is, the second Veronese subring $K[x, y, z]^{(2)} = K[x^2, xy, xz, y^2, yz, z^2]$ of $K[x, y, z]$.

Unless otherwise specified, from now on K will be any infinite field (i.e. not necessarily algebraically closed of characteristic 0). Let us start with the algebras of the class 1).

3.1 Quadric hypersurfaces

We have:

Proposition 3.1 *Any quadric hypersurface is u-Koszul.*

Proof. Let $R = K[x_1, \dots, x_n]/(f)$ where $\deg(f) = 2$. We may argue by induction on n. The case $n = 1$ is trivial. Let $z \neq 0$ be a linear form in R. The ring $R/(z)$ is either a quadric hypersurface with embedding dimension $n - 1$ or a polynomial ring and hence it is u-Koszul by induction. By virtue of 1.5 it is now enough to show that $0 : (z)$ is generated by linear forms. Let $z = y + (f)$ with y a linear form in $K[x_1, \ldots, x_n]$. If $y \nmid f$ then $0 : (z) = 0$ while if $y|f$, say $f = yy_1$, then $0 : (z)$ is generated by the class of y_1. □

Remark 3.2 A complete intersection of quadrics is not u-Koszul in general. For instance, the coordinate ring of four general points of $\mathbf{P}^2$ is a complete intersection of 2 quadrics and by 1.8 is not u-Koszul. Explicitly, the ring $R = K[x, y, z]/I$ where

$$I = (x, y) \cap (x, z) \cap (y, z) \cap (x - z, y - z) = (xz - yz, xy - yz)$$

is a c.i. of two quadrics and in R the ideal $0 : (x + y)$ is generated by one quadric. Another example is $K[x, y, z]/(x^2, yz)$ where one has $0 : (x + y) = (xz)$.

As far as Artinian c.i. of quadrics are concerned, if the codimension is 2 then they are u-Koszul by 2.1, while in codimension > 2 they are not u-Koszul in general, see 1.10.

3.2 Rational normal scrolls

We have:

Theorem 3.3 *Let R_a be the coordinate ring of the rational normal scroll of type $a = (a_1, a_2, \ldots, a_k)$. Then R_a is u-Koszul if and only if either $k = 1$ (a rational normal curve) or $k = 2$ and $a_1 = a_2$.*

Let us split the proof of Theorem 3.3 in three parts and start by showing:

Lemma 3.4 *Let R_a be the coordinate ring of the rational normal scroll of type $a = (a_1, a_2, \ldots, a_k)$ with either $k > 2$ or $k = 2$ and $a_1 \neq a_2$. Then R_a is not u-Koszul.*

Proof. Assume first that $k = 2$ and $a_1 \neq a_2$. Say $a_1 > a_2$. Consider the semigroup presentation of R_a, i.e. $R_a = K[x^{a_1}s_1, x^{a_1-1}ys_1, \ldots, y^{a_1}s_1, x^{a_2}s_2, x^{a_2-1}ys_2, \ldots, y^{a_2}s_2]$. We claim that the ideal $(x^{a_1}s_1) : (y^{a_1}s_1)$ of R_a is not generated by linear forms in the ring R_a. This shows that R_a is not u-Koszul. To prove the claim we note first that the ideal $(x^{a_1}s_1) : (y^{a_1}s_1)$ contains exactly one element of degree 1, namely $x^{a_1}s_1$. Set $t = \min\{i \in \mathbf{N} : i \geq a_1/a_2\}$. Then by construction the element $x^{a_1}y^{ta_2-a_1}s_2^t$ is in $(x^{a_1}s_1) : (y^{a_1}s_1)$, it is not a multiple of $x^{a_1}s_1$ and its degree in R_a is $t > 1$. This concludes the proof of the claim and hence of the case $k = 2$ and $a_1 \neq a_2$.

