Journal of Algebra **246**, 311–330 (2001)
doi:10.1006/jabr.2001.8964, available online at http://www.idealibrary.com on IDEAL®

Algebras of Minors

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Communicated by Craig Huneke

Received December 21, 2000

DEDICATED TO JÜRGEN HERZOG ON THE OCCASION OF HIS 60TH BIRTHDAY

INTRODUCTION

Let I be an ideal of a Noetherian ring R. The Rees algebra of I is the graded R-algebra $\oplus_{k=0}^{\infty} I^k$. The study of the properties of Rees algebras (and of the other blow-up algebras) has attracted the attention of many researchers in the last three decades. For a detailed account of the theories that have been developed and of the results that have been proved, the reader should consult the monograph of Vasconcelos [V]. In this paper we treat a special and interesting case: the Rees algebras of determinantal ideals and their special fibers.

Let $X = (x_{ij})$ be a generic matrix of size $m \times n$ over a field K and let S be the polynomial ring $K[x_{ij}]$. Let I_t be the ideal of S generated by the minors of size t of X. Finally, let $\mathcal{R}_t$ be the Rees algebra of I_t, and let A_t be the subalgebra of S generated by the t-minors of X. In the case of maximal minors, i.e., $t = \min(m, n)$, the algebra A_t is nothing but the homogeneous coordinate ring of the Grassmann variety which is known to be a

0021-8693/01 $35.00

Gorenstein factorial domain; for instance, see Bruns and Vetter [BV]. Furthermore, for maximal minors, Eisenbud and Huneke proved in [EH] that $\mathcal{R}_t$ is normal and Cohen–Macaulay. Their approach is based on the notion of algebra with straightening law. Later Bruns proved that $\mathcal{R}_t$ and A_t are Cohen–Macaulay and normal for any t if char $K = 0$ using invariant theory methods; see [B]. In [BC1] we have shown, by using Sagbi basis deformation and the Knuth–Robinson–Schensted correspondence, that $\mathcal{R}_t$ and A_t are Cohen–Macaulay and normal domains for any t and for any nonexceptional characteristic. Recall that (in our terminology) K has nonexceptional characteristic if either char $K = 0$ or char $K > \min(t, m-t, n-t)$. For exceptional characteristic, $\mathcal{R}_t$ and A_t can be (and perhaps are always) very far from being Cohen–Macaulay; see the example in [B].

For maximal minors, the divisor class group and the canonical class of $\mathcal{R}_t$ have been determined. They are the "expected" ones, that is, those of the Rees algebra of a prime ideal with primary powers in a regular local ring: $\mathrm{Cl}(\mathcal{R}_t)$ is free of rank 1 with generator $\mathrm{cl}(I_t\mathcal{R}_t)$ and the canonical class is $(2-\text{height } I_t)\,\mathrm{cl}(I_t\mathcal{R}_t)$; see Herzog and Vasconcelos [HV] or Bruns, Simis, and Trung [BST]. On the other hand, for nonmaximal minors, $\mathrm{Cl}(\mathcal{R}_t)$ is free of rank t [B].

Our goal is to determine the canonical class of $\mathcal{R}_t$, the divisor class group of A_t, and the canonical class of A_t in all the remaining cases, that is, $t < \min(m, n)$. The case $t = 1$ is anyway easy, and also the study of A_t in the case $m = n = t + 1$ is trivial since that ring is a polynomial ring.

Our main tools are the γ-functions that allow us to describe all relevant ideals in $\mathcal{R}_t$ and A_t and the Sagbi deformation by which we will lift the canonical module from the initial algebras of $\mathcal{R}_t$ and A_t to the algebras $\mathcal{R}_t$ and A_t themselves. In the following we consider A_t and $\mathcal{R}_t$ as subalgebras of the polynomial ring $S[T]$, A_t being generated by the elements of M_tT, where M_t is the set of t-minors, and $\mathcal{R}_t$ being generated by M_tT and the entries of X.

It turns out that the canonical class of $\mathcal{R}_t$ is given by

$$\mathrm{cl}(\omega(\mathcal{R}_t)) = \sum_{i=1}^{t}(2-(m-i+1)(n-i+1)+t-i)\,\mathrm{cl}(P_i)$$

$$= \mathrm{cl}(I_t\mathcal{R}_t) + \sum_{i=1}^{t}(1-\text{height } I_i)\,\mathrm{cl}(P_i).$$

Here P_i is the prime ideal of $\mathcal{R}_t$ determined by γ_i; it has the property $P_i \cap S = I_i$, and it is well known that height $I_i = (m-i+1)(n-i+1)$. Moreover, $\mathrm{cl}(I_t\mathcal{R}_t) = \sum_{i=1}^{t}(t-i+1)\,\mathrm{cl}(P_i)$. The classes $\mathrm{cl}(P_i)$ generate $\mathrm{Cl}(\mathcal{R}_t)$ freely.

The divisor class group of A_t is also free; its rank is 1 with generator the class $\mathrm{cl}(\mathfrak{q})$ of the ideal $\mathfrak{q} = (f)S[T] \cap A_t$, where f is a minor of size $t+1$

of X. The canonical class of A_t is given by

$$(mn - mt - nt)\,\mathrm{cl}(\mathfrak{q}).$$

It follows that, apart from the trivial or known cases listed above, A_t is Gorenstein if and only if $mn = t(m+n)$.

There are many open questions concerning the algebras $\mathcal{R}_t$ and A_t. Just to mention a few of them:

(a) In general, we do not know (the degrees of) the defining equations of $\mathcal{R}_t$ and A_t. It is known that $\mathcal{R}_t$ and A_t are defined by quadrics in the maximal minors case or if $t = 1$ or if $m = n = t + 1$ and that, in general, higher-degree relations are needed. Since A_t has a rational singularity, it has a negative a-invariant. This implies that the defining equations of A_t have degree $\leq \dim A_t$ and $\dim A_t = mn$ for nonmaximal minors. But this bound is not at all sharp. There are indications that quadrics and cubics are enough for A_t and $\mathcal{R}_t$ if $t = 2$.

(b) The Hilbert function of A_t can be computed by the hook length formula (for example, see Stanley [St, 7.21.6]). We have in fact computed these Hilbert functions for a substantial number of cases, but we do not know an explicit formula, not even, say, for the multiplicity of A_t.

(c) For maximal minors, the algebra A_t has an isolated singularity since the Grassmann variety is a homogeneous variety and, hence, smooth; for instance, see [BV]. But in general, we do not know (the dimension of) the singular locus of A_t.

In principle, the results of this paper can be extended to the other generic determinantal ideals (minors of a symmetric matrix, pfaffians of an alternating matrix, etc.). In the last section we do this in detail for a generic Hankel matrix.

Most of this work was done during the visit of the first author to the University of Genova in the framework of a project supported by the Italian Indam-Gnsaga.

1. NOTATION AND GENERALITIES

We keep the notation of the introduction. We will always assume that the characteristic of the field K is not exceptional, which means that it is either 0 or larger than $\min(t, m-t, n-t)$. We will study the algebras $\mathcal{R}_t$ and A_t. As mentioned above, both algebras can be embedded into the polynomial ring $S[T]$, where T is an auxiliary indeterminate. Denote by M_t the set of the t-minors of X. Then A_t is (isomorphic to) the K-subalgebra of $S[T]$ generated by the elements of the set M_tT, and $\mathcal{R}_t$ is the subalgebra generated by the elements of X and of M_tT. Clearly, $A_t \subset \mathcal{R}_t$ and both algebras are $\mathbb{N}$-graded in a natural way. Moreover, $\mathcal{R}_t/(X) \cong A_t$, where (X) is the ideal of $\mathcal{R}_t$ generated by the x_{ij}. Denote by V_t the K-subalgebra of $S[T]$ generated by monomials that have degree t in the variables x_{ij} and degree 1 in T. Note that $A_t \subset V_t$ and that V_t is a normal semigroup ring isomorphic to the tth Veronese subring of the polynomial ring S. The rings $\mathcal{R}_t$ and A_t are known to be normal Cohen–Macaulay domains. The goal of the paper is to determine their divisor class group and canonical class and to discuss the Gorenstein property of these rings. As we have mentioned in the introduction, these invariants are already known in some special cases. Therefore, we will concentrate our attention on the remaining cases, that is, we make the following

ASSUMPTIONS. (a) When studying the Rees algebra $\mathcal{R}_t$, we will assume that $t < \min(m, n)$.

