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The dimension of certain catalecticant varieties

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Abstract

Let $V_{s,t}$ be the rank $\leq s$ locus in $\mathbb{P}^{\binom{2t+2}{2}-1}$ of the generic catalecticant matrix Cat$(t,t;3)$. This matrix has rather more symmetry than a generic symmetric matrix; this implies codim$V_{s,t} \leq \binom{\widetilde{s}+1}{2}$, where $\widetilde{s} := \binom{t+2}{2} - s$.

In this paper, given the integer t, we explicitly determine an integer N, depending on t, with the property that codim$V_{s,t} = \binom{\widetilde{s}+1}{2}$ if and only if $\tilde{s} \leq N$.

1. Introduction

Let $R = k[X_1, \dots, X_n] = \oplus_{t\geq 0} R_t$ with $k = \overline{k}$ an algebraically closed field of characteristic zero. Fix positive integers d, i, j such that $d = i + j$ and consider the bilinear map, given by multiplication,

$$R_i \times R_j \to R_d.$$

One keeps track of this multiplication in a matrix whose rows are indexed by the monomials of R_i (say in the lexicographic order) and whose columns are indexed by the monomials of R_j. In each place of the matrix one enters a new variable $Y_{\underline{a}}$ where $\underline{a}$ is the multiindex of length d corresponding to the monomial which is the result of multiplying the appropriate row monomial by the appropriate column monomial.

The resulting matrix of variables is denoted by $Cat(i, j; n)$ and called the (i, j)-*catalecticant* matrix of R.

The size of this matrix is $\binom{n+i-1}{i} \times \binom{n+j-1}{j}$ and the entries of the matrix are variables taken from the polynomial ring $k[Y_{\underline{a}}]$ in $\binom{n+d-1}{d}$ variables, where $d = i + j$.

In this paper we are concerned with the special case $i = j$ and $n = 3$. The matrix $Cat(t, t; 3)$ has size $\binom{t+2}{2} \times \binom{t+2}{2}$ and it is a symmetric matrix with entries in

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a polynomial ring in $\binom{2t+2}{2}$ variables. It is a matrix of indeterminates but it is not generic since the same variable can be repeated in the matrix.

For example $Cat(1,1;3)$ is the matrix

$$Cat(1,1;3) = \begin{pmatrix} Y_{200} & Y_{110} & Y_{101} \\ Y_{110} & Y_{020} & Y_{011} \\ Y_{101} & Y_{011} & Y_{002} \end{pmatrix}$$

which is the generic symmetric 3×3 matrix. But $Cat(2,2;3)$ is not the generic symmetric 6×6 matrix, namely

$$Cat(2,2;3) = \begin{pmatrix} Y_{400} & Y_{310} & Y_{301} & Y_{220} & Y_{211} & Y_{202} \\ Y_{310} & Y_{220} & Y_{211} & Y_{130} & Y_{121} & Y_{112} \\ Y_{301} & Y_{211} & Y_{202} & Y_{121} & Y_{112} & Y_{103} \\ Y_{220} & Y_{130} & Y_{121} & Y_{040} & Y_{031} & Y_{022} \\ Y_{211} & Y_{121} & Y_{112} & Y_{031} & Y_{022} & Y_{013} \\ Y_{202} & Y_{112} & Y_{103} & Y_{022} & Y_{013} & Y_{004} \end{pmatrix}$$

For every positive integer t and any integer s such that $0 \leq s < \binom{t+2}{2}$, we can consider the ideal $I_{s+1,t}$ generated by the $(s+1) \times (s+1)$ minors of $Cat(t,t;3)$. It defines the rank $\leq s$ locus of the matrix $Cat(t,t;3)$ in the projective space $\mathbb{P}^N$ where $N = \binom{2t+2}{2} - 1$. This projective variety is denoted by $V_{s,t}$ and it is not empty if $s \geq 1$. When s and t vary, we get the catalecticant varieties we refer to in the title.

It is well known that the codimension of the ideal generated by the $m \times m$ minors of a generic symmetric $q \times q$ matrix is $\binom{q-m+2}{2}$. Hence the codimension of $V_{s,t}$ is bounded above by

$$\begin{aligned} \text{codim}V_{s,t} &\leq \min\left\{\binom{2t+2}{2} - 1, \binom{\binom{t+2}{2} - (s+1) + 2}{2}\right\} \\ &= \min\left\{\binom{2t+2}{2} - 1, \binom{\widetilde{s}+1}{2}\right\} \end{aligned}$$

where we let

$$\widetilde{s} := \binom{t+2}{2} - s.$$

Notice that, since $s < \binom{t+2}{2}$, we have $\widetilde{s} \geq 1$. Further

$$\dim V_{s,t} = \binom{2t+2}{2} - 1 - \text{codim}V_{s,t} \geq \max\left\{0, \binom{2t+2}{2} - 1 - \binom{\widetilde{s}+1}{2}\right\}.$$

Hence we will say that

$$\text{exdim}V_{s,t} := \max\left\{0, \binom{2t+2}{2} - 1 - \binom{\widetilde{s}+1}{2}\right\}$$

is the *expected dimension* of $V_{s,t}$.

In [4] Diesel made the conjecture that $\dim V_{s,t} = \mathrm{exdim} V_{s,t}$ if $s \geq \binom{t+1}{2}$ or equivalently $\widetilde{s} \leq t+1$.

The conjecture as stated is false, as shown by Y. Cho and B. Jung in [2]. In this paper we give a numerical criterion for the equality $\dim V_{s,t} = \mathrm{exdim} V_{s,t}$: we fix t and we determine an integer $N = N(t)$ such that $V_{s,t}$ has the expected dimension if and only if $\tilde{s} \leq N$.

We describe now the contents of this paper. First we remark in Section 2 that the locus $V_{s,t}$ is the union of certain (smooth) irreducible varieties $\mathbf{Gor}(T)$ parametrizing graded artinian quotients of $k[X_1, X_2, X_3]$ having Hilbert Function T.

In a previous article [3] we had given a compact formula for the dimension of $\mathbf{Gor}(T)$, and here we manipulate this formula to prove in Theorem 2.4 that $\dim V_{s,t} = \max\{\rho(\Gamma)\}$, where Γ runs among the codimension two artinian Hilbert Functions of socle degree at most t and multiplicity s, and where, for $\Gamma = \{a_0, \cdots, a_t\}$, we define

$$\rho(\Gamma) := 2s - \sum_{i=0}^{t-2} a_i(a_{i+2} - a_{i+1}) + 2a_{t-1}a_t - \frac{a_t(a_t+3)}{2}.$$

Using this formula, in Section 3, we first prove (Corollary 3.4) that if $V_{s,t}$ has the expected dimension then, for every integer $a \geq 1$ such that $\binom{a+2}{2} \leq \tilde{s}$, we must have $\tilde{s} \leq f_t(a)$ where

$$f_t(X) = \frac{16tX + X^4 + 6X^3 + 11X^2 + 30X + 8}{4(X+1)(X+2)}.$$

This inequality is proved by looking at some special codimension two artinian Hilbert Functions Γ_a which we call **towers** and which are defined, in the case $\tilde{s} \leq t+1$, for every non negative integer a such that $\binom{a+2}{2} \leq \tilde{s}$.

The last part of this section is devoted to prove that the converse of the above Corollary holds, namely that $V_{s,t}$ has the expected dimension if and only if $\widetilde{s} \leq f_t(a)$ for every $a \geq 1$ such that $\binom{a+2}{2} \leq \tilde{s}$ (Theorem 3.11).

This is an easy consequence of the more subtle result of this paper (Theorem 3.9) which states that, in the case $\tilde{s} \leq \min\{t, \frac{2t}{3} + 4\}$, the dimension of $V_{s,t}$, which is the maximum of the integers $\rho(\Gamma)$, is achieved on one of the towers.

In the last section of the paper, we study the behavior of the rational function $f_t(X)$ in order to determine explicitly the integer $N(t)$. More precisely, we prove in Theorem 4.2 that if $t \leq 42$ then $N(t) = \frac{2t+7}{3}$, while in Theorem 4.6 we prove that if $t \geq 43$ then $N(t) = f_t(\bar{a})$ where $\bar{a}$ is the integer defined by the inequalities $w(\overline{a}) < t \leq w(\overline{a}+1)$, with

$$w(X) := \frac{X^4 + 4X^3 + 5X^2 - 10X + 16}{8(X-2)}.$$

2. A new formula for $\dim V_{s,t}$

In this section we first recall that $V_{s,t}$ is the union of certain (smooth) irreducible varieties $\mathbf{Gor}(T)$ parametrizing graded artinian quotients of $k[X_1, X_2, X_3]$ having Hilbert Function T. Hence its dimension is the dimension of its biggest irreducible components. Using a compact formula for the dimension of $\mathbf{Gor}(T)$ proved in [3], we get a new formula for $\dim V_{s,t}$ which will be crucial for the main result of the paper.