Now assume that $k > 2$, say $a = (a_1, a_2, a_3, \ldots, a_k)$ with $a_1 \geq a_2 \geq \cdots \geq a_k$. If R_a would be u-Koszul then by 1.6 any ring of the form R_a/I, where I is an ideal generated by elements of degree 1, would be u-Koszul. But among these rings there are also rings of type R_b with $b = \{b_1, b_2\}$ with $b_1 \neq b_2$: just kill all the variables of the blocks with index > 3 (if any) and identify the last variable of the first block with the first variable of the second one. Since we know already that these rings are not u-Koszul, we may conclude that R_a itself is not u-Koszul. $\square$

Next we show:

Theorem 3.5 *The coordinate ring of the rational normal curve of* $\mathbf{P}^n$ *(i.e. the* n*-th Veronese subring of* $K[x, y]$*) is u-Koszul for every* n*.*

We will present two proofs of this theorem. The first is based on the semigroup presentation of the ring under consideration and on the Hilbert-Burch theorem. The second does not use the semigroup presentation and it is valid also for a 2-dimensional homogeneous Cohen-Macaulay domain of minimal multiplicity which is not a Veronese subring of $K[x, y]$ (they can exist if K is not algebraically closed).

Proof of 3.5–(1) Let R be the coordinate ring of the rational normal curve of $\mathbf{P}^n$, i.e. R is the n-th Veronese subring $K[x, y]^{(n)}$ of $S = K[x, y]$. By virtue of 1.4 it suffices to show that if I is an ideal of R generated by linear forms and z is a linear form not in I then the ideal $I :_R (x)$ is generated by linear forms. Note that R is a direct summand of the polynomial ring S and hence $I :_R (z) = (IS :_S (z)) \cap R$. Since linear forms in R are forms of degree n in S, it turns out that it is enough to show that if I is an ideal generated by forms of degree n in S and z is a form of degree n then the ideal $I :_S (z)$ is generated by forms of degree $\leq n$. If $I = 0$ then the assertion is trivial. If the codimension of the ideal $I + (z)$ is 1, then there exists a homogeneous polynomial, say g, which is a common factor of I and z, say $I = gI'$ and $z = gz'$. Then $I :_S (z) = I' :_S (z')$ and hence we may assume without loss of generality that the codimension of ideal $I + (z)$ is 2. Let m be the minimal number of generators of I. By the Hilbert-Burch theorem $I + (z)$ has a presentation $S^m \xrightarrow{A} S^m \oplus S \to I + (z)$ and the maximal minors of the matrix A are a system of minimal generators of $I + (z)$. The matrix A has size $m \times (m+1)$ and each of its rows contains elements of a given degree, say the elements of the i-th row have degree d_i. Since $\sum_i d_i = n$ it follows that $d_i \leq n$. By construction the elements of the last column of the matrix A generates $I : (z)$. It follows that $I :_S (z)$ is generated in degree $\leq n$.

Proof of 3.5–(2) Let R be a 2-dimensional homogeneous Cohen-Macaulay domain of minimal multiplicity. Since R is a domain, according to 1.5 it suffices to show that R/z is u-Koszul for every non-zero linear form in R. Hence the assertion is a consequence of the following lemma.

Lemma 3.6 *A one-dimensional Cohen-Macaulay homogeneous* K*-algebra of minimal multiplicity is u-Koszul.*

Proof. By assumption the Hilbert function of R is $H(R,0)=1$ and $H(R,i)=n$ for $i>0$. Let y be a linear form which is a non-zero-divisor on R. One has $yR_k=R_{k+1}$ for all $k\geq 1$ and hence the multiplication by y from R_k to R_{k+1} is bijective for all $k\geq 1$. Let us denote by J_i the degree i component of a homogeneous ideal J. Let $I\in\mathcal{L}(R)$ and $x\in R_1\setminus I$. Since I is generated by linear forms, one has $I_{k+1}=I_1R_k=I_1R_{k-1}y=I_ky$ for all $k\geq 2$.

