COMMUNICATIONS IN ALGEBRA, 28(3), 1371–1386 (2000)

Betti numbers and lifting of Gorenstein codimension three ideals

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Abstract

In this paper we consider homogeneous Gorenstein ideals of codimension three in a polynomial ring and determine their graded Betti numbers in terms of their Hilbert function. For such ideals we prove also a lifting theorem in the vein of a classical result of Hartshorne concerning monomial ideals.

Introduction

Let I be a homogeneous ideal in a polynomial ring $R = K[X_1, \dots, X_n]$. The graded Betti numbers of I determine its Hilbert function but not the other way round. For special classes of ideals one would like to describe all the possible sets of Betti numbers in terms of the Hilbert function. This has been done by Campanella [2] for the perfect ideals of codimension 2. In the first part of the paper we present such a description for the ideals which are Gorenstein of codimension 3. The cornerstone of our approach is the Buchsbaum-Eisenbud structure theorem. For the class of ideals under investigation, the graded Betti numbers are determined by the degrees of the minimal generators. Hence it suffices to describe the sets of the degrees of the minimal generators in terms of the Hilbert function. In [3] Diesel presented an algorithmic description of these degrees. In Theorem 1.7 we present an explicit description of them in terms of the h-polynomial and of its differences.

A sequence of integers $\underline{a} = a_1, \dots, a_{2g+1}$ is said to be *admissible* if there exists a homogeneous ideal I which is Gorenstein of codimension 3 and such that the degrees of its minimal generators are exactly $a_1, \dots, a_{2g+1}$. The ideal I is called a *model* for the sequence $\underline{a}$, and one says that I realizes $\underline{a}$. For every admissible sequence $\underline{a}$ there exists a "canonical" zero-dimensional model $I \subset K[X, Y, Z]$, which is given by the ideal of Pfaffians of a skew-symmetric matrix M whose entries are either zero's or powers of the indeterminates, see Section 1. In Section 2 we show that by deforming the entries of M and by taking the ideal of Pfaffians of the resulting matrix one obtains a 1-dimensional reduced model of $\underline{a}$ in $K[X,Y,Z,T]$. The deformation of M consists in replacing every power of an indeterminate, say X^h, with the product of h linear factors $X-\mu_i T$ where the μ_i are "generic" field coefficients. Here the field K is assumed to be perfect and infinite and the genericity of the coefficients means that they belong to a non-empty Zariski open subset of an affine space. This kind of deformation was used by Hartshorne [6] to lift any monomial ideal of $K[X_1,\dots,X_n]$ to a radical ideal of $K[X_1,\dots,X_{n+1}]$. The difference between the two approches is that in the monomial case one deforms directly the generators of the ideal while we deform the entries of the matrix whose Pfaffians generate the ideal. It follows that every admissible sequence has a reduced model, see Theorem 2.2. If the base field is algebraically closed then the reduced model is indeed an ideal of points of $\mathbf{P}^3$, a result proved by Geramita and Migliore in [4]. The Zariski open subset defined in the deformation process is described explicitly, hence one can construct, in concrete examples, Gorenstein sets of points in $\mathbf{P}^3$ with prescribed resolution, see 2.7.

1 Graded Betti numbers

In the following I will always denote an homogeneous ideal of R which is Gorenstein of codimension three and which is contained in $(X_1,\dots,X_n)^2$. In this section we describe the possible degrees of the generators and the others graded Betti numbers of I in terms of the h-polynomial of R/I.

A non-decreasing sequence of positive integers $(a_1,\dots,a_{2g+1})$ is called *admissible* if there exist a Gorenstein codimension three ideal I in R such that the degrees of a minimal set of generators of I are exactly $(a_1,\dots,a_{2g+1})$. Let us first recall the characterization of admissible sequences.

By virtue of the structure theorem of Buchsbaum and Eisenbud [1], there exists a skew-symmetric matrix $A=(F_{ij})$ of size $(2g+1)\times(2g+1)$ with homogeneous entries such that I is minimally generated by the $2g$-Pfaffians of M and the degrees of these Pfaffians are $a_1,a_2,\dots,a_{2g+1}$. Further then $g\le a_1$ and R/I has a minimal free resolution

$$0\to R(-c)\to\oplus_{i=1}^{2g+1}R(-b_i)\to\oplus_{i=1}^{2g+1}R(-a_i)\to R\to R/I\to 0 \qquad (1)$$

where the matrix of the map $\oplus_{i=1}^{2g+1}R(-b_i)\to\oplus_{i=1}^{2g+1}R(-a_i)$ is exactly A. Since the resolution is self-dual, for every $i=1,\dots,2g+1$ we have $b_i=c-a_i$ so that $b_1\ge b_2\ge\cdots\ge b_{2g+1}$.

Denote by $H_{R/I}(t)$ the Hilbert function of R/I and by $P_{R/I}(z)$ the Hilbert series. The Hilbert series can be expressed as a rational function $P_{R/I}(z)=h(z)/(1-z)^{n-3}$ where $h(z)\in\mathbf{Z}[z]$ and $h(1)\neq 0$. The polynomial $h(z)$ is called the *h-polynomial* of R/I (sometime improperly the h-polynomial of I). We clearly have

$$h(z)=\frac{1-\sum_{i=1}^{2g+1}z^{a_i}+\sum_{i=1}^{2g+1}z^{b_i}-z^c}{(1-z)^3}.$$

It follows that the first derivative of the numerator of this last fraction vanishes at 1, and this gives the equality $gc=\sum_{i=1}^{2g+1}a_i$. This implies that:

Remark 1.1 *The degrees of a minimal set of generators of a Gorenstein codimension three ideal I completely determine the graded Betti numbers of R/I.*
Hence to describe the family of the possible Betti numbers in terms of the h-polynomial it suffices to describe the family of the possible degrees of a system of generators. For $i, j = 1, \dots, 2g+1$ set $u_{ij} = b_i - a_j$ and $D = (u_{ij})$. It is clear that D is a symmetric matrix whose entries are the degrees of the entries of the matrix A (by convention the polynomial 0 has arbitrary degree). Hence D is called the degree matrix. Since the resolution is minimal one has $F_{ij} = 0$ if $u_{ij} \leq 0$. We remark that the entries of the matrix D do not decrease if we move up or left inside the matrix. From this and from the minimality of the resolution, one has that $u_{ij} > 0$ if $i + j = 2g + 3$.

