Collectanea Mathematica (electronic version): http://www.imub.ub.es/collect

Collect. Math. **54**, 2 (2003), 137–152

Castelnuovo-Mumford regularity of products of ideals

Aldo Conca

Dipartimento di Matematica, Universitá di Genova
Via Dodecaneso 35, I-16146 Genova, Italy

E-mail: conca@dima.unige.it

Jürgen Herzog

Fachbereich 6, Mathematik, Universität GH-Essen
D-45117 Essen, Germany

E-mail: juergen.herzog@uni-essen.de

Received June 5, 2002

Abstract

The Castelnuovo-Mumford regularity $\operatorname{reg}(M)$ is one of the most important invariants of a finitely generated graded module M over a polynomial ring R. For instance, it measures the amount of computational resources that working with M requires. In general one knows that the regularity of a module can be doubly exponential in the degrees of the minimal generators and in the number of the variables. On the other hand, in many situations one has or one conjectures a much better behavior. One may ask, for instance, whether the Castelnuovo-Mumford regularity $\operatorname{reg}(IM)$ of the product of an ideal I with a module M is bounded by the sum $\operatorname{reg}(I)+\operatorname{reg}(M)$. In general this is not the case. But we show that it is indeed the case if either $\dim R/I \leq 1$ or I is generic (in a very precise sense). Further we show that products of ideals of linear forms have always a linear resolution and that the same is true for products of determinantal ideals of a generic Hankel matrix.

Introduction

Let R be a polynomial ring over a field K and let $\mathfrak{m}$ be its graded maximal ideal. Let I be a graded ideal of R and M a finitely generated graded R-module. The highest

Keywords: Castelnuovo-Mumford regularity, linear resolutions, ideals of linear forms.
MSC2000: 13D02, 13P10.

degree of a generator of the product IM is bounded above by the sum of the highest degree of a generator of M and the highest degree of a generator of I. One may wonder whether the same relation holds also for the Castelnuovo-Mumford regularity, that is, whether

$$\operatorname{reg}(IM) \leq \operatorname{reg}(M) + \operatorname{reg}(I). \tag{1}$$

This is not the case in general. There are examples already with $M = I$ such that $\operatorname{reg}(I^2) > 2\operatorname{reg}(I)$, see Sturmfels [15] and Terai [16]. On the other hand, Chandler [5] and Geramita, Gimigliano and Pitteloud [11] have shown that $\operatorname{reg}(I^k) \leq k\operatorname{reg}(I)$ holds for ideals with $\dim R/I \leq 1$. In general one has that $\operatorname{reg}(I^k)$ is asymptotically a linear function of k, see [14, 8]. If $I = \mathfrak{m}$ and M is any finitely generated graded R-module, then $\operatorname{reg}(\mathfrak{m}) = 1$ and it is easy to see that $\operatorname{reg}(\mathfrak{m}M) \leq \operatorname{reg}(M) + 1$ holds. So it is natural to ask whether (1) holds whenever I is generated by a regular R-sequence or at least by a sequence of linear forms. Unfortunately this is also not the case, even when M is a monomial ideal with a linear resolution and I is generated by a subset of the variables, see Example 2.1. The purpose of this note is to describe some cases where (1) is nonetheless valid.

In Section 1 we recall some generalities about regularity and show in Section 2 that (1) is valid for ideals generated by sequences which are almost regular with respect to M and regular with respect to R, see 2.3. We also show the validity of (1) whenever the Krull dimension of R/I is at most 1. The argument is similar as in the corresponding result of Chandler. It follows that (1) holds whenever I is generated by any number of generic forms.

More surprising is the fact, proved in Section 3 (Theorem 3.1), that any product of ideals of linear forms has a linear resolution. This is obtained as a consequence of a description of a primary decomposition of such an ideal, see 3.2.

In Section 4 we consider ideals with linear quotients, that is, ideals which can be generated by a minimal system of generators whose successive colon ideals are generated by linear forms. Examples of such ideals are stable, and squarefree stable ideals in the sense of Eliahou-Kervaire [10] and Aramova-Herzog-Hibi [1], as well as polymatroidal ideals, as noted in [13]. Again it turns out that the property of having linear quotients is not preserved under taking products or powers. However we show in Section 5 that products of polymatroidal ideals are again polymatroidal, and hence have again linear quotients. This is also implied by the fact that discrete polymatroids are just the integer vectors of an integral polymatroid (see [12, Theorem 3.4]) and a theorem on polymatroidal sums [17, Theorem 3].

Let X be a generic Hankel matrix and let I_t be the ideal of the minors of size t of X. It has been shown in [6] that I_2^k has a linear resolution for all k. Furthermore, it follows from results in [2] and [7] that I_t^k has a linear resolution for all k and for all t. As an application of the concept of ideals with linear quotients we show in the last section that any product $I_{t_1} \cdots I_{t_k}$ of ideals of minors of a generic Hankel matrix has a linear resolution.

Some of the results of this paper have been conjectured after explicit computations performed by using the computer algebra system CoCoA[4]. We would like to thank

J. Abbott, A. Bigatti and M. Caboara for their help and suggestions in doing these computations.

1. Generalities

Let K be a field and let R be a polynomial ring over K. Let $M = \oplus_{i\in\mathbb{Z}} M_i$ be a finitely generated graded R-module. For every $i \in \mathbb{N}$ one defines

$$t_i^R(M) = \max\left\{j \mid \beta_{ij}^R(M) \neq 0\right\}$$

where $\beta_{ij}^R(M)$ is the ijth graded Betti number of M as an R-module, i.e.

$$\beta_{ij}^R(M) = \dim_K \operatorname{Tor}_i^R(M, K)_j$$

and $t_i^R(M) = -\infty$ if it happens that $\operatorname{Tor}_i^R(M, K) = 0$. The *Castelnuovo-Mumford regularity* $\operatorname{reg}(M)$ of M is given by

$$\operatorname{reg}(M) = \sup\left\{t_i^R(M) - i \,:\, i \in \mathbb{N}\right\}.$$

The *initial degree* of a non-zero finitely generated graded R-module M is the least i such that $M_i \neq 0$. A finitely generated graded R-module M has a *linear resolution* if its regularity is equal to its initial degree. In other words, M has linear resolution if its minimal generators have all the same degree and the matrices of the minimal free resolution of M over R have entries of degree 1.

