

KRS and determinantal ideals

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1 INTRODUCTION

Let K be a field and X an $m \times n$ matrix of indeterminates. The determinantal ideals in $K[X]$ are the ideals I_t generated by the t -minors of X , $1 \leq t \leq \min(m, n)$, and ideals related to them.

The Knuth–Robinson–Schensted correspondence (KRS) is a powerful tool for the computation of Gröbner bases of determinantal ideals. For this purpose it has first been used by Sturmfels [16]. Then Herzog and Trung [12] have considerably extended the class of ideals to which KRS can be applied. In a different direction Sturmfels' method has been generalized by Bruns and Conca [4] and Bruns and Kwieciński [5]. While Herzog and Trung use Gröbner bases in order to derive numerical results, the papers [4] and [5] aim at structural information, mainly on powers of determinantal ideals and the corresponding Rees algebras.

The crucial point in the application of KRS to Gröbner bases is to show the equality $\text{in}(I) = \text{KRS}(I)$ for the ideals I under consideration. We call these ideals *in-KRS*. Here $\text{in}(I)$ is the initial ideal of I with respect to a so-called diagonal term order on $K[X]$, and $\text{KRS}(I)$ is the image of I under the automorphism of the polynomial ring $K[X]$ induced by KRS – in the strict sense KRS is a bijection from the set of standard bitableaux (or standard monomials) \mathcal{S} to the set of monomials \mathcal{M} of $K[X]$. Both \mathcal{S} and \mathcal{M} are K -bases of $K[X]$: for \mathcal{S} this is asserted by the straightening law of Doubilet–Rota–Stein. Since $\text{in}(I)$ is a monomial ideal, one must assume that I has a basis of standard bitableaux.

The first three sections of the paper are an expanded version of the first author's lecture in the conference. Section 2 recapitulates the straightening law, and Section



3 introduces KRS. Section 4 explains the common ideas underlying the results of [4] and [5]. For this purpose we develop a conceptual framework in which KRS *invariants* play the central role. Such invariant is a function $F : \mathcal{D} \rightarrow \mathbb{N}$ defined on the set \mathcal{D} of all bitableaux (or products of minors) that, roughly speaking, is compatible with the straightening law and, moreover, satisfy the condition

$$F(\Sigma) = \max\{F(\Delta) : \Delta \in \mathcal{D}, \text{in}(\Delta) = \text{KRS}(\Sigma)\}.$$

It is then easy to see that each of the ideals $I_k(F)$ generated by all standard bitableaux Σ with $F(\Sigma) \geq k$ satisfies the condition $\text{in}(I_k(F)) = \text{KRS}(I_k(F))$. Even more is true: $I_k(F)$ is G-KRS, i. e. in addition to being in-KRS, $I_k(F)$ has a Gröbner basis of bitableaux. The class of G-KRS ideals is closed under sums, that of in-KRS ideals I is closed under sums and intersections, and therefore one then obtains many G-KRS or at least in-KRS ideals.

It has been shown in [4] that the functions γ_t introduced by De Concini, Eisenbud and Procesi [9] are KRS invariants. This fact allows one to compute the Gröbner bases, or at least the initial ideals of the symbolic powers of the I_t and products $I_{t_1} \cdots I_{t_s}$. The family α_k of KRS invariants found by Greene has been used in [5] for the analysis of the ideal underlying MacPherson's graph construction in the generic case.

In Section 5 we show that all ideals that are generated by products of minors and do not "prefer any rows or columns" of the matrix X are in-KRS, at least if $\text{char } K$ is 0 or $> \min(m, n)$. In characteristic 0 this is exactly the class of ideals that have a standard monomial basis and are stable under the natural action of $\text{GL}(m, K) \times \text{GL}(n, K)$ on $K[X]$. In fact all these ideals can be written as sums of intersections of symbolic powers of the ideals I_t , and the symbolic powers are G-KRS, as stated above.

Section 6 characterizes those among all the ideals of Section 5 that are even G-KRS. We show that these are essentially the sums of the ideals $J(k, d)$ introduced in [5] and for which Greene's theorem yields the property of being G-KRS. Since each KRS invariant can be derived from a family of G-KRS ideals, this shows that Greene's functions α_k are truly basic KRS invariants, at least if one considers functions $F : \mathcal{D} \rightarrow \mathbb{N}$ for which $F(\Delta)$ only depends on the shape of Δ .

Section 7 complements the results of [4]. We show that the formation of initial ideal and symbolic power commute for the ideals I_t . This result can be interpreted as a description of the semigroup of monomials in the initial algebra of the symbolic Rees algebra by linear inequalities.

In Section 8 we turn to a potential new KRS invariant γ_δ related with the ideal $I(X, \delta)$ cogenerated by a minor δ . Except in the case in which $I_\delta = I_t$, these do not only depend on shape and therefore constitute an interesting new class of functions. Though [12] gives some information on γ_δ , we have not yet been able to show that these are KRS invariants.

The application of KRS to determinantal ideals has also been investigated by Abhyankar and Kulkarni [1, 2]. Furthermore, variants of the KRS can be used to study ideals of symmetric matrices of indeterminates (Conca [7]) or ideals generated by Pfaffians of alternating matrices ([12], Baetjica [3], De Negri [8]).

There are now excellent discussions of KRS available in textbooks; see Fulton [10] and Stanley [15].



2 THE STRAIGHTENING LAW

Let K be a field and X an $m \times n$ matrix of indeterminates over K . For a given positive integer $t \leq \min(m, n)$, we consider the ideal $I_t = I_t(X)$ generated by the t -minors (i. e. the determinants of the $t \times t$ submatrices) of X in the polynomial ring $R = K[X]$ generated by all the indeterminates X_{ij} .

From the viewpoint of algebraic geometry R should be regarded as the coordinate ring of the variety of K -linear maps $f: K^m \rightarrow K^n$. Then $V(I_t)$ is just the variety of all f such that $\text{rank } f < t$, and R/I_t is its coordinate ring.

The study of the determinantal ideals I_t and the objects related to them has numerous connections with invariant theory, representation theory, and combinatorics. For a detailed account we refer the reader to Bruns and Vetter [6].

Almost all of the approaches one can choose for the investigation of determinantal rings use standard bitableaux and the straightening law. The principle governing this approach is to consider all the minors of X (and not just the 1-minors X_{ij}) as generators of the K -algebra R so that products of minors appear as “monomials”. The price to be paid, of course, is that one has to choose a proper subset of all these “monomials” as a linearly independent K -basis: the standard bitableaux are a natural choice for such a basis, and the straightening law tells us how to express an arbitrary product of minors as a K -linear combination of the basis elements. (In [4], [5] and [6] standard bitableaux were called *standard monomials*; however, we will have to consider the ordinary monomials in $K[X]$ so often that we reserve the term monomial for products of the X_{ij} .)

In the following

$$[a_1, \dots, a_t \mid b_1, \dots, b_t]$$

stands for the determinant of the submatrix $(X_{a_i b_j} : i = 1, \dots, t, j = 1, \dots, t)$.

The letter Δ always denotes a product $\delta_1 \cdots \delta_w$ of minors, and we assume that the sizes $|\delta_i|$ (i. e. the number of rows of the submatrix X' of X such that $\delta_i = \det(X')$) are *descending*, $|\delta_1| \geq \dots \geq |\delta_w|$. By convention, the empty minor $[]$ denotes 1. The *shape* $|\Delta|$ of Δ is the sequence $(|\delta_1|, \dots, |\delta_w|)$. If necessary we may add factors $[]$ at the right hand side of the products, and extend the shape accordingly.