We remark that we also have $c > b_1$ since, otherwise, $c \leq b_1 = c - a_1$ would imply $a_1 \leq 0$.

Now suppose we are given a sequence of integers $(2 \leq a_1 \leq a_2 \leq \cdots \leq a_{2g+1})$ with the following two properties:

(a) There exists an integer c such that $\sum_{i=1}^{2g+1} a_i = cg$.

(b) Set $b_i = c - a_i$ and $u_{ij} = b_i - a_j$. Then $u_{ij} > 0$ whenever $i + j = 2g + 3$. This condition can be visualized by the following "shifts table":

$$\begin{array}{ccccccccccc} & & & & b_{2g+1} & \leq & b_{2g} & \leq & \cdots & \leq & b_2 & \leq & b_1 \\ & & & & \vee & & \vee & & & & \vee & & \\ 2 & \leq & a_1 & \leq & a_2 & \leq & a_3 & \leq & \cdots & \leq & a_{2g+1} & & \end{array} \tag{2}$$

For simplicity of notation let us set

$$\begin{array}{ll} u_i = b_i - a_j & \text{for } i = 1, \dots g, j = 2g + 1 - i, \\ v_i = b_i - a_j & \text{for } i = 1, \dots g, j = 2g + 2 - i, \\ w_i = b_{i+1} - a_j & \text{for } i = 1, \dots g, j = 2g + 2 - i. \end{array}$$

Now consider the skew-symmetric matrix $M = (F_{ij})$ where

$$F_{ij} = \begin{cases} X^{u_i} & \text{if } i + j = 2g + 1, j > i \\ Y^{v_i} & \text{if } i + j = 2g + 2, j > i \\ Z^{w_{i-1}} & \text{if } i + j = 2g + 3, j > i \\ 0 & \text{if } i + j \neq 2g + 1, 2g + 2, 2g + 3 \text{ and } j > i \end{cases}$$

For instance, if $g = 2$ the matrix is:

$$M = \begin{pmatrix} 0 & 0 & 0 & X^{u_1} & Y^{v_1} \\ 0 & 0 & X^{u_2} & Y^{v_2} & Z^{w_1} \\ 0 & -X^{u_2} & 0 & Z^{w_2} & 0 \\ -X^{u_1} & -Y^{v_2} & -Z^{w_2} & 0 & 0 \\ -Y^{v_1} & -Z^{w_1} & 0 & 0 & 0 \end{pmatrix}$$

It is well known that the ideal $I \subset R = K[X, Y, Z]$ of the Pfaffians of order $2g$ of M is Gorenstein of codimension three and its minimal free resolution is (1). The ideal I is the 0-dimensional canonical model of $a_1, \dots, a_{2g+1}$.

The above remarks show that a sequence is admissible if and only if it verifies conditions (a) and (b).

We remark that every sequence $(2 \leq a_1 \leq a_2 \leq a_3)$ of three elements is admissible, while for example the sequence $(3,3,3,3,3)$ is not admissible.

We are now looking for some other numerical properties of admissible sequences. Let $\underline{a} = (a_1, \ldots, a_{2g+1})$ be an admissible sequence and let c be the integer such that $\sum a_i = cg$. As before, for every $i = 1, \ldots, 2g+1$ we set $b_i = c - a_i$. Then we define

$$p_{\underline{a}}(z) = 1 - \sum_{i=1}^{2g+1} z^{a_i} + \sum_{i=1}^{2g+1} z^{b_i} - z^c \tag{3}$$

Since $c = a_1 + b_1$ and $a_1 \geq 2$, we have $c > b_1$ so that the degree of $p_{\underline{a}}(z)$ is c. Further $c - a_1 = b_1 > a_1$ so that

$$c \geq 2a_1 + 1. \tag{4}$$

We have already remarked that the polynomial $p_{\underline{a}}(z)$ is a multiple of $(1-z)^3$, so that the polynomial

$$q_{\underline{a}}(z) = p_{\underline{a}}(z)/(1-z) \tag{5}$$

has integer coefficients and verifies

$$q_{\underline{a}}(1) = q'_{\underline{a}}(1) = 0. \tag{6}$$

Whenever there is no danger of confusion we write $p(z)$ and $q(z)$ instead of $p_{\underline{a}}(z)$ and $q_{\underline{a}}(z)$. We let $p(z) = \sum_{i=0}^{c} p_i z^i$ and $q(z) = \sum_{i=0}^{c-1} q_i z^i$. We clearly have for every $1 \leq i \leq c-1$,

$$p_i = q_i - q_{i-1} \tag{7}$$

and

$$q_i = 1 + \aleph\{m \mid b_m \leq i\} - \aleph\{m \mid a_m \leq i\} \tag{8}$$

where $\aleph A$ denotes the cardinality of the set A. The admissible sequence $\underline{a} = (a_1, \ldots, a_{2g+1})$ can be codified in a different way. For every integer j let

$$\alpha_j = \aleph\{m \mid a_m = j\}.$$

Then it is clear that the sequence $(\alpha_j)_{j \geq 0}$ describes $\underline{a}$ in a complete way.

For example the sequence $\underline{a} = (5,5,7,7,8)$ is admissible with $g = 2$ and $c = 16$. It can be codified with the sequence $(\alpha_5 = 2, \alpha_7 = 2, \alpha_8 = 1, \alpha_j = 0)$ for every $j \neq 5,7,8$.

In the same manner we can define for every integer $j \geq 0$ the integers $\beta_j = \aleph\{m \mid b_m = j\}$. It is clear that for every $j \geq 0$

$$\beta_j = \alpha_{c-j} \tag{9}$$

while, by (3), for every $0 < i < c$ we have

$$p_i = \beta_i - \alpha_i = \alpha_{c-i} - \alpha_i. \tag{10}$$

This shows immediately that

Remark 1.2 *The polynomial $p_{\underline{a}}(z)$ is c-antisymmetric, i.e. $p_i + p_{c-i} = 0$ for every $i = 0, \ldots, c$. Furthermore $q_{\underline{a}}(z)$ is $(c-1)$-symmetric, i.e. $q_i = q_{c-1-i}$ for every $i = 0, \ldots, c-1$.*

We have also

Lemma 1.3 *For every r in the range $a_1 + 1 \le r \le c - a_1 - 1 = b_1 - 1$ we have*

$$\aleph\{m \mid b_m \le r\} < \aleph\{m \mid a_m \le r - 1\}.$$

Proof. Let $v = \aleph\{m \mid a_m \le r-1\}$. Since $a_1 \le r - 1$, one has $v \ge 1$. On the other hand from the assumption $r \le c - a_1 - 1$, we have $b_1 \ge r + 1$, hence the conclusion follows if $v = 2g + 1$. So let $1 \le v \le 2g$ and $a_{v+1} \ge r$. Then $r \le a_{v+1} < b_{2g+3-v-1} = b_{2g+2-v}$ and the desider conclusion follows. ⊙

In the next proposition we collect some relevant numerical properties of an admissible sequence $\underline{a}$ and of the associated polynomials $p_{\underline{a}}(z)$ and $q_{\underline{a}}(z)$.