Set $J=I:(x)$. Let $f\in J_k$ with $k\geq 2$. Then $f=yg$ with $g\in R_{k-1}$. Since $fx\in I_{k+1}=yI_k$ we have $ygx=fx=yh$ with $h\in I_k$ and therefore $y(gx-h)=0$. It follows that $gx=h$ and then $g\in J$. We have shown that $J_k=yJ_{k-1}$ for all $k\geq 2$ and hence $J\in\mathcal{L}(R)$. □

As a corollary of 3.6 we have:

Proposition 3.7 *The algebra $R=K[x_1,\dots,x_n]/(x_ix_j : 1\leq i<j\leq n)$ is u-Koszul. In particular the coordinate ring of $n+1$ points of $\mathbf{P}^n$ in general position is u-Koszul.*

Proof. It follows from 3.6 since R is a one-dimensional Cohen-Macaulay ring with minimal multiplicity. Alternatively one may note that R is the fiber product of $K[x_1],\dots,K[x_n]$, and hence the statement follows from 1.6. □

To conclude the proof of 3.3 it remains to treat the case of the scrolls of type (a,a). To this end we need some preliminary observations and results.

The ring $R_{(a,a)}$ is the Segre product of two copies of the Veronese subring $K[x,y]^{(a)}$ of $K[x,y]$. Hence its semigroup presentation is given by the algebra $K[x^as, x^{a-1}ys,\dots,xy^{a-1}s, y^as, x^au, x^{a-1}yu,\dots,xy^{a-1}u, y^au]$. It is a subalgebra of the polynomial ring $S=K[x,y][s,u]$ equipped with the normalized bigraded structure given by $\deg(x)=\deg(y)=(1,0)$ and $\deg(s)=\deg(u)=(0,1)$. The algebra $R_{(a,a)}$ is a direct summand of S and the linear forms of $R_{(a,a)}$ are bihomogeneous forms of bidegree $(a,1)$ in S. We will need the following technical lemma:

Lemma 3.8 *Let $S=K[x,y][s,u]$ be the polynomial ring equipped with the bigraded structure given by* $\deg(x)=\deg(y)=(1,0)$ *and* $\deg(s)=\deg(u)=(0,1)$. *Let f_1,f_2 be a regular sequence of forms of bidegree $(b,1)$ and let $A=S/(f_1,f_2)$. Let $a\geq b$ and $S_\Delta=\oplus_{i\in\mathbf{Z}}S_{(ia,i)}$ and $M=\oplus_{i\in\mathbf{Z}}A_{(ia+b,i+1)}$. Then:*
1) M is an S_Δ-module of positive depth (indeed, it is a Cohen-Macaulay module of dimension 1),
2) Let D be an ideal of A generated by elements of bidegree $(b,1)$ and let z be an element of bidegree $(b,1)$. Set $J=D:(z)$. Then for every $a\geq b$ and for every $k>1$ one has $J_{(ka,k)}=J_{(a,1)}A_{(ka-a,k-1)}$.

Proof. 1) For every bigraded S-module $T=\oplus_{(i,j)\in\mathbf{Z}^2}T_{(i,j)}$ we will denote by T_Δ the S_Δ-graded module $\oplus_{i\in\mathbf{Z}}T_{(ia,i)}$. Furthermore $S(h,k)$ denotes the ring S shifted by (h,k), i.e. the component of bidegree (i,j) of $S(h,k)$ is $S_{(i+h,j+k)}$.

The bigraded minimal free resolution of A as an S-module is given by

$$0 \to S(-2b, -2) \to S(-b, -1)^2 \to S \to A \to 0$$

Taking into consideration only those homogeneous components of bidegree $(ia + b, i + 1)$ we get an exact sequence of S_Δ-modules:

$$0 \to S(-b, -1)_\Delta \to S_\Delta^2 \to S(b, 1)_\Delta \to M \to 0$$

By virtue of [CHTV, Prop.3.4] S_Δ is a Cohen-Macaulay ring of dimension 3 and $S(-b, -1)_\Delta$ is Cohen-Macaulay S_Δ-module of dimension 3. On the other hand $S(b, 1)_\Delta$ has positive depth (it is not CM in general). It follows that

$$\operatorname{depth} M \geq \min\{\operatorname{depth} S(b, 1)_\Delta, \operatorname{depth} S_\Delta^2 - 1, \operatorname{depth} S(-b, -1)_\Delta - 2\} \geq 1.$$