(b) When studying A_t, we will assume that $1 < t < \min(m, n)$ and that $m \neq n$ if $t = \min(m, n) - 1$.

Our approach to the study of the algebras under investigation makes use of Sagbi basis deformations and of the straightening law for generic minors. For generalities on the former, we refer the reader to Conca, Herzog, and Valla [CHV]. As for the latter, the reader can consult Bruns and Vetter [BV], Doubilet, Rota, and Stein [DRS], or De Concini, Eisenbud, and Procesi [DEP].

Here we just recall its main features. The minor of X with row indices $a_1, \dots, a_t$ and column indices $b_1, \dots, b_t$ is denoted by

$$[a_1, \dots, a_t | b_1, \dots, b_t].$$

We will assume that the row and the column indices are given in ascending order $a_i < a_{i+1}$ and $b_i < b_{i+1}$. The set of the minors of X is partially ordered in a natural way,

$$[a_1, \dots, a_t | b_1, \dots, b_t] \leq [c_1, \dots, c_s | d_1, \dots, d_s],$$

if and only if $t \geq s$ and $a_i \leq c_i$, $b_i \leq d_i$ for all $i = 1, \dots, s$.

Let $\delta_1, \delta_2, \dots, \delta_p$ be minors of size $t_1, t_2, \dots, t_p$, respectively. Consider the product $\Delta = \delta_1 \cdots \delta_p$. The shape of Δ is by definition the sequence of numbers $t_1, t_2, \dots, t_p$ and Δ is said to be a standard monomial if $\delta_1 \leq \cdots \leq \delta_p$. The straightening law asserts the following.

THEOREM 1.1. *The standard monomials form a K-vector space basis of S.*

We recall now the definition of the functions γ_t. Given a sequence of numbers $s = s_1, \dots, s_p$ and a number t, we define

$$\gamma_t(s) = \sum_{i=1}^{p} \max(s_i + 1 - t, 0).$$

Then we extend the definition to products of minors by setting

$$\gamma_t(\Delta) = \gamma_t(s),$$

where s is the shape of Δ. Finally, for ordinary monomials M of S, we define

$$\rho_t(M) = \sup\{\gamma_t(\Delta)\colon \Delta \text{ is a product of minors of } X \text{ with } \operatorname{in}(\Delta) = M\},$$

where $\operatorname{in}(f)$ denotes the initial term of a polynomial f with respect to a diagonal term order, i.e., a term order such that $\operatorname{in}(\delta) = x_{a_1b_1}\cdots x_{a_tb_t}$ for every minor $\delta = [a_1,\ldots,a_t|b_1,\ldots,b_t]$ of X. In our previous papers [BC1, BC2], the function ρ_t was also denoted by γ_t; this could create some ambiguities in the present paper, so we have changed the notation.

The functions γ_t were introduced in [DEP] to describe the symbolic powers of I_t and the primary decomposition of the powers of I_t. We have shown in [BC1] that the functions ρ_t describe instead the initial ideals of the symbolic and ordinary powers of I_t. Let us recall the precise statements.

THEOREM 1.2. (a) *$I_t^{(k)}$ is the K-vector space generated by the standard monomials Δ with $\gamma_t(\Delta) \geq k$, and it contains all products of minors Δ with $\gamma_t(\Delta) \geq k$.*

(b) *The initial ideal $\operatorname{in}(I_t^{(k)})$ is the K-vector space generated by the ordinary monomials M with $\rho_t(M) \geq k$.*

THEOREM 1.3. (a) *$I_t^k = \bigcap_{i=1}^t I_i^{(k(t+1-i))}$ is a primary decomposition of I_t^k. In particular, I_t^k is generated by the standard monomials Δ with $\gamma_i(\Delta) \geq k(t+1-i)$ for all $i = 1,\ldots,t$, and it contains all products of minors Δ satisfying these conditions. Moreover, this primary decomposition is irredundant if $t < \min(m,n)$ and $k \geq (v-1)/(v-t)$, where $v = \min(m,n)$.*

(b) *The initial ideal $\operatorname{in}(I_t^k)$ is the K-vector space generated by the ordinary monomials M with $\rho_i(M) \geq k(t+1-i)$ for all $i = 1,\ldots,t$.*

We may define the value of function γ_t for any polynomial f of S as follows. Let $f = \sum_{i=1}^p \lambda_i\Delta_i$ be the unique representation of f as a linear combination of standard monomials Δ_i with coefficients $\lambda_i \neq 0$. Then we set

$$\gamma_t(f) = \inf\{\gamma_t(\Delta_i)\colon i = 1,\ldots,p\}$$

and $\gamma_t(0) = +\infty$. This definition is consistent with the one above in the sense that if Δ is a product of minors of shape s which is nonstandard, then $\gamma_t(\Delta) = \gamma_t(s)$.

The function γ_t is indeed a discrete valuation on S (with values in $\mathbb{N}$), that is, the following conditions are satisfied for every f and g in S:

(a) $\gamma_t(f+g) \geq \min(\gamma_t(f), \gamma_t(g))$, and equality holds if $\gamma_t(f) \neq \gamma_t(g)$,

(b) $\gamma_t(fg) = \gamma_t(f) + \gamma_t(g)$.

Note that (a) follows immediately from the definition, and (b) follows from the fact that the associated graded ring of a symbolic filtration is a domain if the base ring is regular. We may further extend γ_t to the field of fractions $Q(S)$ of S by setting

$$\gamma_t(f/g) = \gamma_t(f) - \gamma_t(g),$$

so that S is a subring of the valuation ring associated with each γ_t.

The next step is to extend the valuation γ_t to the polynomial ring $S[T]$ and to its field of fractions. We want to do this in a way such that the subalgebras $\mathcal{R}_t$ and A_t of $S[T]$ will then be described in terms of these functions. So, from now on, let us fix a number t with $1 \leq t \leq \min(m, n)$. For every polynomial $F = \sum_{j=0}^{p} f_j T^j \neq 0$ of $S[T]$, we set

$$\gamma_i(F) = \inf\{\gamma_i(f_j) - j(t+1-i)\colon j = 0, \dots, p\};$$

in particular

$$\gamma_i(T) = -(t+1-i).$$

Then we have:

Proposition 1.4. *The function γ_i defines a discrete valuation on the field of fractions $Q(S[T])$ of $S[T]$.*

Proof. This is a general fact; see Bourbaki [Bo, Ch. VI, §10, no. 1, lemme 1]. ∎

Note that $F = \sum f_j T^j \in S[T]$ has $\gamma_i(F) \geq 0$ if and only if $f_j \in I_i^{(j(t+1-i))}$ for every j. It follows from Theorem 1.3 that the Rees algebra $\mathcal{R}_t$ of I_t has the following description.

