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Generic initial ideals of points and curves

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Abstract

Let I be the defining ideal of a smooth complete intersection space curve C with defining equations of degrees a and b. We use the partial elimination ideals introduced by Mark Green to show that the lexicographic generic initial ideal of I has Castelnuovo–Mumford regularity $1 + ab(a - 1)(b - 1)/2$ with the exception of the case $a = b = 2$, where the regularity is 4. Note that $ab(a - 1)(b - 1)/2$ is exactly the number of singular points of a general projection of C to the plane. Additionally, we show that for any term ordering τ, the generic initial ideal of a generic set of points in $\mathbb{P}^r$ is a τ-segment ideal.

Keywords: Generic initial ideal; Curves; Points; Lexicographic term order

1. Introduction

Let $S = k[x_0, \dots, x_r]$ where k is an algebraically closed field of characteristic zero and let τ be a term ordering on S. Let $I \subset S$ be a homogeneous ideal. There is a monomial ideal canonically associated with I, its generic initial ideal with respect to τ, denoted by $\text{gin}_\tau(I)$, or simply $\text{gin}_\tau I$. In this paper we study lexicographic generic initial ideals of curves and points via Green's partial elimination ideals.

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For a smooth complete intersection curve C in $\mathbb{P}^3$, we show that the complexity of its lexicographic generic initial ideal, as measured by Castelnuovo–Mumford regularity, is governed by the geometry of a generic projection of C to $\mathbb{P}^2$.

Theorem 1.1. *Let C be a smooth complete intersection of hypersurfaces of degrees $a, b > 1$ in $\mathbb{P}^3$. The regularity of the lexicographic generic initial ideal of C is equal to*

$$\begin{cases} 1 + \frac{a(a-1)b(b-1)}{2} & \text{if } (a,b) \neq (2,2) \\ 4 & \text{if } (a,b) = (2,2). \end{cases}$$

Note that, apart for the exceptional case $a = b = 2$, the regularity of the lexicographic generic initial ideal is 1+ the number of nodes of the generic projection of C to $\mathbb{P}^2$. The statement of Theorem 1.1 generalizes Example 6.10 in Green (1998) which treats the special case where $a = b = 3$.

Macaulay's characterization of Hilbert functions, see for instance Theorem 4.2.10 in Bruns and Herzog (1993), implies that any ideal J is generated in degrees bounded by the largest degree of a generator of the corresponding lex-segment $\mathrm{Lex}(J)$. Much more is true—Bigatti (1993), Hulett (1993) and Pardue (1996) showed the Betti numbers of J are bounded by those of $\mathrm{Lex}(J)$. Let I be the ideal of C in Theorem 1.1. For such an ideal I one can compute the largest degree of a generator of $\mathrm{Lex}(I)$. This has been done, for instance, by Bayer in his Ph.D. thesis (Proposition in II.10.4, Bayer, 1982) and by Chardin and Moreno-Socías (2002), and it turns out to be $\frac{a(a-1)b(b-1)}{2} + ab$. So the lexicographic generic initial ideal in Theorem 1.1 is not equal to the lex-segment ideal but nearly achieves the worst-case regularity for its Hilbert function. Moreover, as shown in Bermejo and Lejeune-Jalabert (1999), the extremal bound can only be achieved if C lies in a plane.

We also study the generic initial ideals of finite sets of points. Surprisingly, when X is a set of generic points its generic initial ideal is an *initial segment*.

Theorem 1.2. *Let I be the ideal of s generic points of $\mathbb{P}^n$. Then $\mathrm{gin}_\tau I$ is equal to the τ-segment ideal $\mathrm{Seg}_\tau(I)$ for all term orders τ. In particular, $\mathrm{gin}_{\mathrm{lex}} I$ is a lex-segment ideal.*

The genericity required in Theorem 1.2 is quite explicit: the conclusion holds for a set X of s points if there is a system of coordinates such that the defining ideal of X does not contain non-zero forms supported on $\leq s$ monomials. A special case of the result when $\tau =$ revlex is proved in Marinari and Ramella (1999).

For an introduction to generic initial ideals see Section 15.9 in Eisenbud (1995). Here we just recall:

Theorem 1.3 (*Galligo, Bayer–Stillman*). *Given a homogeneous ideal I and a term ordering τ on the monomials of S, there exists a dense open subset $U \subseteq \mathrm{GL}_{r+1}(k)$ such that $\mathrm{gin}_\tau I := \mathrm{in}_\tau(g \cdot I)$ is constant over all $g \in U$ and $\mathrm{gin}_\tau I$ is Borel-fixed.*

Recall also that, in characteristic 0, an ideal J is Borel-fixed if it is monomial and satisfies:

$$\text{if } m \text{ is a monomial, } x_i m \in J \implies x_j m \in J, \ \forall j \leq i.$$

From this property one easily shows that the regularity of a Borel-fixed ideal J is the maximum degree of a minimal generator. A minimal resolution of such an ideal was constructed in Eliahou and Kervaire (1990).

As τ varies over all term orderings, both the regularity and the minimal number of generators of $\operatorname{gin}_\tau I$ may vary greatly. The generic initial ideals with respect to the reverse lexicographic (revlex) term ordering have the minimum level of complexity possible.

Theorem 1.4 (*Bayer and Stillman, 1987*). *If I is a homogeneous ideal of S, and $J = \operatorname{gin}_{\text{revlex}} I$ then*

$$\operatorname{reg} I = \operatorname{reg} J = \text{ max degree of a minimal generator of } J.$$

The paper is organized as follows. In Section 2, we set up notation and review terminology. We introduce partial elimination ideals, their basic properties, and algorithms for their computation in Section 3. We focus on the case of complete intersection curves in Section 4 and on the case of points in Section 5.

2. Notation and terminology

Let $S = k[x_0, \ldots, x_r]$ where k is an algebraically closed field of characteristic zero. Denote by $\mathfrak{m}$ the irrelevant maximal ideal of S. For an element $\alpha = (\alpha_0, \ldots, \alpha_r) \in \mathbb{N}^{r+1}$ we let x^α denote $x_0^{\alpha_0} \cdots x_r^{\alpha_r}$. In this section we briefly recall notions related to term orderings and Castelnuovo–Mumford regularity. For a comprehensive introduction to general notions related to Gröbner bases see Cox et al. (1997) and Kreuzer and Robbiano (2000).