Let $j \geq 2$ and $T = (1, h_1, \ldots, h_{j-1}, 1)$ be a symmetric sequence of integers with $h_1 \leq 3$. We say that T is a **Gorenstein sequence** if T is the Hilbert function of a standard Gorenstein Artinian graded algebra $A = k[X_1, X_2, X_3]/I$. The integer j is called the socle degree of T.

Given a Gorenstein sequence T of socle degree j, let us consider the ideal

$$I_T := \sum_{i=1}^{j-1} I_{h_i+1}(Cat(i, j-i; 3))$$

in the polynomial ring $k[Y_{\underline{a}}]$ with $\binom{j+2}{2}$ variables. This ideal defines a variety in $\mathbb{P}^{\binom{j+2}{2}-1}$ which is denoted by $\mathbf{Gor}_{\leq}(T)$.

It is clear that

$$\mathbf{Gor}_{\leq}(T) = \left\{ P \in \mathbb{P}^{\binom{j+2}{2}-1} \mid rank_P Cat(i, j-i; 3) \leq h_i,\ i = 1, \ldots, j-1 \right\}.$$

We can think of $\mathbb{P}^{\binom{j+2}{2}-1}$ as $\mathbb{P}(S_j)$ where $S = k[Y_1, Y_2, Y_3]$ and we identify the points of $\mathbb{P}^{\binom{j+2}{2}-1}$ with the corresponding forms of degree j in S.

Recall that every form F of degree j in $k[Y_1, Y_2, Y_3]$ corresponds, up to scalars, to an artinian Gorenstein graded algebra $A = k[X_1, X_2, X_3]/I_F$ of socle degree j, through the so called *inverse system* of Macaulay. Further, if $A = R/I_F$ is the Gorenstein algebra corresponding to the form F, the Hilbert function of A is given by the formula

$$H_A(i) = rank_F Cat(i, j-i; 3)$$

for every $i \geq 0$. This crucial result is due to Macaulay and a proof can be found in [5] Lemma 2.14.

Hence the elements of $\mathbf{Gor}_{\leq}(T)$ can be identified with Gorenstein Artinian graded algebras $A = k[X_1, X_2, X_3]/I$ with socle degree j and Hilbert function $H_A \leq T$, where the inequality is coefficientwise.

We can partially order the Gorenstein sequences of socle degree j coefficientwise. If $T' \leq T$, then $I_T \subseteq I_{T'}$ so that

$$\mathbf{Gor}_{\leq}(T') \subseteq \mathbf{Gor}_{\leq}(T).$$

Hence, given a Gorenstein sequence T, we can consider the open subset $\mathbf{Gor}(T)$ of $\mathbf{Gor}_{\leq}(T)$ defined as

$$\mathbf{Gor}(T) := \mathbf{Gor}_{\leq}(T) \setminus \bigcup_{T' < T} \mathbf{Gor}_{\leq}(T').$$

We clearly have

$$\mathbf{Gor}(T) = \{F \in \mathbb{P}(S_j) \mid rank_F Cat(i, j-i; 3) = h_i,\ i = 1, \dots, j-1\}.$$

Hence we can say that $\mathbf{Gor}(T)$ parametrizes Artinian Gorenstein graded algebra $A = k[X_1, X_2, X_3]/I$ with socle degree j and Hilbert function $H_A = T$.

In [4] Diesel proved that $\mathbf{Gor}(T)$ is irreducible for every Gorenstein sequence T.

Given the positive integer t, let $1 \le s < \binom{t+2}{2}$. We define Δ to be the set of sequences $T = (1, h_1, \dots, h_{2t-1}, 1)$ which are the Hilbert Functions of Gorenstein Artinian graded algebras $A = k[X_1, X_2, X_3]/I$ of socle degree $2t$ and with $h_t = s$.

If we can consider the union of irreducible strata

$$U_{s,t} := \cup_{T \in \Delta} \mathbf{Gor}(T), \tag{1}$$

it is clear that we can describe $U_{s,t}$ as

$$U_{s,t} = \big\{F \in \mathbb{P}(S_{2t}) \mid rank_F Cat(t, t; 3) = s\big\}.$$

This identifies $U_{s,t}$ as an open subset of $V_{s,t}$, which was defined as the rank $\le s$ locus of $Cat(t, t; 3)$ in $\mathbb{P}^{\binom{2t+2}{2}-1}$.

By Lemma 3.5 pg. 75 in [5], $U_{s,t}$ is in fact a dense open subset of $V_{s,t}$, hence, using (1), we get

$$\dim V_{s,t} = \dim U_{s,t} = \max_{T \in \Delta} \big\{ \dim \mathbf{Gor}(T) \big\}. \tag{2}$$

We recall here that each Gorenstein sequence $T \in \Delta$ is $2t$-symmetric in the sense that, for $i \le t$, $h_{2t-i} = h_i$.

Further, for every $T \in \Delta$ we have $h_1 = 3$, save for the sequences of embedding dimension ≤ 2 which are either $(1, 1, \cdots, 1, 1)$ if $s = 1$, or $(1, 2, \cdots, s, s, s, \cdots, 2, 1)$ if $2 \le s \le t+1$.

Later in this section we will give a formula for computing $\dim \mathbf{Gor}(T)$ for every Gorenstein sequence T of embedding dimension 3, but let us first consider the case of embedding dimension ≤ 2.

It is clear that a graded algebra $A = k[X_1, X_2, X_3]/I$ has embedding dimension ≤ 2 if and only if I is a complete intersection ideal generated by three forms of degree $1, s, 2t - s + 2$.

If $s = 1$, I is a complete intersection of forms of degree $1, 1, 2t + 1$. Hence $\dim \mathbf{Gor}(T)$ equals the dimension of the Grassmannian $Gr(2, 3)$ of 2-dimensional linear subspaces of the 3-dimensional vector space $k[X_1, X_2, X_3]_1$. Hence

$$\dim \mathbf{Gor}(T) = \dim Gr(2, 3) = 2, \tag{3}$$

(this is case 6 in Section 4.6 of Diesel paper [4]).

If $s = t + 1$, then I is a complete intersection of forms of degree $1, t+1, t+1$, and it is clear that

$$\dim \mathbf{Gor}(T) = \dim Gr(1, 3) + \dim Gr(2, s + 1) = 2s$$

(this is case 3 in Section 4.6 of Diesel paper [4]).

Finally, if $2 \le s \le t$, we have

$$\dim \mathbf{Gor}(T) = \dim Gr(1,3) + \dim Gr(1, s+1) + \dim Gr(1, 2t-s+3-(2t-2s+3))$$

where the last summand is like that because, if F is a form of degree s in two variables, then $\dim(F)_{2t-s+2} = 2t-2s+3$. Thus, if $2 \le s \le t$, we get

$$\dim \mathbf{Gor}(T) = 2 + s + (s-1) = 2s+1 \tag{4}$$

(this is case 5 in Section 4.6 of Diesel paper [4]).

We can use these results to compute $\dim V_{1,t}$ and $\dim V_{2,t}$ for every t. Namely if $s = 1$, then $\Delta = \{(1, 1, \cdots, 1, 1)\}$ so that, by (1) and (3), we get

$$\dim V_{1,t} = 2.$$

If $s = 2$, then $\Delta = \{(1, 2, \cdots, 2, 1)\}$ so that, by (1) and (4), we get

$$\dim V_{2,t} = 5.$$

We give now an easy formula for $\dim \mathbf{Gor}(T)$ when T is a Gorenstein sequence of socle degree $2t$ such that $h_1 = 3$ and $h_t = s$.