Proposition 1.4 *Let $\underline{a} = (a_1, \ldots, a_{2g+1})$ be an admissible sequence. Then:*
i)

$$\begin{cases} q_j = 1, & \text{if } 0 \le j \le a_1 - 1 \text{ or } b_1 \le j \le c-1 \\ q_{a_1} = 1 - \alpha_{a_1} & \\ q_j \le -\alpha_j & \text{if } a_1 + 1 \le j \le b_1 - 1 \end{cases}$$

ii)

$$\begin{cases} \alpha_j = 0, & \text{if } 0 \le j \le a_1 - 1 \text{ or } b_1 \le j \\ \alpha_{a_1} = 1 - q_{a_1} & \\ \max(0, -p_j) \le \alpha_j \le -q_j & \text{if } a_1 + 1 \le j \le b_1 - 1. \end{cases}$$

Moreover if c is even, say $c = 2l$, then $q_l + \alpha_l$ is even.

iii) Let us assume that $c = a_m + a_n$ for some integers m and n. Then the sequence $\underline{a}'$ obtained from $\underline{a}$ by deleting a_m and a_n is admissible. Moreover $p_{\underline{a}}(z) = p_{\underline{a}'}(z)$ and in particular the degree of $p_{\underline{a}'}(z)$ is c.

Proof. i) Note that for $0 \le j \le a_1 - 1$, we have $0 = \aleph\{m \mid a_m \le j\} = \aleph\{m \mid b_m \le j\}$ so that by Eq.(8) we have $q_j = 1$.

Since $q_{\underline{a}}(z)$ is $(c-1)$-symmetric we have also $q_j = 1$ for $b_1 \le j \le c - 1$. Further, since

$$\aleph\{m \mid b_m \le a_1\} = 0 \quad \text{and} \quad \aleph\{m \mid a_m \le a_1\} = \alpha_{a_1},$$

we get $q_{a_1} = 1 - \alpha_{a_1}$. Finally, if $a_1 + 1 \leq j \leq b_1 - 1$, then we can use ii) and Eq.(8) to get $q_j < 1 + \aleph\{m \mid a_m \leq j-1\} - \aleph\{m \mid a_m \leq j\} = 1 - \alpha_j$.

ii) It is clear by the definition that $\alpha_j = 0$ if $j \leq a_1 - 1$ or $j \geq b_1$. Further by i) $\alpha_{a_1} = 1 - q_{a_1}$ and $\alpha_j \leq -q_j$ if $a_1 + 1 \leq j \leq b_1 - 1$.

But by (10) we have also $\alpha_j = -p_j + \beta_j$ for every j, which implies $\alpha_j \geq -p_j$ and $\alpha_j \geq \max(0, -p_j)$. At this point we need only to prove that $q_l + \alpha_l$ is even if $c = 2l$ is even. But we have

$$q_l + \alpha_l = 1 + \aleph\{m \mid a_m \geq l\} - \aleph\{m \mid a_m \leq l-1\} =$$

$$1 + 2g + 1 - 2\aleph\{m \mid a_m \leq l-1\} = 2g + 2 - 2\aleph\{m \mid a_m \leq l-1\}.$$

iii) Since $a_2 + a_m \leq a_2 + a_{2g+1} < c$, we have $n \geq 3$. Hence $a_n + a_m = c > a_n + a_{2g+3-n}$ so that $a_m > a_{2g+3-n}$ and

$$m \geq 2g + 4 - n.$$

Let $(a'_1, \ldots, a'_{2g-1})$ be the new vector obtained by deleting a_n and a_m. We have

$$\sum_{i=1}^{2g-1} a'_i = \sum_{i=1}^{2g+1} a_i - c = cg - c = c(g-1)$$

and hence

$$1 - \sum z^{a_i} + \sum z^{b_i} - z^c = 1 - \sum z^{a'_i} + \sum z^{b'_i} - z^c.$$

Therefore we need only to prove that $a'_i + a'_j < c$ if $i + j = 2g+1$. But if $i+j = 2g+1$, then $i < n$ or $j < m$ otherwise $2g + 1 = i + j \geq n + m \geq 2g + 4$. So, if $j < m$, then

$$a'_i + a'_j \leq a_{i+1} + a_{j+1} < c,$$

while if $i < n$ then

$$a'_i + a'_j \leq a_i + a_{j+2} < c.$$

The conclusion follows. ⊙

We have proved that if $\underline{a}$ is admissible, the polynomial $p_{\underline{a}}(z)/(1-z)$ verifies the properties as in iii). Now the relevant remark is that the converse holds.

Proposition 1.5 *Let $d \geq 4$ and $f(z) = \sum_{i=0}^{d} f_i z^i$ be a d-symmetric polynomial with integer coefficients such that $f'(1) = f(1) = 0$. Let further assume that for some integer t with $4 \leq 2t \leq d$, we have*

$$\begin{cases} f_j = 1, & \text{if } 0 \leq j \leq t-1 \text{ or } d-t+1 \leq j \leq d \\ f_j \leq 0, & \text{if } t \leq j \leq d-t. \end{cases}$$

Then the sequence

$$\underline{a} = (t, \overbrace{t, \ldots, t}^{-f_t}, \overbrace{t+1, \ldots, t+1}^{-f_{t+1}}, \ldots, \overbrace{d-t, \ldots, d-t}^{-f_{d-t}})$$

is admissible and verifies $p_{\underline{a}}(z) = (1-z)f(z)$.