Lemma 1.5.

$$\mathcal{R}_t = \{F \in S[T]\colon \gamma_i(F) \geq 0 \textit{ for } i = 1, \dots, t\}.$$

Similarly, $A_t = \{F \in V_t\colon \gamma_i(F) \geq 0$ for $i = 2, \dots, t\}$. But this description is redundant:

Lemma 1.6.

$$A_t = \{F \in V_t\colon \gamma_2(F) \geq 0\}.$$

Proof. We have to show that if Δ is a standard monomial with deg $\Delta = kt$ and $\gamma_2(\Delta) \geq k(t-1)$, then $\gamma_i(\Delta) \geq k(t+1-i)$ for all $j = 3, \dots, t$. For $i = 1, \dots, m$, let a_i denote the number of factors of Δ which are minors of size i. By assumption we have that

(a) $\sum_{i=1}^{m} i a_i = kt$,

(b) $\sum_{i=2}^{m}(i-1)a_i \geq k(t-1)$.

In view of condition (a), condition (b) can be rewritten as

(c) $\sum_{i=1}^{m} a_i \leq k$.

We have to show that

(d) $\sum_{i=j}^{m} a_i(i+1-j) \geq k(t+1-j)$.

Note that

$$\sum_{i=j}^{m} a_i(i+1-j) = \sum_{i=1}^{m} a_i(i+1-j) - \sum_{i=1}^{j-1} a_i(i+1-j)$$
$$= kt + \sum_{i=1}^{m} a_i(1-j) + \sum_{i=1}^{j-1} a_i(j-i-1).$$

Therefore (d) is equivalent to

$$\left(k - \sum_{i=1}^{m} a_i\right)(j-1) + \sum_{i=1}^{j-1} a_i(j-i-1) \geq 0,$$

which is true since the left-hand side is the sum of nonnegative terms. ∎

Similarly, the initial algebras of $\mathcal{R}_t$ and A_t have a description in terms of the functions ρ_i. To this end we extend the definition of the function ρ_i to monomials of $S[T]$ by setting

$$\rho_i(MT^k) = \rho_i(M) - k(t+1-i),$$

where M is a monomial of S. Furthermore, for a polynomial $F = \sum_{j=1}^{p} \lambda_j N_j$ of $S[T]$ where the N_i are monomials and the λ_j are nonzero elements of K, we set

$$\rho_i(F) = \inf\{\rho_i(N_j)\colon j = 1, \ldots, p\}.$$

Now Theorem 1.3 implies:

LEMMA 1.7. (1) *The initial algebra* $\operatorname{in}(\mathcal{R}_t)$ *of* $\mathcal{R}_t$ *is given by*

$$\operatorname{in}(\mathcal{R}_t) = \{F \in S[T]\colon \rho_i(F) \geq 0 \text{ for } i = 1, \ldots, t\}.$$

The set of the monomials N of $S[T]$ such that $\rho_i(N) \geq 0$ for $i = 1, \ldots, t$ form a K-vector space basis of $\operatorname{in}(\mathcal{R}_t)$.

(2) *The initial algebra $in(A_t)$ of A_t is given by*

$$\operatorname{in}(A_t) = \{F \in V_t\colon \rho_2(F) \geq 0\}.$$

The set of the monomials N of $S[T]$ such that $\rho_1(N) = 0$ and $\rho_2(N) \geq 0$ form a K-vector space basis of $\operatorname{in}(A_t)$.

The major difference between the functions γ_i and ρ_i is that the latter is not a valuation. Nevertheless, it will turn out that ρ_i is an “intersection” of valuations; see Section 3.

2. THE DIVISOR CLASS GROUPS OF $\mathcal{R}_t$ AND A_t

The divisor class group of $\mathcal{R}_t$ has already been determined in [B, Cor. 2.4]. In this section we will present a (slightly) different approach to it, which will also be used to determine the divisor class group of A_t. We will need the following general facts.

LEMMA 2.1. *Let A be a normal domain. Suppose $v_1, \dots, v_t$ are discrete valuations on A. Let $B = \{x \in A: v_i(x) \geq 0$ for $i = 1, \dots, t\}$ and $P_i = \{x \in B: v_i(x) \geq 1\}$. Then*:

(1) *Assume that there exist elements $y_1, \dots, y_t \in A$ with $v_i(y_i) < 0$ and $v_j(y_i) \geq 0$ for $j \neq i$. Then P_i is a height-1 ideal of the Krull domain B whose symbolic power $P_i^{(j)}$ is $\{x \in B: v_i(x) \geq j\}$ for every $j \geq 1$.*

(2) *Assume that* $\operatorname{height}(P_i A) > 1$ *for $i = 1, \dots, t$. Then $A = \bigcap B_Q$, where the intersection is taken over all height-1 prime ideals Q of B with $Q \notin \{P_1, \dots, P_t\}$.*

Proof. (1) Let V_i be the valuation ring of v_i. Then evidently

$$B = V_1 \cap \cdots \cap V_t \cap \bigcap_{\text{height } Q=1} A_Q,$$

and B is obviously a Krull domain. All essential valuation overrings of B occur in the above intersection. Since, by assumption, y_i is in A and in all the V_j with $j \neq i$, but not in B, we have that V_i is not redundant in the above description of B. Then V_i is an essential valuation overring of B. The rest is clear by standard results about essential valuations of Krull domains.

(2) Let f be an arbitrary element of A. Let Q be a height-1 prime ideal of B with $Q \notin \{P_1, \dots, P_t\}$. We choose an element $g \in P_1 \cap \cdots \cap P_t \setminus Q$. Since $v_i(g) \geq 1$ for all i, there is a positive integer r such that $v_i(fg^r) = v_i(f) + rv_i(g) \geq 0$ for all i. It follows that $fg^r \in B$. Hence $f \in B_Q$. So we get $A \subseteq B_Q$. To prove $A \supseteq \bigcap B_Q$, we first note that $A = \bigcap A_{\mathfrak{Q}}$, where $\mathfrak{Q}$ is taken over all height-1 prime ideal $\mathfrak{Q}$ of A. So we only need to show that if $Q = \mathfrak{Q} \cap B$, then Q is a height-1 prime ideal of B and $Q \notin \{P_1, \dots, P_t\}$. That $Q \notin \{P_1, \dots, P_t\}$ follows from the assumption $\operatorname{height}(P_i A) > 1$. To prove $\operatorname{height}(Q) = 1$, it is sufficient to show that $B_Q = A_{\mathfrak{Q}}$. Let $f \in A \setminus \mathfrak{Q}$. Then $f \notin QB_Q$ because otherwise there is an element $g \in B \setminus Q$ such that $fg \in Q \subset \mathfrak{Q}$, contradicting the facts that $f \notin \mathfrak{Q}$, $g \notin \mathfrak{Q}$. Thus, f is an invertible element in B_Q. Hence we can conclude that $A_{\mathfrak{Q}} \subseteq B_Q$, so that $A_{\mathfrak{Q}} = B_Q$. ∎

We are grateful to the referee for the above lemma, which has simplified our original treatment. For $i = 1, \dots, t$, we set $P_i = \{F \in \mathcal{R}_t: \gamma_i(F) \geq 1\}$.

LEMMA 2.2. *The ideal P_i is a height-*1 *prime ideal of $\mathcal{R}_t$ for $i = 1, \ldots, t$ and $P_i^{(j)} = \{F \in \mathcal{R}_t \colon \gamma_i(F) \geq j\}$ for every $j > 0$.*

Proof. We apply Lemma 2.1(1) with $A = S[T]$ and $v_i = \gamma_i$. For $i = 1, \ldots, t$, set

$$y_i = g^a f^{t-i} T^{m-i},$$

where g is an $(i-1)$-minor, f is an m-minor, and a is a large enough integer. The reader may check that these elements have the right γ-values, i.e., $\gamma_j(y_i) \geq 0$ for $j = 1, \ldots, t$ and $j \neq i$ and $\gamma_i(y_i) < 0$. ∎

We can now describe the polynomial ring as a subintersection of the Rees algebra.