For a general Gorenstein sequence T of socle degree $j \ge 2$, Kleppe proved in [6] that $\mathbf{Gor}(T)$ is smooth; hence the dimension of $\mathbf{Gor}(T)$ is equal to the dimension of the tangent space to $\mathbf{Gor}(T)$ in any point. We may apply Theorem 3.9 in [5] to get

$$\dim \mathbf{Gor}(T) = H_{R/I^2}(j) - 1 = H_{I/I^2}(j)$$

for every ideal I such that $A := k[X_1, X_2, X_3]/I$ is Gorenstein and $H_A = T$. If we write the Hilbert series of A as

$$P_A(z) = h(z) = 1 + h_1 z + h_2 z^2 + \ldots + h_{j-2} z^{j-2} + h_{j-1} z^{j-1} + z^j,$$

we proved in [3] Theorem 4.1 that the Hilbert Series of I/I^2 is

$$P_{I/I^2}(z) = (1+z)^3 h(z^2)/2 - (1-z)^3 h(z)^2/2 - z^{j+3} h(z).$$

Hence $\dim \mathbf{Gor}(T)$ is equal to the coefficient of z^j in the polynomial

$$\frac{(1+z)^3 h(z^2) - (1-z)^3 h(z)^2}{2}.$$

In the case $j = 2t$, the coefficient of z^{2t} in $\frac{(1+z)^3 h(z^2)}{2}$ is

$$\frac{h_t + 3h_{t-1}}{2}$$

and that of z^{2t} in

$$\frac{(1-z)^3h(z)^2}{2} = \frac{(1-z)h(z)}{2}\frac{(1-z)^2h(z)}{2}$$

is

$$\sum_{i=0}^{2t} \frac{a_i b_{2t-i}}{2}$$

where we let

$$\sum a_i z^i := (1-z)h(z), \quad \sum b_i z^i := (1-z)^2 h(z)$$

to be the first and second difference of $h(z)$.

Summing up we get

$$\dim \mathbf{Gor}(T) = \frac{h_t + 3h_{t-1} - \sum_{i=0}^{2t} a_i b_{2t-i}}{2}.$$

In the case $h_t = s$, we have $a_t = s - h_{t-1}$ so that

$$h_t + 3h_{t-1} = s + 3(s - a_t) = 4s - 3a_t.$$

As we have seen before, $h(z)$ is $2t$-symmetric, hence $(1-z)h(z)$ is $(2t+1)$-antisymmetric and $(1-z)^2h(z)$ is $(2t+2)$-symmetric. This means

$$a_{t+k} = -a_{t+1-k}, \quad b_j = b_{2t+2-j}.$$

We get

$$\dim \mathbf{Gor}(T) = \frac{4s - 3a_t - \sum_{i=0}^{t} a_i b_{i+2} + \sum_{i=t+1}^{2t} a_{2t+1-i} b_{2t-i}}{2}.$$

It is easy to see that we have

$$a_{j+1}b_j + a_{j-1}b_{j+1} = a_j(a_{j+1} - a_{j-1})$$

for every $j \geq 1$, so that

$$\begin{aligned}
&\sum_{i=0}^{t} a_i b_{i+2} - \sum_{i=t+1}^{2t} a_{2t+1-i} b_{2t-i} \\
=&\sum_{i=0}^{t} a_i b_{i+2} - \sum_{i=t+1}^{2t-1} a_{2t-i}(a_{2t-i+1} - a_{2t-i-1}) + \sum_{i=t+1}^{2t-1} a_{2t-i-1} b_{2t-i+1} - a_1 b_0 \\
=&\sum_{i=0}^{t} a_i b_{i+2} - a_{t-1}a_t + a_0 a_1 + \sum_{i=0}^{t-2} a_i b_{i+2} - a_1 b_0 \\
=&2\sum_{i=0}^{t-2} a_i b_{i+2} + a_{t-1}b_{t+1} + a_t b_{t+2} - a_{t-1}a_t \\
=&2\sum_{i=0}^{t-2} a_i b_{i+2} - 4a_{t-1}a_t + a_t^2.
\end{aligned}$$

By easy computation, we get from the above formula the following result:

Proposition 2.1

Let $T = \{1, 3, h_2, \ldots, h_{2t-2}, 3, 1\}$ be a Gorenstein sequence of socle degree $2t$ and with $h_t = s$; let $a_i := h_i - h_{i-1}$ for every i. Then

$$\dim \mathbf{Gor}(T) = 2s - \sum_{i=0}^{t-2} a_i(a_{i+2} - a_{i+1}) + 2a_{t-1}a_t - \frac{a_t(a_t+3)}{2}.$$

We recall now that Stanley proved in [7] that a symmetric sequence of socle degree $2t$, say $\{1, 3, \ldots, h_i, \ldots, 3, 1\}$, is a Gorenstein sequence if and only if half of its first difference $(1, 2, h_2 - 3, \ldots, h_t - h_{t-1})$ is a codimension two admissible sequence, which means a sequence which is the Hilbert function of an Artinian graded algebra $k[X_1, X_2]/J$ of embedding dimension two and socle degree at most t.

Notice that if we let as before $a_i := h_i - h_{i-1}$, then $\sum_{i=o}^{t} a_i = h_t$.

We can easily describe the codimension two admissible sequences of socle degree at most t. They are sequences $\Gamma = (a_0 = 1, a_1 = 2, \ldots, a_t)$ of $t+1$ non negative integers with the property that for some integer m, $2 \le m \le t+1$

1) $a_i = i + 1$ for $0 \le i \le m - 1$,
2) $0 \le a_{i+1} \le a_i$ for $m - 1 \le i \le t$.

The integer m is called the *initial degree* of Γ.

DEFINITION 2.2. We say that **a sequence** $\Gamma = (a_0 = 1, 2, \ldots, a_t)$ **is in** $V_{s,t}$ and, by abuse of notation, we write $\Gamma \in V_{s,t}$, if Γ verifies the above conditions 1) and 2) and moreover has multiplicity s, which means $\sum a_i = s$.

For $\Gamma \in V_{s,t}$ we define

$$\rho(\Gamma) := 2s - \sum_{i=0}^{t-2} a_i(a_{i+2} - a_{i+1}) + 2a_{t-1}a_t - \frac{a_t(a_t+3)}{2}.$$

Using Proposition 2.1 we can prove now the following well known lemma.

Lemma 2.3

Let $1 \le s \le \binom{t+1}{2}$. Then $\dim V_{s,t} \ge 3s - 1$.

Proof. If $s = 1, 2$ we have already seen that $\dim V_{s,t} = 3s - 1$. Let $3 \le s \le \binom{t+1}{2}$. It is clear that there exists an integer m such that $\binom{m+1}{2} \le s < \binom{m+2}{2}$; this forces $2 \le m \le t$ and $m = t$ if and only if $s = \binom{t+1}{2}$.

Let us consider the sequence

$\Gamma :=$

0	1	...	m-1	m	m+1	...	t
1	2	...	m	$s - \binom{m+1}{2}$	0	...	0

Since $0 \le s - \binom{m+1}{2} \le m+1$, we get $\Gamma \in V_{s,t}$ and

$$\rho(\Gamma) = 2s - \left[\sum_{j=1}^{m-2} j + (m-1)\left(s - \binom{m+1}{2} - m\right) + m\left(\binom{m+1}{2} - s\right)\right] = 3s - 1.$$

By (2) and Proposition 2.1 we get $\dim V_{s,t} \ge \rho(\Gamma) = 3s - 1$ and the conclusion follows. □

A corollary of this easy result is the following crucial formula for $\dim V_{s,t}$.

Theorem 2.4

Let $t \ge 2$ and $3 \le s < \binom{t+2}{2}$; then

$$\dim V_{s,t} = \max\left\{\rho(\Gamma)\right\},$$

where the maximum is over the sequences $\Gamma \in V_{s,t}$ as described in Definition 2.2.

Proof. If $s \ge t+2$ the result is clear because for every $T \in \Delta$ we have $h_1 = 3$. If $s \le t+1$ then $s \le \binom{t+1}{2}$ and the conclusion follows by the Lemma because the unique sequence in Δ with $h_1 \le 2$ is $T = (1, 2, \ldots, s-1, s, \ldots, s, s-1, \ldots, 2, 1)$ for which $\dim \mathbf{Gor}(T) \le 2s + 1 < 3s - 1$, as we have pointed out before. □

For example, if $\widetilde{s} = 1$, then $s = \binom{t+2}{2} - 1$ and $V_{s,t}$ is an hypersurface in $\mathbb{P}^{\binom{2t+2}{2}-1}$. Hence we have

$$\dim V_{s,t} = \binom{2t+2}{2} - 2 = 2t^2 + 3t - 1$$

which is the expected dimension. Let us compute $\dim V_{s,t}$ when $t \ge 2$ by using Theorem 2.4. It is clear that $s \ge t+2$ and the unique sequence in $V_{s,t}$ is

$$\Gamma := (1, 2, \ldots, t, t).$$

We have

$$\begin{aligned} \rho(\Gamma) &= 2s - [1 + 2 + \ldots + (t-2)] + 2t^2 - \frac{t(t+3)}{2} \\ &= 2\binom{t+2}{2} - 2 - \binom{t-1}{2} + 2t^2 - \frac{t(t+3)}{2} = 2t^2 + 3t - 1. \end{aligned}$$

If $t = 1$, then $s = 2$ and the hypersurface $V_{2,1}$ is the zero locus of the determinant of the generic symmetric 3×3 matrix, namely the secant line variety to the Veronese surface in $\mathbb{P}^5$.