Definition 2.1. We say that a total ordering τ on the monomials of S is a *term ordering* if it is a well-ordering satisfying

$$x^\alpha >_\tau x^\beta \Rightarrow x^\gamma \cdot x^\alpha >_\tau x^\gamma \cdot x^\beta \ \forall, \gamma \in \mathbb{N}^{r+1}.$$

A term ordering τ on S allows us to assign to each non-zero element $f \in S$ an *initial term* $\operatorname{in}_\tau(f)$ and to any ideal I an initial ideal $\operatorname{in}_\tau(I)$.

In what follows we will work exclusively with homogeneous ideals and we will always require that the term ordering is degree compatible: $m > n$ if $\deg(m) > \deg(n)$.

The *lexicographic* and (degree) *reverse lexicographic* term orderings feature prominently in the literature. If x^α and x^β are two monomials of the same degree, then $x^\alpha >_{\text{lex}} x^\beta$ if the left-most non-zero entry of $\alpha - \beta$ is positive and $x^\alpha >_{\text{revlex}} x^\beta$ if the right-most non-zero entry of $\alpha - \beta$ is negative.

Although our primary motivation for studying partial elimination ideals is to understand lexicographic initial ideals, partial elimination ideals also provide a mechanism for studying initial ideals with respect to any *elimination* order.

Definition 2.2. An *elimination order* for the first t variables of S is a term order τ such that if f is a polynomial whose initial term $\operatorname{in}_\tau(f)$ does not involve variables $x_0, \ldots, x_{t-1}$, then f itself does not involve variables $x_0, \ldots, x_{t-1}$.

As we shall see in Proposition 3.4, one may use an elimination order for the variable x_0 to compute partial elimination ideals. If τ is an elimination order for x_0, then it is equivalent to a $(1, r)$ *product order* which first sorts monomials by powers of x_0 and then sorts the remaining variables by an arbitrary term ordering τ_0.

We will use the notion of *Castelnuovo–Mumford regularity* as a rough measure of the complexity of our computations.

Definition 2.3. Let M be a finitely generated graded S-module, and let

$$0 \to \oplus_j S(-a_{lj}) \to \cdots \oplus_j S(-a_{1j}) \to \oplus_j S(-a_{0j}) \to M \to 0$$

be a minimal graded free resolution of M. We say that M is d*-regular* if $a_{ij} \leq d + i$ for all i and j, and that the *regularity* of M, denoted reg M, is the least d such that M is d-regular.

One may also formulate the definition of regularity in terms of vanishings of local cohomology with respect to $\mathfrak{m}$. The vanishing of the zeroth local cohomology group is related to the notion of *saturation* which plays an important role in the study of regularity.

Definition 2.4. Let $I \subseteq S$ be a homogeneous ideal. The *saturation* of I, denoted I^{sat} is defined to be $I :_S \mathfrak{m}^\infty$. Note that $I_d = I_d^{\mathrm{sat}}$ for all $d \gg 0$. We say that I is d-saturated if I agrees with its saturation in degrees d and higher. The minimum degree for which I is d-saturated is the *saturation degree* (also the *satiety index* in Green (1998)) of I.

3. Partial elimination ideals

Let $S = k[x_0, \dots, x_r]$, and let $\overline{S} = k[x_1, \dots, x_r]$. Let τ be an arbitrary elimination order on S that eliminates the variable x_0 and hence induces a term order, denoted by τ_0, on $\overline{S}$. In this section we set up the theory of partial elimination ideals over a polynomial ring in $r + 1$ variables as introduced in Green (1998). Much of the material in Sections 3.1 and 3.2 appears either explicitly or implicitly in Green (1998), but we give proofs here both to keep the presentation self-contained and to present a more algebraic point of view.

We represent any non-zero polynomial f in S as

$$f = f_0 x_0^p + f_1 x_0^{p-1} + \cdots + f_p$$

with $f_i \in \overline{S}$ and $f_0 \neq 0$. The polynomial f_0 is called the initial coefficient of f with respect to x_0 and is denoted by $\mathrm{incoef}_{x_0}(f)$. The integer p is called the x_0-degree of f and is denoted by $\deg_{x_0}(f)$.

3.1. Definitions and basic facts

In this section we define the partial elimination ideals and describe their basic algebraic and geometric properties. We begin with the definition:

Definition 3.1 (*Definition 6.1 in Green (1998)*). Let I be a homogeneous ideal in S. The p*-th partial elimination ideal* of I is defined to be the ideal

$$K_p(I) := \{\mathrm{incoef}(f) \mid f \in I \text{ and } \deg_{x_0} f = p\} \cup \{0\}$$

in the polynomial ring $\overline{S} = k[x_1, \ldots, x_r]$.

It is easy to see that if I is homogeneous then $K_p(I)$ is also homogeneous.

In Lemma 3.2 we gather together some elementary algebraic facts about the partial elimination ideals. We leave the proof to the reader. The decomposition of $\mathrm{in}_\tau I$ given in part (1) is one of the motivations for the definition.

Lemma 3.2. *Let I be a homogeneous ideal.*

(1) $\mathrm{in}_\tau I = \sum_p x_0^p \, \mathrm{in}_{\tau_0} K_p(I)$.
(2) *Taking K_p commutes with taking initial ideals: $K_p(\mathrm{in}_\tau I) = \mathrm{in}_{\tau_0} K_p(I)$.*
(3) *The partial elimination ideals are an ascending chain of ideals, i.e., $K_i(I) \subseteq K_{i+1}(I)$ for all i.*

One expects that if I is in generic coordinates, then the partial elimination ideals $K_p(I)$ are already in generic coordinates. Proposition 3.3 shows that this is indeed the case.

Proposition 3.3. *Let $I \subset S$ be a homogeneous ideal. If I is in generic coordinates then* $\mathrm{in}_{\tau_0} K_p(I) = \mathrm{gin}_{\tau_0} K_p(I)$.

