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Universally Koszul Algebras Defined by Monomials.

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Let K be a field and let $R = \bigoplus_{i \in N} R_i$ be a homogeneous K-algebra, that is, an algebra of the form $K[x_1, \ldots, x_n]/I$ where I is a homogeneous ideal. The minimal R-free resolution of a graded R-module M is said to be linear if the matrices that represent the maps of the resolution have entries of degree 1. Recall that R is said to be Koszul if K has a linear R-free resolution. More generally, one says that R is universally Koszul (uk for short) if all the ideals of R generated by elements of degree 1 have a linear R-free resolution. For an updated survey on Koszul algebras we refer the reader to the recent paper of Fröberg [F]. For generalities on uk algebras we refer the reader to [C].

Our goal is to classify the uk algebras defined by monomials. We recall first a few facts. Given two homogeneous K-algebras $R = K[x_1, \ldots, x_n]/I$ and $S = K[y_1, \ldots, y_m]/J$ the fiber product $R \circ S$ of R and S is $K[x_1, \ldots, x_n, y_1, \ldots, y_m]/H$ where $H = I + J + (x_i y_j\colon i = 1, \ldots, n \text{ and } j = 1, \ldots, m)$. One has [Lemma 1.6, C]:

LEMMA 1. (*a*) *A polynomial extension* $R[x]$ *of an algebra* R *is uk if and only if* R *is uk.*

(*b*) *The fiber product* $R \circ S$ *of algebras* R *and* S *is uk if and only if* R *and* S *are both uk.*

(*c*) *If* R *is uk and* I *is an ideal of* R *generated by elements of degree* 1 *then* R/I *is uk.*

Lemma 1.5 and Proposition 2.2 in [C] give two sufficient conditions for an algebra to be uk:

LEMMA 2. *Let* R *be a homogeneous algebra.*

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(*a*) *Assume that for every non-zero element z of degree* 1 *in R one has that the algebra $R/(z)$ is uk and that the ideal* $0: z$ *is generated by elements of degree* 1. *Then R is uk.*

(*b*) *Assume that for every non-zero element z of degree* 1 *in R one has hat*

$$\{x \in R_1 : xz = 0\}\, R_1 = R_2.$$

Then R is uk.

Let I be an ideal generated by monomials of degree 2 in a set of variables X. The restriction of I to a subset $Y \subset X$ is the ideal J generated by those monomial generators of I which involve only elements of the set Y. If $R = K[X]/I$ and H is the ideal of R generated by the elements in the set $X \backslash Y$, then $R/H = K[Y]/J$. Hence, by Lemma 1(*c*), if I defines a uk algebra, then its restrictions define uk algebras as well.

Given an integer $n \geqslant 0$ let us denote by $H(n)$ the algebra $K[x_1, \ldots, x_n]/I$ where $I = (x_1, \ldots, x_{n-1})^2 + (x_n^2)$. Note that $H(0)$ is simply K and $H(1) = K[x_1]/(x_1)^2$. One has:

LEMMA 3. (*a*) *$H(n)$ is uk for all n.*

(*b*) *Let K be a field of characteristic* 2 *and $R = K[x_1, \ldots, x_n]/I$ be such that $x_i^2 \in I$ for all i. Let x be an indeterminate. Then R is uk if and only if $R[x]/(x^2)$ is uk.*

PROOF. To prove (*a*) we apply the criterion (*b*) of Lemma 2 to $R = H(n)$. To this end, let $z = a_1 x_1 + \ldots + a_{n-1} x_{n-1} + a_n x_n$ be a non-zero element of degree 1 in R and let V be the degree 1 part of $0: z$. Note that if $a_n = 0$ then V contains $x_1, \ldots, x_{n-1}$ and if $a_n \neq 0$ then V contains $a_1 x_1 + \ldots + a_{n-1} x_{n-1} - a_n x_n$. This is enough to conclude that $R_1 V = R_2$.