2) One easily shows that $\dim A_{(i,j)} = 2b$ if $i \geq 2b$ and $j \geq 2$ and also that $\dim A_{(b,1)} = 2b$. Since by 1) we know that M has positive depth, there exists an element $g \in A_{(a,1)}$ such that the multiplication by g from $A_{(b,1)}$ to $A_{(b+a,2)}$ is injective. But then it is also surjective. It follows that for all $(h, k) \geq (b + a, 2)$ one has

$$A_{(h,k)} = A_{(h-b-a,k-2)} A_{(b+a,2)} = A_{(h-b-a,k-2)} A_{(b,1)} g = A_{(h-a,k-1)} g.$$

From now one the argument is similar to that of 3.6. Let $F \in J_{(ka,k)}$ with $k > 1$. Since $a \geq b$, we have $(ka, k) \geq (b + a, 2)$ and hence $F = gF_1$ with $F_1 \in A_{(ka-a,k-1)}$. By assumption

$$Fz \in D_{(ka+b,k+1)} = D_{(b,1)} A_{(ka,k)} = D_{(b,1)} A_{(ka-a,k-1)} g = D_{(b+ka-a,k)} g$$

and hence $Fz = Bg$ with $B \in D_{(b+ka-a,k)}$. Then $(F_1 z - B)g = 0$ and $\deg(F_1 z - B) = (b + ka - a, k)$. But g is a non-zero-divisor on M, and hence $F_1 z - B = 0$ which in turn implies $F_1 \in J_{(ka-a,k-1)}$ and we are done by induction on k. □

Lemma 3.9 *Let* $S = K[x, y][s, u]$ *be the polynomial ring equipped with the bigraded structure given by* $\deg(x) = \deg(y) = (1, 0)$ *and* $\deg(s) = \deg(u) = (0, 1)$. *Let* V *be a non-zero space of forms of bidegree* $(a, 1)$. *Then there exists a decomposition of* V *as* $V = W + \langle z \rangle$ *such that if we set* $J = (W) : (z)$, *then* $J_{(ka,k)} = J_{(a,1)} S_{(ka-a,k-1)}$ *for all* $k > 1$.

Proof. Let d be the dimension of V. If $d = 1$ then the assertion is trivial. So assume $d > 1$. Let g be the greatest common divisor of the elements of V (g is possibly 1) and let $(\alpha, \beta) \leq (a, 1)$ its bidegree. Then we may write $V = gV'$ where V' is a space of forms of bidegree $(a - \alpha, 1 - \beta)$ and the codimension of the ideal (V') is 2. If $\beta = 1$ then we can argue has in the (first) proof of 3.5 and the conclusion follows from the Hilbert-Burch theorem. Hence we may assume that $\beta = 0$. We may then find $f_1, f_2 \in V'$ of bidegree $(a - \alpha, 1)$ such that they form a regular sequence in S.

If $d = 2$ then $V = g\langle f_1, f_2\rangle$ and we may clearly take $W = \langle gf_1\rangle$ and $z = gf_2$ so that $J = (f_1)$ and the assertion follows.

If $d > 2$ then let W be any space of forms of bidegree bidegree $(a, 1)$, such that $\dim W = d - 1$ and W contains gf_1, gf_2. Let z be an element of V such that $V = W + \langle z\rangle$. We claim that this decomposition has the desired property. To this end note that by construction $W = gW'$ and $z = gz'$ so that $J = (W) : (z) = (W') : (z')$. Set $A = S/(f_1, f_2)$. Note that by by construction $(f_1, f_2) \subseteq J$ and $J/(f_1, f_2) = (W')/(f_1, f_2) : (\overline{z'})$ where $\overline{z'}$ is the residue class of z' in A. Then the claim follows directly from 3.8. □