Proof. Let $\mathrm{GL}_r(k)$ act on $\overline{S}$ in the usual way and extend this to an action on S in the trivial fashion by letting elements of $\mathrm{GL}_r(k)$ fix x_0.

We know that the ideal I determines a dense open subset $U \subset \mathrm{GL}_{r+1}(k)$ with the property that $g \in U$ implies that $\mathrm{in}_\tau(gI) = \mathrm{gin}_\tau I$. We show that for each $g \in U$ there is a dense open subset $U' \subset \mathrm{GL}_r(k)$ so that for all $h \in U'$

(1) $\mathrm{gin}_{\tau_0}(K_p(gI)) = \mathrm{in}_{\tau_0}(hK_p(gI))$
(2) hg is again a generic change of coordinates for I.

Consider the space $\mathrm{GL}_r(k) \times \mathrm{GL}_{r+1}(k)$ with projection maps π_1 and π_2 onto the first and second factors, respectively. The map

$$\phi : \mathrm{GL}_r(k) \times \mathrm{GL}_{r+1}(k) \to \mathrm{GL}_{r+1}(k)$$

given by $\phi(h, g) = hg$ is regular. The inverse image of U under the map ϕ is a dense open subset of $\mathrm{GL}_r(k) \times \mathrm{GL}_{r+1}(k)$. For each $g \in U$ the set $W := \pi_1(\pi_2^{-1}(g) \cap \phi^{-1}(U))$ is a dense open subset of $\mathrm{GL}_r(k)$. The element g determines a dense open set $V \subset \mathrm{GL}_r(k)$ such that $h \in V$ satisfies (2), i.e., each $h \in V$ is a set of generic coordinates for $K_p(gI)$. Then any $h \in U' := W \cap V$ has the property that hg is a set of generic coordinates for I.

For h and g chosen as above, we have $hK_p(gI) = K_p(hgI)$. Thus, $\mathrm{gin}_{\tau_0}(K_p(gI)) = \mathrm{in}_{\tau_0} K_p(hgI)$. By Lemma 3.2(2), $\mathrm{in}_{\tau_0} K_p(I) = K_p(\mathrm{in}_\tau I)$, which implies that $\mathrm{gin}_{\tau_0}(K_p(gI)) = K_p \, \mathrm{in}_\tau(hgI)$.

Since hg is again generic, $\mathrm{gin}_\tau I = \mathrm{in}_\tau gI = \mathrm{in}_\tau hgI$. So we have $\mathrm{gin}_{\tau_0}(K_p(gI)) = K_p(\mathrm{in}_\tau gI)$. Using Lemma 3.2(2) again, we obtain $\mathrm{gin}_{\tau_0}(K_p(gI)) = \mathrm{in}_{\tau_0} K_p(gI)$ and this proves the assertion. □

The partial elimination ideals of an arbitrary homogeneous ideal I can be recovered in an easy way from a Gröbner basis for I. In practice one may want to take a $(1, r)$ product order with the reverse lexicographic ordering on the last r variables in order to minimize computations.

Proposition 3.4. *Let G be a Gröbner basis for I with respect to an elimination ordering τ. Then the set*

$$G_p = \{\mathrm{incoef}_{x_0}(g) \mid g \in G \text{ and } \deg_{x_0}(g) \le p\}$$

is a Gröbner basis for $K_p(I)$.

Proof. Note that if $g \in I$ and $\deg_{x_0}(g) = p$ then $\mathrm{incoef}_{x_0}(g) \in K_p(I)$ by definition. By Lemma 3.2(3) we have that the elements of G_p are in $K_p(I)$. We will show that their initial terms generate $\mathrm{in}_{\tau_0} K_p(I)$. Suppose that m is a monomial in the ideal $\mathrm{in}_{\tau_0} K_p(I)$. This implies that there exists $f \in I$ such that $\mathrm{in}_\tau(f) = mx_0^p$ and hence there exists $g \in G$ such that $\mathrm{in}_\tau(g) | \mathrm{in}_\tau(f)$. Set $h = \mathrm{incoef}_{x_0}(g)$. It follows that $\deg_{x_0} g \le p$, so that $h \in G_p$, and $\mathrm{in}_{\tau_0} h|m$. □

By part (3) of Lemma 3.2 we know that the subscheme cut out by the p-th partial elimination ideal is contained in the subscheme defined by the $(p-1)$-th partial elimination ideal. The following result gives the precise relationship between the partial elimination ideals and the geometry of the projection map from $\mathbb{P}^r$ to $\mathbb{P}^{r-1}$.

Theorem 3.5 (*Proposition 6.2 in Green (1998)*). *Let Z be a reduced subscheme of $\mathbb{P}^r$ not containing $[1 : 0 : \cdots : 0]$ and let $I = I(Z)$ be the homogeneous ideal of Z. Let*

$$\pi : \mathbb{P}^r \to \mathbb{P}^{r-1}$$

be the projection from the point $[1 : 0 : \cdots : 0]$. Set-theoretically, $K_p(I)$ is the ideal of

$$\{z \in \pi(Z) \mid |\pi^{-1}(z)| > p\},$$

where $|\pi^{-1}(z)|$ denotes the length of the scheme-theoretic fiber above p.

Proof. We prove the theorem by reducing to the affine case. We begin by introducing some notation. If $J \subseteq S$ is a homogeneous ideal, let $J_{(x_i)}$ denote its dehomogenization in $k[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_r}{x_i}]$.

To show that $K_p(I)$ cuts out the $(p+1)$-fold points set-theoretically it suffices to show that $K_p(I)_{(x_i)}$ cuts out the $(p+1)$-fold points in each of the standard affine open patches of $\mathbb{P}^{r-1}$ for $i = 1, \ldots, r$. If we consider the ideal $I_{(x_i)} \subseteq k[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_r}{x_i}]$ with the term ordering induced by τ in the natural way on the monomials in $\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_r}{x_i}$, then by Lemma 4.8.3 in Haiman (2001), $K_p(I_{(x_i)})$ is set-theoretically the ideal of the $(p+1)$-fold points lying in this affine patch.