A short exact sequence

$$0 \to N \to M \to P \to 0$$

of finitely generated graded R-modules yields a long exact sequence of Tor-modules

$$\cdots \to \operatorname{Tor}_{i+1}^R(P, K) \to \operatorname{Tor}_i^R(N, K) \to \operatorname{Tor}_i^R(M, K) \to \operatorname{Tor}_i^R(P, K) \to \cdots$$

It follows that

$$\begin{aligned} \operatorname{reg}(M) &\leq \max\left\{\operatorname{reg}(P), \operatorname{reg}(N)\right\}, \\ \operatorname{reg}(N) &\leq \max\left\{\operatorname{reg}(M), \operatorname{reg}(P) + 1\right\}, \\ \operatorname{reg}(P) &\leq \max\left\{\operatorname{reg}(N) - 1, \operatorname{reg}(M)\right\}. \end{aligned} \tag{2}$$

Let N be a graded module of finite length. We set $s(N) = \max\{s \colon N_s \neq 0\}$. One has (see [9, Corollary 20.19]):

Lemma 1.1

Let N be a graded R-module of finite length. Then:

(a) $\operatorname{reg}(N) = s(N)$

(b) *If $0 \to N \to M \to P \to 0$ is a short exact sequence of finitely generated graded modules then* $\operatorname{reg}(M) = \max\{\operatorname{reg}(P), s(N)\}$.

Let M be a finitely generated graded R-module. A homogeneous element $x \in R$ of degree d is called *almost regular on* M if the multiplication map $x\colon M_{i-d} \to M_i$ is injective for all $i \gg 0$. Let $N = H^0_{\mathfrak{m}}(M)$, i.e. $N = \{a \in M : \mathfrak{m}^k a = 0 \quad \text{for some} \quad k\}$. Then x is almost regular for M if and only if x is a non-zerodivisor on M/N.

A sequence $x_1, \dots, x_m$ of homogeneous elements of R is called an *almost regular M-sequence* if x_i is almost regular on $M/(x_1, \dots, x_{i-1})M$ for $i = 1, \dots, n$.

Proposition 1.2

Let M be a finitely generated graded R-module and $x \in R$ an almost regular element on M of degree d. Set $N = H^0_{\mathfrak{m}}(M)$. Then

$$\operatorname{reg}(M) = \max\big\{\operatorname{reg}(M/xM) - d + 1, s(N)\big\}.$$

Proof. Set $a = \operatorname{reg}(M), b = \operatorname{reg}(M/xM), c = \operatorname{reg}(xM)$ and $s = s(N)$. We have to show that $a = \max\{b - d + 1, s\}$. Let $W = (0 :_M x)$; then $W \subset N$ and $s(W) = s(N) = s$. We obtain two exact sequences

$$0 \to W(-d) \to M(-d) \to xM \to 0,$$

and

$$0 \to xM \to M \to M/xM \to 0.$$

By virtue of 1.1 and of the first exact sequence we have

$$(i) \quad a = \max\{c - d, s\}$$

while from the second exact sequence and (2) we have:

$$(ii) \quad c \le \max\{a, b+1\}, \qquad (iii) \quad b \le \max\{a, c-1\}.$$

By (i) and (ii) we have $a = \max\{c - d, s\} \le \max\{a - d, b + 1 - d, s\}$ which implies $a \le \max\{b+1-d, s\}$. By (iii) and (i) we have $b \le \max\{a, c-1\} = \max\{c-d, s, c-1\} = \max\{s, c-1\}$. Hence $\max\{b+1-d, s\} \le \max\{s+1-d, c-d, s\} = \max\{c-d, s\} = a$. □

Given a homogeneous ideal I in a polynomial ring R and a finitely generated graded R-module M, one defines the saturation $(IM)^{sat}$ of IM as follows:

$$(IM)^{sat} = \{x \in M\colon \mathfrak{m}^k x \subset IM \quad \text{for some} \quad k\}$$

and the *saturation degree* $\operatorname{sat}(IM)$ the smallest index j such that IM and $(IM)^{sat}$ coincide from degree j on. In other words, $\operatorname{sat}(IM) = s((IM)^{sat}/IM) + 1$. Note that $H^0_{\mathfrak{m}}(M/IM) = (IM)^{sat}/IM$, and hence $\operatorname{sat}(IM)$ is the smallest index j such that $H^0_{\mathfrak{m}}(M/IM)$ vanishes from degree j on. As a consequence of 1.2 we have

Corollary 1.3

Let $I \subset R$ be a homogeneous ideal, and let $x \in R$ be a linear form which is almost regular on R/I. Then $\operatorname{reg}(I) = \max\{\operatorname{reg}(I + (x)), \operatorname{sat}(I)\}$.

2. Regularity of products of ideals and modules

Given a finitely generated graded R-module M and a homogeneous ideal $I \subset R$, the purpose of this section is to discuss cases in which the inequality (1) holds. We mentioned already in the introduction that this is not always the case. On the other hand, if one takes $I = \mathfrak{m}$, where $\mathfrak{m}$ is the graded maximal ideal of R, then $\operatorname{reg}(\mathfrak{m}M) \leq \operatorname{reg}(M) + 1$ and hence (1) holds. So it is natural to ask whether (1) holds in case I is generated by an R-regular sequence. Unfortunately this is not true, even when I is generated by linear forms, as the following example shows.

EXAMPLE 2.1: Let $R = K[a,b,c,d]$, and let $J = (a^2b, abc, bcd, cd^2)$. The resolution of J is

$$0 \to R^3(-4) \to R^4(-3) \to J \to 0.$$

It follows that $\operatorname{reg}(J) = 3$. If we take $I = (b,c)$ then the resolution of IJ is

$$0 \to R(-8) \to R^3(-6) \oplus R^2(-7) \to R^{10}(-5) \oplus R(-6) \to R^8(-4) \to IJ \to 0.$$

The non-linear minimal syzygy among the generators of IJ is $a^2(bcd^2) - d^2(a^2bc) = 0$. Anyway, $\operatorname{reg}(IJ) = 5$.