PROPOSITION 2.3. *Set $R = \mathcal{R}_t$. Then*

$$S[T] = \bigcap R_Q,$$

*where the intersection is extended over all the height-*1 *prime ideals Q of $\mathcal{R}_t$ different from $P_1, \ldots, P_t$.*

Proof. We apply Lemma 2.1(2) with $A = S[T]$ and $v_i = \gamma_i$. To this end, it suffices to note that $I_i S[T] \subset P_i S[T]$ for all i. ∎

The ideal $I_t\mathcal{R}_t$ is a height-1 ideal of $\mathcal{R}_t$. This follows, for instance, from the fact that $\mathcal{R}_t$ has dimension $\dim S + 1$ and the associated graded ring $\mathcal{R}_t/I_t\mathcal{R}_t$ has dimension $\dim S$. Furthermore, $I_t\mathcal{R}_t$ is divisorial since it is isomorphic to $\oplus_{k>0} I_t^k T^k$, which is a height-1 prime ideal.

As a consequence of Lemma 2.2 and Theorem 1.3 (or [B, Thm. 2.3]), we obtain

PROPOSITION 2.4. *One has*

$$I_t\mathcal{R}_t = \bigcap_{i=1}^{t} P_i^{(t-i+1)},$$

*and this is an irredundant primary decomposition of $I_t\mathcal{R}_t$. In particular, P_i is a height-*1 *prime ideal of $\mathcal{R}_t$ for $i = 1, \ldots, t$.*

Proof. By virtue of Lemma 2.2, the equality $I_t\mathcal{R}_t = \cap_{i=1}^{t} P_i^{(t-i+1)}$ is simply a reinterpretation of Theorem 1.3. Since we assume that $t < \min(m, n)$, the primary decomposition of I_t^k given in Theorem 1.3 is irredundant for $k \gg 0$. It follows that the primary decomposition of $I_t\mathcal{R}_t$ is also irredundant. ∎

Remark 2.5. Since one can prove directly that P_1 is a height-1 prime ideal with primary powers, one can also use the standard localization argument in order to show Lemma 2.2 and Proposition 2.4 (see [B]). However, this argument is not available for the Hankel matrices discussed in Section 6.

THEOREM 2.6. *The divisor class group* $\mathrm{Cl}(\mathcal{R}_t)$ *of* $\mathcal{R}_t$ *is free of rank* t,

$$\mathrm{Cl}(\mathcal{R}_t) \cong \mathbb{Z}^t,$$

with basis $\mathrm{cl}(P_1), \dots, \mathrm{cl}(P_t)$.

Proof. The conclusion follows from Proposition 2.4 and from a general result of Simis and Trung [ST, Thm. 1.1]. Let us mention that one can derive the result also directly from Fossum [F, Thm. 7.1] and Proposition 2.3. ∎

We have similar constructions and results for A_t. First define

$$\mathfrak{p} = \{F \in A_t\colon \gamma_2(F) \geq 1\}.$$

It is clear that $\mathfrak{p}$ is a prime ideal of A_t.

LEMMA 2.7. *The ideal* $\mathfrak{p}$ *is prime of height* 1. *Moreover, one has* $\mathfrak{p}^{(j)} = \{F \in A_t\colon \gamma_2(F) \geq j\}$.

Proof. We apply Lemma 2.1(1), where $A = V_t$ and $v_1 = \gamma_2$. It suffices to take $f = x_{11}^t T \in V_t$ and note that $\gamma_2(f) = -t + 1 < 0$. ∎

PROPOSITION 2.8. *Set* $A = A_t$. *Then*

$$V_t = \bigcap A_P,$$

where the intersection is extended over all the height-1 *prime ideals* P *of* A_t *with* $P \neq \mathfrak{p}$.

Proof. We apply Lemma 2.1(2), where $A = V_t$ and $v_1 = \gamma_2$. It suffices to prove that $\mathfrak{p}A$ has height > 1. To this end, take f_1 and f_2 distinct $(t+1)$-minors (they exist by our assumptions). Set $g_j = f_j^t T^{t+1}$. Then $\gamma_2(g_j) = 1$ and hence $g_j \in \mathfrak{p}V_t$. Since the f_i are prime elements, $(f_1^t, f_2^t)S$ has height 2. Since V_t is a direct summand of the polynomial ring, it follows that $(g_1, g_2)V_t$ has height 2. ∎

Let f be a $(t+1)$-minor of X. Set $g = f^t T^{t+1}$. By construction, $g \in V_t$ and $\gamma_2(g) = 1$ so that $g \in A_t$. Set

$$\mathfrak{q} = (f)S[T] \cap A_t.$$

In other words, $\mathfrak{q} = \{fa \in A_t\colon a \in S[T]\}$. Since f is a prime element in $S[T]$, the ideal $\mathfrak{q}$ is prime. Furthermore, we have $\mathfrak{p}\mathfrak{q}^t \subset (g) \subset \mathfrak{p} \cap \mathfrak{q}$. The second inclusion is trivial. As for the first, note that any generator of $\mathfrak{p}\mathfrak{q}^t$ can be written in the form gb, and then just evaluate γ_2 to show that b is in A_t.

Note also that:

COROLLARY 2.9. *The ideal* $\mathfrak{q}$ *is a height*-1 *prime ideal of* A_t. *Furthermore,* $\mathfrak{q}^{(j)} = (f^j)S[T] \cap A_t$.

Proof. It suffices to check that $\mathfrak{q}$ does not contain $\mathfrak{p}$. Let f_1 be a $(t+1)$-minor of X different from f (it exists by our assumptions). Then $f_1^t T^{t+1}$ is in $\mathfrak{p}$, but not in $\mathfrak{q}$. The second statement is easy. ∎

It follows that:

THEOREM 2.10. *The divisor class group of* A_t *is free of rank* 1,

$$\mathrm{Cl}(A_t) \cong \mathbb{Z},$$

with basis $\mathrm{cl}(\mathfrak{q})$. *Furthermore, we have* $\mathrm{cl}(\mathfrak{p}) = -t\,\mathrm{cl}(\mathfrak{q})$.

Proof. By [F, Thm. 7.1] and Proposition 2.8 there is an exact sequence:

$$0 \to \mathbb{Z}\,\mathrm{cl}(\mathfrak{p}) \to \mathrm{Cl}(A_t) \to \mathrm{Cl}(V_t) \to 0. \tag{1}$$

It is well known that $\mathrm{Cl}(V_t)$ is isomorphic to $\mathbb{Z}/t\mathbb{Z}$ and that it is spanned by the class of the prime ideal P generated by the elements of the form $x_{11}\mu T$, where μ is a monomial in the x_{ij} of degree $t-1$. Note that $\mathrm{cl}(\mathfrak{p})$ is a torsion-free element in $\mathrm{Cl}(A_t)$. This is because $\mathfrak{p}^{(j)} = \{F \in A_t\colon \gamma_2(F) \geq j\}$ cannot be principal; in fact, it contains all the elements of the form g^j, where $g = f^t T^{t+1}$ and f is a $(t+1)$-minor. Now fix a $(t+1)$-minor f and set $P_1 = (f)S[T] \cap V_t$, that is, P_1 is the ideal generated by all the elements of the form $f\mu T^2$, where μ is a monomial of degree $t-1$. Evidently, P_1 is isomorphic to P and hence $\mathrm{cl}(P_1)$ generates $\mathrm{Cl}(V_t)$. But $\mathfrak{q}$ is $P_1 \cap A_t$ and hence $\mathrm{cl}(\mathfrak{q})$ is the preimage of $\mathrm{cl}(P_1)$ with respect to the map $\mathrm{Cl}(A_t) \to \mathrm{Cl}(V_t)$ of (1). It follows that $\mathrm{cl}(\mathfrak{p})$ and $\mathrm{cl}(\mathfrak{q})$ generate $\mathrm{Cl}(A_t)$. By evaluating the function γ_2, one shows that

$$\mathfrak{q}^{(t)} \cap \mathfrak{p} = (g)A_t.$$

Hence $\mathrm{cl}(\mathfrak{p}) = -t\,\mathrm{cl}(\mathfrak{q})$. Consequently, $\mathrm{Cl}(A_t)$ is generated by $\mathrm{cl}(\mathfrak{q})$, and $\mathrm{Cl}(A_t)$ is torison free. ∎

3. CANONICAL MODULES OF $\mathrm{in}(\mathscr{R}_t)$ AND $\mathrm{in}(A_t)$

The goal of this section is to describe the canonical modules of the semigroup rings $\mathrm{in}(\mathscr{R}_t)$ and $\mathrm{in}(A_t)$. We know that $\mathrm{in}(\mathscr{R}_t)$ and $\mathrm{in}(A_t)$ are normal (see [BC1]) and hence their canonical modules are the vector spaces spanned by all monomials represented by integral points in the relative interiors of the corresponding cones.