Also the case $\widetilde{s} = 2, 3, 4$ are easy to handle. If $\widetilde{s} = 2$ and $t = 1$, then $s = 1$ and $\dim V_{1,1} = 2$ which is the expected dimension. If $t \ge 2$, then $s = \binom{t+2}{2} - 2 \ge t + 2$. It is clear that there is a unique sequence in $V_{s,t}$ and it is

$$\Gamma = (1, 2, \ldots, t, t-1).$$

We have

$$\rho(\Gamma) = 2s - [1+2+\ldots+(t-2)-(t-1)] + 2t(t-1) - \frac{(t-1)(t+2)}{2} = 2t^2+3t-3.$$

The expected dimension is

$$\binom{2t+2}{2} - 1 - 3 = 2t^2+3t-3.$$

If $t=1$, then $s=1$ and the surface $V_{1,1}$ is the zero locus of the ideal generated by the 2×2 minors of the generic symmetric 3×3 matrix, namely the Veronese surface in $\mathbb{P}^5$.

In the case $\widetilde{s}=3$ we have $s=\binom{t+2}{2}-3$ so that $t\geq 2$; if $t=2$, then $s=3$ and $\dim V_{3,2}=8$. This is the expected dimension.

If $t\geq 3$, we have two sequences in $V_{s,t}$, namely

$$\Gamma = (1,2,\ldots,t-1,t,t-2), \qquad \Lambda = (1,2,\ldots,t-1,t-1,t-1).$$

We have

$$\rho(\Gamma) = 2t^2+3t-6, \quad \rho(\Lambda) = 2t^2+t-4,$$

hence $\dim V_{s,t} = 2t^2+3t-6$. The expected dimension is

$$\binom{2t+2}{2} - 1 - 6 = 2t^2+3t-6.$$

Finally in the case $\widetilde{s}=4$, we have $s=\binom{t+2}{2}-4$ so that $t\geq 2$; if $t=2$, then $s=2$ and $\dim V_{2,2}=5$. The expected dimension is 4 so that this is the first example where the dimension is bigger than the expected dimension.

If $t\geq 3$, we have two sequences in $V_{s,t}$, namely

$$\Gamma = (1,2,\ldots,t-1,t,t-3), \quad \Lambda = (1,2,\ldots,t-1,t-1,t-2).$$

By easy computation we get

$$\rho(\Gamma) = 2t^2+3t-10 > \rho(\Lambda) = 2t^2+t-5,$$

hence $\dim V_{s,t} = 2t^2+3t-10$, which is the expected dimension.

Let us consider the case $t=2$. We have seen that $\dim V_{1,2}=2$, $\dim V_{2,2}=5$, $\dim V_{3,2}=8$, $\dim V_{4,2}=11$ ($\widetilde{s}=2$), $\dim V_{5,2}=13$ ($\widetilde{s}=1$). Hence $V_{s,2}$ has the expected dimension if and only if $s\geq 3$. This is the kind of result we are looking for when $t\geq 3$.

3. The main result

In this section we focus on some special sequences $\Gamma_a \in V_{s,t}$ which are defined, in the case $\widetilde{s}\leq t+1$, for every non negative integer a such that $\binom{a+2}{2}\leq \widetilde{s}$. These sequences will be called the **towers** for $V_{s,t}$; their relevance will be clear when we prove that, in the case $\widetilde{s} \leq \min(t, \frac{3t}{3}+4)$, the maximum of the integers $\rho(\Gamma)$, which is the dimension of $V_{s,t}$, is achieved on one of the towers.

As a consequence we will get an explicit criterion for $V_{s,t}$ having the expected dimension.

In the following, to avoid trivial cases already considered, we assume that $t \geq 3$ and s is an integer such that $3 \leq s < \binom{t+2}{2}$. First of all we prove that $\widetilde{s} \leq t+1$ is a necessary condition for $V_{s,t}$ to have the expected dimension.

Proposition 3.1

If $\widetilde{s} \geq t+2$, *then* $\dim\ V_{s,t} > \text{exdim}\ V_{s,t}$.

Proof. We have $\widetilde{s} \geq t+2$ so that $s \leq \binom{t+1}{2} - 1$, hence, by Lemma 2.3, we get $\dim\ V_{s,t} \geq 3s-1$.

We claim that $3s-1$ is strictly bigger than the expected dimension, namely

$$3s-1 > \max\left\{0, \binom{2t+2}{2} - 1 - \binom{\widetilde{s}+1}{2}\right\}.$$

We have

$$3s-1 > \binom{2t+2}{2} - 1 - \binom{\widetilde{s}+1}{2}$$

if and only if

$$\widetilde{s}^2 - 5\widetilde{s} + 4 - t^2 + 3t > 0.$$

Since $\widetilde{s} \geq t+2$, we must prove that

$$t+2 > \frac{5+\sqrt{9+4t^2-12t}}{2}$$

which is equivalent to $8(t-1) > 0$. The conclusion follows. $\square$

A consequence of this result is that, when $\widetilde{s} \geq t+2$, one should better take $3s-1$ for the expected dimension of $V_{s,t}$. Namely in the paper [2] some instances where $\dim\ V_{s,t} = 3s-1$ are presented. For example it is shown that this is the case when $t \geq 9$ and $t+1 \leq s \leq 4t-3$.

We used Theorem 2.4 for computing $\dim\ V_{s,t}$ in the case $t = 17$ and $s = 150$ ($\widetilde{s} = 21$). We got $\dim\ V_{150,17} = 459 > 3s-1 = 449$, but of course here $s = 150 > 4t-3 = 65$.

It would be interesting to determine, in the case $\widetilde{s} \geq t+2$ ($s \leq \binom{t+1}{2}$), when $\dim\ V_{s,t} = 3s-1$.

We remark that if $\widetilde{s} = t+1$, then $3s-1 = \binom{2t+2}{2} - 1 - \binom{\widetilde{s}+1}{2}$.

We also notice that if $\widetilde{s} \leq t+1$, then $\text{exdim} V_{s,t} = \binom{2t+2}{2} - 1 - \binom{\widetilde{s}+1}{2}$.

If $\widetilde{s} \leq t+1$ and $a \geq 0$ is an integer such that $\binom{a+2}{2} \leq \widetilde{s}$, then $\binom{a+2}{2} \leq t+1$ so that $a \leq t-2$. Since $t-a \geq t+1-\widetilde{s}+\binom{a+1}{2}$, the sequence

$\Gamma_a :=$

0	1	...	t-a-1	...	t-1	t
1	2	...	t-a	...	t-a	t+1-$\widetilde{s} + \binom{a+1}{2}$

is in $V_{s,t}$. In particular Γ_0 is always in $V_{s,t}$, because $\tilde{s} \geq 1$.

DEFINITION 3.2. Let $\widetilde{s} \leq t+1$. For every non negative integer a such that $\binom{a+2}{2} \leq \widetilde{s}$, let Γ_a to be the sequence

$\Gamma_a :=$

0	1	...	t-a-1	...	t-1	t
1	2	...	t-a	...	t-a	t+1-$\widetilde{s} + \binom{a+1}{2}$

Such a sequence is in $V_{s,t}$ and is called a **tower** for $V_{s,t}$.

For example, if $t = 12$ and $\widetilde{s} = 10$, then $s = 81$ and the towers for $V_{81,12}$ are the following four sequences:

	0	1	...	7	8	9	10	11	12
Γ_0	1	2	...	8	9	10	11	12	3
Γ_1	1	2	...	8	9	10	11	11	4
Γ_2	1	2	...	8	9	10	10	10	6
Γ_3	1	2	...	8	9	9	9	9	9

We always have

$$\begin{aligned}\rho(\Gamma_0) &= 2s - \left[\binom{t-1}{2} + (t-1)(1-\widetilde{s})\right] + 2t(t+1-\widetilde{s}) - \frac{(t+1-\widetilde{s})(t+4-\widetilde{s})}{2} \\ &= 2t^2 + 3t - \binom{\widetilde{s}+1}{2} = \binom{2t+2}{2} - 1 - \binom{\widetilde{s}+1}{2},\end{aligned}$$

hence

$$\mathrm{exdim} V_{s,t} = \rho(\Gamma_0).$$

This proves that Diesel conjecture, even if not true, is consistent. This was already remarked by Diesel in [4], pg. 385.

In the following we will use the equality:

$$\rho(\Gamma_0) = \binom{2t+2}{2} - 1 - \binom{\widetilde{s}+1}{2} = 2s + t^2 - 2 + 3\widetilde{s}/2 - \widetilde{s}^2/2. \tag{5}$$

which is easy to prove.