On the other hand, one has

Theorem 2.2

Let M be a finitely generated graded R-module and let I be an ideal of R generated by an almost regular M-sequence $x_1, \ldots, x_m$. Set $\deg x_i = d_i$. Then

$$\operatorname{reg}(IM) \leq \operatorname{reg}(M) + d_1 + d_2 + \cdots + d_m - m + 1.$$

Proof. By 1.2 we have

$$\operatorname{reg}(M/(x_1, \ldots, x_{i-1})M) \geq \operatorname{reg}(M/(x_1, \ldots, x_i)M) - d_i + 1$$

for all $i = 1, \ldots, m$. This implies that

$$\operatorname{reg}(M/IM) \leq \operatorname{reg}(M) + d_1 + \cdots + d_m - m.$$

Now

$$\begin{aligned} \operatorname{reg}(IM) &\leq \max\{\operatorname{reg}(M/IM) + 1, \operatorname{reg}(M)\} \\ &\leq \max\{\operatorname{reg}(M) + d_1 + \cdots + d_m - m + 1, \operatorname{reg}(M)\} \\ &= \operatorname{reg}(M) + d_1 + \cdots + d_m - m + 1. \ \square \end{aligned}$$

Corollary 2.3

Suppose that, in addition to the assumptions of 2.2, $x_1, \ldots, x_m$ is a regular R-sequence. Then $\operatorname{reg}(IM) \leq \operatorname{reg} M + \operatorname{reg}(I)$.

Proof. For the proof we just note that $\operatorname{reg}(I) = d_1 + d_2 + \cdots + d_m - m + 1$ if $x_1, \ldots, x_m$ is a regular R-sequence. □

The following result generalizes a theorem of [5] and [11], and is another case in which the inequality (1) holds.

Theorem 2.4

Let I be a graded ideal of R with $\dim R/I \leq 1$. Then for any finitely generated graded R-module M we have

$$\operatorname{reg}(IM) \leq \operatorname{reg}(M) + \operatorname{reg}(I).$$

Proof. The proof follows very much the line of arguments of [5].

Let $x \in R_1$ be an element which is almost regular on M, M/IM and R/I. We first show that

$$\operatorname{sat}(IM) \leq \operatorname{reg}(M) + \operatorname{reg}(I). \tag{3}$$

We set $r = \operatorname{reg}(M)$ and $t = \operatorname{reg}(I)$. Since $(IM)^{sat}/IM$ and $(IM :_M x)/IM$ have the same socle, it suffices to show that if $f \in M$ is homogeneous of degree $\geq r + t$ with $xf \in IM$, then $f \in IM$.

Suppose that $f = \sum_i f_i m_i$ and $xf = \sum_i g_i m_i$ with $g_i \in I$. Then $\sum_i (xf_i - g_i)m_i = 0$. Consider the exact sequence

$$0 \longrightarrow U \to F \xrightarrow{\varepsilon} M \longrightarrow 0$$

where F is free with basis $e_1, \ldots, e_k$ and $\varepsilon(e_i) = m_i$. Then $\sum_i (xf_i - g_i)e_i \in U$. Let $u_1, \ldots, u_l$ be a homogeneous system of generators of U, and $u_j = \sum_i a_{ij} e_i$. Then

$$\sum_i (xf_i - g_i)e_i = \sum_j k_j u_j = \sum_i \Big(\sum_j a_{ij} k_j\Big) e_i,$$

so that $xf_i - g_i = \sum_j a_{ij} k_j$. Note that $\deg k_j \geq r + t + 1 - \deg u_j \geq t$. Hence, $k_j \in I + (x)$, since $(I + (x))_i = R_i$ for $i \geq t$. Thus $k_j = xp_j + q_j$ with $q_j \in I$. This yields

$$x\Big(f_i - \sum_j a_{ij} p_j\Big) = g_i + \sum_j a_{ij} q_j.$$

This equation implies that $f_i - \sum_j a_{ij} p_j \in I^{sat}$. However, since $\operatorname{sat}(I) \leq \operatorname{reg}(I) = t$ and $\deg(f_i - \sum_j a_{ij} p_j) \geq t$, it follows that $f_i - \sum_j a_{ij} p_j \in I$. We conclude that $f = \sum_i (f_i - \sum_j a_{ij} p_j) m_i \in IM$. This concludes the proof of (3).

In order to prove the theorem we assume first that $\dim M/IM = 0$. By (2) we have $\operatorname{reg}(IM) \leq \max\{\operatorname{reg}(M), \operatorname{reg}(M/IM) + 1\}$. Hence it suffices to show that $\operatorname{reg}(M/IM) \leq \operatorname{reg}(M) + \operatorname{reg}(I) - 1$. Since $\operatorname{reg}(M/IM) = s(M/IM)$ by 1.1, and since $s(M/IM) = \operatorname{sat}(IM) - 1$, this follows from (3).

Now we assume that $\dim M/IM = 1$. Set $N = M/xM$. Then Proposition 1.2 implies

$$\operatorname{reg}(M/IM) = \max\{\operatorname{reg}(N/IN), \operatorname{sat}(IM) - 1\}. \tag{4}$$

By 1.1 we also have $\operatorname{reg}(N/IN) \leq \max\{\operatorname{reg}(IN)-1, \operatorname{reg}(N)\}$, and since N/IN is 0-dimensional we conclude from the first part of the proof that $\operatorname{reg}(IN) \leq \operatorname{reg}(N) + \operatorname{reg}(I)$, so that

$$\begin{aligned}\operatorname{reg}(N/IN) &\leq \max\{\operatorname{reg}(N)+\operatorname{reg}(I)-1, \operatorname{reg}(N)\}\\ &= \operatorname{reg}(N)+\operatorname{reg}(I)-1 \leq \operatorname{reg}(M)+\operatorname{reg}(I)-1.\end{aligned}$$

The last inequality holds since x is almost regular on M. Thus together with (4) we obtain

$$\operatorname{reg}(M/IM) \leq \max\{\operatorname{reg}(M)+\operatorname{reg}(I)-1, \operatorname{sat}(IM)-1\}. \tag{5}$$

Notice further that

$$\operatorname{reg}(IM) \leq \max\{\operatorname{reg}(M), \operatorname{reg}(M/IM)+1\}. \tag{6}$$

We may assume that $\operatorname{reg}(IM) > \operatorname{reg}(M)$, because otherwise nothing is to prove. But then (6) implies that $\operatorname{reg}(IM) \leq \operatorname{reg}(M/IM)+1$. Hence together with (5) we get

$$\operatorname{reg}(IM) \leq \max\{\operatorname{reg}(M)+\operatorname{reg}(I), \operatorname{sat}(IM)\}.$$

The desired inequality follows from (3). □

As a Corollary of 2.3 and 2.4 we obtain:

Corollary 2.5

Let M be a finitely generated graded R-module and let I be an ideal generated by a generic sequence of homogeneous forms. Then $\operatorname{reg}(IM) \leq \operatorname{reg}(M) + \operatorname{reg}(I)$.