A product of minors is also called a *bitableau*. The choice of this term bitableau is motivated by the graphical description of a product Δ as a pair of Young tableaux as in Figure 1: Every product of minors is represented by a bitableau and, conversely,

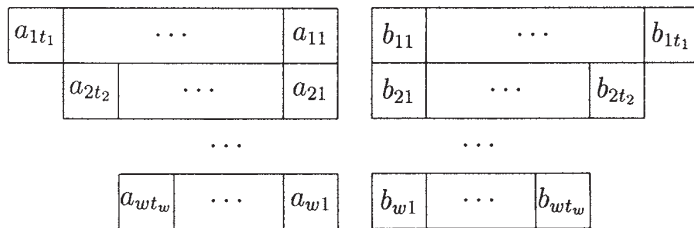


Figure 1: A bitableau

every bitableau stands for a product of minors if the length of the rows is decreasing from top to bottom, the entries in each row are strictly increasing from the middle to the outmost box, the entries of the left tableau are in $\{1, \dots, m\}$ and those of the right tableau are in $\{1, \dots, n\}$. These conditions are always assumed to hold.

For formal correctness one should consider the bitableaux as purely combinatorial objects and distinguish them from the ring-theoretic objects represented by them, but since there is no real danger of confusion, we simply identify them.

Whether Δ is a standard bitableau is controlled by a partial order of the minors, namely

$$[a_1, \dots, a_t \mid b_1, \dots, b_t] \preceq [c_1, \dots, c_u \mid d_1, \dots, d_u] \\ \iff t \geq u \quad \text{and} \quad a_i \leq c_i, \quad b_i \leq d_i, \quad i = 1, \dots, u.$$

A product $\Delta = \delta_1 \cdots \delta_w$ is called a *standard bitableau* if

$$\delta_1 \preceq \cdots \preceq \delta_w,$$

in other words, if in each column of the bitableau the indices are non-decreasing from top to bottom. The letter Σ is reserved for standard bitableaux.

The fundamental straightening law of Doubilet–Rota–Stein says that every element of R has a unique presentation as a K -linear combination of standard bitableaux (for example, see Bruns and Vetter [6]):

THEOREM 2.1. (a) *The standard bitableaux are a K -vector space basis of $K[X]$.*

(b) *If the product $\delta_1 \delta_2$ of minors is not a standard bitableau, then it has a representation*

$$\delta_1 \delta_2 = \sum x_i \varepsilon_i \eta_i, \quad x_i \in K, \quad x_i \neq 0,$$

where $\varepsilon_i \eta_i$ is a standard bitableau, $\varepsilon_i \prec \delta_1, \delta_2 \prec \eta_i$ (here we must allow that $\eta_i = 1$).

(c) *The standard representation of an arbitrary bitableau Δ , i.e. its representation as a linear combination of standard bitableaux Σ , can be found by successive application of the straightening relations in (b).*

(d) *Moreover, at least one Σ with $|\Sigma| = |\Delta|$ appears with a non-zero coefficient in the standard representation of Δ .*

Let e_1, \dots, e_m and f_1, \dots, f_n denote the canonical \mathbb{Z} -bases of \mathbb{Z}^m and \mathbb{Z}^n respectively. Clearly $K[X]$ is a $\mathbb{Z}^m \oplus \mathbb{Z}^n$ -graded algebra if we give X_{ij} the “vector bidegree” $e_i \oplus f_j$. All minors are homogeneous with respect to this grading, and therefore the straightening relations must preserve the multiplicities with which row and column indices occur on the left hand side.

The straightening law implies that the ideals I_t have a K -basis of standard bitableaux:

COROLLARY 2.2. *The standard bitableaux $\Sigma = \delta_1 \cdots \delta_w$ such that $|\delta_1| \geq t$ form a K -basis of I_t .*



All these standard bitableaux are elements of I_t since $\delta_1 \in I_t$ if $|\delta_1| \geq t$. Conversely, every $x \in I_t$ can be written as a K -linear combination of products δM where δ is a minor of size t and M is a monomial. Properties (b) and (c) of the straightening law imply that the standard bitableaux in the standard presentation of δM have the required property.

We say that an ideal $I \subset R$ has a *standard basis* if I is the K -vector space spanned by the standard bitableaux $\Sigma \in I$.

3 THE KNUTH–ROBINSON–SCHENSTED CORRESPONDENCE

Let Σ be a standard bitableau. The Knuth–Robinson–Schensted correspondence (see Fulton [10] or Stanley [15]) sets up a bijective correspondence between standard bitableaux and monomials in the ring $K[X]$. We use the version of KRS given by Herzog and Trung [12].

If one starts from bitableaux, the correspondence is constructed from the *deletion* algorithm. Let $\Sigma = (a_{ij}|b_{ij})$ be a non-empty standard bitableau. Then one constructs a pair of integers (ℓ, r) and a standard bitableau Σ' as follows.

- (a) One chooses the largest entry ℓ in the *left* tableau of Σ ; suppose that $\{(i_1, j_1), \dots, (i_u, j_u)\}$, $i_1 < \dots < i_u$, is the set of indices (i, j) such that $\ell = a_{ij}$.
- (b) Then the boxes at the pivot position $(p, q) = (i_u, j_u)$ in the left and the right tableau are removed.
- (c) The entry $\ell = a_{pq}$ of the removed box in the left tableau is the first component of the output, and b_{pq} is stored in s .
- (d) The second and third components of the output are determined by a “push out” procedure on the *right* tableau as follows:
 - (i) if $p = 1$, then $r = s$ is the second component of the output, and the third is the standard bitableau Σ' that has now been created;
 - (ii) otherwise the entry b_{pq} is moved one row up and pushes out the right most entry b_{p-1k} such that $b_{p-1k} \leq b_{pq}$ whereas b_{p-1k} is stored in s .
 - (iii) one replaces p by $p - 1$ and goes to step (i).

It is now possible to define KRS recursively: One sets $\text{KRS}([|]) = 1$, and $\text{KRS}(\Sigma) = \text{KRS}(\Sigma')X_{\ell r}$ for $\Sigma \neq [|]$.

We give an example in Figure 2. The circles in the left tableau mark the pivot position, those in the right mark the chains of “push-outs”: In this example we have

$$\text{KRS}(\Sigma) = X_{14}X_{21}X_{32}X_{45}X_{56}XX_{63}.$$

It is often more convenient to denote the output by a two row array instead of a monomial by making the row indices of the factors the upper row and the column indices the lower row; in the example

$$\text{krs}(\Sigma) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 2 & 5 & 6 & 3 \end{pmatrix}.$$

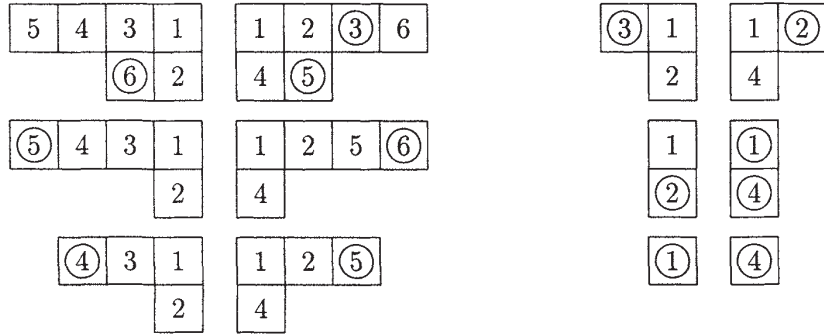


Figure 2: The KRS algorithm

In general we set

$$\text{krs}(\Sigma) = \begin{pmatrix} u_1 & \dots & u_w \\ v_1 & \dots & v_w \end{pmatrix}.$$

In both rows indices may appear several times; however, the indices u_i in the upper row are non-decreasing from left to right, and if $u_i = u_{i+1}$, then $v_i \geq v_{i+1}$ for the indices in the lower row, as is easily checked.