To compute the difference $\rho(\Gamma_a) - \rho(\Gamma_0)$ for $a \geq 1$, we need to introduce the following functions

$$D(X) := \frac{X^4 + 6X^3 + 11X^2 + 30X + 8}{4},$$

and for every positive integer t

$$f_t(X) := \frac{4Xt + D(X)}{(X+1)(X+2)} = \frac{16tX + X^4 + 6X^3 + 11X^2 + 30X + 8}{4(X+1)(X+2)}. \tag{6}$$

Lemma 3.3

Let Γ_a be a tower for $V_{s,t}$ with $a \geq 1$; then

$$\rho(\Gamma_a) - \rho(\Gamma_0) = \binom{a+2}{2}\big[\widetilde{s} - f_t(a)\big].$$

Proof. If we let $r := t + 1 - \widetilde{s} + \binom{a+1}{2}$, we have

$$\begin{aligned}\rho(\Gamma_a) &= 2s - \left[\binom{t-a-1}{2} + (t-a)(r-t+a)\right] + 2r(t-a) - \frac{r(r+3)}{2} \\ &= 2s - \binom{t-a-1}{2} + (t-a)^2 + r(t-a) - \frac{r(r+3)}{2}.\end{aligned}$$

When we use the equality $r := t + 1 - \widetilde{s} + \binom{a+1}{2}$, we get

$$\rho(\Gamma_a) = 2s - \frac{a^4}{8} - \frac{3a^3}{4} - \frac{11a^2}{8} - \frac{15a}{4} + t^2 - 2at + \frac{a^2\widetilde{s}}{2} + \frac{3a\widetilde{s}}{2} - \frac{\widetilde{s}^2}{2} + \frac{5\widetilde{s}}{2} - 3.$$

Hence, by (5), we get

$$\begin{aligned}\rho(\Gamma_a) - \rho(\Gamma_0) &= -\frac{a^4}{8} - \frac{3a^3}{4} - \frac{11a^2}{8} - \frac{15a}{4} - 2at + \frac{a^2\widetilde{s}}{2} + \frac{3a\widetilde{s}}{2} + \widetilde{s} - 1 \\ &= \binom{a+2}{2}\widetilde{s} - 2at - \frac{a^4 + 6a^3 + 11a^2 + 30a + 8}{8} \\ &= \binom{a+2}{2}\widetilde{s} - \left[2at + \frac{D(a)}{2}\right] = \binom{a+2}{2}\big[\widetilde{s} - f_t(a)\big]. \ \square\end{aligned}$$

As a trivial consequence of the above lemma, we get a necessary condition for $V_{s,t}$ having the expected dimension.

Corollary 3.4

If $V_{s,t}$ has the expected dimension, then for every $a \geq 1$ such that $\binom{a+2}{2} \leq \tilde{s}$, we have

$$\widetilde{s} \leq f_t(a).$$

Proof. By assumption $\dim V_{s,t} = \text{exdim} V_{s,t} = \rho(\Gamma_0)$, so that by Theorem 2.4 $\rho(\Gamma_a) \leq \rho(\Gamma_0)$ for every tower Γ_a. The conclusion follows by the lemma. □

Thus, for example, $V_{15,5}$ cannot have the expected dimension because $\widetilde{s} = 6$ and $\binom{1+2}{2} = 3 \leq \widetilde{s} = 6$, but $f_5(1) = 17/3 < 6$.

We want to prove now that the converse of the above statement holds. This will be a consequence of the fact that when $\widetilde{s} \leq \min(t, \frac{2t}{3} + 4)$ then $\dim V_{s,t}$ is achieved on a tower.

We will prove this last result by using the following strategy. Starting from a sequence in $V_{s,t}$ we will reach a tower along a path of sequences in $V_{s,t}$ in such a way that, at each step, the function ρ does not decrease. We need some preparatory result.

Proposition 3.5

Let $\Gamma = (\dots, d, a, b, c)$ be a sequence in $V_{s,t}$ such that $c > 0$ and $a \neq b < t$. Then $\Lambda = (\dots, d, a, b+1, c-1)$ is in $V_{s,t}$. Further, if $\widetilde{s} \leq (2/3)t + 4$, then

$$\rho(\Lambda) \geq \rho(\Gamma).$$

Proof. If $a < b$, then $a = t-1$ and $b = t$. Hence $a > b$ so that $\Lambda \in V_{s,t}$.

It is clear that for a suitable integer K we can write

$$\begin{aligned} \rho(\Lambda) - \rho(\Gamma) = &\Big\{ 2s - \big[K + d(b+1-a) + a(c-1-b-1)\big] \\ &+ 2(b+1)(c-1) - \frac{(c-1)(c+2)}{2} \Big\} \\ &- \Big\{ 2s - \big[K + d(b-a) + a(c-b)\big] + 2bc - \frac{c(c+3)}{2} \Big\} \\ = &\ 2a - 2b + 3c - d - 1. \end{aligned}$$

We let $j := t - d$, hence $j \geq 2$ since $d \leq t-2$. We clearly have

$$s \leq \binom{t-2}{2} + d + a + b + c = \binom{t-2}{2} + t - j + a + b + c$$

so that, with the assumption $\widetilde{s} \leq (2/3)t + 4$, we get

$$(2/3)t + 4 \geq \widetilde{s} = \binom{t+2}{2} - s \geq \binom{t+2}{2} - \binom{t-2}{2} - t + j - (a+b+c),$$

which implies

$$t \leq \frac{3(a+b+c) - 3j + 18}{7}. \tag{7}$$

But $b + 1 \leq a \leq t - 1$, hence

$$a \leq t - 1 \leq \frac{3(a+b+c) - 3j + 18 - 7}{7} \leq \frac{3(2a+c-1) - 3j + 11}{7}$$

which implies

$$a \leq 3c - 3j + 8. \tag{8}$$

We need to prove

$$t - j \leq 2a - 2b + 3c - 1.$$

By (7) we have

$$t - j \leq \frac{3(a+b+c) - 3j + 18 - 7j}{7},$$

so that we only need to prove

$$17b \leq 11a + 18c + 10j - 25.$$

But $b \leq a - 1$, hence

$$17b \leq 17(a-1) = 11a + 6a - 17 \leq 11a + 6(3c - 3j + 8) - 17 = 11a + 18c - 18j + 31,$$

where the inequality follows by (8).

It remains to prove that

$$11a + 18c - 18j + 31 \leq 11a + 18c + 10j - 25.$$

But this is equivalent to $56 \leq 28j$ which is true because $j \geq 2$. □

The assumption $\widetilde{s} \leq (2/3)t + 4$ in the above Proposition is crucial.

Let $t = 13$, $s = 92$ so that $\widetilde{s} = 13 \leq t$ but $\widetilde{s} = 13 > (2/3)t + 4 = 38/3$. With $\Gamma = (\ldots, 10, 11, 12, 11, 3)$ and $\Lambda = (\ldots, 10, 11, 12, 12, 2)$, we have $\rho(\Gamma) = 293$ and $\rho(\Lambda) = 292$.

Lemma 3.6

Let us assume $r \geq 2$, $\widetilde{s} \leq t$, and suppose that the sequence

$\Gamma :=$

...	*t-r*	...	*t-1*	*t*
...	*b*	...	*b*	*c*

is in $\in V_{s,t}$. Then

$$3b \geq t + \binom{r+1}{2}.$$

Proof. Since $\widetilde{s} \leq t$, $c \leq b$ and $a_{t-r-1} \leq t - r$, we have

$$\binom{t+1}{2} < s \leq \binom{t-r+1}{2} + rb + c \leq \binom{t-r+1}{2} + (r+1)b$$

hence

$$(r+1)b \geq \binom{t+1}{2} + 1 - \binom{t-r+1}{2} = rt - \binom{r}{2} + 1. \tag{9}$$

On the other hand, $a_{t-r} = b \leq t - r + 1$ so that

$$(r+1)(t-r+1) \geq (r+1)b \geq rt - \binom{r}{2} + 1$$

which implies

$$t \geq \binom{r+1}{2}.$$

By (9), we must prove

$$\frac{rt - \binom{r}{2} + 1}{r+1} \geq \frac{t + \binom{r+1}{2}}{3}.$$

We have

$$\frac{rt - \binom{r}{2} + 1}{r+1} - \frac{t + \binom{r+1}{2}}{3} = \frac{t(4r-2) - (r^3 + 5r^2 - 2r - 6)}{6(r+1)}$$

so that, since $t \geq \binom{r+1}{2}$, we only need to prove that

$$\binom{r+1}{2} \geq \frac{r^3 + 5r^2 - 2r - 6}{4r - 2}.$$

This is equivalent to

$$r^3 - 4r^2 + r + 6 \geq 0$$

so that the conclusion follows because $r \geq 2$ and

$$r^3 - 4r^2 + r + 6 = (r-2)(r-3)(r+1). \square$$

The assumption $\widetilde{s} \leq t$ in the Lemma is essential. If $t = 5$ and $\widetilde{s} = 6$, we get $s = 15$ and $(1, 2, 3, 3, 3, 3) \in V_{15,5}$ but $9 < 5 + \binom{4}{2} = 11$.