To simplify notation, we identify monomials of $S[T]$ with integral points of $\mathbb{R}^{mn+1}$. To a subset G of the lattice $[1, \ldots, m] \times [1, \ldots, n]$ we associate the ideal $P_G = (x_{ij}: (i, j) \notin G)$ of S and a linear form l_G on $\mathbb{R}^{mn}$ defined by $l_G(x_{ij}) = 1$ if $(i, j) \notin G$ and $l_G(x_{ij}) = 0$ otherwise.

The initial ideal $\operatorname{in}(I_i)$ of I_i is the square free monomial ideal generated by the initial terms (i.e., main diagonal products) of the i-minors. Therefore, $\operatorname{in}(I_i)$ is the Stanley–Reisner ideal of a simplicial complex Δ_i. Let $\mathbf{F}_i$ be the set of the facets of Δ_i. Then $\operatorname{in}(I_i) = \cap_{F \in \mathbf{F}_i} P_F$. The elements of $\mathbf{F}_i$ are described by Herzog and Trung [HT] in terms of families of nonintersecting paths. It turns out that Δ_i is a pure (even shellable) simplicial complex. We have shown in [BC2] that

$$\operatorname{in}\big(I_i^{(k)}\big) = \bigcap_{F \in \mathbf{F}_i} P_F^k \tag{2}$$

and that

$$\operatorname{in}\big(I_t^k\big) = \bigcap_{i=1}^{t} \bigcap_{F \in \mathbf{F}_i} P_F^{k(t+1-i)}. \tag{3}$$

For every $F \in \mathbf{F}_i$, we extend the linear form l_F to the linear form L_F on $\mathbb{R}^{mn+1}$ by setting $L_F(T) = -(t+1-i)$. Then Eqs. (2) and (3) imply

LEMMA 3.1. *A monomial N belongs to $\operatorname{in}(\mathcal{R}_t)$ if and only if it has nonnegative exponents and $L_F(N) \geq 0$ for every $F \in \mathbf{F}_i$ and $i = 1, \ldots, t$.*

This is the description of the semigroup in terms of linear homogeneous inequalities. It follows from the general theory (see Bruns and Herzog [BH, Ch. 6]) that the canonical module $\omega(\operatorname{in}(\mathcal{R}_t))$ of $\operatorname{in}(\mathcal{R}_t)$ is the semigroup ideal of $\mathcal{R}_t$ generated by the monomials N with all exponents ≥ 1 and $L_F(N) \geq 1$ for every $F \in \mathbf{F}_i$ and $i = 1, \ldots, t$.

Let $\mathcal{X}$ denote the product of all the variables x_{ij} with $(i, j) \in [1, \ldots, m] \times [1, \ldots, n]$. The canonical module $\omega(\operatorname{in}(\mathcal{R}_t))$ has a description in terms of $\mathcal{X}$ and the functions ρ_i:

LEMMA 3.2. *The canonical module $\omega(\operatorname{in}(\mathcal{R}_t))$ of $\operatorname{in}(\mathcal{R}_t)$ is the ideal*

$$\{F \in S[T]: \mathcal{X}T \mid F \text{ in } S[T] \text{ and } \rho_i(F) \geq 1 \text{ for every } i = 1, \ldots, t\}$$

of $\operatorname{in}(\mathcal{R}_t)$.

Proof. Let $N = MT^k$ be a monomial, where M is a monomial in the variables x_{ij}. Then, for a given i, $L_F(N) \geq 1$ for every $F \in \mathbf{F}_i$ if and only if $l_F(M) \geq k(t+1-i)+1$ for every $F \in \mathbf{F}_i$. By Eq. (2), this is equivalent to $M \in \operatorname{in}(I_i^{(k(t+1-i)+1)})$, which in turn is equivalent to $\rho_i(M) \geq k(t+1-i)+1$. Summing up, $L_F(N) \geq 1$ for every $F \in \mathbf{F}_i$ and $i = 1, \ldots, t$ if and only if $\rho_i(N) \geq 1$ for every $i = 1, \ldots, t$. ∎

Similarly, the canonical module $\omega(\mathrm{in}(A_t))$ has a description in terms of the function ρ_2:

LEMMA 3.3. *The canonical module* $\omega(\mathrm{in}(A_t))$ *of* $\mathrm{in}(A_t)$ *is the ideal*

$$\{F \in V_t\colon \mathscr{X}T \mid F \text{ in } S[T] \text{ and } \rho_2(F) \geq 1\}$$

of $\mathrm{in}(A_t)$.

For later application, we record the following lemma. Its part (2) asserts that $\mathscr{X}$ is a "linear" element for the functions ρ_i.

LEMMA 3.4. (1) $\rho_i(\mathscr{X}) = (m-i+1)(n-i+1)$.

(2) *Let M be any monomial in the x_{ij}. Then* $\rho_i(\mathscr{X}M) = \rho_i(\mathscr{X}) + \rho_i(M)$ *for every* $i = 1, \dots, \min(m,n)$.

Proof. Let M be a monomial in the x_{ij}. We know that $\rho_i(M) \geq k$ if and only if $M \in \mathrm{in}(I_i^{(k)})$. From Eq. (2) we may deduce that

$$\rho_i(M) = \inf\{l_F(M)\colon F \in \mathbf{F}_i\}.$$

Note that Δ_i is a pure simplicial complex of dimension equal to the dimension of the determinantal ring defined by I_i minus 1. It follows that $l_F(\mathscr{X}) = (m-i+1)(n-i+1)$ for every facet F of Δ_i. In particular, $\rho_i(\mathscr{X}) = (m-i+1)(n-i+1)$.

Since $l_F(NM) \geq l_F(N) + l_F(M)$ for all monomials N, M and for every F, we have $\rho_i(MN) \geq \rho_i(N) + \rho_i(M)$. Conversely, let G be a facet of Δ_i such that $\rho_i(M) = l_G(M)$. Then $l_G(\mathscr{X}M) = l_G(\mathscr{X}) + l_G(M) = \rho_i(\mathscr{X}) + \rho_i(M)$. Hence $\rho_i(MN) \leq \rho_i(N) + \rho_i(M)$, too. ∎

4. THE DEFORMATION LEMMA

We have been able to identify the canonical modules of $\mathrm{in}(\mathscr{R}_t)$ and $\mathrm{in}(A_t)$ and we would like to use this information to describe the canonical modules of $\mathscr{R}_t$ and A_t. To this end, we need a deformation lemma.

LEMMA 4.1. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring equipped with a term order τ and with a grading induced by positive weights* $\deg(x_i) = v_i$. *Let B be a finitely generated K-subalgebra of R generated by homogeneous polynomials and let J be a homogeneous ideal of B. Denote by* $\mathrm{in}(B)$ *and* $\mathrm{in}(J)$ *the initial algebra and the initial ideal of B and J, respectively. Then we have*:

(1) *If* $\mathrm{in}(B)$ *is finitely generated and* $\mathrm{in}(B)/\mathrm{in}(J)$ *is Cohen–Macaulay, then B/J is Cohen–Macaulay.*

(2) *If* $\mathrm{in}(B)$ *is finitely generated and Cohen–Macaulay and* $\mathrm{in}(J)$ *is the canonical module of* $\mathrm{in}(B)$ (*up to shift*), *then* B *is Cohen–Macaulay and* J *is the canonical module of* B (*up to the same shift*).