Proof. The sequence $\underline{a}$ has $1-\sum_{i=t}^{d-t} f_i$ components. Since $0=f(1)=2t+\sum_{i=t}^{d-t} f_i$, we have $1-\sum_{i=t}^{d-t} f_i = 2t+1$. The sum of the components of this sequence is

$$t-\sum_{i=t}^{d-t} if_i = t+td=(d+1)t$$

where we used the equality $f'(1)=0=td+\sum_{i=t}^{d-t} if_i$.

Hence $c=d+1$ and if we let $b_i=d+1-a_i$ it is clear, by the d-symmetry of $f(z)$, that the corresponding shifts table is

$$\begin{array}{cccccccccccc} & t+1 & \dots & t+1 & t+2 & \dots & t+2 & \dots & d-t+1 & \dots & d-t+1 & d-t+1 \\ t & t & \dots & t & t+1 & \dots & t+1 & \dots & d-t & \dots & d-t & \end{array}$$

and it verifies the admissibility condition.

Finally we have

$$\begin{aligned} &p_{\underline{a}}(z) = 1-\sum z^{a_i}+\sum z^{b_i}-z^c = \\ &1-(-f_t+1)z^t-\textstyle\sum_{i=t+1}^{d-t}(-f_i)z^i+\sum_{i=t+1}^{d-t}(-f_{i-1})z^i+(1-f_{d-t})z^{d-t+1}-z^{d+1} = \\ &1+(f_t-1)z^t+\textstyle\sum_{i=t+1}^{d-t}(f_i-f_{i-1})z^i+(1-f_{d-t})z^{d-t+1}-z^{d+1} = \\ &(1-z)f(z). \end{aligned}$$

This gives the conclusion. ⊙

In the following we say that a polynomial $h(z)$ is *admissible* if there exists a Gorenstein homogeneous codimension three ideal $I \subset R = k[X_1,\dots,X_n]$ such that

$$P_{R/I}(z) = h(z)/(1-z)^{n-3}.$$

As a consequence of Proposition 1.5 we have immediately the well known characterization of admissible h-polynomial (see [8]).

Theorem 1.6 *Let $h(z)$ be a polynomial of degree $s \geq 2$ with integer coefficients and let $q(z)=\sum q_i z^i = h(z)(1-z)^2$ be its second difference. Then $h(z)$ is admissible if and only if $h(z)$ is s-symmetric and for some integer t with $4 \leq 2t \leq s+2$, we have*

$$\begin{cases} q_j = 1, & \text{if } 0 \leq j \leq t-1 \text{ or } s+3-t \leq j \leq s+2 \\ q_j \leq 0, & \text{if } t \leq j \leq s+2-t. \end{cases}$$

The integer t in Theorem 1.6 is the *initial degree* of $h(z)=\sum h_i z^i$, which is by definition the least of the integers m such that

$$h_m < \binom{m+2}{2}.$$

One can easily translate the conditions of 1.6 into conditions on the coefficients of $h(z)$ as in [8].

Let now $h(z)$ be an admissible polynomial. It is clear that we may have different admissible sequences $\underline{a}$ such that

$$p_{\underline{a}}(z) = h(z)(1-z)^3.$$

This corresponds to the fact that ideals with different graded Betti numbers can have the same Hilbert function.

Let $h(z)$ be an admissible polynomial and $\underline{a} = (a_1, \dots, a_{2g+1})$, be an admissible sequence. We say that $\underline{a}$ is *$h(z)$-admissible* if

$$p_{\underline{a}}(z) = h(z)(1-z)^3.$$

In other words, $\underline{a}$ is $h(z)$-admissible if there exists a Gorenstein codimension three ideal I in R such that $h(z)$ is the h-polynomial of R/I and $(a_1, \dots, a_{2g+1})$ are the degrees of the generators of I.

The main result of this section is the following theorem where all the $h(z)$-admissible sequences of a given admissible polynomial $h(z)$ are described. The description is in terms of the sequence of the α_j's, where $\alpha_j = \aleph\{m \mid a_m = j\}$.

Theorem 1.7 *Let $h(z)$ be an admissible polynomial of degree s and initial degree t; further let $q(z) = \sum q_i z^i = h(z)(1-z)^2$ and $p(z) = \sum p_i z^i = h(z)(1-z)^3$ be its second and third difference respectively. Then an admissible sequence $\underline{a}$ is $h(z)$-admissible if and only if the following conditions hold:*

a) $\alpha_j = 0$, *if* $0 \le j \le t-1$ *or* $s-t+3 \le j$.

b) $\alpha_t = -q_t + 1$.

c) $\max(0, -p_j) \le \alpha_j \le -q_j$ *if* $t+1 \le j \le s-t+2$.

d) $\alpha_{s+3-j} - \alpha_j = p_j$ *for* $t+1 \le j \le s-t+2$.

e) If $s+3 = 2l$ *is even, then* $\alpha_l + q_l$ *is even.*

Proof. Let $\underline{a}$ be an admissible sequence. Then $p_{\underline{a}}(z) = h(z)(1-z)^3 = p(z)$ hence

$$q_{\underline{a}}(z) = p_{\underline{a}}(z)/(1-z) = h(z)(1-z)^2 = q(z).$$

Since c is the degree of $p_{\underline{a}}(z)$, we get $c = s+3$. On the other hand, it is clear that the initial degree t of $h(z)$ is also the least integer i such that $q_i \le 0$. Hence, by Proposition 1.4 i), we get $t = a_1$. Now a), b), c) and e) follow immediately from Proposition 1.4 iv), while d) follows by (9) and (10).

As for the converse, we must prove that any sequence verifying conditions a) to e) is $h(z)$-admissible.

In order to prove this, we first remark that the sequence corresponding to the maximum value of α_j is the sequence

$$\underline{a} = (t, \overbrace{t, \dots, t}^{-q_t}, \overbrace{t+1, \dots, t+1}^{-q_{t+1}}, \dots, \overbrace{s-t+2, \dots, s-t+2}^{-q_{s-t+2}})$$

which by Theorem 1.6 and Proposition 1.5 is admissible and verifies

$$p_{\underline{a}}(z) = (1-z)q(z) = h(z)(1-z)^3.$$

Hence it is $h(z)$-admissible. For this sequence we have $c = s+3$, hence by iii) of Proposition 1.4, every sequence obtained from $\underline{a}$ by removing for every i the pair $(t+i, s+3-t-i)$ as many times as possible, is $h(z)$-admissible.