Now we are ready to prove:

Theorem 3.10 *The coordinate ring $R_{(a,a)}$ of the rational normal scroll of type (a, a) is u-Koszul.*

Proof. By 1.4 it suffices to show that $\mathcal{L}(R_{(a,a)})$ is a Koszul filtration of $R_{(a,a)}$. Since $R_{(a,a)}$ is a direct summand in $S = K[x, y][s, u]$ this follows immediately from 3.9. □

Remark 3.11 We want to point out that there is an important difference between the proof of the u-Koszul property for the rational normal curves and that for the scrolls of type (a, a). In the case of the rational normal curves the result is a consequence of the fact that if I is an ideal of $K[x, y]$ generated by elements of degree n and z is an element of degree n then the ideal $I : (z)$ is generated in degree $\leq n$. When dealing with the case (a, a) one has the same sort of situation. In order to show that the scroll of type (a, a) is u-Koszul it would be enough to prove that given an ideal $I \subset K[x, y][s, u]$ generated by forms of bidegree $(a, 1)$ and an element z of bidegree $(a, 1)$, then the ideal $I : (z)$ is generated by forms of bidegree $\leq (a, 1)$. Unfortunately this is not the case. For instance if $a = 2$, $I = (x^2 s, xyt) \subset S = K[x, y][s, u]$, and $z = xys - x^2 t$ then the ideal $I : (z)$ is minimally generated by (xy, xs, yt, st). The bidegree st is $(0, 2) \not\leq (2, 1)$. But the important fact is that these “bad" generators do not give contributions in the relevant bidegrees. This is what we have proved in 3.9. In the example, if we multiply st with $S_{(4,0)}$ to get to bidegree $(4, 2)$ we obtain something which is in the span of the other generators. Already $S_{(1,0)}st$ is in (xy, xs, yt).

3.3 The Veronese surface

For the algebra of the class 3) in Bertini-Del Pezzo’s classification we have:

Theorem 3.12 *The coordinate ring of the Veronese embedding of $\mathbf{P}^2 \to \mathbf{P}^5$ (i.e. $K[x, y, z]^{(2)}$) is u-Koszul.*

Proof. Set $S = K[x, y, z]$ and denote by R the second Veronese subring of S. We prove that $\mathcal{L}(R)$ is a Koszul filtration. Let V be a non-zero space of linear forms in R (i.e. a space of quadrics in S). Set $d = \dim V$. We have to show that there exist a subspace W of V of dimension $d-1$ such that $(W) :_R (V)$ is generated by elements of R of degree 1. If $d = 1$ then the assertion is trivial. If $d = 2$ then $V = \langle f, g\rangle$ and $(f) :_R (g) = ((f) :_S (g)) \cap R$ and the assertion is clearly true. So assume $d > 2$. If the ideal (V) has codimension ≥ 2 in S then we may find a regular sequence f, g of elements of V. Since R is a direct summand in S, the elements f, g form a regular sequence in R also. The $R/(f, g)$ is a 1-dimensional Cohen-Macaulay homogeneous K-algebra of minimal multiplicity. Hence $R/(f, g)$ is u-Koszul by 3.6. It follows that any subspace W of V containing f, g has the desired property. If instead the ideal (V) has codimension 1 in S then V is necessarily of the form lS_1 where l is a non-zero linear form in S. We may assume $l = x$. Then take $W = \langle x^2, xy\rangle$ and note that $(W) :_R (V) = (x^2, xy) :_R (xz) = (x^2, xy, y^2, xz, yz)$. □

We have seen that a 2-dimensional Cohen-Macaulay homogeneous algebra R which is a domain of minimal multiplicity is u-Koszul. One may wonder whether the domain assumption can be replaced by, say, the assumption that R is reduced. This is not possible. The ring $R = K[x, y, z, t]/(xy, yz, zt)$ is a 2-dimensional Cohen-Macaulay homogeneous reduced algebra with minimal multiplicity which is not u-Koszul.

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