Proof. (1) Let $f_1, \dots, f_k$ be a Sagbi basis of B and let $g_1, \dots, g_h$ be a Gröbner basis of J. We may assume that these polynomials are monic and homogeneous. Consider the presentation $K[y_1, \dots, y_k]/I \cong B$ of B obtained by mapping y_i to f_i and the presentation $K[y_1, \dots, y_k]/I_1 \cong \mathrm{in}(B)$ of $\mathrm{in}(B)$ obtained by mapping y_i to $\mathrm{in}(f_i)$. Let $h_1, \dots, h_p$ be a system of binomial generators for the toric ideal I_1, say $h_i = y^{a_i} - y^{b_i}$. Then, for each i, we have expressions $f^{a_i} - f^{b_i} = \sum \lambda_{ij} f^{c_{ij}}$ with $\lambda_{ij} \in K\setminus\{0\}$ and $\mathrm{in}(f^{c_{ij}}) < \mathrm{in}(f^{a_i})$ for every i, j. It is known that the polynomials

$$y^{a_i} - y^{b_i} - \sum \lambda_{ij} y^{c_{ij}} \tag{4}$$

generate I. For each g_i, we may take a presentation $g_i = f^{d_i} + \sum \delta_{ij} f^{d_{ij}}$ with $\delta_{ij} \in K\setminus\{0\}$ and $\mathrm{in}(f^{d_{ij}}) < \mathrm{in}(f^{d_i})$ for all i, j. Then, by construction, the preimage of J in $K[y]/I$ is generated by the elements

$$y^{d_i} + \sum \delta_{ij} y^{d_{ij}}, \tag{5}$$

and the preimage of $\mathrm{in}(J)$ in $K[y]/I_1$ is generated by the y^{d_i}. Hence B/J is the quotient of $K[y]$ defined by the ideal H that is generated by the polynomials (4) and (5), and $\mathrm{in}(B)/\mathrm{in}(J)$ is the quotient of $K[y]$ defined by the ideal H_1 that is generated by the polynomials h_i and y^{d_i}.

If we can find a positive weight w on $K[y]$ such that $\mathrm{in}_w(H) = H_1$, then there is a one-parameter flat family with special fiber $\mathrm{in}(B)/\mathrm{in}(J)$ and general fiber B/J. This implies that B/J is Cohen–Macaulay, provided $\mathrm{in}(B)/\mathrm{in}(J)$ is. Let us define w. First consider a positive weight α on $K[x]$ such that $\mathrm{in}_\alpha(f^{c_{ij}}) < \mathrm{in}_\alpha(f^{a_i})$ for every i, j and $\mathrm{in}_\alpha(f^{d_{ij}}) < \mathrm{in}_\alpha(f^{d_i})$ for every i and j. That such an α exists is a well-known property of monomial orders; for instance, see Sturmfels [S, Proof of Cor. 1.11]. Then we define w as the "preimage" of α in the sense that we put $w(y_i) = \alpha(\mathrm{in}(f_i))$. It is clear, by construction, that the initial forms of the polynomials (4) and (5) with respect to w are exactly the h_i and the y^{d_i}.

This proves that $\mathrm{in}_w(H)$ contains H_1. But H_1 and H have the same Hilbert function by construction, and H and $\mathrm{in}_w(H)$ have the same Hilbert function because they have the same initial ideal if we refine w to a term order; for instance, see [S, Prop. 1.8]. Here we consider Hilbert functions with respect to the original graded structure induced by the weights v_i. It follows that $\mathrm{in}_w(H) = H_1$ and we are done.

(2) That B is Cohen–Macaulay follows from [CHV]. Since B is a Cohen–Macaulay positively graded K-algebra which is a domain, to prove that J is the canonical module of B it suffices to show that J is a maximal

Cohen–Macaulay module whose Hilbert series satisfies the relation $H_J(t) = (-1)^d t^k H_B(t^{-1})$ for some integer k, where $d = \dim B$ [BH, Thm. 4.4.5, Cor. 4.4.6].

The relation $H_J(t) = (-1)^d t^k H_B(t^{-1})$ holds since, by assumption, the corresponding relation holds for the initial objects and Hilbert series do not change by taking initial terms. So it is enough to show that J is a maximal Cohen–Macaulay module. But $\operatorname{in}(J)$ is a height-1 ideal since it is the canonical module [BH, Prop. 3.3.18], and hence also J has height 1. Therefore, it suffices to show that B/J is a Cohen–Macaulay ring. But this follows from (1) since $\operatorname{in}(B)/\operatorname{in}(J)$ is Cohen–Macaulay (it is even Gorenstein) [BH, Prop. 3.3.18]. ∎

5. THE CANONICAL CLASSES OF $\mathcal{R}_t$ AND A_t

In this section we will describe the canonical modules of $\mathcal{R}_t$ and of A_t and determine the canonical classes.

Assume for simplicity that $m \le n$. Let us consider a product of minors D such that $\operatorname{in}(D) = \mathcal{X}$ and $\gamma_i(D) = \rho_i(\mathcal{X})$. Since we have already computed $\rho_i(\mathcal{X})$ (see Lemma 3.4), we can determine the shape of D, which turns out to be $1^2, 2^2, \dots, (m-1)^2, m^{(n-m+1)}$. In other words, D must be the product of two minors of size 1, two minors of size $2, \dots,$ two minors of size $m-1$, and $n-m+1$ minors of size m. It is then not difficult to show that D is uniquely determined, the 1-minors are $[m|1]$ and $[1|n]$, the 2-minors are $[m-1, m|1, 2]$ and $[1, 2|n-1, n]$, and so on.

We have:

THEOREM 5.1. (1) *The canonical module of $\mathcal{R}_t$ is the ideal*

$$J = \{F \in S[T]\colon DT \mid F \text{ in } S[T] \text{ and } \gamma_i(F) \ge 1 \text{ for } i = 1, \dots, t\}.$$

(2) *The canonical module of A_t is the ideal*

$$J_1 = \{F \in V_t\colon DT \mid F \text{ in } S[T] \text{ and } \gamma_2(F) \ge 1\}.$$

Proof. By virtue of Lemma 4.1, it suffices to show that $\operatorname{in}(J)$ and $\operatorname{in}(J_1)$ are the canonical modules of $\operatorname{in}(\mathcal{R}_t)$ and $\operatorname{in}(A_t)$, respectively. A description of the canonical modules of $\operatorname{in}(\mathcal{R}_t)$ and $\operatorname{in}(A_t)$ has been given in Section 3. Therefore, it is enough to check that $\operatorname{in}(J)$ is exactly the ideal described in Lemma 3.2 and $\operatorname{in}(J_1)$ is the ideal described in Lemma 3.3. Note that we may write

$$J = DT\{F \in S[T]\colon \gamma_i(F) \ge 1 - \gamma_i(DT) \text{ for } i = 1, \dots, t\}$$

and

$$J_1 = DT\{F \in S[T]\colon \gamma_2(F) \ge 1 - \gamma_2(DT)\} \cap V_t.$$

Furthermore, by virtue of Lemma 3.4,

$$\omega(\mathrm{in}(\mathcal{R}_t)) = \mathcal{X}T\{F \in S[T]\colon \rho_i(F) \geq 1 - \rho_i(\mathcal{X}T) \text{ for } i = 1, \dots, t\}$$

and

$$\omega(\mathrm{in}(A_t)) = \mathcal{X}T\{F \in S[T]\colon \rho_2(F) \geq 1 - \rho_2(\mathcal{X}T)\} \cap V_t.$$

Since, by the very definition of D, we have $\mathrm{in}(D) = \mathcal{X}$ and $\gamma_i(D) = \rho_i(\mathcal{X})$, it suffices to show that

$$\mathrm{in}(\{F \in S[T]\colon \gamma_i(F) \geq j\}) = \{F \in S[T]\colon \rho_i(F) \geq j\}.$$

But this has (essentially) been proved in [BC1]. ∎

Now we determine the canonical class of $\mathcal{R}_t$.