Proof. If I is generated by $\leq \dim R$ elements then its generators, being generic, form a regular sequence on R which is almost regular sequence on M. Then the result follows from 2.3. On the other hand, if I is generated by $\geq \dim R$ elements, then $\dim R/I = 0$ and the desired inequality follows from 2.4. □

3. Regularity of products of ideals of linear forms

The goal of this section is to prove the following:

Theorem 3.1

Let $I_1, I_2, \dots, I_d$ be non-zero ideals of R generated by linear forms. Then the product $I_1 I_2 \cdots I_d$ has a linear resolution, i.e.

$$\operatorname{reg}(I_1 I_2 \cdots I_d) = d.$$

To prove the theorem we need some preliminary results. Let us fix some notation. For a subset A of $\{1, \dots, d\}$ we will set $I_A = \sum_{j \in A} I_j$ and denote by $|A|$ the cardinality of A. We have:

Lemma 3.2

Let $I_1, I_2, \dots, I_d$ be non-zero ideals of R generated by linear forms. Then

$$I_1 \cdots I_d = \cap_A I_A^{|A|}$$

is a (possibly redundant) primary decomposition of $I_1 \cdots I_d$. Here the intersection is extended to all the non-empty subsets A of $\{1, \dots, d\}$.

As a Corollary of 3.2 we have:

Corollary 3.3

Let $I_1, I_2, \ldots, I_d$ be non-zero ideals of R generated by linear forms. Then

$$\operatorname{sat}(I_1 I_2 \cdots I_d) \leq d.$$

Proof. of 3.3: Set $J = I_1 I_2 \cdots I_d$. By virtue of 3.2 $J^{sat} = \cap_A I_A^{|A|}$ where the intersection is extended to all the non-empty subsets A of $\{1, \ldots, d\}$ such that $I_A \neq \mathfrak{m}$. It follows that $J = J^{sat} \cap \mathfrak{m}^d$ if $\sum I_i = \mathfrak{m}$ and $J = J^{sat}$, otherwise. This implies that $\operatorname{sat}(J) \leq d$. □

Now we prove 3.2:

Proof. The ideal $I_A^{|A|}$ is obviously I_A-primary and hence it suffices to prove that $I_1 \cdots I_d = \cap_A I_A^{|A|}$. Set $J = I_1 I_2 \cdots I_d$. Let J_i be the product of the I_j with $j \neq i$. By induction on d, it is enough to show that:

$$J = J_1 \cap \ldots \cap J_d \cap \Big(\sum_{i=1}^{d} I_i \Big)^d .$$

We prove this equality by induction on d and on $\dim R$. The critical inclusion is $\supseteq$. We may assume that $\sum I_i = \mathfrak{m}$ (otherwise all the ideals live in a smaller polynomial ring). It is also harmless to assume that the residue field is infinite. Summing up, what we have to prove is that if f is an element in $J_1 \cap \ldots \cap J_d$ of degree $\geq d$ then $f \in J$. As J_i is a product of $(d-1)$ ideals of linear forms, by induction we know that Corollary 3.3 holds for J_i and hence $\operatorname{sat}(J_i) \leq d-1$ for all i. Let x be a linear form which is a non-zerodivisor on R/J_j^{sat} for all the J_j of positive dimension. The ideals $J + (x)/(x)$ of $R/(x)$ is the product of ideals of linear forms $I_i + (x)/(x)$. So, arguing modulo x and using induction on $\dim R$, we see that $f \in J + (x)$. Write $f = h + x f_1$, with $h \in J$. Replacing f with $f - h$ we may assume from the really beginning that $f = x f_1$. Since $f = x f_1 \in J_i$ and $\operatorname{sat}(J_i) \leq d-1$, by the choice of x we may deduce that f_1 itself is in J_i for all i. Now since the sum of the I_i is $\mathfrak{m}$ we may write $x = \sum_i x_i$ with $x_i \in I_i$. Then we have $f = x f_1 = \sum_i x_i f_1$ and each $x_i f_1 \in I_i J_i = J$ so that $f \in J$. □

We are ready to prove 3.1

Proof. Set $J = I_1 \ldots I_d$. Since J is generated in degree d our task is to prove that $\operatorname{reg}(J) \leq d$. We prove it by induction on the dimension of R and on d. The claim is trivial if $\dim R = 1$. If $\dim R/J = 0$ then the assertion is also trivial. We may hence assume that $\dim R/J > 0$. Let x be a linear form which is a non-zerodivisor modulo J^{sat}. By 1.3 we have that $\operatorname{reg}(J) = \max\{\operatorname{reg}(J + (x)), \operatorname{sat}(J)\}$. Note that $\operatorname{reg}(J + (x)) = 1 + \operatorname{reg}(R/J + (x))$. Since $\operatorname{reg}(R/J + (x))$ can be interpreted as the regularity of $R/J + (x)$ as an $R/(x)$-module and the ideal $J + (x)/(x)$ of $R/(x)$ is a product of ideals of linear forms we have $\operatorname{reg}(R/J + (x)) = d - 1$. It follows that $\operatorname{reg}(I + (x)) = d$. Since by 3.3 $\operatorname{sat}(J) \leq d$, we are done. □

The primary decomposition of 3.2 is in general far from being irredundant. For example we have:

Proposition 3.4

Let $V_1, \ldots, V_d$ be a family of subspaces of R_1 which is linearly general, i.e. one has

$$\dim \sum_{i \in A} V_i = \min \left\{ \dim R_1, \sum_{i \in A} \dim V_i \right\}$$

for all the non-empty subsets A of $\{1, \ldots, d\}$. Assume that $\sum_{i=1}^{d} V_i = R_1$. Let I_i be the ideal generated by V_i. Then

$$I_1 \cdots I_d = I_1 \cap \ldots \cap I_d \cap \mathfrak{m}^d$$

is a primary decomposition of $I_1 \cdots I_d$.