Conversely, if we are given a monomial, then, by arranging its factors in a suitable order, there is always a unique way to represent it as a two rowed array satisfying the condition just given. The reader may check that one can set up an *insertion* algorithm exactly inverting the deletion procedure above (what was deleted last, must be inserted first). In combinatorics one most often uses standard bitableaux for the investigation of sequences (or two row arrays). Then insertion is more important than deletion.

Since insertion and deletion are inverse operations, one obtains

THEOREM 3.1. *The map KRS is a bijection between the set of standard bitableaux on $\{1, \dots, m\} \times \{1, \dots, n\}$ and the monomials of $K[X]$.*

This theorem proves half of part (a) of the straightening law: it is enough to check that every element of $K[X]$ can be written as a linear combination of standard bitableaux. Using the straightening law one can now *extend KRS to a K -linear automorphism* of $K[X]$: with the standard representation $x = \sum a_\Sigma \Sigma$ one sets $\text{KRS}(x) = \sum a_\Sigma \text{KRS}(\Sigma)$.

The automorphism KRS does not only preserve the total degree, but even the $\mathbb{Z}^m \oplus \mathbb{Z}^n$ degree introduced above: in fact, no column or row index gets lost. Note also that KRS it is not a K -algebra isomorphism: it acts as the identity on polynomials of degree 1 but it is not the identity map. It would be interesting to have some insight on the property of KRS as a linear map like, for instance, its eigenvalues and eigenspaces.

REMARK 3.2. We note two important properties of KRS:

- (a) KRS commutes with transposition of the matrix X : Let X' be a $n \times m$ matrix of indeterminates, and let $\tau : K[X] \rightarrow K[X']$ denote the K -algebra isomorphism induced by the substitution $X_{ij} \mapsto X'_{ji}$; then $\text{KRS}(\tau(f)) = \tau(\text{KRS}(f))$ for all $f \in K[X]$. Note that it suffices to prove the equality when f is a standard bitableau. Then the statement follows from [12, Lemma 1.1].
- (b) All the powers Σ^k of a standard bitableau are again standard, and one has

$$\text{KRS}(\Sigma^k) = \text{KRS}(\Sigma)^k.$$

4 KRS INVARIANTS AND GRÖBNER BASES

The power of KRS in the study of Gröbner bases for determinantal ideals was detected by Sturmfels [16]. He applied Schensted's theorem:

THEOREM 4.1. *Let (t_1, \dots, t_w) be the shape of the standard bitableau Σ . Then t_1 is the length of the longest strictly increasing subsequence in the lower row of $\text{krs}(\Sigma)$.*

If $(v_{i_1}, \dots, v_{i_q})$ is a strictly increasing subsequence of the lower row, then the subsequence $(u_{i_1}, \dots, u_{i_q})$ of the upper row must also be strictly increasing. Therefore

$$\text{KRS}(\Sigma) = M \cdot \text{diag}[u_{i_1}, \dots, u_{i_q} \mid v_{i_1}, \dots, v_{i_q}] \tag{*}$$

where M is a monomial $\text{diag}(\delta)$ denotes the product of all the indeterminates in the diagonal of the minor δ .

Once and for all we now introduce a *diagonal term order* on the polynomial ring $K[X]$. With respect to such a term order the initial monomial $\text{in}(\delta)$ is $\text{diag}(\delta)$. There are various choices for a diagonal term order, For example one can take the lexicographic order induced by the total order of the X_{ij} that coincides with the lexicographic order of the (i, j) .

Schensted's theorem implies through its equivalent (*) that for a standard bitableau $\Sigma \in I_t$ there exists a t -minor δ such that

$$\text{in}(\delta) = \text{diag}(\delta) \mid \text{KRS}(\Sigma),$$

and, in particular, $\text{KRS}(\Sigma) \in \text{in}(I_t)$: if $q > t$, then we can simply write

$$\text{diag}[u_{i_1}, \dots, u_{i_q} \mid v_{i_1}, \dots, v_{i_q}] = M'' \text{diag}[u_{i_1}, \dots, u_{i_t} \mid v_{i_1}, \dots, v_{i_t}].$$

Since I_t has a basis of standard bitableaux, it follows that $\text{KRS}(I_t) \subset \text{in}(I_t)$. The K -vector space $\text{KRS}(I_t)$ has the same Hilbert function as I_t with respect to total degree since KRS preserves total degree. But $\text{in}(I_t)$ also has the same Hilbert function as I_t . This implies:

THEOREM 4.2. *The t -minors of X form a Gröbner basis of I_t , and $\text{KRS}(I_t) = \text{in}(I_t)$.*



In fact, the equation $\text{KRS}(I_t) = \text{in}(I_t)$ has just been observed, and if $M \in \text{in}(I_t)$ is a monomial, then it must be of the form $\text{KRS}(\Sigma)$ for some standard bitableau $\Sigma \in I_t$. But then M is divisible by $\text{in}(\delta)$ for some t -minor δ . Exactly this condition must be satisfied for the set of t -minors to form a Gröbner basis. It is worth formulating the idea behind the proof of Theorem 4.2 as a lemma:

LEMMA 4.3. (a) Let I be an ideal of $K[X]$ which has a K -basis, say B , of standard bitableaux, and let S be a subset of I . Assume that for all $\Sigma \in B$ there exists $s \in S$ such that $\text{in}(s) \mid \text{KRS}(\Sigma)$. Then S is a Gröbner basis of I and $\text{in}(I) = \text{KRS}(I)$.

(b) Let I and J be homogeneous ideals such that $\text{in}(I) = \text{KRS}(I)$ and $\text{in}(J) = \text{KRS}(J)$. Then $\text{in}(I) + \text{in}(J) = \text{in}(I + J) = \text{KRS}(I + J)$ and $\text{in}(I) \cap \text{in}(J) = \text{in}(I \cap J) = \text{KRS}(I \cap J)$.

The proof of part (a) has been explained for the special case of $I = I_t$. For (b) one uses

$$\begin{aligned} \text{KRS}(I + J) &= \text{KRS}(I) + \text{KRS}(J) = \text{in}(I) + \text{in}(J) \subseteq \text{in}(I + J), \\ \text{KRS}(I \cap J) &= \text{KRS}(I) \cap \text{KRS}(J) = \text{in}(I) \cap \text{in}(J) \supseteq \text{in}(I \cap J), \end{aligned}$$

and concludes equality from the Hilbert function argument.

DEFINITION 4.4. Let I be an ideal with a standard basis. Then we say that I is *in-KRS* if $\text{in}(I) = \text{KRS}(I)$; if, in addition, the bitableaux $\Delta \in I$ form a Gröbner basis, then I is *G-KRS*.

In slightly different words, an ideal I with a standard basis is *in-KRS* if for each $\Sigma \in I$ there exists $x \in I$ with $\text{KRS}(\Sigma) = \text{in}(x)$; it is *G-KRS* if x can always be chosen as a bitableau. As a consequence of Lemma 4.3 one obtains

LEMMA 4.5. Let I and J be ideals with a basis of standard bitableaux.

(a) If I and J are *G-KRS*, then $I + J$ is also *G-KRS*.

(b) If I and J are *in-KRS*, then $I + J$ and $I \cap J$ are also *in-KRS*.

In general the property of being *G-KRS* is not inherited by intersections as we will see below.