It remains for us to show that for any $i = 1, \ldots, r$,

$$K_p(I)_{(x_i)} = K_p(I_{(x_i)}).$$

It is clear that $K_p(I)_{(x_i)} \subseteq K_p(I_{(x_i)})$. For the opposite inclusion, note that $\frac{x_0}{x_i}$ appears in the dehomogenization of a monomial m precisely as many times as x_0 appears in m, and apply the definitions. □

In the situation of Theorem 3.5, we can see that $K_0(I)$ is in fact radical. The ideal $K_0(I)$ is just equal to $I \cap \overline{S}$. On the other hand, the higher $K_p(I)$ need not be radical even if I is a prime complete intersection of codimension 2 in generic coordinates; see Example 4.3.

3.2. Partial elimination ideals for codimension 2 complete intersection

Let

$$f = x_0^a + f_1 x_0^{a-1} + \cdots + f_{a-1}x_0 + f_a$$

and

$$g = x_0^b + g_1 x_0^{b-1} + \cdots + g_{b-1}x_0 + g_b$$

where $f_1, \ldots, f_a$ and $g_1, \ldots, g_a$ are indeterminates.

We wish to describe the partial elimination ideals of the ideal $I_{a,b}$ generated by f and g in $S = k[x_0, \ldots, x_r]$ after specializing the f_i and the g_i to homogeneous elements of $\overline{S} = k[x_1, \ldots, x_r]$ of degree i.

As we saw in Section 3.1, the partial elimination ideals of an arbitrary homogeneous ideal I can be recovered from a Gröbner basis for I and, vice versa, give information on that Gröbner basis. In this section we discuss a result of Eisenbud and Green showing that $K_p(I_{a,b})$ is generated by the minors of a truncation of the Sylvester matrix as long as the forms f and g are generic enough. Both Theorem 3.6 and Lemma 3.7 are well-known to experts, but we give proofs for completeness.

Theorem 3.6 (*Proposition 6.8(3) in Green (1998)*). *Assume that the f_i and the g_j are independent indeterminates and that $p < a \le b$. Let*

$$R = k[f_1, \ldots, f_a, g_1, \ldots, g_b, x_0],$$

where k is an arbitrary field. Define $\mathrm{Syl}_p(f, g)$ to be the matrix consisting of the first $a + b - p$ rows of the Sylvester matrix of f and g, i.e.

$$\mathrm{Syl}_p(f,g) = \begin{pmatrix} 1 & 0 & & 0 & 1 & 0 & & 0 \\ f_1 & 1 & & 0 & g_1 & 1 & & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ f_a & f_{a-1} & \ddots & 1 & & & \ddots & 1 \\ 0 & f_a & \ddots & \vdots & 0 & g_b & \ddots & \vdots \\ \vdots & \vdots & \ddots & f_{a-p-1} & \vdots & \vdots & \ddots & g_{b-p-1} \\ 0 & 0 & & f_{a-p} & 0 & 0 & & g_{b-p} \end{pmatrix}.$$

Then the ideal $K_p(f, g) \subset R$ is generated by the maximal minors of the matrix $\mathrm{Syl}_p(f, g)$.

Proof. Let $R_{\le t}$ denote the vector space of polynomials in R with $\deg_{x_0} \le t$. To compute $K_p(f, g)$ we want to find all $A, B \in R$ such that

$$Af + Bg = c_0 x_0^p + c_1 x_0^{p-1} + \cdots + c_{p-1}x_0 + c_p. \tag{1}$$

with $c_i \in k[f_1, \ldots, f_a, g_1, \ldots, g_b]$. Note that it suffices to find all A, B satisfying the Eq. (1) where $\deg_{x_0} A \le b - 1$ and $\deg_{x_0} B \le a - 1$.

The matrix $\mathrm{Syl}_p(f,g)$ gives a linear map

$$R_{\le b-1} \oplus R_{\le a-1} \to R_{\le a+b-1}/(1, x_0, \dots, x_0^{p-1}).$$

The kernel of $\mathrm{Syl}_{p+1}(f,g)$ consists of the set of all $(A,B) \in R_{\le b-1} \oplus R_{\le a-1}$ that satisfy Eq. (1). The image of $\ker \mathrm{Syl}_{p+1}(f,g)$ under $\mathrm{Syl}_p(f,g)$ is exactly the set

$$\{c_0 x_0^p \mid c_0 x_0^p + c_1 x_0^{p-1} + \cdots + c_{p-1} x_0 + c_p \in (f,g)\}.$$

We will show that the maximal minors of $\mathrm{Syl}_p(f,g)$ generate the image of $\ker \mathrm{Syl}_{p+1}(f,g)$ under the map $\mathrm{Syl}_p(f,g)$ as long as $\mathrm{Syl}_{p+1}(f,g)$ drops rank in the expected codimension. The proof that $\mathrm{Syl}_{p+1}(f,g)$ does indeed drop rank in codimension $p+2$ will be given in Lemma 3.7.

If $\mathrm{Syl}_{p+1}(f,g)$ drops rank in the expected codimension, then since R is Cohen–Macaulay we conclude that the Buchsbaum–Rim complex resolves the cokernel of $\mathrm{Syl}_{p+1}(f,g)$. (See Eisenbud (1995) A2.6 for details.)

Using the Buchsbaum–Rim complex we can give explicit formulas for elements of $\ker \mathrm{Syl}_{p+1}(f,g)$ indexed by $T \subseteq \{1,\dots,a+b\}$ with $|T| = a+b-p$. Define $\mathrm{Syl}_{p+1}(f,g)^T$ to be the $(a+b-p-1) \times (a+b-p)$ matrix consisting of all of the columns of $\mathrm{Syl}_{p+1}(f,g)$ indexed by elements of T. Define W_T to be the vector of length $a+b$ whose i-th entry is 0 if $i \notin T$ and $\mathrm{sign}(i) \det \mathrm{Syl}_{p+1}(f,g)^{T-\{i\}}$ if $i \in T$ where $\mathrm{sign}(i) = 1$ if the number of elements of T less than i is even, and -1 if the number of elements of T less than i is odd. The Buchsbaum–Rim complex is a resolution precisely when the vectors W_T generate the kernel of $\mathrm{Syl}_{p+1}(f,g)$.