Proposition 3.7

Let $r \geq 2$ and let

$\Gamma :=$

...	*t-r-2*	*t-r-1*	*t-r*	...	*t-1*	*t*
...	*d*	a	b	...	b	c

If $\Gamma \in V_{s,t}$, $\widetilde{s} \leq t$, $b < a$ and $c \geq b - 2$, then the sequence

$\Lambda :=$

...	*t-r-2*	*t-r-1*	*t-r*	...	*t-1*	*t*
...	*d*	a	*b+1*	...	*b+1*	*c-r*

is in $V_{s,t}$ and

$$\rho(\Lambda) \geq \rho(\Gamma).$$

Proof. In order to prove that $\Lambda \in V_{s,t}$, we only need to prove that $c \geq r$. But if $c < r$ the sequence

...	t-r-2	t-r-1	t-r	...	t-r+c-1	t-r+c	...	t-1	t
...	d	a	b+1	...	b+1	b	...	b	0

would be in $V_{s,t}$, a contradiction to the assumption $s > \binom{t+1}{2}$.

It is clear that for a suitable integer K we have

$$\rho(\Lambda) = 2s - [K + d(b+1-a) + (b+1)(c-r-b-1)]$$

$$+ 2(b+1)(c-r) - \frac{(c-r)(c-r+3)}{2}$$

$$\rho(\Gamma) = 2s - [K + d(b-a) + b(c-b)] + 2bc - \frac{c(c+3)}{2}.$$

An easy computation shows that $\rho(\Lambda) - \rho(\Gamma) = -d + b(2-r) + c(r+1) - \binom{r}{2} + 1$, hence we need to prove

$$b(2-r) + c(r+1) - \binom{r}{2} + 1 \geq d.$$

Since $c \geq b-2$ and $d \leq t-r-1$, it is enough to prove that

$$3b - 2(r+1) - \binom{r}{2} + 1 \geq t - r - 1$$

which is the same as

$$3b \geq t - r - 1 + 2(r+1) + \binom{r}{2} - 1 = t + \binom{r+1}{2}.$$

This is true by the above lemma. □

The assumption $c \geq b-2$ in the above Proposition is crucial. Let $t = 22$ and $s = 254$ so that $\widetilde{s} = 22$. If $\Gamma = (\ldots, 19, 20, 19, 19, 6)$ and $\Lambda = (\ldots, 19, 20, 20, 20, 4)$, then $\rho(\Gamma) = 804$, while $\rho(\Lambda) = 803$.

Lemma 3.8

Let $r \geq 2$ and

$\Gamma :=$

...	*t-r-3*	*t-r-2*	*t-r-1*	*t-r*	...	*t-1*	*t*
...	*e*	*d*	*a*	*b*	...	*b*	*c*

If $\Gamma \in V_{s,t}$, $a \neq b \neq c$ and $b \leq t - r$, then

$\Lambda :=$

...	*t-r-3*	*t-r-2*	*t-r-1*	*t-r*	...	*t-1*	*t*
...	*e*	*d*	*a-1*	*b*	...	*b*	*c+1*

is in $V_{s,t}$ and

$$\rho(\Lambda) - \rho(\Gamma) = e - d + b - c - 2.$$

Proof. If $a < b$, then $a = t - r$ and $b = t - r + 1$, a contradiction. Hence $a > b$. Further $b \geq c$, so that $c < b$. This proves that $\Lambda \in V_{s,t}$.

For a suitable integer K we can write

$$\rho(\Lambda) = 2s - [K + e(a - 1 - d) + d(b - a + 1) + b(c + 1 - b)]$$

$$+ 2b(c + 1) - \frac{(c + 1)(c + 4)}{2}$$

$$\rho(\Gamma) = 2s - [K + e(a - d) + d(b - a) + b(c - b)] + 2bc - \frac{c(c + 3)}{2}.$$

An easy computation shows that

$$\rho(\Lambda) - \rho(\Gamma) = e - d + b - c - 2. \ \Box$$

We are ready to prove the main result of the paper.

Theorem 3.9

If $\widetilde{s} \leq \min\{t, \frac{2t}{3} + 4\}$, *then*

$$\dim V_{s,t} = \max\{\rho(\Gamma_a)\}$$

where the maximum is over the towers for $V_{s,t}$.

Proof. By Theorem 2.4 we know that $\dim V_{s,t} = \max_{\Gamma \in V_{s,t}}\{\rho(\Gamma)\}$, hence it suffices to show that, given a sequence $\Gamma \in V_{s,t}$, one can find a tower Γ_a for $V_{s,t}$ such that $\rho(\Gamma_a) \geq \rho(\Gamma)$.

Let $\Gamma = (\dots, n, b, m)$ be an element of $V_{s,t}$. If $n < b$ then $n = t - 1$, $b = t$ and $\Gamma = \Gamma_0$. Hence we may assume that $n \geq b$ so that $b < t$; since $s > \binom{t+1}{2}$ we have also $m > 0$. If $b < n$, by Proposition 3.5 the sequence $\Lambda = (\dots, n, b + 1, m - 1)$ is in $V_{s,t}$ and $\rho(\Lambda) \geq \rho(\Gamma)$. Going on in this way, we may assume that

$\Gamma :=$

...	t-r-3	t-r-2	t-r-1	t-r	...	t-1	t
...	e	d	a	b	...	b	c

with $r \geq 2$ and $a \neq b$.

If $a < b$, then $a = t - r$, $b = t - r + 1$ and $\Gamma = \Gamma_{r-1}$ is a tower. So let $a > b$, which implies also $b \leq t - r$. If $c \geq b - 2$, by Proposition 3.7, the sequence

$N :=$

...	t-r-1	t-r	...	t-1	t
...	a	b+1	...	b+1	c-r

is in $V_{s,t}$ and $\rho(N) \geq \rho(\Gamma)$.

Otherwise, $c \le b-3$ so that

$$e-d+b-c-2 \ge e-d+c+3-c-2 = e-d+1 \ge 0$$

and by Lemma 3.8 the sequence

$M :=$

...	t-r-3	t-r-2	t-r-1	t-r	...	t-1	t
...	e	d	a-1	b	...	b	c+1

is in $V_{s,t}$ and $\rho(M) \ge \rho(\Gamma)$.

In both cases we moved from Γ to a sequence in $V_{s,t}$ with the property that the difference between the integer in position $t-r-1$ and that in position $t-r$ decreases by one. It is now clear that, after a finite number of steps, we will reach a tower Γ_a for $V_{s,t}$ such that $\rho(\Gamma_a) \ge \rho(\Gamma)$. □

We made some computations with CoCoa when $t \le 72$ and $\widetilde{s} \le t$ and it turns out that only 16 cases do not verify the conclusion of the theorem. The case corresponding to the smallest value of t is $t = 13$, $\widetilde{s} = 13$, so that $s = 92$. We have $\dim V_{92,13} = \rho(\Gamma)$ where $\Gamma = (\dots, 10, 11, 12, 11, 3)$ is obviously not a tower.

The case corresponding to the highest value of t is $t = 25$, $\widetilde{s} = 21$, so that $s = 330$. We have $\dim V_{330,25} = \rho(\Gamma)$ where $\Gamma = (\dots, 22, 23, 24, 23, 7)$ is not a tower.

We remark that, if $\widetilde{s} \le t$, there is no counterexample to the equality $\dim V_{s,t} = \max\{\rho(\Gamma_a)\}$.

Hence we make the following conjecture:

Conjecture 3.10

If $t \ge 26$ and $\widetilde{s} \le t$, then

$$\dim V_{s,t} = \max\{\rho(\Gamma_a)\}$$

where the maximum is over the towers for $V_{s,t}$.

As a consequence of the above theorem, we can prove the converse of Corollary 3.4.

Theorem 3.11

Let $f_t(X)$ be the rational function defined as in (6). Then $V_{s,t}$ has the expected dimension if and only if

$$\widetilde{s} \le f_t(a)$$

for every integer $a \ge 1$ such that $\binom{a+2}{2} \le \tilde{s}$.