The *KRS* correspondence could be used for the proof of Theorem 4.2 since, by Schensted's theorem, the length of the first row of a standard bitableau is a *KRS* invariant:

DEFINITION 4.6. Let \mathcal{D} be the set of all bitableaux on the matrix X and $F : \mathcal{D} \rightarrow \mathbb{N}$ a function on \mathcal{D} . Then we define a function on the set \mathcal{M} of monomials, also called F , by

$$F(M) = \max\{F(\Delta) : \Delta \in \mathcal{D}, M = \text{in}(\Delta)\}.$$

Of course $F(M)$ is well-defined since there are only finitely many $\Delta \in \mathcal{D}$ with $M = \text{in}(\Delta)$. We say F is a *KRS-invariant* if the following conditions are satisfied:



- (a) $F(\Delta)$ is the minimum of $F(\Sigma)$ where Σ runs through the standard bitableaux in the standard representation of a bitableau Δ ; moreover, if Σ' appears in the standard representation of $x\Delta$ for some $x \in R$, then $F(\Sigma') \geq F(\Delta)$.
- (b) $F(\Sigma) = F(\text{KRS}(\Sigma))$ for all standard bitableaux $\Sigma \in \mathcal{D}$.

If just condition (a) is satisfied, then we say that we say that F is *str-monotone*.

In order to interpret condition (b) combinatorially we write

$$\text{krs}(\Sigma) = \begin{pmatrix} u_1 & \dots & u_w \\ v_1 & \dots & v_w \end{pmatrix}.$$

Then the bitableaux Δ such that $\text{KRS}(\Sigma) = \text{in}(\Delta)$ correspond bijectively to the decompositions of the lower row of $\text{krs}(\Sigma)$ into strictly increasing sequences, called *inc-decompositions*. In fact, if the sequence v_{i_1}, \dots, v_{i_t} is strictly increasing, then the same holds for u_{i_1}, \dots, u_{i_t} , or equivalently, $X_{u_{i_1}v_{i_1}} \cdots X_{u_{i_t}v_{i_t}}$ is the diagonal product of a t -minor. Note that u_{i_1}, \dots, u_{i_t} is always non-decreasing, and therefore $X_{u_{i_1}v_{i_1}} \cdots X_{u_{i_t}v_{i_t}}$ can only be a diagonal product if v_{i_1}, \dots, v_{i_t} is strictly increasing.

Thus condition (b) requires that $F(\Sigma)$, a number associated with the standard bitableau, is encoded in the sequence of integers forming the lower row of $\text{krs}(\Sigma)$.

It is now an easy exercise to show

PROPOSITION 4.7. *Let F be a KRS-invariant, and let k be an integer. Let $I_k(F)$ be the ideal generated by all bitableaux Δ such that $F(\Delta) \geq k$. Then*

- (a) $I_k(F)$ has a standard basis formed by all standard bitableaux Σ such that $F(\Sigma) \geq k$.
- (b) Moreover, $I_k(F)$ is G-KRS.

Part (a) follows immediately from str-monotonicity and part (b) follows from 4.3. In general, it does not suffice to take the standard bitableaux in $I_k(F)$ to obtain a Gröbner basis of $I_k(F)$; we will discuss an example below.

Starting from the ideals $I_k(F)$ and applying 4.5 one can now find new ideals that are G-KRS or at least in-KRS.

We have seen that the length of the first row is a KRS-invariant. In order to apply KRS to a wider class of ideals one has to find other (or more general) KRS invariants. One such family of invariants are the functions γ_t defined as follows. For an integer s and a sequence s_1, \dots, s_w of integers one sets

$$\gamma_t(s) = (s - t + 1)_+ \quad \text{and} \quad \gamma_t(s_1, \dots, s_w) = \sum_{i=1}^w \gamma_t(s_i).$$

Here we have used the notation $(k)_+ = \max(0, k)$. One then defines this function for bitableaux $\Delta = \delta_1 \cdots \delta_w$ by

$$\gamma_t(\Delta) = \gamma_t(|\Delta|).$$

The invariants γ_t are of interest since they describe the symbolic powers of the ideals I_t . Provided the characteristic of the field is 0 or $\geq \min(m, n)$ (we then say K has *non-exceptional* characteristic), all products $I_{t_1} \cdots I_{t_r}$ have a primary decomposition as intersections of such symbolic powers, and can therefore be described in terms of the γ_t :

- (a) $\Delta \in I_t^{(k)} \iff \gamma_t(\Delta) \geq k$;
- (b) the standard bitableaux Σ with $\gamma_t(\Sigma) \geq k$ are a K -basis of $I_t^{(k)}$;
- (c)

$$I_{t_1} \cdots I_{t_r} = \bigcap_{j=1}^s I_j^{(g_j)}, \quad g_j = \gamma_j(t_1, \dots, t_r).$$

See [6, Section 10] and [9]. The straightening law shows that γ_t is str-monotone. In [4] we have proved

THEOREM 4.8. *The functions γ_t are KRS-invariants.*

As a consequence of this theorem and Lemma 4.5 one obtains that all ideals $I_t^{(k)}$ are G-KRS and that all products of ideals of minors are in-KRS if $\text{char } K = 0$ or $\text{char } K \geq \min(m, n)$. Furthermore one can then show that the “initial algebras” of the symbolic and ordinary Rees algebras of the ideals I_t are normal semigroup rings. In particular this implies that these algebras are Cohen–Macaulay. Another object accessible to this approach is the subalgebra A_t of $K[X]$ generated by the t -minors of X . See [4] for a detailed discussion.

EXAMPLES 4.9. (a) We choose $m, n \geq 3$. By the above discussion the ideal $I_2^{(2)}$ is G-KRS. We want to show that the standard bitableaux in $I_2^{(2)}$ do not form a Gröbner basis. The monomial $M = x_{12}x_{23}x_{21}x_{32}$ is the initial term of a bitableau of shape $(2, 2)$ and hence $M \in \text{in}(I_2^{(2)})$. If the standard bitableaux in $I_2^{(2)}$ were a Gröbner basis, then M would be divisible by the initial of a standard bitableau in $I_2^{(2)}$. The standard bitableaux in $I_2^{(2)}$ of degree ≤ 4 have shape $(3), (3, 1)$ and $(2, 2)$ and clearly their initial term cannot divide M .

(b) Suppose that I and J are G-KRS and let Σ be a standard bitableau in $I \cap J$. Then we can find bitableaux $\Delta_1 \in I$ and $\Delta_2 \in J$ such that $\text{KRS}(\Sigma) = \text{in}(\Delta_1) = \text{in}(\Delta_2)$. In general it can happen that $\Delta_1 \notin J$ and $\Delta_2 \notin I$, and $I \cap J$ need not be G-KRS. An example is $I = I_3^{(2)}, J = I_4$ for a matrix of size at least 6×6 . In fact, let $\Sigma = [1\ 3\ 4\ 5 \mid 1\ 2\ 3\ 6][2\ 6 \mid 4\ 5]$. (This is the example considered in Section 3). Then $\Sigma \in J$ is obvious, and $\Sigma \in I$ since $\gamma_2(\Sigma) \geq 2$. Since $I \cap J$ is in-KRS, it follows that $\text{KRS}(\Sigma) = X_{14}X_{21}X_{32}X_{45}X_{56}XX_{63} \in \text{in}(I \cap J)$. It is however impossible to write this monomial as the initial monomial of a bitableau Δ in $I \cap J$. The bitableaux of degree 6 in $I \cap J$ are those of shapes $(6), (5, 1)$, and $(4, 2)$. (By Schensted’s theorem, only the last shape would be possible.)

Let T be a new indeterminate and define the ideal J in the extended polynomial ring $K[X][T]$ by

$$J = I_m + I_{m-1}T + \cdots + I_1T^{m-1} + (T^m);$$

we assume that $m \leq n$. This ideal and its Rees algebra $\bigoplus_{i=0}^{\infty} J^i$ is fundamental for the generic case of MacPherson’s graph construction; see [5]. If one “expands” the power J^k into a “polynomial” in T , then the “coefficient” of T^{km-d} is

$$J(k, d) = \sum I_0^{e_0} I_1^{e_1} \cdots I_m^{e_m},$$

$$e_0 + e_1 + \cdots + e_m = k, \quad e_1 + 2e_2 + \cdots + me_m = d.$$

For a non-increasing sequence s_1, \dots, s_w of non-negative integers let us define

$$\alpha_k(s_1, \dots, s_w) = \sum_{i=1}^k s_i$$

where $s_i = 0$ if $i > w$. Then we can set

$$\alpha_k(\Delta) = \alpha_k(|\Delta|)$$

for every bitableau Δ . The straightening law shows that α_k is str-monotone, and it follows easily that

- (a) $\Delta \in J(k, d) \iff \alpha_k(\Delta) \geq d$;
- (b) the standard bitableaux Σ with $\alpha_k(\Sigma) \geq d$ are a K -basis of $J(k, d)$.