Finally, we apply $\mathrm{Syl}_p(f,g)$ to the elements W_T constructed above. The dot product of W_T with each of the first $a+b-p-1$ rows of $\mathrm{Syl}_p(f,g)$ is zero since W_T is in the kernel of $\mathrm{Syl}_{p+1}(f,g)$. The dot product of W_T with the last row is just the expansion of the maximal minor of $\mathrm{Syl}_p(f,g)$ corresponding to the columns indexed by T by this final row. Therefore,

$$\mathrm{Syl}_p(f,g) \cdot W_T = \det \mathrm{Syl}_p(f,g)^T x_0^p. \quad \square$$

Lemma 3.7. *If f_i and g_j are independent indeterminates and $p < a \le b$ the matrix* $\mathrm{Syl}_p(f,g)$ *drops rank in the expected codimension* $p+1$.

Proof. We will show that the set where $\mathrm{Syl}_p(f,g)$ fails to have maximal rank, that is, where $\dim_k \ker \mathrm{Syl}_p(f,g) \ge p+1$, has codimension $p+1$ in the space of all f and g where the f_i and g_i take values in k. The result follows if we can show that for any specialization of the indeterminates f_i and g_j to values in k, $\dim_k \ker \mathrm{Syl}_p(f,g) \ge p+1$ if and only if f and g have a common factor of degree $p+1$.

It is clear that if f and g have a common factor of degree $p+1$ then $\dim_k \ker \mathrm{Syl}_p(f,g) \ge p+1$, since we can use the $(p+1)$ common factors to construct $(p+1)$ syzygies on f and g with distinct degrees.

To prove the other direction, we will use induction on p. Suppose that $p=0$. Then

$$\dim_k \ker \mathrm{Syl}_0(f,g) > 0$$

if and only if $\mathrm{Res}(f, g) = 0$. It is well-known (see Cox et al. (1997)) that $\mathrm{Res}(f, g) = 0$ if and only if f and g have a common factor of degree at least one.

We treat the case where $p > 0$. Our assumption implies that we can find $p + 1$ linearly independent elements $(A_0, B_0), \ldots, (A_p, B_p)$ of the kernel of $\mathrm{Syl}_p(f, g)$. Since

$$A_0 f + B_0 g, \ldots, A_p f + B_p g \in \mathrm{span}(1, \ldots, x_0^{p-1}),$$

there is a nontrivial linear relation $\sum \lambda_i(A_i f + B_i g) = 0$. Hence, f and g must have a common factor so that $f = (x - \alpha) f'$ and $g = (x - \alpha)g'$. By induction, we will be done if we can show that the dimension of

$$\{(A, B) \in k[x_0]_{\le b-2} \oplus k[x_0]_{\le a-2} \mid Af' + Bg' \in \mathrm{span}(1, x_0, \ldots, x_0^{p-2})\}$$

is $\ge p$. But, we can assume (after reordering and cancelling leading terms) that for $i \ge 1$, $\deg A_i \le b - 2$ and $\deg B_i \le a - 2$. Consequently, for $i \ge 1$,

$$A_i f' + B_i g' \in \mathrm{span}(1, x_0, \ldots, x_0^{p-2}). \quad \square$$

Note that the first b rows of $\mathrm{Syl}_p(f, g)$ contain constants. Recall the following fact:

Lemma 3.8 (*pg. 10 in Bruns and Vetter (1988)*). *Suppose that $M = (m_{i,j})$ is a $p \times q$ matrix with entries in a commutative ring. If $m_{p,q}$ is a unit, then the ideal generated by the maximal minors of M is the same as the ideal generated by the maximal minors of the $(p-1) \times (q-1)$ matrix N with entries*

$$n_{i,j} = m_{i,j} - m_{p,j} m_{i,q} m_{p,q}^{-1} \qquad 1 \le i \le p-1, \ \ 1 \le j \le q-1.$$

We have the following:

Corollary 3.9 (*See the Remark Following Proposition 6.9 in Green (1998)*). *Let $a \le b$ and assume that the f_i and g_i are sufficiently general homogeneous polynomials of degree i in variables $x_1, \ldots, x_r$. Assume also that $p < a$.*

(1) *The ideal of maximal minors of $\mathrm{Syl}_p(f, g)$ is always contained in $K_p(f, g)$. It has the expected codimension, $p + 1$, if $p \le r - 1$.*
(2) *Assume $p \le r - 2$. Then we have:*
   (a) *$K_p(f, g)$ is equal to the ideal of maximal minors of $\mathrm{Syl}_p(f, g)$.*
   (b) *$K_p(f, g)$ is also the ideal generated by the maximal minors of a matrix of size $(a - p) \times a$ whose (i, j)-th entry is either 0 or has degree $b + i - j$.*
   (c) $\mathrm{reg}\, K_p(f, g) = ab + \binom{a-p+1}{2} - \binom{a+1}{2} + p(a - p - 1)$.

Proof. Let $R = k[f_1, \ldots, f_a, g_1, \ldots, g_b, x_0]$ where the f_i and g_j are indeterminates as in Theorem 3.6. Generators for $K_p(f, g)$ as an ideal in R also generate the p-th partial elimination ideal of the ideal generated by f and g in the ring $R \otimes_k k[x_1, \ldots, x_r]$, which we will denote by $K_p(f, g) \otimes k[x_1, \ldots, x_r]$. An elementary argument shows that if $p+1 \le r$, then for sufficiently general forms $f_i, g_j \in k[x_1, \ldots, x_r]$, the specialization of the matrix $\mathrm{Syl}_p(f, g)$ still drops rank in the expected codimension.

Thus, (1) and (2, a) follow from the proof of Theorem 3.6 and from Lemma 3.7. Part (b) of (2) follows from (2, a) and from iterated use of Lemma 3.8. Finally (2, c) follows from (2, b) and from Lemma 3.11. $\square$

Remark 3.10. The above corollary is sharp in the sense that, in general, $K_{r-1}(f, g)$ strictly contains the ideal of maximal minors of $\mathrm{Syl}_{r-1}(f, g)$. For instance, one can check with CoCoA that this happens if $r = 3$ and $a = b = 4$.