Since in $\underline{a}$ we have $-q_{t+i}$ components equal to $t+i$ and $-q_{s+3-t-i}$ components equal to $s+3-t-i$, the pair $(t+i, s+3-t-i)$ can be deleted at most $\min(-q_{t+i}, -q_{s+3-t-i})$ times.

Since $q(z)$ is $(s+2)$-symmetric we have

$$\min(-q_{t+i}, -q_{s+3-t-i}) = min(-q_{t+i}, -q_{t+i-1}).$$

Hence, for every sequence deduced by $\underline{a}$, we have

$$-q_{t+i} \geq \alpha_{t+i} \geq -q_{t+i} - \min(-q_{t+i}, -q_{t+i-1}) = \max(0, -p_{t+i}).$$

Further, since the component $t+i$ and $s+3-t-i$ are deleted the same number r of times, we also have

$$\alpha_{s+3-t-i} - \alpha_{t+i} = -q_{s+3-t-i} - r - (-q_{t+i} - r) = -q_{s+3-t-i} + q_{t+i} = -q_{t+i-1} + q_{t+i} = p_{t+i}.$$

Finally if $s+3 = 2l$ is even, we can delete l an even number $2r$ of times, so that $\alpha_l = -q_l - 2r$ and $\alpha_l + q_l$ is even.

We have proved that the set of sequences verifying a) to e) in the theorem coincides with the set of sequences obtained from the fundamental sequence $\underline{a}$ by cancellation of pairs $(t+i, s+3-t-i)$. This proves that every sequence verifying the assumptions is $h(z)$-admissible as desired. ⊙

We remark that the analogous result in the perfect codimension two case has been proved by Campanella in [2].

We want to illustrate now the result by a specific example.

Example 1.8 Let us consider the admissible polynomial $h(z)$ corresponding to the h-vector

$$(1, 3, 6, 10, 15, 19, 22, 22, 19, 15, 10, 6, 3, 1).$$

We have $s = 13$, $s+3 = 16$, $t = 5$, $s-t+2 = 10$; $h(z)$ and its first three differences can be visualized as follows:

1	3	6	10	15	19	22	22	19	15	10	6	3	1			
1	2	3	4	5	4	3	0	−3	−4	−5	−4	−3	−2	−1		
1	1	1	1	1	−1	−1	−3	−3	−1	−1	1	1	1	1	1	
1	0	0	0	0	−2	0	−2	0	2	0	2	0	0	0	0	−1

Here the fundamental vector is $(5,5,6,7,7,7,8,8,8,9,10)$. All the other vectors can be obtained by deleting from this vector the pairs $(6,10)$, $(7,9)$ and $(8,8)$ as many times as possible. We get the following family of admissible sequences

$$(5,5,6,7,7,7,8,8,8,9,10)$$

$$(5,5,7,7,7,8,8,8,9) \qquad (5,5,6,7,7,8,8,8,10) \qquad (5,5,6,7,7,7,8,9,10)$$

$$(5,5,7,7,8,8,8) \qquad (5,5,6,7,7,8,10) \qquad (5,5,7,7,7,8,9)$$

$$(5,5,7,7,8).$$

In terms of the α's this family of $h(z)$-admissible sequences is described by the following data.

$$\begin{cases} \alpha_j = 0, & \text{if } 0 \le j \le 4 \text{ or } j \ge 11 \\ \alpha_5 = -q_5 + 1 = 2 & \\ 0 \le \alpha_6 \le 1 & \alpha_9 - \alpha_7 = p_7 = -2 \\ 2 \le \alpha_7 \le 3 & \alpha_{10} - \alpha_6 = p_6 = 0 \\ 0 \le \alpha_8 \le 3 & \alpha_8 + q_8 = \alpha_8 - 3 \text{ is even.} \end{cases}$$

S.Diesel presented in [3] an algorithm to determine all the admissible $h(z)$-sequences of a given h-polynomial $h(z)$. Roughly speaking, she knows how to determine the shortest $h(z)$-admissible sequence and she describes a procedure to construct all the other $h(z)$-admissible sequences starting from the shortest one, but she cannot predict when does the procedure stop. In other words, she does not describe the longest $h(z)$-admissible sequence in terms of $h(z)$.

Let us illustrate the construction of the fundamental sequence in a parametric example.

Example 1.9 Let t and j be positive integers such that $\binom{t+2}{2} \le j < \binom{t+3}{2}$. For every $m \ge 4$ we consider the polynomial

$$T(z) = \sum_{i=0}^{t} \binom{i+2}{2} z^i + \sum_{i=t+1}^{t+m-1} j z^i + \sum_{i=t+m}^{2t+m} \binom{2t+m-i+2}{2} z^i.$$

The coefficients of T and of its first two differences are the following:

$$\begin{array}{cccccccccccccccc} 1 & 3 & \dots & \binom{t+1}{2} & \binom{t+2}{2} & j & j & j & \dots & j & \binom{t+2}{2} & \binom{t+1}{2} & \binom{t}{2} & \dots & 3 & 1 \\ 1 & 2 & \dots & t & t+1 & f & 0 & 0 & \dots & 0 & -f & -(t+1) & -t & \dots & -3 & -2 \\ 1 & 1 & \dots & 1 & 1 & g & -f & 0 & \dots & 0 & -f & g & 1 & \dots & 1 & 1 \end{array}$$

where $f = j - \binom{t+2}{2}$ and $g = f - (t+1)$.

The initial degree of $T(z)$ is $t+1$ and $s = 2t+m$. The fundamental sequence can be constructed as in Proposition (1.5) by using the second difference

$$(1-z)^2T(z) = \sum_{i=0}^{t} z^i + gz^{t+1} - fz^{t+2} - fz^{t+m} + gz^{t+m+1} + \sum_{i=t+m+2}^{2t+m+2} z^i$$

of $T(z)$. It turns out that the fundamental sequence is

$$(t+1, \overbrace{t+1,\ldots,t+1}^{-g}, \overbrace{t+2,\ldots,t+2}^{f}, \overbrace{t+m,\ldots,t+m}^{f}, \overbrace{t+m+1,\ldots,t+m+1}^{-g}).$$

The reader should compare our construction with the one in [5, Sect.1] where the fundamental sequence is detected by using an idea of T.Harima which involves configurations of points of $\mathbf{P}^2$.