THEOREM 4.10. *The functions α_k are KRS-invariants.*

An analysis of α_k in terms of inc-decompositions shows that $\alpha_k(KRS(\Sigma)) \geq d$ if and only if the lower sequence of $\text{krs}(\Sigma)$ contains a subsequence of length d that itself can be decomposed into k increasing subsequences. Thus Theorem 4.10 is just a re-interpretation of Greene’s theorem [11]: $\alpha_k(\Sigma)$ is the maximal length of a subsequence that has an inc-decomposition into k parts.

For the “determinantal” consequences of Theorem 4.10 we refer the reader to [5]. The relationship between the KRS invariants γ_t and α_k is analyzed in Section 6.

5 IDEALS DEFINED BY SHAPE

We say that an ideal $I \subset K[X]$ is *defined by shape* if it is generated as an ideal by a set of bitableaux, and, moreover, it depends only on $|\Delta|$ whether a bitableau Δ belongs to I . In this section we want to characterize the ideals defined by shape in the case in which the characteristic of K is big enough. In particular we will see that all these ideals are in-KRS.

The following “balancing lemma” is a crucial argument; it is a simplified version of [6, (10.10)].

LEMMA 5.1. *Let π and ρ be minors of X , and set $u = |\rho|$, $v = |\pi|$ (we include the case $\pi = 1$, in which $u = 0$). Suppose that $u < v$ and $\text{char } K = 0$ or $\text{char } K > \min(u + 1, m - (u + 1), n - (u + 1))$. Then $\pi\rho \in I_{u+1}I_{v-1}$.*



The case in which $u = 0$ is just Laplace expansion. In general the lemma says that a product $\pi\rho$ of minors can be expressed as a linear combination of minors that are “more balanced” in size. By repeated application of the lemma we see that $\pi\rho$ is even a linear combination of products $\delta\varepsilon$ such that $|\delta| + |\varepsilon| = |\pi| + |\rho|$ and $|\delta| \leq |\varepsilon| \leq |\delta| + 1$.

The group $\text{GL} = \text{GL}(m, K) \times \text{GL}(n, K)$ operates as a group of linear substitutions on $R = K[X]$ in a natural way: for $M \in \text{GL}(m, K)$ and $N \in \text{GL}(n, K)$ one substitutes X_{ij} by the corresponding entry of MXN^{-1} . Therefore R is an interesting object for the representation theory of GL , and representation theory offers another approach to the theory of determinantal rings. It is clear that an ideal defined by shape is GL -stable since each element $g \in \text{GL}$ transforms a minor into a linear combination of minors of the same size.

Let σ be a shape. Then the ideal $I^{(\sigma)}$ is generated by all bitableaux Δ with $\gamma_t(\Delta) \geq \gamma_t(\sigma)$ for all t (see [6, Section 11]). If all these inequalities hold, then

$$|\Delta| \geq \sigma.$$

If even $|\Delta|_k \geq \sigma_k$ for all k , then we write $|\Delta| \supset \sigma$.

The ideals $I^{(\sigma)}$ are evidently defined by shape. By definition

$$I^{(\sigma)} = \bigcap_t I_t^{(\gamma_t(\sigma))}.$$

Therefore $I^{(\sigma)}$ has a standard basis and is in-KRS.

The following theorem should have been contained in [6, Section 11].

THEOREM 5.2. *Suppose that $\text{char } K = 0$. Then the following are equivalent for an ideal $I \subset K[X]$:*

- (a) *I is defined by shape;*
- (b) *I is a sum of ideals of type $I^{(\sigma)}$.*
- (c) *I has a standard basis, and I is stable under the action of GL on $K[X]$.*

Proof. (a) \Rightarrow (b): I is obviously contained in the sum of all ideals $I^{(|\Delta|)}$ where Δ runs through the generators of I .

On the other hand, let Σ be a (standard) bitableau contained in $I^{(|\Delta|)}$, that is, $|\Sigma| \geq |\Delta|$. If even $|\Sigma| \supset |\Delta|$, then we can apply Laplace expansion and write Σ as a linear combination of bitableaux of the same shape as Δ . Suppose that $|\Sigma| \not\supset |\Delta|$, and let k be the smallest index such that $|\Sigma|_k < |\Delta|_k$. Then there must be an index $j < k$ such that $|\Sigma|_j > |\Delta|_j$. Now one applies the balancing lemma above, increasing the k -th row at the expense of the j -th. After finitely many balancing steps we have written Σ as a K -linear combination of bitableaux Ξ such that $|\Xi| \supset |\Delta|$.

(b) \Rightarrow (a): This is evident, as well as (b) \Rightarrow (c).

(c) \Rightarrow (b): We have to show that $\Xi \in I$ for all standard bitableaux Ξ such that $|\Xi| \geq |\Sigma|$ for some standard bitableau $\Sigma \in I$.



Set $\sigma = |\Sigma|$. Suppose first that $|\Xi| = |\Sigma|$. By [6, (11.10)] there is a decomposition of GL stable subspaces

$$I^{(\sigma)} = M_\sigma \oplus I_{>}^{(\sigma)}$$

where $I_{>}^{(\sigma)}$ is generated by all standard bitableaux Θ such that $|\Theta| > \sigma$; in this decomposition M_σ is irreducible and the unique GL-stable complement of $I_{>}^{(\sigma)}$ in $I^{(\sigma)}$. Thus we can write

$$\Xi = x + y$$

where $x \in M_\sigma, y \in I_{>}^{(\sigma)}$. Since $\Xi \notin I_{>}^{(\sigma)}$, we deduce that $x \neq 0$.

For $\Xi = \Sigma$ it follows that $M_\sigma \subset I$. In fact, I has a unique decomposition as a direct sum of irreducible GL-modules, and these are exactly the M_τ ; since the projection to M_σ is non-trivial, it must appear in the decomposition of I .

For general Ξ it now follows that $x \in I$, and therefore all the standard bitableaux in the standard representation of x must belong to I . However, since $y \in I_{>}^{(\sigma)}$, Ξ must appear in this standard representation.

Suppose now that $|\Xi| > \sigma$. If we find some standard bitableau Γ of shape $|\Xi|$ such that $\Gamma \in I$, then the argument just given shows that $\Xi \in I$. Since we can connect σ and $|\Xi|$ by a chain of shapes with respect to the partial order \leq , it is enough to consider the case in which $|\Xi|$ is an upper neighbor of σ . One obtains the upper neighbors by either inserting a new box below the bottom of the diagram (and thereby increasing γ_1 by 1) or by removing an "outer corner" box of the Young diagram of shape σ and inserting it at an "inner corner" in such a way that the box travels one row up or one column to the right. (At the end of the first row is also an inner corner, provided its length is $< \min(m, n)$). Figure 3 illustrates the three cases.

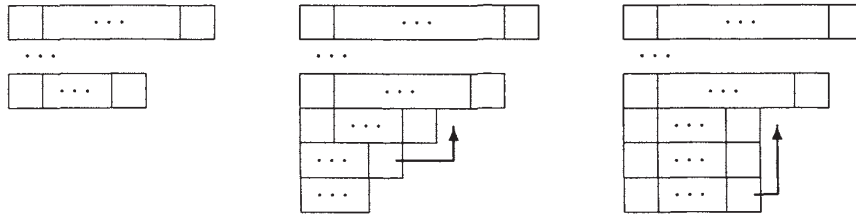


Figure 3: Upper neighbors with respect to \leq

The case in which a new box is added, is trivial. In fact, we fill it with X_{mn} , and $X_{mn}\Sigma$ is standard of the right shape.