Proof. We have to show that all the terms $I_A^{|A|}$ with $1 < |A| < d$ in the primary decomposition 3.2 are superfluous. For such an A we distinguish two cases. If $\sum_{i \in A} \dim V_i \leq \dim R_1$ then by assumption $\dim \sum_{i \in A} V_i = \sum_{i \in A} \dim V_i$ which implies that $\cap_{i \in A} I_i = \Pi_{i \in A} I_i$. Hence $I_A^{|A|}$ contains $\cap_{i=1}^{d} I_i$ and it is therefore superfluous. If instead $\sum_{i \in A} \dim V_i > \dim R_1$, then by assumption $I_A = \mathfrak{m}$ and hence $I_A^{|A|} \supset \mathfrak{m}^d$. □

On the other hand there are cases where all the $2^d - 1$ ideals appearing in the primary decomposition 3.2 are essential.

EXAMPLE 3.5: Let $R = K[x_1, \ldots, x_d, y]$ and consider $I_i = (x_i, y)$. Set $J = I_1 \cdots I_d$. It is not difficult to show that for any subset $A \subseteq \{1, \ldots, d\}$ one has $J : m = (y) + (x_i : i \in A) = I_A$ where $m = y^{|A|-1} \Pi_{i \notin A} x_i$. Hence each I_A is an associated prime of J. Therefore the primary decomposition given in 3.2 is irredundant in this case.

Question 3.6 After 3.1 it is natural to ask whether

$$\operatorname{reg}(I_1 I_2 \cdots I_d) \leq \operatorname{reg}(I_1) + \operatorname{reg}(I_2) + \ldots + \operatorname{reg}(I_d)$$

holds for ideals I_i generated by regular sequences. By 2.5, this is true if each I_i is generated by generic forms.

4. Modules with linear quotients

We say that a finitely generated graded R-module M has *linear quotients* if M admits a minimal system of generators $m_1, \ldots, m_k$ such that for every $t = 1, \ldots, k$ one has that $\langle m_1, \ldots, m_{t-1} \rangle :_R m_t$ is an ideal of R generated by linear forms.

Examples of ideals with linear quotients are strongly stable and squarefree strongly stable ideals. Other important classes will be considered in the next sections.

Lemma 4.1

If M has linear quotients then

$$\operatorname{reg}(M) = \max \{\deg m \colon m \text{ is a minimal generator of } M\}.$$

In particular, if all generators of M have the same degree, then M has a linear resolution over R.

Proof. Let $m_1, \ldots, m_k$ be as in the definition of module with linear quotients. Set $M_t = \langle m_1, \ldots, m_t \rangle$. We have an exact sequence

$$0 \to M_{t-1} \to M_t \to M_t/M_{t-1} \to 0$$

and M_t/M_{t-1} is of the form $R/I[-\deg(m_t)]$ with I an ideal of R generated by linear forms. Since $\operatorname{reg}(R/I) = 0$ it follows that $\operatorname{reg}(M_t) \leq \max\{\operatorname{reg}(M_{t-1}), \deg(m_t)\}$ and hence, by induction, the assertion follows. □

EXAMPLE 4.2: The ideal $J = (a^2b, abc, bcd, cd^2)$ of 2.1 has linear quotients, the successive colons being:

$$(0), \quad (a), \quad (a), \quad (b).$$

On the other hand there are ideals with linear resolution and without linear quotients. The easiest example is the ideal I of 2-minors of the matrix

$$\begin{pmatrix} a & b & c \\ b & c & d \end{pmatrix}.$$

I has a linear resolution but it cannot have linear quotients since it is a prime ideal and hence $(f) : (g) = (f)$ for each $f \in I$ with $\deg(f) = 2$.

Note that for a monomial ideal I to have linear quotients (with respect to the monomial generators) is a purely combinatorial property and hence does not depend on the characteristic of the base field. On the other hand the minimal free resolution of a monomial ideal, and hence its linearity, depends, in general, on the characteristic of the base field. This shows that also for monomial ideals to have linear quotients is a stronger property than to have a linear resolution. The (famous) example of the Stanley-Reisner ideal of a triangulation of the real projective plane (see for example [3, pag. 236]) gives an example of square free monomial ideal that, if the characteristic of K is not 2, has a linear resolution and does not have linear quotients.

We have seen that the property of having a linear resolution is not preserved by taking products or powers of ideals. The same thing can happen for the property of having linear quotients:

EXAMPLE 4.3: We know from 4.2 that $J = (a^2b, abc, bcd, cd^2)$ has linear quotients, but as we have seen in 2.1, $(b, c)J$ does not even have a linear resolution. Also, the ideal $I = (a^2b, a^2c, ac^2, bc^2, acd)$ has linear quotients, the quotients being

$$(0), \quad (b), \quad (a), \quad (a), \quad (c, a).$$

But the minimal resolution of $I2$ begins with

$$R^{24}(-7) \oplus R(-8) \to R^{15}(-6) \to I^2 \to 0$$

and hence I^2 cannot have linear quotients.

Question 4.4 We have seen that a product of ideals of linear forms has a linear resolution. One may ask whether such an ideal has even linear quotients. In the next section we will see that this is the case for products of ideals of variables, see 5.4. For the general case, we have tested many examples with CoCoA, starting with generic and with special ideals of linear forms. We have always found ideals with linear quotients.

5. Polymatroidal ideals

In this section we consider a class of monomial ideals with linear quotients which is closed under the operation of taking products. The theorems presented here correspond to analogue theorems in matroid theory.

Let $R = K[x_1, \dots, x_n]$ be the polynomial ring. For a monomial ideal $I \subset R$ we denote by $G(I)$ the unique minimal set of monomial generators, and for a monomial $u = x_1^{a_1} \dots x_n^{a_n}$ we set $\nu_i(u) = a_i$ for $i = 1, \dots, n$. Given monomials u, v we set

$$[u, v] = \mathrm{GCD}(u, v).$$

DEFINITION 5.1. A monomial ideal $I \subset R$ is said to be *polymatroidal* if all its generators have the same degree and if it satisfies the following exchange property:

for all $u, v \in G(I)$ and all i with $\nu_i(u) > \nu_i(v)$, there exists an integer j with $\nu_j(v) > \nu_j(u)$ such that $x_j(u/x_i) \in G(I)$.

The name is explained by the fact that the elements of $G(I)$ correspond to the basis of a polymatroid, as defined in [17]. If I is a squarefree ideal, then this set corresponds to the basis of a matroid. Hence squarefree polymatroidal ideals are also called *matroidal*.

For the convenience of the reader we reproduce from [13] the proof of the following important property of polymatroidal ideals.

Theorem 5.2

A polymatroidal ideal I has linear quotients with respect to the reverse lexicographical order of the generators.