To prove (*b*) set $S = R[x]/(x^2)$. Since $S/(x) = R$, the «if» part follows from Lemma 1(*c*). For the other implication we apply the criterion (a) of Lemma 2 and argue by induction on n. Let z be an element of degree 1 is S, say $z = L_1 + ax$ with L_1 an element of degree 1 in R and $a \in K$. We have to show that $S/(z)$ is uk and that $0 : z$ is generated by elements of degree 1. We discuss first the case $a = 0$. Then $S/(z) = R/(L_1)[x]/(x^2)$ and by induction we know that this ring is uk. Furthermore $0 : z = (0 :_R L_1)S$ and $(0 :_R L_1)$ is generated in degree 1 since R is uk. Now assume $a \neq 0$. We may assume $a = 1$. Then $S/(z) = R/(L_1^2) = R$ since, by assumption, $L_1^2 = 0$ in R and hence $S/(z)$ is uk. We show now that $0 : z = (z)$. As $z^2 = 0$ in S the inclusion $\supset$ holds. For the other inclusion let $f \in S$ be an element in $0 : z$. Clearly f can be written (in a unique way) as $f = h + xg$ with $h, g \in R$. Since $fz = 0$ we have that $h + gL_1 = 0$. That is, $h = gL_1$ and then $f = gz$. ■

We have also

LEMMA 4. *Let I be one of the following ideals:*

(1) (xy, z^2),

(2) (x^2, xy, z^2),

(3) *a monomial ideal whose squarefree generators are xy, zt,*

(4) *a monomial ideal whose squarefree generators are xy, yz, zt.*

Then the algebra R defined by I is not uk. Furthermore the same conclusion holds if the characteristic of the base field is $\neq 2$ and I is equal to

(5) (x^2, y^2, z^2).

PROOF. In the cases (1) and (2) we claim that (the class of) xz is a minimal generator of $0 : (y + z)$ in R. This implies that R is not uk. That $xz \in 0 : (y + z)$ is clear. It is easy to see that there are no elements of degree 1 in $0 : (y + z)$. Hence xz is a minimal generator of $0 : (y + z)$.

In the cases (3) and (4) we claim that xt is a minimal generator of $0 : (y + z)$ in R. That $xt \in 0 : (y + z)$ is clear. To prove that xt is a minimal generator we may assume that I is the largest possible, i.e. $I = (xy, yz, zt, x^2, y^2, z^2, t^2)$. It is easy to see that the space of the elements of degree 1 in $0 : (y + z)$ is generated by y and z. As xt is not in the ideal generated by y, z in R we may conclude that xt is minimal generator of $0 : (y + z)$.

Finally (5) has been observed in [Example 1.10, C]. ■

We are in the position to state our result. For a base field of characteristic $\neq 2$ we have:

THEOREM 5. *Let R be an algebra defined over a field K of characteristic $\neq 2$ by an ideal I generated by monomials of degree 2 in a set of variables X. The following are equivalent:*

(1) *R is uk,*

(2) *R is obtained from the algebras $H(n)$ by iterated polynomial extensions and fiber products.*

(3) *The restriction of I to any subset of variables of X does not give an ideal of type* (1)-(5) *of the list of Lemma* 4.

In characteristic 2 we have:

THEOREM 6. *Let R be an algebra defined over a field K of characteristic 2 by an ideal I generated by monomials of degree 2 in a set of variables X. The following are equivalent:*

(1) *R is uk,*

(2) *R is obtained from the field K by iterated polynomial extensions, fiber products and extension of the type of Lemma 3(b).*

(3) *The restriction of I to any subset of variables of X does not give an ideal of type $(1)-(4)$ of the list of Lemma 4.*