We now assume that a box travels one row up, say from row k to row $k - 1$. With $\sigma = (s_1, \dots, s_w)$ let $p = s_k$ and $q = s_{k-1}$. One forms the standard bitableau $\Theta = \theta_1 \cdots \theta_w$ where

$$\begin{aligned} \theta_j &= [1, \dots, s_j | 1, \dots, s_j], & j \neq k, \\ \theta_k &= [1, \dots, p - 1, q + 1 | 1, \dots, p - 1, q + 1] \end{aligned}$$

of shape σ . Since Θ is a standard bitableau of shape σ , it belongs to I , as was shown above. Note that $s_{k+1} < p < q < s_{k-2}$ (where $s_{k+1} = 0$ if $k = w$ and $s_{k-2} = \infty$ if $k = 2$).

Now we apply the cyclic permutation $\pi \in (p\ p+1 \dots q\ q+1)$ to both the rows and columns of the matrix X ; the transformation π belongs to GL . Therefore $\pi(\Theta) \in I$. All the factors of Θ except θ_{k-1} and θ_k are invariant under π , whereas

$$\pi(\theta_{k-1}) = [1, \dots, p-1, p+1, \dots, q+1 \mid 1, \dots, p-1, p+1, \dots, q+1] \quad (1)$$

$$\pi(\theta_k) = [1, \dots, p-1, p \mid 1, \dots, p-1, p]. \quad (2)$$

If we straighten this product and multiply its standard presentation with the remaining factors of Θ , then we obtain the standard representation of $\pi(\Theta)$. Therefore it is enough that a standard bitableau of shape $(q+1, p-1)$ appears in the standard representation of $\pi(\theta_{k-1})\pi(\theta_k)$. (Whereas the proof of (a) \Rightarrow (b) is based on “balancing”, we now need the “unbalancing” effect of straightening.) Mapping all the indeterminates X_{ij} with $i \neq j$, $i < p$ or $j < p$, to 0 and $X_{11}, \dots, X_{p-1, p-1}$ to 1, one reduces the claim to the assertion that in the standard representation of $[1 \mid 1][2, \dots, r \mid 2, \dots, r]$ with $r \geq 3$ the minor $[1, \dots, r \mid 1, \dots, r]$ shows up, and that is immediate from Laplace expansion.

The case in which the box travels one column to the right is similar and left to the reader. Essentially it is the case in which σ consists of a single column. \square

One should note that the implications (b) \Rightarrow (a) and (b) \Rightarrow (c) are true over arbitrary fields (actually, over all rings of coefficients), whereas (a) \Rightarrow (b) needs only that $\text{char } K > \min(m, n)$. The implication (c) \Rightarrow (b) uses the hypothesis that K has characteristic 0 more profoundly: the ideal $I_1^{p+1} + (X_{11}^p, \dots, X_{mn}^p)$ satisfies (c) if $\text{char } K = p > 0$, but is not a sum of ideals $I^{(\sigma)}$ (provided that X is not just a 1×1 matrix).

COROLLARY 5.3. *Suppose that K is a field of characteristic 0. Then all GL -stable ideals that have a standard basis are in-KRS.*

This follows from Lemma 4.5 since the ideals $I^{(\sigma)}$ are in-KRS.

Note that there are ideals with a standard basis which are not in-KRS. For instance, let Σ be standard bitableau with $\text{KRS}(\Sigma) \neq \text{in}(\Sigma)$ and set $d = \deg \Sigma$. Then $I = (\Sigma) + (X_{ij} : 1 \leq i \leq m, 1 \leq j \leq n)^{d+1}$ has a standard basis and $\text{KRS}(I) \neq \text{in}(I)$. As we have seen in the previous section, there are in-KRS ideals that are not G-KRS. The G-KRS ideals among those considered in Corollary 5.3 will be characterized in the next section.

6 BASIC KRS-INVARIANTS

In this section we want to show that the functions α_k are basic KRS-invariants, as far as functions $F : \mathcal{D} \rightarrow \mathbb{N}$ are considered that depend only on shape. First we prove a converse of Proposition 4.7:

PROPOSITION 6.1. *Let $F : \mathcal{D} \rightarrow \mathbb{N}$ be a str-monotone function, and let k be an integer. Let $I_k(F)$ be the ideal generated by all bitableaux Δ such that $F(\Delta) \geq k$. Then the following hold:*



- (a) $I_k(F)$ has a standard basis, and $\Delta \in I_k(F)$ (if and) only if $F(\Delta) \geq k$;
- (b) if $I_k(F)$ is G-KRS for all k , then F is a KRS-invariant.

Proof. (a) It follows immediately from the definition of str-monotonicity that $I_k(F)$ is the K -vector space generated by the bitableaux Δ with $F(\Delta) \geq k$, and that the standard bitableaux $\Sigma \in I_k(F)$ form a K -basis.

(b) We have to show that $F(\Sigma) = F(\text{KRS}(\Sigma))$ for every standard bitableau Σ . First note that $F(\Sigma) \leq F(\text{KRS}(\Sigma))$. Set $k = \text{KRS}(\Sigma)$. Since by assumption $I_k(F)$ is G-KRS we have that $\text{KRS}(\Sigma) = \text{in}(\Delta)$ for some $\Delta \in I_k(F)$ and consequently $F(\text{KRS}(\Sigma)) \geq k$. It follows that for every t one has

$$\text{KRS}(I_t(F)) \subseteq \langle M : F(M) \geq t \rangle \subseteq \text{in}(I_t(F)) = \text{KRS}(I_t(F))$$

and hence

$$\text{KRS}(I_t(F)) = \langle M : F(M) \geq t \rangle.$$

Now let M be a monomial and set $t = F(M)$. Then $\Sigma = \text{KRS}^{-1}(M)$ is in $I_t(F)$ which proves that $F(\Sigma) \geq F(\text{KRS}(\Sigma))$. \square

Now we can show that the KRS-invariance of the functions γ_t follows from that of the α_k :

PROPOSITION 6.2. For all t , $1 \leq t \leq \min(m, n)$ and all $r \geq 1$ one has

$$I_t^{(r)} = \sum_{k \geq 1} J(k, r + k(t - 1)).$$

Proof. Let $\Sigma \in I_t^{(r)}$ be a (standard) bitableau, and suppose that k is the biggest index such that $|\Sigma|_k \geq t$. Then obviously $\Sigma \in J(k, r + (t - 1))$.

The verification of the inclusion \supseteq is likewise simple. \square

Since by Greene's theorem the ideals $J(k, d)$ are G-KRS, it follows immediately that the ideals $I_t^{(r)}$ are G-KRS. In conjunction with Proposition 6.1 we therefore obtain that the γ_t are KRS-invariants. From hindsight, Theorem 4.8 is an easy consequence of Greene's theorem.

The most difficult part of [4] is the proof that the ideals I_t^r are G-KRS in non-exceptional characteristics. Actually $I_t^r = J(r, rt)$ so that the property of being G-KRS is no longer surprising for I_t^r .

Not every shape defined, G-KRS ideal is the sum of ideals $J(k, d)$. As an example one can take $I_1^4 \cap I_3 = I_1 I_3$. Another exception occurs for square matrices. For example, if $m = n = 4$, then $I_2 I_4$ is G-KRS, but it loses this property for bigger matrices. Nevertheless, the next theorem shows that the functions α_k are truly basic KRS-invariants:

THEOREM 6.3. Let I be a shape defined ideal. If I is G-KRS, then it is the sum of ideals of type $J(k, d) \cap I_1^u$ and, if $m = n$, $(J(k, d) \cap I_1^u) I_n^v$.