Lemma 3.11. *Let $X = (h_{ij})$ be an $m \times n$ matrix of forms with $m \leq n$. Assume $a_1, \ldots, a_m$ and $b_1, \ldots, b_n$ are integers such that $\deg(h_{ij}) = a_i + b_j > 0$ whenever $h_{ij} \neq 0$. Assume that the ideal I_m of maximal minors of X has the expected codimension $n - m + 1$. Then*

$$\operatorname{reg} I_m = \sum_i a_i + \sum_j b_j + (\max(a_i) - 1)(n - m).$$

Proof. The Eagon–Northcott complex gives a resolution of I_m which is minimal since the entries of the matrices in the resolution are (up to sign) the entries of X and 0. Keeping track of the shifts one obtains the formula above. The same formula can be derived from the result (Bruns and Herzog, 1992, Corollary 1.5). Another formula for the regularity appears in Budur et al. (2004). □

In particular we have:

Corollary 3.12. *Let I be the ideal of a smooth complete intersection C in $\mathbb{P}^3$ defined by two forms f and g of degrees $a, b > 1$. Assume that I is in generic coordinates. We have:*

(a) *$K_1(f, g)$ is equal to the ideal of maximal minors of $\mathrm{Syl}_1(f, g)$ and has codimension 2 in $k[x_1, x_2, x_3]$.*
(b) *$K_2(f, g)$ contains the ideal of maximal minors of $\mathrm{Syl}_2(f, g)$ and both ideals have codimension 3 in $k[x_1, x_2, x_3]$.*

Proof. We will use a geometric argument to show that if f and g are in sufficiently general coordinates, then $\mathrm{Syl}_1(f, g)$ has codimension 2 and $\mathrm{Syl}_2(f, g)$ has codimension 3 in $k[x_1, x_2, x_3]$. Since these codimensions are the expected values for those determinantal ideals, the conclusion will follow by Corollary 3.9.

Recall the classical fact that a generic projection of a smooth irreducible curve in $\mathbb{P}^3$ has only nodes as singularities. (See Theorem IV.3.10 in Hartshorne (1977).) It follows that after a generic change of coordinates, the image of the projection from the point $[1 : 0 : 0 : 0]$ will have only nodes as singularities. As a consequence, we see that for each point $q \in \mathbb{P}^2$, the fiber of the projection of the curve C will contain at most two points, and the set of q with $\pi^{-1}(q) = 2$ is finite. In other words, $\deg \gcd(f(x_0, q), g(x_0, q)) \leq 2$ and equality holds for only finitely many q. From the proof of Lemma 3.7, we can see $\mathrm{Syl}_p(f, g)$ drops rank at q if and only if $f(x_0, q)$ and $g(x_0, q)$ have a common factor of degree $\geq p + 1$. Therefore, we see that $\mathrm{Syl}_1(f, g)$ drops rank at a finite set of points and and $\mathrm{Syl}_2(f, g)$ does not drop rank at any point in $\mathbb{P}^2$. □

4. The lexicographic gin of a complete intersection curve in $\mathbb{P}^3$

Let $I_{a,b}$ be a codimension 2 complete intersection ideal in the polynomial ring $S = k[x_0, x_1, x_2, x_3]$ defined by two forms of degrees $a, b > 1$. Let $C = V(I_{a,b})$ be the curve in $\mathbb{P}^3$ defined by $I_{a,b}$. We will assume that C is smooth and in generic coordinates. In other words, we assume that $I_{a,b}$ is prime, that the singular locus of $S/I_{a,b}$ consists solely of the homogeneous maximal ideal and that $I_{a,b}$ is in generic coordinates. We have that C has degree ab and genus $ab(a+b-4)/2+1$. From Theorem 3.5 we know that $K_0(I_{a,b})$ is the radical ideal of the projection $\pi : C \to \mathbb{P}^2$ from the point $[1:0:0:0]$. Since C is in generic coordinates by assumption, the projection π is generic. Proposition 4.1 describes additional numerical data associated with $\pi(C)$.

Proposition 4.1. *The ideal $K_0(I_{a,b})$ is generated by a single polynomial of degree ab. It cuts out a degree ab curve with $ab(a-1)(b-1)/2$ nodes.*

Proof. We already know that $K_0(I_{a,b})$ is the radical ideal of $\pi(C)$ which has degree ab. So it remains to show that $\pi(C)$ has $ab(a-1)(b-1)/2$ nodes.

Since a general projection of any space curve has only nodes as singularities, we have that $\pi(C)$ is a plane curve with only nodes as singularities. Since C is the normalization of $\pi(C)$ and C has genus $ab(a+b-4)/2+1$, $\pi(C)$ has

$$\frac{(ab-1)(ab-2)}{2} - \left(\frac{ab(a+b-4)}{2}+1\right) = \frac{a(a-1)b(b-1)}{2}$$

nodes (see Remark 3.11.1 in Hartshorne (1977)). $\square$

Already, we can begin to describe the generators of $\operatorname{gin}_{\text{lex}} I_{a,b}$:

Corollary 4.2. *The ideal $\operatorname{gin}_{\text{lex}} I_{a,b}$ contains x_1^{ab} and this is the only generator that is not divisible by x_0.*

Proof. The generators of $\operatorname{gin}_{\text{lex}} I_{a,b}$ are elements of $x_0^p \operatorname{gin}_{\text{lex}} K_p(I_{a,b})$ for various p. So clearly, the generators of $\operatorname{gin}_{\text{lex}} K_0(I_{a,b})$ are the only generators of $\operatorname{gin}_{\text{lex}} I_{a,b}$ not containing a factor of x_0. But $K_0(I_{a,b})$ is principal, generated by a form of degree ab in generic coordinates. The leading term of such a form is x_1^{ab}. $\square$

We are ready to prove the main result of the paper:

Proof (Theorem 1.1). Set $I = I_{a,b}$, $K_p = K_p(I)$. By virtue of Lemma 3.2 and since $x_0^a \in \operatorname{gin}_{\text{lex}} I$ we have

$$\operatorname{gin}_{\text{lex}} I = \sum_{p=0}^{a} x_0^p \operatorname{gin}_{\text{lex}} K_p.$$

From Proposition 4.1 we know that $\operatorname{gin}_{\text{lex}} K_0 = (x_1^{ab})$. The proof consists of three steps. First, we compute the regularity of $\operatorname{gin}_{\text{lex}} K_1$ explicitly. Then we show that the regularity of $\operatorname{gin}_{\text{lex}} K_p - p \le 1 + \operatorname{reg} \operatorname{gin}_{\text{lex}} K_1$ for $2 \le p \le a-1$. Finally, we will show that $\operatorname{gin}_{\text{lex}} I$ actually requires a generator of degree $\frac{1}{2}a(a-1)b(b-1)+1$, which will complete the proof.