Proof. Let $u \in G(I)$, and let J be the ideal generated by all $v \in G(I)$ with $v > u$ (in the reverse lexicographical order). Then

$$J : u = (v/[v, u] \colon v \in J).$$

Thus in order to prove that $J : u$ is generated by monomials of degree 1, we have to show that for each $v > u$ there exists $x_j \in J : u$ such that x_j divides $v/[v, u]$.

In fact, let $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$. Since $v > u$, there exists an integer i with $a_i > b_i$ and $a_k = b_k$ for $k = i+1, \dots, n$, and hence an integer j with $b_j > a_j$ such that $u' = x_j(u/x_i) \in G(I)$. Since $j < i$, we see that $u' \in J$, and from the equation $x_i u' = x_j u$ we deduce that $x_j \in J : u$. Finally, since $\nu_j(v/[u, v]) = b_j - \min\{b_j, a_j\} = b_j - a_j > 0$, we have that x_j divides $v/[v, u]$. $\square$

Though products of ideals with linear quotients need not to have linear quotients, we nevertheless have

Theorem 5.3

Let I and J be polymatroidal monomial ideals. Then IJ is polymatroidal.

Proof. Let u and v be two monomials of same degree. We set

$$d(u,v) = \frac{1}{2}\sum_i |\nu_i(u) - \nu_i(v)|.$$

Note that this is an integer. We call $d(u,v)$ the distance between u and v. This function satisfies the usual rules of a distance function. In particular, one has $d(u,v) = 0$ if and only if $u = v$.

Now let $u_1, u \in G(I)$ and $v_1, v \in G(J)$ and suppose that $\nu_i(u_1v_1) > \nu_i(uv)$. Then we may assume that $\nu_i(u_1) > \nu_i(u)$. Hence there exists an integer j_1 such that $\nu_{j_1}(u) > \nu_{j_1}(u_1)$ and $u_2 = x_{j_1}(u_1/x_i) \in G(I)$. Moreover we have $d(u_2, u) < d(u_1, u)$.

If $\nu_{j_1}(v) \geq \nu_{j_1}(v_1)$ we are done, because then $\nu_{j_1}(uv) > \nu_{j_1}(u_1v_1)$, and

$$x_{j_1}(u_1v_1/x_i) = u_2v_1 \in G(IJ).$$

Otherwise $\nu_{j_1}(v_1) > \nu_{j_1}(v)$. Hence there exists k_1 with $\nu_{k_1}(v) > \nu_{k_1}(v_1)$ and such that $v_2 = x_{k_1}(v_1/x_{j_1}) \in G(J)$. Moreover we have $d(v_2, v) < d(v_1, v)$.

If $\nu_{k_1}(u) \geq \nu_{k_1}(u_2)$, then $\nu_{k_1}(uv) > \nu_{k_1}(u_2v_1) = \nu_{k_1}(x_{j_1}(u_1v_1/x_i))$. Thus if $k_1 \neq i$, then $\nu_{k_1}(uv) > \nu_{k_1}(u_1v_1)$, and we are done since

$$x_{k_1}(u_1v_1/x_i) = u_2v_2 \in G(IJ).$$

On the other hand, if $k_1 = i$, then $u_1v_1 = u_2v_2$, and by induction we may assume that the exchange property holds since $d(u_2, u) < d(u_1, u)$ and $d(v_2, v) < d(v_1, v)$.

Otherwise $\nu_{k_1}(u_2) > \nu_{k_1}(u)$. Hence there exists j_2 with $\nu_{j_2}(u) > \nu_{j_2}(u_2)$ and such that $u_3 = x_{j_2}(u_2/x_{k_1}) \in G(I)$. If $\nu_{j_2}(v) \geq \nu_{j_2}(v_2)$, then $\nu_{j_2}(uv) > \nu_{j_2}(u_2v_2) = \nu_{j_2}(x_{k_1}(u_1v_1/x_i))$. Thus if $j_2 \neq i$, then $\nu_{j_2}(uv) > \nu_{j_2}(u_1v_1)$, and we are done since

$$x_{j_2}(u_1v_1/x_i) = u_3v_2 \in IJ.$$

On the other hand, if $j_2 = i$, then $u_3v_2 = u_1v_1$, and by induction on the distance we have the desired exchange property. Otherwise $\nu_{j_2}(v_2) > \nu_{j_2}(v)$.

We may proceed in this way. Suppose we have already constructed sequences $x_{j_1}, \ldots, x_{j_r}$, $x_{k_1}, \ldots, x_{k_{r-1}}$, and $u_1, \ldots, u_{r+1} \in G(I)$, $v_1, \ldots, v_r \in G(J)$ such that for $p = 1, \ldots, r$ we have

(i) $x_{k_{p-1}}$ divides u_p and x_{j_p} divides v_p,
(ii) $u_{p+1} = x_{j_p}(u_p/x_{k_{p-1}})$ and $v_p = x_{k_{p-1}}(v_{p-1}/x_{j_{p-1}})$,
(iii) $d(u_{p+1}, u) < d(u_p, u)$ and for $p \neq r$, $d(v_{p+1}, v) < d(v_p, v)$,
(iv) $\nu_{j_p}(u) > \nu_{j_p}(u_p)$ and $\nu_{k_p}(v) > \nu_{k_p}(v_p)$.

Here we have set $k_0 = i$ for systematic reasons. Notice that

$$u_{r+1} = x_{j_r} \cdots x_{j_1}(u_1/x_i x_{k_1} \cdots x_{k_{r-1}}) \quad \text{and} \quad v_r = x_{k_{r-1}} \cdots x_{k_1}(v_1/x_{j_1} \cdots x_{j_{r-1}}).$$

If $\nu_{j_r}(v) \geq \nu_{j_r}(v_r)$, then by (iv), $\nu_{j_r}(uv) > \nu_{j_r}(u_rv_r) = \nu_{j_r}(x_{k_{r-1}}(u_1v_1/x_i))$. Thus, if $j_r \neq i$, then $\nu_{j_r}(uv) > \nu_{j_r}(u_1v_1)$, and we are done since

$$x_{j_r}(u_1v_1/x_i) = u_{r+1}v_r \in G(IJ).$$

On the other hand, if $j_r = i$, and then $u_1v_1 = u_{r+1}v_r$ and by induction on the distance we have the desired exchange property.