THEOREM 5.2. *The canonical class of $\mathcal{R}_t$ is given by*

$$\mathrm{cl}(\omega(\mathcal{R}_t)) = \sum_{i=1}^{t}(2 - (m-i+1)(n-i+1) + t - i)\,\mathrm{cl}(P_i)$$

$$= \mathrm{cl}(I_t\mathcal{R}_t) + \sum_{i=1}^{t}(1 - \text{height } I_i)\,\mathrm{cl}(P_i).$$

Proof. The second formula for $\omega(\mathcal{R}_t)$ follows from the first, since height $I_i = (m-i+1)(n-i+1)$ and $\mathrm{cl}(I_t\mathcal{R}_t) = \sum_{i=1}^{t}(t-i+1)\,\mathrm{cl}(P_i)$ by Proposition 2.4.

We have seen that

$$\omega(\mathcal{R}_t) = DT\{F \in S[T]\colon \gamma_i(F) \geq 1 - \gamma_i(DT) \text{ for } i = 1, \dots, t\}.$$

We can get rid of DT and obtain a representation of $\omega(\mathcal{R}_t)$ as a fractional ideal, namely,

$$\omega(\mathcal{R}_t) = \{F \in S[T]\colon \gamma_i(F) \geq 1 - \gamma_i(DT) \text{ for } i = 1, \dots, t\}.$$

It follows that $\omega(\mathcal{R}_t) = \bigcap P_i^{(1-\gamma_i(DT))}$. As $\gamma_i(DT) = (m-i+1)(n-i+1) - (t+1-i)$, we are done. ∎

For A_t, the situation is slightly more difficult since $D \notin A_t$ in general. Therefore, we need an auxiliary lemma.

LEMMA 5.3. *Let f_j be a j-minor of X and let $\mathfrak{q}_j = (f_j)S[T] \cap A_t$. Then $\mathfrak{q}_j$ is a height-1 prime ideal of A_t and* $\mathrm{cl}(\mathfrak{q}_j) = (j-t)\,\mathrm{cl}(\mathfrak{q})$.

Proof. Note that $\mathfrak{q}_{t+1} = \mathfrak{q}$ by definition. Let Δ be a product of minors. We say that Δ has *tight shape* if its degree is divisible by t and it has exactly $\deg(\Delta)/t$ factors. In other words, Δ has tight shape if $\gamma_1(\Delta T^k) = 0$ and $\gamma_2(\Delta T^k) = 0$, where $k = \deg(\Delta)/t$. Let Δ be a product of minors with tight shape and let $k = \deg(\Delta)/t$. Note that

$$(\Delta)S[T] \cap A_t = (\Delta T^k)A_t.$$

Now fix $j \leq t$ and set $k = t - j + 1$; the product $\Delta = f_j f_{t+1}^{t-j}$ has tight shape and hence $\mathfrak{q}_j \cap \mathfrak{q}_{t+1}^{(t-j)} = (\Delta T^k)A_t$. Note that, for obvious reasons, $\mathfrak{q}_j$ does not contain $\mathfrak{q}$. It follows that $\mathfrak{q}_j$ is a prime ideal of height 1 and that $\mathrm{cl}(\mathfrak{q}_j) = (j - t)\,\mathrm{cl}(\mathfrak{q})$. Now take $j > t$ and set $k = j - t + 1$; the product $\Delta = f_j f_{t-1}^{j-t}$ has tight shape and hence, as above, we conclude that $\mathfrak{q}_j$ is prime of height 1 and $\mathrm{cl}(\mathfrak{q}_j) = (t - j)\,\mathrm{cl}(q_{t-1})$. Since we know already that $\mathrm{cl}(q_{t-1}) = -\mathrm{cl}(\mathfrak{q})$, we are done. ∎

Now we can prove

THEOREM 5.4. *The canonical class of A_t is given by*

$$\mathrm{cl}(\omega(A_t)) = (mn - tm - tn)\,\mathrm{cl}(\mathfrak{q}).$$

Proof. Assume that $m \leq n$. Set $W = (DT)S[T] \cap A_t$. We have seen that

$$\omega(A_t) = W \cap \mathfrak{p}.$$

Consequently

$$\mathrm{cl}(\omega(A_t)) = \mathrm{cl}(W) + \mathrm{cl}(\mathfrak{p}) = \mathrm{cl}(W) - t\,\mathrm{cl}(\mathfrak{q}).$$

Note that W can be written as the intersection of $(D)S[T] \cap A_t$ and $(T)S[T] \cap A_t$. But the latter is the irrelevant maximal ideal of A_t, whence $W = (D)S[T] \cap A_t$. Further, $(D)S[T] \cap A_t$ can be written as an intersection of ideals $\mathfrak{q}_j$. Taking into consideration the shape of D and Lemma 5.3, we have

$$\mathrm{cl}(W) = \left(\sum_{j=1}^{m-1} 2(j - t) + (n - m + 1)(m - t)\right)\mathrm{cl}(\mathfrak{q}).$$

Summing up, we get the desired result. ∎

As a corollary we have

THEOREM 5.5. *The ring A_t is Gorenstein if and only if $mn = t(m + n)$.*

Note that we assume $1 < t < \min(m, n)$ and $m \neq n$ if $t = \min(m, n) - 1$ in the theorem. We have observed in the introduction that A_t is indeed Gorenstein (and even factorial) in the remaining cases.

Remark 5.6. It is also possible to derive Theorem 5.4 from Theorem 5.2 by a suitable generalization of [BV, (8.10)].

6. ALGEBRAS OF MINORS OF GENERIC HANKEL MATRICES

Let S be the polynomial ring $K[x_1, \dots, x_n]$ over some field K. Choose a Hankel matrix X with distinct entries $x_1, \dots, x_n$; this means that X is an $a \times b$ matrix (x_{ij}) with $x_{ij} = x_{i+j-1}$ and $a + b - 1 = n$. Let I_t be the ideal generated by the minors of size t of X. It is known that I_t does not depend on the size of the matrix X (provided, of course, X contains t-minors); it depends only on t and n. For a given n, it follows that t may vary from 1 to m, where $m = [(n+1)/2]$ is the integer part of $(n+1)/2$. All the properties of the generic determinantal ideals that we have used in the previous sections to study their algebras of minors hold also for the determinantal ideals of Hankel matrices. This has been shown in Conca [C]. In particular:

(1) The symbolic powers $I_t^{(k)}$ and the primary decomposition of I_t^k are described in terms of the γ-functions.

(2) The initial algebras are described in terms of the corresponding functions for monomials, the ρ-functions.

(3) The ρ-functions can be described in terms of the linear forms associated to the facets of the simplicial complex Δ_i of $\mathrm{in}(I_i)$; moreover, Δ_i is a pure simplicial complex.

(4) The determinantal ideal I_t is prime and height $I_t = n - 2t + 2$; in particular, the minors of X are irreducible polynomials.

(5) The product of all the variables is a linear element for the ρ functions; that is, if M is a monomial and $\mathscr{X}$ is the product of all the variables, then $\rho_i(M\mathscr{X}) = \rho_i(M) + \rho_i(\mathscr{X})$.