By Corollary 3.12 we have that K_1 is the ideal of maximal minors of a matrix of size $(a-1)\times a$ whose ij entry has degree $b+i-j$. The resolution of K_1 is given by the Hilbert–Burch complex. It is then easy to determine the degree of K_1 from the numerical data of the resolution. We obtain that K_1 is unmixed and of degree $\frac{1}{2}a(a-1)b(b-1)$. We also know that the radical of K_1 is the ideal of definition of $\frac{1}{2}a(a-1)b(b-1)$ points. It follows that K_1 itself is the radical ideal defining $\frac{1}{2}a(a-1)b(b-1)$ points. We can conclude from Corollary 5.3 that $\operatorname{reg}\operatorname{gin}_{\text{lex}}(K_1) = \frac{1}{2}a(a-1)b(b-1)$.

We now prove that for $p > 1$, the degrees of the generators of $x_0^p \operatorname{gin}_{\text{lex}} K_p$ are bounded above by $1+\operatorname{reg}\operatorname{gin}_{\text{lex}} K_1$, that is, by $1+\frac{1}{2}a(a-1)b(b-1)$. This will imply that $\operatorname{reg}\operatorname{gin}_{\text{lex}} I$ is $\max(ab, 1+\frac{1}{2}a(a-1)b(b-1))$.

From Corollary 3.12(2) we have that the ideal, say J, of the maximal minors of $\operatorname{Syl}_2(f, g)$ is Artinian (i.e. $K[x_1, x_2, x_3]/J$ is Artinian) and is contained in K_2 and that J is contained in K_p for $p > 1$. The regularity of an Artinian ideal D is given by the smallest k such that the k-th power of the maximal ideal is contained in the ideal D and hence does not change when passing to the initial ideal. It follows that $\operatorname{reg}\operatorname{gin}_{\text{lex}} K_p \le \operatorname{reg} J$ for every $p > 1$. So the generators of $x_0^p \operatorname{gin}_{\text{lex}} K_p$ are in degrees $\le p + \operatorname{reg} J$. Taking into consideration that $K_a = (1)$, it is enough to show that $\operatorname{reg} J \le \frac{1}{2}a(a-1)b(b-1)+1-p$ for all $p = 2, \dots, a-1$. So we may assume $a > 2$ and we have to show that $\operatorname{reg} J \le \frac{1}{2}a(a-1)b(b-1)+2-a$. To compute the regularity of J we first use Lemma 3.8 to get rid of the units in the matrix defining J and then we use Lemma 3.11. We get $\operatorname{reg}(J) = ab + \binom{a-1}{2} - \binom{a+1}{2} + 2(a-3)$.

So it remains to show that

$$ab + \binom{a-1}{2} - \binom{a+1}{2} + 2(a-3) \le \frac{1}{2}a(a-1)b(b-1) + 2 - a$$

that is

$$1/2a^2b^2 - 1/2a^2b - 1/2ab^2 - 1/2ab - a + 7 \ge 0$$

for all $3 \le a \le b$. This is a simple calculus exercise.

To finish the proof, we will show that if m is a minimal generator of $\operatorname{gin}_{\text{lex}} K_1$, of degree $\frac{1}{2}a(a-1)b(b-1)$, then $x_0 m$ is a minimal generator of $\operatorname{gin}_{\text{lex}} I$. If $x_0 m$ is not a minimal generator of $\operatorname{gin}_{\text{lex}} I$, then it must be divisible by some monomial n that is a minimal generator of $\operatorname{gin}_{\text{lex}} I$. This implies that $n \mid x_0 m$ and that n must be in $\operatorname{gin}_{\text{lex}} K_0$. However, this means that $n \mid m$ and $n \in \operatorname{gin}_{\text{lex}} K_1$ since $K_0 \subseteq K_1$. This contradicts our choice of m as a minimal generator. We conclude that $x_0 m$ must be a minimal generator of $\operatorname{gin}_{\text{lex}} I$. $\square$

Example 4.3. One can check (using CoCoA, for instance) that $I = (x^3 - yz^2, y^3 - z^2 t)$ defines an irreducible complete intersection curve C with just one singular point and that $K_1(gI)$ with g a generic change of coordinates is not radical. Indeed, $K_1(gI)$ has degree 18 and it defines only 11 points, namely the 11 singular points of the generic projection of C to $\mathbb{P}^2$. In this case, the regularity of $\operatorname{gin}_{\text{lex}}(I)$ is 16 and not 19 as in the smooth case.

5. The regularity of gins of points

Set $S = k[x_0, \dots, x_r]$. We start with the following well-known lemma:

Lemma 5.1. *Let I be a homogeneous ideal of S such that S/I has Krull dimension* 1 *and* $\deg(S/I) = e$. *Set* $c = \min\{j \mid \dim[S/I]_i = e$ *for all* $i \ge j\}$. *Then* $\operatorname{reg}(I) \le \max\{e, c\}$.