Otherwise $\nu_{j_r}(v_r) > \nu_{j_r}(v)$, and there exists k_r with $\nu_{k_r}(v) > \nu_{k_r}(v_r)$ and such that $v_{r+1} = x_{k_r}(v_r/x_{j_r}) \in G(J)$. Moreover we have $d(v_{r+1}, v) < d(v_r, v)$. Thus the new elements x_{k_r} and v_{r+1} satisfy again the properties (i)-(iv).

If $\nu_{k_r}(u) \geq \nu_{k_r}(u_{r+1})$, then by (iv), $\nu_{k_r}(uv) > \nu_{k_r}(u_{r+1}v_r) = \nu_{k_r}(x_{j_r}(u_1v_1/x_i))$. Thus, if $k_r \neq i$, then $\nu_{k_r}(uv) > \nu_{k_r}(u_1v_1)$, and we are done since

$$x_{k_r}(u_1v_1/x_i) = u_{r+1}v_{r+1} \in G(IJ).$$

On the other hand, if $k_r = i$, and then $u_1v_1 = u_{r+1}v_{r+1}$ and by induction on the distance we have the desired exchange property.

Otherwise $\nu_{k_r}(u_{r+1}) > \nu_{k_r}(u)$, and there exists j_{r+1} with $\nu_{j_r}(u) > \nu_{j_r}(u_{r+1})$ and such that $u_{r+2} = x_{j_{r+1}}(u_{r+1}/x_{k_r}) \in G(I)$. Moreover, $d(u_{r+2}, u) < d(u_{r+1}, u)$. Thus we have the conditions (i)-(iv) as before but r replaced by $r+1$. Condition (iii) implies that the process must terminate. This proves the theorem. □

Since ideals generated by subsets of the variables are obviously polymatroidal, Theorem 5.3 implies

Corollary 5.4

Let $I_1, \ldots, I_d$ be ideals generated by subsets of the variables. Then $I = I_1 \cdots I_d$ has linear quotients.

Let I and J be matroidal ideals. We let $I * J$ be the ideal which is generated by all monomials uv with $u \in G(I)$ and $v \in G(J)$ such that uv is squarefree. We call $I * J$ the squarefree product of I and J. Analogously to 5.3 we have

Theorem 5.5

Let I and J be matroidal ideals. Then $I * J$ is matroidal.

The proof of this theorem similar to that of 5.3. We leave it to the reader.

As a particular case of 5.5 one has that the squarefree product of ideals generated by variables is matroidal. The corresponding matroid is usually called *transversal*.

6. Products of ideals defined by Hankel matrix

In this section we use the notion of ideals with linear quotients to show that products of ideals of minors of a Hankel matrix have a linear resolution.

Let R be the polynomial ring $K[x_1, \ldots, x_n]$ over some field K. Let X be a Hankel matrix with distinct entries $x_1, \ldots, x_n$; this means that X is an $a \times b$ matrix (y_{ij}) with $y_{ij} = x_{i+j-1}$ and $a + b - 1 = n$. Let I_t be the ideal generated by the minors of size t of X. It is known that I_t does not depend on the size of the matrix X (provided, of course, X contains t-minors); it depends only on t and n. For a given n it follows that t may vary from 1 to m, where $m = [(n+1)/2]$ is the integer part of $(n+1)/2$. It is known that the powers of I_2 have a linear resolution, [7]. Blum [2, 3.6] has recently shown that if the Rees algebra $R(I)$ of an ideal I is Koszul then all the powers of I have linear resolutions. As we know that $R(I_t)$ is Koszul [7], we have that I_t^k has a linear resolution for all t and k. We prove here a stronger result:

Theorem 6.1

Let X be a generic Hankel matrix. Let $t_1, \ldots, t_p$ be integers and I be the product of $I_{t_1} \cdots I_{t_p}$. Then I has a linear resolution.

We recall some definitions and results from [6]. Let τ be the lexicographic term order on the monomials of R and $>_1$ the partial order on $x_1, \ldots, x_n$ defined by $x_j >_1 x_i$ if and only if $j - i > 1$. A $>_1$-chain is a monomial $x_{i_1} \cdots x_{i_k}$ such that $x_{i_1} <_1 x_{i_2} <_1 \ldots <_1 x_{i_k}$. Denote by J the initial ideal of $I = I_{t_1} \cdots I_{t_p}$ and by J_k that of I_k. We know that

$$J_k = (m : m \text{ is a } >_1\text{-chain of degree } k\}$$

and that

$$J = J_{t_1} \cdots J_{t_p}.$$

Since the regularity can only increase by passing to the initial ideal, it suffices to show that

Proposition 6.2

The ideal J has linear quotients.

Before proving 6.2 we will describe the generators of J. They have a description in terms of the γ-functions associated to the canonical decomposition of any monomial of R. Let us recall how. Any monomial m of R has a canonical decomposition $m = m_1 \cdots m_k$ as a product of $>_1$-chains. The monomial m_1 is defined to be the largest, with respect to τ, among all the $>_1$-chains which divide m. Similarly, m_2 is the largest among all the $>_1$-chains which divide m/m_1 and so on. The shape of a monomial m is the sequence of integers $s(m) = \deg(m_1), \ldots, \deg(m_k)$ where $m = m_1 \cdots m_k$ is the canonical decomposition of m. By the very definition, the shape of m is a weakly decreasing sequence. For any t and for any sequence of integers $s = s_1, \ldots, s_p$ one defines

$$\gamma_t(s) = \sum_{i=1}^{p} \max(s_i - t + 1, 0).$$

Furthermore, if m is a monomial then we set:

$$\gamma_t(m) = \gamma_t(s(m)).$$

EXAMPLE 6.3: Let $m = x_1^2 x_2^3 x_3^2 x_5^3 x_6 x_7 x_8^3$. Then

$$m = (x_1x_3x_5x_7)(x_1x_3x_5x_8)(x_2x_5x_8)(x_2x_6x_8)(x_2)$$

is the canonical decomposition of m. Its shape is $s(m) = 4, 4, 3, 3, 1$ and its γ-values are $\gamma_1(m) = 15, \gamma_2(m) = 10, \gamma_3(m) = 6, \gamma_4(m) = 2$, and $\gamma_t(m) = 0$ for $t > 4$.

Given the numbers $t_1, \dots, t_p$, let us denote by Ω the set of the monomials m such that $\deg(m) = \sum_{j=1}^{p} t_j$ and $\gamma_i(m) \geq \gamma_i(t_1, \dots, t_p)$ for every i. In [6] it is proved:

Proposition 6.4

(1) Ω is a system of generators of J,

(2) Let m be a monomial with a decomposition (canonical or not) $m = n_1 \cdots n_v$ where the n_i are $>_1$-chains. Set $s = \deg(n_1), \dots, \deg(n_v)$. Then $\gamma_i(m) \geq \gamma_i(s)$ for every i.