Now fix a number t, $1 \le t \le m = [(n+1)/2]$. Denote by $\mathscr{R}_t$ the Rees algebra of I_t and by A_t the algebra generated by the minors of size t. These algebras have been studied in [C, Section 4]. It turns out that $\mathscr{R}_t$ and A_t are Cohen–Macaulay normal domains and that the dimension of A_t is n, unless $t = m$. In the latter case, the minors are algebraically independent. The arguments of the previous sections apply also to the present situation. One deduces the following theorem.

THEOREM 6.1. (1) *If $t < m$, then the divisor class group $\mathrm{Cl}(\mathscr{R}_t)$ of $\mathscr{R}_t$ is free of rank t with basis $\mathrm{cl}(P_1), \dots, \mathrm{cl}(P_t)$. Here P_i is the prime ideal of $\mathscr{R}_t$ associated with the valuation γ_i and $P_i \cap S = I_i$. The canonical class is*

$$\begin{aligned}\mathrm{cl}(\omega(\mathscr{R}_t)) &= \sum_{i=1}^{t}(-n + t + i)\,\mathrm{cl}(P_i)\\ &= \mathrm{cl}(I_t\mathscr{R}_t) + \sum_{i=1}^{t}(1 - \text{height } I_i)\,\mathrm{cl}(P_i).\end{aligned}$$

(2) *If* $t = m$ *and* n *is even, then* $\mathrm{Cl}(\mathcal{R}_t)$ *is free of rank* 1 *with basis* $\mathrm{cl}(P)$, *where* $P = I_m\mathcal{R}_t$. *In this case,* $\mathcal{R}_t$ *is a complete intersection and* $\mathrm{cl}(\omega(\mathcal{R}_t)) = 0$.

(3) *If* $t = m$ *and* n *is odd, then* I_m *is principal.*

To prove the theorem, one argues as in the generic case. The only difference is that the product of minors D such that $\mathrm{in}(D) = \mathcal{X}$ and $\gamma_i(D) = \rho_i(\mathcal{X})$ now has shape $m, m-1$ if n is odd, and shape m, m if n is even.

For the algebra A_t we have:

THEOREM 6.2. (1) *Assume that* $1 < t < m$ *and* n *is even if* $t = m - 1$. *Then the divisor class group* $\mathrm{Cl}(A_t)$ *of* A_t *is free of rank* 1 *with basis* $\mathrm{cl}(\mathfrak{q})$. *Here* $\mathfrak{q}$ *is the prime ideal of* A_t *defined as* $\mathfrak{q} = (f)S[T] \cap A_t$, *where* f *is a minor of size* $t+1$. *The canonical class is* $\mathrm{cl}(\omega(A_t)) = (n - 3t)\,\mathrm{cl}(\mathfrak{q})$.

(2) *If* $t = m - 1$ *and* n *is odd, then* A_t *is isomorphic to the coordinate ring* $\mathrm{Grass}(m-1, m+1)$ *of the Grassmann variety of the subspaces of dimension* $m - 1$ *in a vector space of dimension* $m + 1$. *In particular,* A_t *is factorial.*

(3) *If* $t = m$ *or* $t = 1$, *then* A_t *is a polynomial ring.*

The only assertion which still needs an argument is (2). To this end, note that I_t is the ideal of maximal minors of the Hankel matrix of size $t \times (n+1) - t$. This induces a surjective map from $\mathrm{Grass}(t, n+1-t)$ to A_t. If we take n odd and $t = m - 1$, then the dimension of $\mathrm{Grass}(t, n+1-t)$ is n, which is also the dimension of A_t. Hence the map $\mathrm{Grass}(t, n+1-t) \to A_t$ is an isomorphism.

THEOREM 6.3. *The ring* A_t *is Gorenstein if and only if one of the following conditions holds:*

(1) $3t = n$,

(2) $t = m - 1$ *and* n *is odd,*

(3) $t = m$ *or* $t = 1$.

Despite the analogy, let us point out that there are some important differences between the Rees algebras $\mathcal{R}_t$ and the algebras A_t for generic matrices and those for Hankel matrices. In the Hankel case, the algebras $\mathcal{R}_t$ and A_t are defined by (Gröbner bases of) quadrics, one has $\mathrm{in}(I_t^k) = \mathrm{in}(I_t)^k$ for all t and k, and all the results hold over an arbitrary field, no matter what the characteristic is.

REFERENCES

[Bo] N. Bourbaki, "Algèbre Commutative," Chap. I–IX, Hermann, Masson, 1961–1983.

[B] W. Bruns, Algebras defined by powers of determinantal ideals, *J. Algebra* **142** (1991), 150–163.

[BC1] W. Bruns and A. Conca, KRS and powers of determinantal ideals, *Compositio Math.* **111** (1998), 111–122.

[BC2] W. Bruns and A. Conca, KRS and determinantal ideals, *in* "Geometric and Combinatorial Aspects of Commutative Algebra, Messina 1999" 67–87, Lecture Notes in Pure and Appl. Math. 217 Dekker, New York, 2001.

[BH] W. Bruns and J. Herzog, "Cohen–Macaulay Rings," Cambridge Studies in Advanced Mathematics, Vol. 39, Cambridge Univ. Press, Cambridge, UK, 1993.

[BST] W. Bruns, A. Simis, and N. V. Trung, Blow-up of straightening-closed ideals in ordinal Hodge algebras, *Trans. Amer. Math. Soc.* **326** (1991), 507–528.

[BV] W. Bruns and U. Vetter, "Determinantal Rings," Lecture Notes in Mathematics, Vol. 1327, Springer, New York, 1988.

[C] A. Conca, Straightening law and powers of determinantal ideals of Hankel matrices, *Adv. Math.* **138** (1998), 263–292.

[CHV] A. Conca, J. Herzog, and G. Valla, Sagbi bases and application to blow-up algebras, *J. Reine Angew. Math.* **474** (1996), 113–138.

[DEP] C. De Concini, D. Eisenbud, and C. Procesi, Young diagrams and determinantal varieties, *Invent. Math.* **56** (1980), 129–165.

[DRS] P. Doubilet, G. C. Rota, and J. Stein, On the foundations of combinatorial theory: IX, Combinatorials methods in invariants theory, *Stud. Appl. Math.* **53** (1974), 185–216.

[EH] D. Eisenbud and C. Huneke, Cohen–Macaulay Rees algebras and their specialization, *J. Algebra* **81** (1983), 202–224.

[F] R. Fossum, "The Divisor Class Group of a Krull Domain," Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 74, Springer-Verlag, New York, Heidelberg, 1973.

[HSV1] J. Herzog, A. Simis, and W. Vasconcelos, On the canonical module of the Rees algebra and the associated graded ring of an ideal, *J. Algebra* **105** (1987), 285–302.

[HSV2] J. Herzog, A. Simis, and W. Vasconcelos, Arithmetic of normal Rees algebras, *J. Algebra* **143** (1991), 269–294.

[HT] J. Herzog and N. V. Trung, Gröbner bases and multiplicity of determinantal and pfaffian ideals, *Adv. in Math.* **96** (1992), 1–37.

[HV] J. Herzog and W. Vasconcelos, On the divisor class group of Rees-algebras, *J. Algebra* **93** (1985), 182–188.

[ST] A. Simis and N. V. Trung, The divisor class group of ordinary and symbolic blow-ups, *Math. Z.* **198** (1988), 479–491.

[St] R. P. Stanley, "Enumerative Combinatorics," Vol. 2, Cambridge Univ. Press, Cambridge, UK, 1999.

[S] B. Sturmfels, "Gröbner Bases and Convex Polytopes," AMS University Lecture Series, Vol. 8, Amer. Math. Soc., Providence, RI, 1995.

[V] W. V. Vasconcelos, "Arithmetic of Blowup Algebras," London Mathematical Society Lecture Note Series, Vol. 195, Cambridge Univ. Press, Cambridge, UK, 1994.