Proof. Let $\Sigma \in I$ be a (standard) bitableau. Suppose that

$$\sigma = |\Sigma| = (s_1, \dots, s_k, 1, \dots, 1)$$

with $s_k \geq 2$, and $s_1 < \max(m, n)$ (equality can only occur if $m = n$). Set $d = \alpha_k(\sigma)$ and $D = \deg(\Sigma)$. Then $J = I(k, d) \cap I_1^D$ is the smallest ideal of type $J(k, d) \cap I_1^u$ containing Σ .

Among the shapes of the elements in the standard basis of J there exists a unique element that is minimal with respect to the partial order defined by the functions γ_t (see Section 5). In fact let $u = \lfloor d/k \rfloor$ and $r = d - uk$. Then the smallest element is

$$\theta = (u + 1, \dots, u + 1, u, \dots, u, 1, \dots, 1)$$

where $u + 1$ appears r times, u appears $d - r$ times, and 1 as often as in σ . We have $J = I^{(\theta)}$. Therefore, if $\sigma = \theta$, the ideal J is contained in I , as follows from Theorem 5.2.

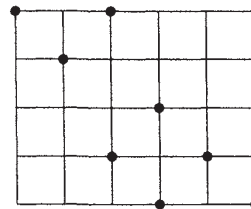
Now suppose that $\sigma > \theta$. Then it is enough to show that there exists a standard bitableau Σ of shape σ such that $|\Delta| < \sigma$ whenever $\text{in}(\Delta) = \text{KRS}(\Sigma)$. Since we have $|\Delta| \leq \sigma$ from the KRS invariance of the functions γ_t (or α_k) it really suffices to find Σ of shape σ such that $|\Delta| \neq \sigma$ for all Δ with $\text{in}(\delta) = \text{KRS}(\Sigma)$.

Instead of constructing Δ directly, we find a monomial M such that $\text{KRS}^{-1}(M)$ has the desired shape, but M cannot be written as $\text{in}(\Delta)$ where $|\Delta| = \sigma$. The shape of $\text{KRS}^{-1}(M)$ can be controlled via the γ or α functions.

Let us first consider $\sigma = (s_1, s_2)$; we set $p = s_1, q = s_2$. Then $\max(m, n) > p \geq q + 2 > 1$ and $\min(m, n) \geq p$. It is harmless to assume $n = \max(m, n)$. With $r = p - q + 2$ let

$$M = X_{11}X_{22}X_{34} \cdots X_{p,p+1} \cdot X_{13} \cdot X_{r3}X_{r+1,4} \cdots X_{p,q+1}.$$

For $p = 5$ and $q = 3$ this monomial has the following “picture”:



The reader can check that $\text{KRS}^{-1}(M)$ indeed as shape (p, q) . However, it is not possible to decompose M into a product $\text{in}(\delta_1)\text{in}(\delta_2)$ where $|\delta_1| = p, |\delta_2| = q$.

In the general case one must multiply M with suitable factors. These are not hard to find. □

7 INITIAL IDEAL OF SYMBOLIC POWERS AND SYMBOLIC POWERS OF THE INITIAL IDEAL

Let S be a polynomial ring and τ a term order on S . Given an ideal J of S and an integer b one denotes by $J_{\leq b}$ the intersection of all the primary components

of height $\leq b$ in a primary decomposition of J ($J_{\leq b}$ is independent of the chosen primary decomposition). One knows that $J_{\leq b} = \{f : \text{height}(J : f) > b\}$ and that

$$\text{in}(J_{\leq b}) \subseteq \text{in}(J)_{\leq b}, \tag{1}$$

see [17]. If J has height c , then we define the k -symbolic power $J^{(k)}$ of J to be:

$$J^{(k)} = (J^k)_{\leq c}.$$

Note that $J^{(1)} = J$ if and only if J has no embedded primes and all the minimal primes have height c . Now by (1)

$$\text{in}(J^{(k)}) \subseteq \text{in}(J^k)_{\leq c}.$$

On the other hand, since $\text{in}(J)^k \subseteq \text{in}(J^k)$, we have

$$\text{in}(J)^{(k)} = (\text{in}(J)^k)_{\leq c} \subseteq \text{in}(J^k)_{\leq c}.$$

Summing up, there are inclusions

$$\text{in}(J^{(k)}) \subseteq \text{in}(J^k)_{\leq c} \supseteq \text{in}(J)^{(k)}. \tag{2}$$

These inclusions are in general strict. In this section we will show that they are equalities when J is the determinantal ideal I_t and τ is a diagonal term order.

THEOREM 7.1. *In non-exceptional characteristics we have*

$$\text{in}(I_t^{(k)}) = \text{in}(I_t^k)_{\leq c} = \text{in}(I_t)^{(k)}$$

where c is the height of I_t , i.e. $c = (m - t + 1)(n - t + 1)$.

The initial ideal $\text{in}(I_t)$ of I_t is the square free monomial ideal generated by the diagonals of the t -minors, simply called t -diagonals in the sequel. Hence it is the Stanley-Reisner ideal of the simplicial complex

$$\Delta_t = \{A \subseteq \{1, \dots, m\} \times \{1, \dots, n\} : A \text{ does not contain } t\text{-diagonals}\}.$$

Denote by \mathbf{F}_t the set of the facets of Δ_t . Then

$$\text{in}(I_t) = \bigcap_{F \in \mathbf{F}_t} P_F$$

where P_F denotes the ideal generated by the x_{ij} with $(i, j) \notin F$. The elements of \mathbf{F}_t are described in [12] in terms of families of non-intersecting paths. It turns out that Δ_t is a pure (even shellable) simplicial complex. Since the powers of the ideals P_F are P_F -primary, it follows that

$$\text{in}(I_t)^{(k)} = \bigcap_{F \in \mathbf{F}_t} P_F^k.$$

We start by proving:



PROPOSITION 7.2. *We have*

$$\text{in}(I_t^{(k)}) = \bigcap_{F \in \mathbf{F}_t} P_F^k.$$

We have already mentioned in Section 4 that $I_t^{(k)}$ is G-KRS, and in particular the initial ideal $\text{in}(I_t^{(k)})$ of $I_t^{(k)}$ is generated by the monomials M with $\gamma_t(M) \geq k$. Now a monomial $M = \prod_{i=1}^s x_{a_i b_i}$ is in P_F^k if and only if the cardinality of $\{i : (a_i, b_i) \notin F\}$ is $\geq k$. Equivalently, M is in P_F^k if and only if the cardinality of $\{i : (a_i, b_i) \in F\}$ is $\leq \deg(M) - k$. If we set

$$w_t(M) = \max\{|A| : A \subseteq [1, \dots, s] \text{ and } \{(a_i, b_i) : i \in A\} \in \Delta_t\}$$

then we have that a monomial M is in $\bigcap_{F \in \mathbf{F}_t} P_F^k$ if and only if $w_t(M) \leq \deg(M) - k$, that is, $\deg(M) - w_t(M) \geq k$. Now Proposition 7.2 follows from:

LEMMA 7.3. *Let M be a monomial. Then $\gamma_t(M) + w_t(M) = \deg(M)$.*

We reduce this lemma to a combinatorial statement on sequences of integers. Given such a sequence b we define $w_t(b)$ to be the cardinality of the longest subsequence of b which does not contain an increasing subsequence of length t , that is,

$$w_t(b) = \max\{\text{length}(c) : c \text{ is a subsequence of } B \text{ and } \gamma_t(c) = 0\}.$$

Let $M = \prod_{i=1}^s x_{a_i b_i}$ be a monomial. We may order the indices such that $a_i \leq a_{i+1}$ for every i and $b_{i+1} \geq b_i$ whenever $a_i = a_{i+1}$. (We have already considered this rearrangement in Section 3.) Then the t -diagonals dividing M correspond to increasing subsequences of length t of the sequence b , and $w_t(M) = w_t(b)$. Since $w_t(M)$ depends only on the sequence b we may assume that $a_i = i$ for every i . Then, by exchanging the role between the a_i 's and the b_i 's we may also assume that the b_i are distinct integers (see Remark 3.2). Summing up, it suffices to show that:

LEMMA 7.4. *One has $\gamma_t(b) + w_t(b) = \text{length}(b)$ for every sequence b of distinct integers.*

Proof. Let P be the tableau obtained from b by the Robinson-Schensted insertion algorithm. We have already discussed the first part of Greene's theorem, namely that the sum $\alpha_k(P)$ of the lengths of the first k rows of P is the length of the longest subsequence of b that has a decomposition into k increasing subsequences. But the theorem contains a second (dual) assertion: the sum $\alpha_k^*(P)$ of the lengths of the first k columns of P is the length of the longest subsequence of b that can be decomposed into k decreasing subsequences.