2 Lifting to radical

We have seen in the first section that for every admissible sequence $a_1 \le a_2 \le \cdots \le a_{2g+1}$ there exists a (canonical) 0-dimensional model. The goal of this section is to show how the 0-dimensional model can be lifted to a 1-dimensional reduced model by deforming the entries of the skew-symmetric matrix of the 0-dimensional model.

Let $a_1 \le a_2 \le \cdots \le a_{2g+1}$ be an admissible sequence and let $c = \sum a_i/g$ and $b_1 = c - a_1, \ldots, b_{2g+1} = c - a_{2g+1}$. Let M be the matrix introduced in Section 1 whose ideal I of $2g$-Pfaffians is the canonical 0-dimensional model of $a_1, \ldots, a_{2g+1}$.

Let now T be a new indeterminate. We deform the matrix M in the following way. For all $i = 1, \ldots, g$, we take vectors $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{iu_i}) \in K^{u_i}$, $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{iv_i}) \in K^{v_i}$, $\delta_i = (\delta_{i1}, \ldots, \delta_{iw_i}) \in K^{w_i}$. For simplicity of notation we will denote by α, γ, δ the vectors $(\alpha_1, \ldots, \alpha_g)$, $(\gamma_1, \ldots, \gamma_g)$,$(\delta_1, \ldots, \delta_g)$, and consider (α, γ, δ) as a vector of K^t, $t = \sum_i (u_i + v_i + w_i)$. We set

$$\begin{aligned} X(\alpha_i) &= \textstyle\prod_{j=1}^{u_i}(X - \alpha_{ij}T), \\ Y(\gamma_i) &= \textstyle\prod_{j=1}^{v_i}(Y - \gamma_{ij}T), \\ Z(\delta_i) &= \textstyle\prod_{j=1}^{w_i}(Z - \delta_{ij}T). \end{aligned}$$

Then we consider the skew-symmetric matrix N obtained from M by replacing $X^{u_i}, Y^{v_i}, Z^{w_i}$ with $X(\alpha_i), Y(\gamma_i), Z(\delta_i)$ respectively. .

For instance, if $g = 2$, the matrix N is:

$$N = \begin{pmatrix} 0 & 0 & 0 & X(\alpha_1) & Y(\gamma_1) \\ 0 & 0 & X(\alpha_2) & Y(\gamma_2) & Z(\delta_1) \\ 0 & -X(\alpha_2) & 0 & Z(\delta_2) & 0 \\ -X(\alpha_1) & -Y(\gamma_2) & -Z(\delta_2) & 0 & 0 \\ -Y(\gamma_1) & -Z(\delta_1) & 0 & 0 & \end{pmatrix}$$

Let J be the ideal of $S = K[X, Y, Z, T]$ generated by the Pfaffians of order $2g$ of N. One has:

Lemma 2.1 *For all the α, γ, δ the ideal J is Gorenstein of codimension 3 and the degrees of its minimal generators are $a_1, a_2, \ldots, a_{2g+1}$. The residue class of T in R/J is a non-zero divisor and $S/(J+T) = R/I$.*

Proof. Let us denote by τ the lexicographic term order induced by the order $Y > X > Z > T$. Denote by D_i the Pfaffian of the submatrix of N obtained by cancelling the i-th row and column. The initial term of D_1 is a power of Z, the initial term of D_{g+1} is a power of Y and that of D_{2g+1} is a power of X. This is enough to conlude the J has codimension 3. The rest follows from [1]. ⊙

We have seen that for every α, γ, δ the ideal J is a 1-dimensional model of the sequence $a_1, \ldots, a_{2g+1}$. We now show that for a generic α, γ, δ the ideal J is radical.

Theorem 2.2 *Assume that K is an infinite and perfect field. Then there exists a non-empty Zarisky-open subset U of K^t such that for all the $(\alpha, \gamma, \delta) \in U$ the ideal J is radical. In other word, for generic α, γ, δ, the ideal J is a 1-dimensional reduced model of $a_1, \ldots, a_{2g+1}$.*

We will use the following well-known

Lemma 2.3 *Let I be a homogeneous ideal in a polynomial ring R. Assume that I is equidimensional and without embedded components (e.g. R/I is Cohen-Macaulay). Let $P_1, \ldots, P_r$ be a set of distinct minimal prime ideals of I, and assume that $\sum_{i=1}^r e(R/P_i) \geq e(R/I)$. Then $\cap_{i=1}^r P_i = I$.*

Proof. It follows easily from the multiplicity formula. ⊙

We will also need the following

Lemma 2.4 *Let $A = (a_{ij})$ be a skew-symmetric matrix of size $(2g+1) \times (2g+1)$ such that $a_{ij} = 0$ for all $i, j \leq g$ and for all $i, j \geq g+2$. Let Q be the ideal of Pfaffians of order $2g$ of A. Denote by A_1 the submatrix of A of the first g rows and last $g+1$ columns and by A_2 the submatrix of A of the first $g+1$ rows and last g columns. One has $Q = I_g(A_1) + I_g(A_2)$.*

Proof. For $i = 1, \ldots, g+1$, the Pfaffian of the matrix obtained from A by deleting the i-th row and columns is equal to the determinant to the matrix obtained from A_2 by deleting the i-th row, while the Pfaffian of the matrix obtained from A by deleting the $g+i$-th row and columns is equal to the determinant to the matrix obtaine from A_1 by deleting the i-th column. ⊙

Proof. [of 2.2]: The multiplicity of the ideal J depends only on the degree matrix and it is known to be :

$$e(S/J) = \sum_{i=1}^{g} (\sum_{j,k=i}^{g} u_j w_k) v_i$$

see [7, Prop.6.3].