We introduce a total order σ on the monomials of R as follows. Let m, n be monomials of R and $m = m_1 \cdots m_k$ and $n = n_1 \cdots n_h$ their canonical decompositions. We set $m >_\sigma n$ if $m_j >_\tau n_j$ for the first index j such that $m_j \neq n_j$. Note that σ is different from τ; for instance $x_1^2 >_\tau x_1x_3$ but $x_1x_3 >_\sigma x_1^2$. Note also that σ is not a term order. Now we are ready to prove:

Proof. of 6.2: We show that J has linear quotients with respect to the set of generators Ω totally ordered by σ. Let m, n be elements of Ω with $m >_\sigma n$. We have to show that there exists $v \in \Omega$ such that $v >_\sigma n$, $v/[v, n]$ divides $m/[m, n]$ and $\deg[v, n] = \deg v - 1$. Let $m = m_1 \cdots m_k$ and $n = n_1 \cdots n_h$ be the canonical decompositions and let j be the smallest index such that $m_j \neq n_j$. Then $m_j >_\tau n_j$. Let $m_j = x_{a_1} \cdots x_{a_r}$ and $n_j = x_{b_1} \cdots x_{b_s}$. Then there exists a index z such that $a_i = b_i$ for $i = 1, \dots, z-1$ and either $a_z < b_z$ or $s = z-1$ and $r \geq z$. In the former case ($a_z < b_z$) we put $v = nx_{a_z}/x_{b_z}$. In the latter case we put $v = nx_{a_z}/x_q$ where x_q is a variable which appear in n_{j+1} (note that $h > j$, since m and n have both degree $\sum t_i$). We have to show that v has the desired properties.

First of all, note that $v/[v, n] = x_{a_z}$. This is clear in the first case while in the second it follows from the fact that q cannot be equal to a_z otherwise the j-th factor in the canonical decomposition of n would be a multiple of $x_{b_1} \cdots x_{b_{z-1}} x_{a_z}$.

Secondly, we claim that x_{a_z} divides $m/[m, n]$. To this end, note that $m/[m, n] = m^{'}/[m^{'}, n^{'}]$ where $m^{'} = m/e$ and $n^{'} = n/e$ and e is the common initial part of the canonical decomposition, i.e. $e = m_1 \cdots m_{j-1} x_{a_1} \cdots x_{a_{z-1}}$. Since x_{a_z} appears in $m^{'}$ and it does not appear in $n^{'}$ (otherwise, as above, the j-th factor in the canonical decomposition of n would be a multiple of $x_{b_1} \cdots x_{b_{z-1}} x_{a_z}$), we may conclude that x_{a_z} divides $m/[m, n]$.

It remains to show that v belongs to Ω and that $v >_\sigma n$. In the case $a_z < b_z$ note that the v has a decomposition into $>_1$-chains $v = n_1 \cdots n_{j-1} u n_{j+1} \cdots n_h$ with $u = n_j x_{a_z}/x_{b_z}$. This need not to be the canonical decomposition, but its shape is equal to that of the canonical decomposition of n and this is enough (by 6.4) to conclude that $v \in \Omega$. Since by construction $u >_\tau n_j$, it is not difficult to check that $v >_\sigma n$. In the case $s = z-1$ and $r \geq z$ note that the v has a decomposition into $>_1$-chains $v = n_1 \cdots n_{j-1} u_1 u_2 n_{j+2} \cdots n_h$ with $u_1 = n_j x_{a_z}$ and $u_2 = n_{j+1}/x_{a_z}$. As above, this need not to be the canonical decomposition. Its shape has been obtained from the shape of n by the operation "increase a larger factor and decrease a shorter". The effect of this operation on the γ-values is clear: the γ-values cannot decrease. This, together with the fact that n is in Ω and 6.4 implies that v is in Ω. As in the other case, since $u_1 >_\tau n_j$ one can also deduce that $v > n$. $\square$

References

1. A. Aramova, J. Herzog, and T. Hibi, Squarefree lexsegment ideals, *Math. Z.* **228** (1998), 353–378.
2. S. Blum, Subalgebras of bigraded Koszul algebras, *J. Algebra* **242** (2001), 795–809.
3. W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, 1993.
4. A. Capani, G. Niesi, and L. Robbiano, CoCoA, a system for doing Computations in Commutative Algebra, Available via from `cocoa.dima.unige.it`.
5. K.A. Chandler, Regularity of the powers of an ideal, *Comm. Algebra* **25** (1997), 3773–3776.
6. A. Conca, Hilbert function and resolution of the powers of the ideal of the rational normal curve, *J. Pure Appl. Algebra* **152** (2000), 65–74.
7. A. Conca, Straightening law and powers of determinantal ideals of Hankel matrices, *Adv. Math.* **138** (1998), 263–292.
8. S.D. Cutkosky, J. Herzog, and N.V. Trung, Asymptotic behaviour of the Castelnuovo-Mumford regularity, *Compositio Math.* **118** (1999), 243–261.
9. D. Eisenbud, *Commutative Algebra, with a View toward Algebraic Geometry*, Springer, 1995.
10. S. Eliahou and M. Kervaire, Minimal resolutions of some monomial ideals, *J. Algebra* **129** (1990), 1–25.
11. A.V. Geramita, A. Gimigliano, and Y. Pitteloud, Graded Betti numbers of some embedded rational n-folds, *Math. Ann.* **301** (1995), 363–380.
12. J. Herzog and T. Hibi, Discrete polymatroids, preprint 2001.
13. J. Herzog and Y. Takayama, Resolutions by mapping cones, *Homology Homotopy Appl.* **4** (2002), 277–294.
14. V. Kodiyalam, Asymptotic behaviour of Castelnuovo-Mumford regularity, *Proc. Amer. Math. Soc.* **128** (2000), 407–411.
15. B. Sturmfels, Four counterexamples in combinatorial algebraic geometry, *J. Algebra* **230** (2000), 282–294.
16. N. Terai, personal communication, see also Pragmatic 1997. Papers from the Summer School held in Catania, May 12–June 14, 1997. Matematiche (Catania) **53** (1998).
17. D.J.A. Welsh, *Matroid Theory,* Academic Press, London-New York, 1976.