It follows that a sequence a has no increasing subsequence of length t if and only if it can be decomposed into $t - 1$ decreasing subsequences. Then $w_t(b)$ is equal to the maximal length of a subsequence of b which can be decomposed into $t - 1$ decreasing subsequences.

Therefore $w_t(b) = \alpha_{t-1}^*(P)$. On the other hand, by Theorem 4.8 we know that $\gamma_t(b)$ is equal to $\gamma_t(P)$ which is the sum of the length of the columns of P of index $\geq t$. Therefore $\gamma_t(b) + w_t(b)$ is equal to the number of entries of P which is the length of b . \square



We know [4, Thm. 3.5] that $\text{in}(I_t^k) = \bigcap_{j=1}^t \text{in}(I_j^{k(t+1-j)})$ (in non-exceptional characteristic) and hence, taking into consideration Proposition 7.2, we have:

$$\text{in}(I_t^k) = \bigcap_{j=1}^t \bigcap_{F \in \mathbf{F}_j} P_F^{k(t+1-j)}. \tag{3}$$

Since the powers of the ideal P_F are P_F -primary we have that (3) is indeed a primary decomposition of $\text{in}(I_t^k)$. Hence $\text{in}(I_t^k)_{<c}$ is equal to $\bigcap_{F \in \mathbf{F}_t} P_F^k$. This concludes the proof of 7.1.

REMARK 7.5. (a) Theorem 7.1 can be interpreted as a description of the normal semigroup of monomials in the initial algebra of the Rees algebra of I_t in terms of linear inequalities.

(b) The argument above shows also that $\text{in}((I_t^k)_{\leq b}) = \text{in}(I_t^k)_{\leq b}$ for every integer b . But in general $(\text{in}(I_t^k)_{\leq b})$ is strictly smaller than $\text{in}(I_t^k)_{\leq b}$. This is because not all the variables appear in the generators of $\text{in}(I_t)$ while the maximal ideal is associated to I_t^k (and hence to $\text{in}(I_t^k)$) for large k .

QUESTION 7.6. What is a primary decomposition of the powers of $\text{in}(I_t)$? Is the Rees algebra of $\text{in}(I_t)$ Cohen-Macaulay? Is it normal?

8 COGENERATED IDEALS

As before, let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates. We consider the set of minors of X equipped with the usual partial order that has been introduced in Section 2. Let $\delta = [a_1, a_2, \dots, a_r | b_1, b_2, \dots, b_r]$ be a minor of X . One defines $I(\delta, X)$ to be the ideal of $K[X]$ generated by all the minors μ such that $\mu \not\leq \delta$, i.e.

$$I(\delta, X) = (\mu : \mu \not\leq \delta).$$

The ideal $I(\delta, X)$ is said to be the ideal cogenerated by δ . For general facts about the ideals cogenerated by minors we refer the reader to [6]. We just recall that $I(\delta, X)$ is a prime ideal and that it has a standard basis. Namely the set

$$B(\delta) = \{\Sigma : \Sigma = \sigma_1 \cdots \sigma_w \text{ is a standard bitableau and } \sigma_1 \not\leq \delta\}$$

is a K -vector space basis of $I(\delta, X)$. Herzog and Trung have shown in [12, Theorem 2.4] that the natural generators of $I(\delta, X)$ (i.e. the minors μ such that $\mu \not\leq \delta$) form a Gröbner basis of $I(\delta, X)$ with respect to the diagonal term order. Their argument makes use of the KRS correspondence and boils down to the study of the KRS image of the elements of $B(\delta)$. We will see that this can be rephrased in terms of properties of a suitable γ -function associated to δ . We will henceforth denote its value on the bitableau Δ by $\gamma_\delta(\Delta)$. We start by defining $\gamma_\delta(\mu)$ for a single minor $\mu = [c_1, \dots, c_k | d_1, \dots, d_k]$, namely

$$\gamma_\delta(\mu) = \max\{(i - j + 1)_+ : 1 \leq i \leq k, 1 \leq j \leq r + 1 \text{ and } (c_i < a_j \text{ or } d_i < b_j)\}$$

where, by definition, $a_{r+1} = b_{r+1} = \infty$. Then we extend, by linearity, the γ_δ -function to product of minors, that is, if $\Delta = \mu_1 \cdots \mu_h$ is a product of minors, then

$$\gamma_\delta(\Delta) = \sum_{i=1}^h \gamma_\delta(\mu_i).$$

The standard bitableaux Σ such that $\gamma_\delta(\Sigma) \neq 0$ are exactly the elements of $B(\delta)$. Note that if one takes $\delta = [1, \dots, t-1 \mid 1, \dots, t-1]$ then $I(\delta, X) = I_t$ and $\gamma_\delta(\mu) = \gamma_t(\mu)$. We may extend, as we have done in Section 4, the definition of the γ_δ -function also to ordinary monomials by setting:

$$\gamma_\delta(M) = \max\{\gamma_\delta(\Delta) : \Delta \text{ is a bitableau and } \text{in}(\Delta) = M\}.$$

In terms of the γ_δ -function Herzog and Trung [12, Lemma 1.2] proved

LEMMA 8.1. *Let Σ be a standard bitableau. Then*

$$\gamma_\delta(\Sigma) \neq 0 \Rightarrow \gamma_\delta(\text{KRS}(\Sigma)) \neq 0$$

and this implies that

THEOREM 8.2. *The ideal $I(\delta, X)$ is G-KRS.*

Note that from 8.2 one has that

$$\gamma_\delta(\Sigma) \neq 0 \iff \gamma_\delta(\text{KRS}(\Sigma)) \neq 0.$$

There are many natural questions concerning the function γ_δ and related ideals. For instance:

QUESTIONS 8.3. Let $J(\delta, k) = I_k(\gamma_\delta)$, that is, the ideal generated by the bitableaux Δ such that $\gamma_\delta(\Delta) \geq k$, and let $B(\delta, k)$ be the set of the standard bitableaux Σ with $\gamma_\delta(\Sigma) \geq k$.

- (a) Is γ_δ a KRS-invariant?
- (b) Is $B(\delta, k)$ a basis of $J(\delta, k)$? i.e. is γ_δ str-monotone?
- (c) Is $J(\delta, k)$ equal to $I(\delta, X)^{(k)}$? The inclusion $J(\delta, k) \subseteq I(\delta, X)^{(k)}$ holds since the symbolic powers form a filtration and for a single minor μ is not difficult to see that $\mu \in I(\delta, X)^{(k)}$ where $k = \gamma_\delta(\mu)$.
- (d) Is $\text{in}(I(\delta, X)^{(k)}) = \text{in}(I(\delta, X))^{(k)}$?

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