Now we are going to describe a family of minimal prime ideals of J. For $k = 1, \dots, g$, we define F_k to be the determinant of the submatrix of first k rows and last k columns of N, that is,

$$F_k = \det \begin{pmatrix} 0 & \dots & 0 & X(\alpha_1) & Y(\gamma_1) \\ 0 & \dots & X(\alpha_2) & Y(\gamma_2) & Z(\delta_1) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ X(\alpha_{k-1}) & Y(\gamma_{k-1}) & Z(\delta_{k-2}) & \dots & 0 \\ Y(\gamma_k) & Z(\delta_{k-1}) & 0 & \dots & 0 \end{pmatrix}$$

By construction F_k is homogeneous of degree $v_1 + v_2 + \dots + v_k$ and $Y^{v_1+v_2+\dots+v_k}$ appears in F_k with coefficient ± 1. For all $i, j = 1, \dots, g$, we define the ideal

$$Q_{ij} = (X(\alpha_i), Z(\delta_j), F_{i\wedge j})$$

where $i \wedge j = \min(i, j)$. Using 2.4 it is easy to see that $J \subset Q_{ij}$. Since $X(\alpha_i) = \prod_{h=1}^{u_h}(X - \alpha_{ih}T)$ and $Z(\delta_j) = \prod_{h=1}^{w_h}(Z - \delta_{jh}T)$, for all $a = 1, \dots, u_i$, and $b = 1, \dots, w_j$, we have:

$$J \subset (X - \alpha_{ia}T, Z - \delta_{jb}T, F_{i\wedge j})$$

Let us denote by $F_{i\wedge j}(a, b)$ the polynomial which is obtained from $F_{i\wedge j}$ by replacing X with $\alpha_{ia}T$ and Z with $\delta_{jb}T$. So we have:

$$J \subset (X - \alpha_{ia}T, Z - \delta_{jb}T, F_{i\wedge j}(a, b))$$

Note that $F_{i\wedge j}(a, b)$ is a homogeneous polynomial of $K[Y, T]$ of degree $v_1 + \dots + v_{i\wedge j}$. The set U is defined by the conditions:

i) the vectors α and δ have no repeated entries, that is, $\alpha_{ij} \neq \alpha_{hk}$ and $\delta_{ij} \neq \delta_{hk}$ if $(i, j) \neq (h, k)$,

ii) For all $i, j = 1, \dots, g$, and for all $a = 1, \dots, u_i$, and $b = 1, \dots, w_j$, the polynomial $F_{i\wedge j}(a, b)$ is square free, that is it is not divisible by the square of an irriducible polynomial.

Condition i) is clearly open. Condition ii) is also open because can be translated into the non-vanishing of the resultant of $F_{i\wedge j}(a, b)$ evaluated at $T = 1$ and its first derivative with respect to Y. Now we show that for all (α, γ, δ) in U the ideal J is radical. Every irriducible factor G of $F_{i\wedge j}(a, b)$ gives rise to a minimal prime of J, namely $(X - \alpha_{ia}T, Z - \delta_{jb}T, G)$. Because of i) the ideals $(X - \alpha_{ia}T, Z - \delta_{jb}T, G)$ are all distinct as i, j, a, b and G vary. Hence by virtue of 2.3 it is enough to show that the sum of the multiplicities of all the $S/(X - \alpha_{ia}T, Z - \delta_{jb}T, G)$ is equal to $e(S/J)$. The multiplicity of $S/(X - \alpha_{ia}T, Z - \delta_{jb}T, G)$ is the degree G. Because of ii) by summing up the multiplicities $S/(X - \alpha_{ia}T, Z - \delta_{jb}T, G)$, as G varies in the set of the irreducible factors of $F_{i\wedge j}(a, b)$, we obtain the degree of $F_{i\wedge j}(a, b)$, that is, $v_1 + \dots + v_{i\wedge j}$. Then summing up as a and b vary, we obtain $(v_1 + \dots + v_{i\wedge j})u_i w_j$. Finally summing up as i, j vary we obtain $\sum_{i,j=1}^{g}(v_1 + \dots + v_{i\wedge j})u_i w_j$ and this is equal to $e(S/J)$ by the above mentioned result.

It remains to show that U is not empty. To this end we need the following:

Lemma 2.5 *Let K be an infinite and perfect field.*
1) Let F, G be polynomials in $K[Y]$. Assume that F is square free. If $char(K) \neq 0$ assume also that $\deg G \leq \deg F + 1$. Then:

a) The set $A = \{a \in K : aF + G$ is not square free $\}$ is finite.

b) The set $B = \{a \in K : (Y - a)F + G$ is not square free$\}$ is finite.

2) Let $F_1, \dots, F_m, G_1, \dots, G_m$, be polynomials in $K[Y]$. Assume that $F_1, \dots, F_m$, are square free. If $char(K) \neq 0$ assume also that $\deg G_j \leq \deg F_j + 1$ for all j. Then for all $n \in \mathbf{N}$ there exist $a_1, \dots, a_n \in K$ such that $\prod_{i=1}^{n}(Y - a_i)F_j + G_j$ is square free for all $j = 1, \dots, m$.

Proof. 1) a) Denote by H' the first derivative of a polynomial H. We distinguish two cases:

Case 1: Assume first that $FG' - F'G = 0$. Since F is square free one has $\gcd(F, F') = 1$, and then F divides G, say $G = HF$. It follows that $H'F^2 = 0$ and hence $H' = 0$. If $char(K) = 0$ then $\deg H = 0$. If $char(K) \neq 0$ one has also $\deg H = 0$ because of the assumption on the degrees of F and G. Then $A = \{-H\}$.

Case 2: Assume now that $FG' - F'G \neq 0$. Let $a \in A$. Then $aF + G$ is not square free and hence $aF + G$ and its first derivative $aF' + G'$ have a common root, say x, in some extension of K. From $aF(x) + G(x) = 0$ and $aF'(x) + G'(x) = 0$ it follows that x is a root of $FG' - F'G$. Since F is square free then either $F(x) \neq 0$ or $F'(x) \neq 0$. One has that either $a = -G(x)/F(x)$ or $a = -G'(x)/F'(x)$ where x is a root of $FG' - F'G$. Hence A is finite.

1) b) follows from a) because $(Y - a)F + G = -aF + (YF + G)$.

2) Take $a_2, \dots, a_n$, distinct elements in K such that $F_j(a_i) \neq 0$ for all $j = 1, \dots, m$, and for all $i = 2, \dots, n$. Then, replacing F_j with $\prod_{i=2}^{n}(Y - a_i)F_j$, we may assume $n = 1$. Then the claim follows from 1)b). ⊙

We fix now α and δ to be any vectors which satisfy condition i). We show that it is possible to find γ such that ii) is satisfied. Note that the polynomial F_k involves only $\gamma_1, \dots, \gamma_k$. Hence we may argue by induction, and prove the following stronger claim:

Claim: Let $1 \leq k \leq g$. Assume that $\gamma_1, \dots, \gamma_{k-1}$, have been already determined such that for all $h = 1 \dots k-1$, for all $i, j = h, \dots, g$, and for all $a = 1, \dots, u_i$, and $b = 1, \dots, w_j$, the polynomial $F_h(a, b)$ is square free. Then there exists a vector γ_k such that for all $i, j = k, \dots, g$, and for all $a = 1, \dots, u_i$, and $b = 1, \dots, w_j$, the polynomial $F_k(a, b)$ is square free.