Proof. We need only to prove the "if"part of the theorem. We have already seen that if $\widetilde{s} \le 2$ then $V_{s,t}$ has the expected dimension. Hence let $\widetilde{s} \ge 3$. Then $\binom{1+2}{2} = 3 \le \widetilde{s}$ so that

$$\widetilde{s} \le f_t(1) = \frac{2t+7}{3} \le \frac{2t}{3} + 4.$$

It is clear that $\widetilde{s} \le \frac{2t+7}{3}$ does imply $\widetilde{s} \le t$ unless $t = 2, 3, 4$ and $\widetilde{s} = t+1$, cases in which it is easy to check that $V_{s,t}$ has the expected dimension by using Theorem 2.3. Hence we may assume $\widetilde{s} \le \min\{t, \frac{2t}{3} + 4\}$ and apply the above theorem to get $\dim V_{s,t} = \max\{\rho(\Gamma_a)\}$.

Now the assumption $\widetilde{s} \le f_t(a)$ for every $a \ge 1$ such that $\binom{a+2}{2} \le \tilde{s}$, implies by Lemma 3.3 that $\rho(\Gamma_a) \le \rho(\Gamma_0)$ for every tower for $V_{s,t}$. Hence

$$\dim V_{s,t} = \rho(\Gamma_0) = \text{exdim} V_{s,t}$$

and the conclusion follows. □

This theorem is quite effective if we know t and s. For example if $t = 36$ and $\widetilde{s} = 26$, then we get $s = \binom{38}{2} - 26 = 677$. We have $\binom{a+2}{2} \le \widetilde{s} = 26$ if and only if $a \le 5$; by using the table at the end of the paper, we see that

$$f_{36}(1) = \frac{79}{3} > 26, \quad f_{36}(2) = \frac{83}{3} > 26, \quad f_{36}(3) = \frac{271}{10} > 26,$$
$$f_{36}(4) = \frac{406}{15} > 26, \quad f_{36}(5) = \frac{586}{21} > 26,$$

so that $V_{677,36}$ has the expected dimension.

With the same $t = 36$, if we let $\widetilde{s} = 27$, then $s = 676$ and we have $\binom{1+2}{2} = 3 \le \widetilde{s}$. Since $f_{36}(1) = \frac{79}{3} < 27$, $V_{676,36}$ has not the expected dimension.

However, a natural and more difficult question is the following: for which s does $V_{s,36}$ have the expected dimension? Of course we can apply the above theorem, but this need a lot of computations because s must range from 1 to 703.

In the next section we will find the right answer: $V_{s,36}$ has the expected dimension if and only if $s \ge 677$ ($\widetilde{s} \le 26$).

4. The conclusion

In this last section, we want to improve Theorem 3.11 in order to give a complete answer to the following problem: given the generic catalecticant matrix $Cat(t, t; 3)$, for which s the ideal generated by the $s+1$ minors has the expected codimension ?

It is clear that we need a deeper knowledge of the rational function

$$f_t(X) = \frac{16tX + X^4 + 6X^3 + 11X^2 + 30X + 8}{4(X+1)(X+2)}.$$

We recall, see (6), that we can write

$$f_t(X) := \frac{4Xt + D(X)}{(X+1)(X+2)}$$

where

$$D(X) := \frac{X^4 + 6X^3 + 11X^2 + 30X + 8}{4}.$$

Let us start with the following remark.

Lemma 4.1

We have $f_t(1) \leq f_t(a)$ for every $a \geq 1$ if and only if $t \leq 41$.

Proof. We have

$$f_t(1) = \frac{2t+7}{3}, \quad f_t(2) = \frac{2t+11}{3}$$

so that $f_t(1) \leq f_t(a)$ for every $a \geq 1$ if and only if $f_t(1) \leq f_t(a)$ for every $a \geq 3$.

We have

$$f_t(a) - f_t(1) = \frac{4at + D(a)}{(a+1)(a+2)} - \frac{2t+7}{3} \geq 0$$

if and only if

$$3(4at + D(a)) \geq (2t+7)(a+1)(a+2)$$

if and only if

$$t\,[2(a+1)(a+2) - 12a] \leq 3D(a) - 7(a+1)(a+2)$$

if and only if

$$t\,[2(a-1)(a-2)] \leq 3D(a) - 7(a+1)(a+2).$$

Hence $f_t(1) \leq f_t(a)$ for every $a \geq 3$ if and only if

$$t \leq \frac{3D(a) - 7(a+1)(a+2)}{2(a-1)(a-2)} = \frac{3a^3 + 21a^2 + 26a + 32}{8(a-2)}.$$

Now it is easy to see that the rational function

$$g(X) := \frac{3X^3 + 21X^2 + 26X + 32}{8(X-2)},$$

verifies

$$g(3) = 95/2 = 47,.. \quad g(4) = 83/2 = 41,5 \quad g(5) = 531/12 = 44,..$$

and is strictly increasing for $X \geq 4$. The conclusion follows. □

This result is no more true if $t = 42$, since we have

$$f_{42}(1) = 91/3 > f_{42}(4) = 454/15.$$

This lemma gives already the solution of our problem for small values of t.

Theorem 4.2

If $t \leq 42$, then $V_{s,t}$ has the expected dimension if and only if $\widetilde{s} \leq \frac{2t+7}{3}$.

Proof. Let $V_{s,t}$ have the expected dimension. If $\widetilde{s} \leq 2$, then $\widetilde{s} \leq \frac{2t+7}{3}$; hence we may assume $\widetilde{s} \geq 3 = \binom{1+2}{2}$. By Theorem 3.11 we get $\widetilde{s} \leq f_t(1) = \frac{2t+7}{3}$ as required.

As for the converse, we have $\widetilde{s} \leq \frac{2t+7}{3} = f_t(1)$, hence if $t \leq 41$, by the above lemma we get $\widetilde{s} \leq f_t(a)$ for every $a \geq 1$. The conclusion follows by Theorem 3.11.

If $t = 42$, we have $\widetilde{s} \leq \frac{2t+7}{3} = 91/3$ so that $\widetilde{s} \leq 30$. As in the above lemma we have

$$f_{42}(a) = 30 + 1/3 + \frac{2(a-1)(a-2)(g(a)-42)}{3(a+1)(a+2)}.$$

Since $g(X)$ is strictly increasing for $X \geq 4$ and $g(5) \geq 42$, we have $g(a) \geq 42$ for every $a \geq 5$ so that $f_{42}(a) \geq 30$ for every $a \geq 5$. Since $f_{42}(3) = 30.7$, and $f_{42}(4) = 30.2$, we get $\widetilde{s} \leq f_t(a)$ for every $a \geq 1$ and the conclusion follows again by Theorem 3.11. $\square$

This result proves for example that $V_{s,36}$ has the expected dimension if and only if $\widetilde{s} \leq \frac{79}{3} = 26, 3..$, as announced at the end of Section 3.

Unfortunately, the above theorem does not hold if $t = 43$. With such t we have $\frac{2t+7}{3} = \frac{86+7}{3} = 31$. If we take $\widetilde{s} = 31$ we get $\binom{4+2}{2} = 15 \leq \widetilde{s}$, so that Γ_4 is a tower. Since $\widetilde{s} - f_{43}(4) = 31 - 154/5 = 1/5$, by Lemma 3.3 we get $\rho(\Gamma_4) > \rho(\Gamma_0)$ so that

$$\dim V_{s,t} \geq \rho(\Gamma_4) > \rho(\Gamma_0) = \mathrm{exdim} V_{s,t}.$$

We come now to the general case.

Lemma 4.3

Let t be a positive integer and $a \geq 3$. We have

$$f_t(a-1) \geq f_t(a) \iff t \geq \frac{aD(a) - (a+2)D(a-1)}{4(a-2)}$$
$$f_t(a+1) \geq f_t(a) \iff t \leq \frac{(a+1)D(a+1) - (a+3)D(a)}{4(a-1)}$$

and equality holds on the left if and only if it holds on the right.