PROOF OF THEOREMS 5 and 6. (2) $\Rightarrow$ (1) follows from Lemma 1 and 2. (1) $\Rightarrow$ (3) follows from Lemma 4. We prove (3) $\Rightarrow$ (2) by induction on the cardinality $\#X$ of X. If $\#X=1$ then the assertion is clearly true. So assume that $\#X>1$. Let V be a subset of X such that for all pairs $x, y \in V$ with $x \neq y$ one has $xy \notin I$ and assume that V is maximal with respect to this property. Say $V=\{v_1, v_2, \ldots, v_k\}$. Set $W = X \backslash V$ and $G_j = \{x \in W\colon xv_j \in I\}$. By the definition of V we have that $W = \bigcup_{i=1}^{k} G_i$. We claim that:

(*a*) For $i=1, \ldots, k$, $x \in G_i$ and $y \in W \backslash G_i$ then $xy \in I$,

(*b*) For $1 \leq i < j \leq k$ then either $G_i \subseteq G_j$ or $G_j \subseteq G_i$.

To prove (*a*), let j be such that $y \in G_j$. Since $i \neq j$ we have that I contains xv_i and yv_j and by construction does not contain $v_i v_j$ and $v_i y$. Hence I must contain also xy otherwise the square free part of its restriction to $\{x, y, v_i, v_j\}$ would be either xv_i, yv_j or $xv_i, yv_j, v_j x$, a contradiction since these are ideals of type (3) and (4) in the list of Lemma 4. To prove (*b*), assume by contradiction that there exist $x \in G_i \backslash G_j$ and $y \in G_j \backslash G_i$ and argue as in case (*a*).

After renumbering if needed, by (*b*) we may assume that

$$G_1 \subseteq G_2 \subseteq \ldots \subseteq G_k = W.$$

If $G_1 \neq \emptyset$, then by (*a*) and definition of the G_i we have for each $x \in G_1$ and $y \in X \backslash G_1$ then $xy \in I$. Then R is the fiber product of the algebra R_1 defined by the restriction of I to G_1 with the algebra R_2 defined by the restriction of I to $X \backslash G_1$. As R_1 and R_2 clearly satisfy condition (3) of the theorem, we may assume by induction that they also satisfy (2). As $R = R_1 \circ R_2$, also R satisfies (2) and we are done.

If instead $G_1 = \emptyset$ and $v_1^2 \notin I$ then R is a polynomial extension, and again we are done by induction.

So we are left with the case in which $G_1 = \emptyset$ and $v_1^2 \in I$. Let h be the largest index such $G_h = \emptyset$. We may also assume that $v_i^2 \in I$ for $i = 1, \ldots, h$ otherwise we conclude as above.

If $W = \emptyset$ (equivalently, $h = k$) then R is equal to $K[v_1, \ldots, v_k]/(v_1^2, \ldots, v_k^2)$. This ring is obtained by iterated extensions of the type of Lemma 3(b) if the characteristic of K is 2. If, instead, the characteristic of K is $\neq 2$, then $k = 2$ (otherwise a restriction would be of type (5)) and R is $H(2)$.

Therefore we may assume $W \neq \emptyset$ (equivalently $h < k$). Let $i > h$ and let $x \in G_i$. Then I contains v_1^2, xv_i and does not contain $v_1 v_i$, xv_1. It follows that v_i^2 and x^2 must be in I otherwise I would have a restriction of type (1) or (2). In particular $x^2 \in I$ for all $x \in X$. But then, if the characteristic of K is 2, R is an extension of the type of Lemma 3(b) (with $x = v_1$). By induction, this concludes the proof in the characteristic 2 case. Assume that the characteristic of K is not 2. Since $x^2 \in I$ for all $x \in X$ and W is not empty, then $h = 1$ and $k = 2$, otherwise there would be a restriction of type (5). Now let $x, y \in W$. Since we know that $x^2, y^2, v_1^2 \in I$ and $xv_1, yv_1 \notin I$ it follows that $xy \in I$. Summing up, R is (isomorphic to) $H(n)$. ■

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