Since $F_1 = Y(\gamma_1) = \prod_{j=1}^{v_1}(Y - \gamma_{1j}T)$, for $k = 1$ the claim is trivial. Assume that $k > 1$. Since

$$F_k = \pm Y(\gamma_k)F_{k-1} \pm X(\alpha_{k-1})Z(\delta_{k-1})F_{k-2},$$

for all $i, j = k, \dots, g$, and for all $a = 1, \dots, u_i$, and $b = 1, \dots, w_j$, we have that:

$$F_k(a, b) = \pm Y(\gamma_k)F_{k-1}(a, b) \pm \rho F_{k-2}(a, b)$$

where

$$\rho = \prod_{h=1}^{u_{k-1}} (\alpha_{ia} - \alpha_{k-1h}) \prod_{h=1}^{w_{k-1}} (\delta_{jb} - \delta_{k-1h}) T^{u_{k-1}+w_{k-1}}$$

Note that $F_k(a,b)$ is homogeneous in $K[Y,T]$ of degree $v_1+\cdots+v_k$ and the coefficient of $Y^{v_1+\cdots+v_k}$ in $F_k(a,b)$ is ± 1. Then to show that there exists γ_k such that all the $F_k(a,b)$ are square free we may specialize $T=1$. Then the claim follows directly from 2.5 since $F_{k-1}(a,b)$ (for $T=1$) is square free and its degree in Y is bigger than that of $F_{k-2}(a,b)$. This conludes the proof of 2.2. ⊙

As a corollary of 2.2 we obtain another proof of the following theorem which is due to Geramita and Migliore [4, 2.1]:

Theorem 2.6 *Let K be an algebraically closed field. Then any admissible sequence can be realized by an ideal of (distinct) points of $\mathbf{P}^3_K$.*

The conditions which define the open set U are explicit and they can be easily checked in concrete cases. For instance:

Example 2.7 Assume for simplicity $K = \mathbf{C}$. Consider the following admissible sequence: $(a_1,a_2,a_3,a_4,a_5) = (2,3,3,4,4)$. Then $g=2$, $c=8$ and $(b_1,b_2,b_3,b_4,b_5) = (6,5,5,4,4)$. It follows that $u_1=2, u_2=2, v_1=2, v_2=1, w_1=1, w_2=1$. The multiplicity in this case is 18. The 0-dimensional model is given by the Pfaffians of order 4 of the matrix

$$M = \begin{pmatrix} 0 & 0 & 0 & X^2 & Y^2 \\ 0 & 0 & X^2 & Y & Z \\ 0 & -X^2 & 0 & Z & 0 \\ -X^2 & -Y & -Z & 0 & 0 \\ -Y^2 & -Z & 0 & 0 & 0 \end{pmatrix}$$

From the proof of 2.2 we know that we may choose $\alpha_1 = (0,1), \alpha_2 = (2,3), \delta_1 = (0), \delta_2 = (1)$. Then we have to choose $\gamma_1 = (\gamma_{11}, \gamma_{12})$, and $\gamma_2 = (\gamma_{21})$ such that condition ii) is satisfied. Since $F_1(a,b) = (Y - \gamma_{11}T)(Y - \gamma_{12}T)$, condition ii) for $i \wedge j = 1$ is equivalent to $\gamma_{11} \neq \gamma_{12}$ and we may choose $\gamma_{11} = 0$ and $\gamma_{12} = 1$. Since in this case $F_2 = -(Y - \gamma_{21}T)Y(Y-T) + X(X-T)Z$, condition ii) for $i = j = 2$ reads as follows: The polynomials

$$F_2(1,1) = -(Y-\gamma_{21}T)Y(Y-T) + 2T(2T-T)T = (Y-\gamma_{21}T)Y(Y-T) + 2T^3$$
$$F_2(2,1) = -(Y-\gamma_{21}T)Y(Y-T) + 3T(3T-T)T = (Y-\gamma_{21}T)Y(Y-T) + 6T^3$$

are square free. In order to check this condition we may specialize $T=1$, and look for a γ_{21} such that $-(Y-\gamma_{21})Y(Y-1)+2$ and $-(Y-\gamma_{21})Y(Y-1)+6$ are square free. We can take for instance $\gamma_{21}=0$. Then the ideal J of the 4-Pfaffians of

$$\begin{pmatrix} 0 & 0 & 0 & X(X-T) & Y(Y-T) \\ 0 & 0 & (X-2T)(X-3T) & Y & Z \\ 0 & (2T-X)(X-3T) & 0 & Z-T & 0 \\ X(T-X) & -Y & T-Z & 0 & 0 \\ Y(T-Y) & -Z & 0 & 0 & 0 \end{pmatrix}$$

is a Gorenstein ideal of 18 points in $\mathbf{P}^3$ which realizes the sequence $(2,3,3,4,4)$. We may also determine the ideals of these points. They are

$$(X,Y,Z),(X,Y-T,Z),(X-T,Y,Z),(X-T,Y-T,Z),$$
$$(X,Y,Z-T),(X,Y-T,Z-T),(X-T,Y,Z-T),(X-T,Y-T,Z-T),$$
$$(X-2T,Y,Z),(X-2T,Y-T,Z),(X-3T,Y,Z),(X-3T,Y-T,Z),$$
$$(X-2T,Y-s_1T,Z-T),(X-2T,Y-s_2T,Z-T),(X-2T,Y-s_3T,Z-T),$$
$$(X-3T,Y-r_1T,Z-T),(X-3T,Y-r_2T,Z-T),(X-3T,Y-r_3T,Z-T)$$

where s_1,s_2,s_3 are the roots of $-Y^2(Y-1)+2$ and r_1,r_2,r_3 are the roots of $-Y^2(Y-1)+6$.

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Received: November 1998

Revised: March 1999