Proof. We have $f_t(a-1) \geq f_t(a)$ if and only if

$$\frac{4(a-1)t + D(a-1)}{a(a+1)} \geq \frac{4at + D(a)}{(a+1)(a+2)}$$

if and only if

$$\frac{4(a-1)t + D(a-1)}{a} \geq \frac{4at + D(a)}{(a+2)}$$

if and only if

$$4(a+2)(a-1)t + (a+2)D(a-1) \geq 4a^2t + aD(a)$$

if and only if

$$4t(a-2) \geq aD(a) - (a+2)D(a-1)$$

if and only if

$$t \geq \frac{aD(a) - (a+2)D(a-1)}{4(a-2)}.$$

The second assertion follows in the same way. □

Now we remark that for every $a \geq 3$ we have

$$\frac{aD(a) - (a+2)D(a-1)}{4(a-2)} = \frac{a^4 + 4a^3 + 5a^2 - 10a + 16}{8(a-2)}.$$

Hence if we consider the rational function

$$w(X) := \frac{X^4 + 4X^3 + 5X^2 - 10X + 16}{8(X-2)}, \tag{10}$$

we have

$$f_t(a-1) \geq f_t(a) \iff t \geq w(a)$$
$$f_t(a+1) \geq f_t(a) \iff t \leq w(a+1)$$

and the equality holds on the left if and only if it holds on the right.

It is easy to see that for $X \geq 3$ the function $w(X)$ is strictly increasing and $w(3) = 55/2$.

This means that, if $t \geq 28$, then $t > w(3)$ and we can find an integer $\overline{a} \geq 3$ such that

$$w(\overline{a}) < t \leq w(\overline{a}+1).$$

We thus have the following result:

Lemma 4.4

If $t \geq 28$, there exists an integer $\overline{a} \geq 3$, such that

$$f_t(\overline{a}-1) > f_t(\overline{a}) \leq f_t(\overline{a}+1).$$

We prove now that the integer $\overline{a}$ verifies the inequality

$$\binom{\overline{a}+2}{2} \leq f_t(\overline{a}).$$

This will be a consequence of the following lemma.

Lemma 4.5

If $a \geq 3$ and $w(a) \leq t$, then

$$\binom{a+2}{2} \leq f_t(a).$$

Proof. We must prove that

$$\frac{4at+D(a)}{(a+1)(a+2)} \geq \binom{a+2}{2}.$$

Since $w(a) \leq t$, we only need to prove that

$$\frac{4aw(a)+D(a)}{(a+1)(a+2)} \geq \binom{a+2}{2}.$$

This is true if and only if

$$\frac{4a\frac{aD(a)-(a+2)D(a-1)}{4(a-2)}+D(a)}{(a+1)(a+2)} \geq \binom{a+2}{2}$$

if and only if

$$\frac{a^2D(a)-a(a+2)D(a-1)+(a-2)D(a)}{(a+1)(a+2)(a-2)} \geq \binom{a+2}{2}$$

if and only if

$$(a^2+a-2)D(a)-a(a+2)D(a-1) \geq (a-2)(a+1)(a+2)\binom{a+2}{2}$$

if and only if

$$(a-1)D(a)-aD(a-1) \geq \frac{(a-2)(a+1)^2(a+2)}{2}.$$

An easy computation shows that this is equivalent to $a^4+2a^3+3a^2+10a \geq 0$, so that the conclusion follows. □

We come prove now to the main result of this section.

Theorem 4.6

Let $f_t(X)$ and $w(X)$ be the rational functions defined as in (6) and (10) respectively. If $t \geq 43$, and we let $\overline{a}$ be the unique integer such that

$$w(\overline{a}) < t \leq w(\overline{a}+1),$$

then $V_{s,t}$ has the expected dimension if and only if $\widetilde{s} \leq f_t(\overline{a})$.

Proof. If $V_{s,t}$ has the expected dimension and, by contradiction, $\widetilde{s} > f_t(\overline{a})$, from the above lemma we get $\binom{\overline{a}+2}{2} \leq f_t(\overline{a}) < \widetilde{s}$, which is absurd by Theorem 3.11.

Let us prove that $\widetilde{s} \leq f_t(\overline{a})$ implies $V_{s,t}$ having the expected dimension. By Theorem 3.11 it is enough to show that

$$f_t(\overline{a}) = \min_{a \geq 1} f_t(a).$$

Since $t \geq 43$, by Lemma 4.1 we have $f_t(2) \geq f_t(1) > f_t(a)$ for some integer $a \geq 3$. Hence it is enough to prove

$$f_t(\overline{a}) = \min_{a \geq 3} f_t(a).$$

Now we remark that

$$\lim_{a \to +\infty} \frac{D(a)}{(a+1)(a+2)} = +\infty,$$

hence, since for every t and $a \geq 1$ we have

$$f_t(a) = \frac{4at + D(a)}{(a+1)(a+2)} \geq \frac{D(a)}{(a+1)(a+2)},$$

there exists an integer m such that

$$f_t(a) \geq f_t(\overline{a})$$

for every $a \geq m$. If $m = 3$ we are done; so let $m \geq 4$ and

$$f_t(m-1) < f_t(\overline{a}) \leq f_t(m).$$

If we would have

$$f_t(2) \leq f_t(3) \leq \ldots \leq f_t(m-2) \leq f_t(m-1)$$

then $f_t(1) = \min_{a \geq 1} f_t(a)$, and $t \leq 41$ by Lemma 4.1. Thus there exists an integer j, $3 \leq j \leq m-1$ such that

$$f_t(j-1) > f_t(j) \leq \ldots \leq f_t(m-1) < f_t(\overline{a}) \leq f_t(m).$$

By Lemma 4.3, we get

$$w(j) < t \leq w(j+1),$$

so that

$$w(j) < t \leq w(\overline{a}+1), \qquad w(\overline{a}) < t \leq w(j+1).$$

Since $w(X)$ is strictly increasing for $X \geq 3$, this implies

$$j < \overline{a} + 1, \quad \overline{a} < j + 1.$$

Thus $j = \overline{a}$, a contradiction because $f_t(j) < f_t(\overline{a})$. $\square$

Here are some of the values of the functions $f_t(X)$ and $w(X)$. We have:

$$f_t(1) = \frac{2t+7}{3}, \quad f_t(2) = \frac{2t+11}{3} \quad f_t(3) = \frac{6t+55}{10}, \quad f_t(4) = \frac{8t+118}{15},$$
$$f_t(5) = \frac{10t+226}{21}, \quad f_t(6) = \frac{12t+397}{28}, \quad f_t(7) = \frac{14t+652}{36},$$
$$f_t(8) = \frac{16t+1015}{45}, \quad f_t(9) = \frac{18t+1513}{55}, \quad f_t(10) = \frac{10t+1088}{33},$$
$$f_t(11) = \frac{22t+3037}{78}, \quad f_t(12) = \frac{24t+4132}{91}, \quad f_t(13) = \frac{26t+5500}{105}.$$

$$w(3) = 55/2 = 27.5, \quad w(4) = 71/2 = 35.5, \quad w(5) = 152/3 = 50.6,$$
$$w(6) = 287/4 = 71.7, \quad w(7) = 991/10 = 99.1, \quad w(8) = 400/3 = 133.3,$$
$$w(9) = 1226/7 = 175.1, \quad w(10) = 901/4 = 225.2, \quad w(11) = 5119/18 = 284.3,$$
$$w(12) = 3533/10 = 353.3, \quad w(13) = 4760/11 = 432.7, \quad w(14) = 6281/12 = 523.4$$

For example, if $t = 100$, then $\overline{a} = 7$, so that $V_{s,100}$ has the expected dimension if and only if $\widetilde{s} \leq f_{100}(7) = \frac{1400+652}{36} = 57$.

If $t = 500$, then $\overline{a} = 13$, so that $V_{s,500}$ has the expected dimension if and only if $\widetilde{s} \leq f_{500}(13) = \frac{(26)(500)+5500}{105} = 176, 1$.

Some of the results of this paper were conjectured after explicit computations performed by the computer algebra system CoCoA ([1]).

References

1. A. Capani, G. Niesi, and L. Robbiano, CoCoA, *a System for Doing Computations in Commutative Algebra*, 1995, Available via anonymous ftp from `cocoa.dima.unige.it`
2. Y.H. Cho and B.E. Jung, The dimension of the determinantal scheme $V_s(t, t, 2)$ of the catalecticant matrix, *Comm. Algebra* **28** (2000), 2423–2443.
3. A. Conca and G. Valla, Hilbert function of powers of ideals of low codimension, *Math. Z.* **230** (1999), 753–784.
4. S.J. Diesel, Irreducibility and dimension theorems for families of height 3 Gorenstein algebras, *Pacific J. Math.* **172** (1996), 365–397.
5. A. Iarrobino and V. Kanev, *Power Sums, Gorenstein Algebras, and Determinantal Loci*, Lecture Notes in Mathematics 1721, Springer-Verlag, Berlin and New York, 1999.
6. J.O. Kleppe, The smoothness and the dimension of $\mathbb{P}$Gor(H) and of the strata of the punctual Hilbert scheme, *J. Algebra* **200** (1998), 606–628.
7. R.P. Stanley, Hilbert functions of graded algebras, *Advances in Math.* **28** (1978), 57–83.