Proof. Let J be the saturation of I. Then S/J is a one-dimensional CM (Cohen–Macaulay) algebra. It is well-known and easy to see that $\operatorname{reg}(J) \le \deg(S/J) = e$ and $\dim[S/J]_i = e$ for all $i \ge e-1$. Let p denote the saturation degree (satiety index) of

I, i.e. the least j such that $I_i = J_i$ for all $i \geq j$. From the characterization of regularity in terms of local cohomology it follows immediately that $\operatorname{reg}(I) = \max\{\operatorname{reg}(J), p\}$. To conclude, it is enough to show that $p \leq \max\{e, c\}$. If $p > e$ then $I_i = J_i$ for all $i \geq p$ and $I_{p-1} \subsetneq J_{p-1}$. Thus, $\dim[S/I]_i = e$ for all $i \geq p$ and $\dim[S/I]_{p-1} > e$. Hence $p = c$ and we are done. □

Corollary 5.2. *Let I be a homogeneous ideal of S such that S/I has Krull dimension* 1. *Assume that the Hilbert function of I is equal to the Hilbert function of a one-dimensional CM ideal (e.g. I is an initial ideal of a one-dimensional CM ideal). Then* $\operatorname{reg}(I) \leq \deg(S/I)$.

Proof. This follows from Lemma 5.1 since the assumption implies that $\dim[S/I]_i = \deg(S/I)$ for all $i \geq \deg(S/I) - 1$. □

Corollary 5.3. *Let I be the ideal of a set X of s points of $\mathbb{P}^r$. Then*

$$\operatorname{reg}\operatorname{gin}_{\text{lex}} I = s.$$

Proof. By Corollary 5.2 we have $\operatorname{reg}\operatorname{gin}_{\text{lex}} I \leq s$. A general projection of X to $\mathbb{P}^1$ will give s distinct points. This implies that x_{r-1}^s is in $\operatorname{gin}_{\text{lex}} I$. Since we work with the lex order, x_{r-1}^s is a minimal generator of $\operatorname{gin}_{\text{lex}} I$. □

We want to show now that for a set of generic points the gin lex and indeed any gin has a very special form: it is a segment ideal. Consider the polynomial ring $S = k[x_0, \ldots, x_r]$ equipped with a term order τ. Assume that $x_0 >_\tau x_1 >_\tau \cdots >_\tau x_r$.

Definition 5.4. A vector space V of forms of degree d is said to be a τ-segment if it is generated by monomials and for every monomial m in V and every monomial n of degree d with $n >_\tau m$ one has $n \in V$.

Given a non-negative integer $u \leq \binom{r+d}{r}$ there exists exactly one τ-segment of forms of degree d and of dimension u: it is the space generated by the u largest monomials of degree d with respect to τ and it will be denoted by $\operatorname{Seg}_\tau(d, u)$. Given a homogeneous ideal I for every d we consider the τ-segment $\operatorname{Seg}_\tau(d, \dim I_d)$ and define

$$\operatorname{Seg}_\tau(I) = \oplus_d \operatorname{Seg}_\tau(d, \dim I_d).$$

By the very definition, $\operatorname{Seg}_\tau(I)$ is a graded monomial vector space and simple examples show that $\operatorname{Seg}_\tau(I)$ is not an ideal in general. But there are important exceptions: Macaulay's numerical characterization of Hilbert functions (Bruns and Herzog, 1993, Theorem 4.2.10) can be rephrased by saying that for every homogeneous ideal I the space $\operatorname{Seg}_{\text{lex}}(I)$ is an ideal. In the following lemma we collect a few simple facts about segments that will be used in the proof of that result.

Lemma 5.5. *Let τ be a term order and let $V \subset S_a$ be a τ-segment with* $\dim S_a/V \leq a$. *Then $S_1 V$ is a τ-segment with* $\dim S_{a+1}/V S_1 = \dim S_a/V$.

Proof. First observe that since $x_{r-1}^a > x_{r-1}^{a-j} x_r^j$ for $j = 1, \ldots, a$ we have that $x_{r-1}^a \in V$ and hence $(x_0, \ldots, x_{r-1})^a \subseteq V$. To prove that $V S_1$ is a τ-segment assume that n is a monomial of degree $a + 1$ such that $x_i m < n$ with m in V; we have to show that

$n \in VS_1$. Let k be the largest index such that x_k divides n, so that $n = x_k n_1$. If $k \geq i$ then $x_i n_1 \geq x_k n_1 > x_i m$. It follows that $n_1 > m$ and hence $n_1 \in V$ so that $n \in VS_1$. If, instead, $k < i$ then $n \in (x_0, \dots, x_{r-1})^{a+1}$ which is contained in VS_1 since we have seen already that $(x_0, \dots, x_{r-1})^a$ is contained in V.

To conclude, it is enough to show that the map ϕ induced by multiplication by x_r is an isomorphism from S_a/V to S_{a+1}/VS_1. We show first that ϕ is injective. If m is a monomial in $S_a \setminus V$, then $mx_r \notin VS_1$. Otherwise, $mx_r = nx_i$ for some $n \in V$ and some i, and then $m > n$, a contradiction. To prove that ϕ is surjective, consider a monomial m in $S_{a+1} \setminus VS_1$. Then $m = x_r n$ since $(x_0, \dots, x_{r-1})^{a+1} \subset VS_1$. Obviously, $n \notin V$. So ϕ is surjective. $\square$

Proposition 5.6. *Let I be the ideal defining s points, say $P_1, \dots, P_s$, of $\mathbf{P}^r$. Assume that there exists a coordinate system $x_0, x_1, \dots, x_r$ such that I does not contain forms of degree $\leq s$ supported on $\leq s$ monomials. Then $\mathrm{gin}_\tau I = \mathrm{Seg}_\tau(I)$ for all term orders τ. In particular $\mathrm{gin}_{\mathrm{lex}} I = \mathrm{Seg}_{\mathrm{lex}}(I)$.*

Proof. It is easy to see that the assumption implies that the Hilbert function of S/I is the expected one, namely $\dim[S/I]_d = \min\{s, \binom{r+d}{r}\}$ for all d. Fix a term order τ. For a given $d \leq s$ consider the set M_d of the smallest (with respect to τ) $\min\{s, \binom{r+d}{r}\}$ monomials of degree d. By assumption these monomials are a basis of S/I in degree d. It follows immediately that $\mathrm{in}_\tau I_d = \mathrm{Seg}_\tau(I)_d$ for every $d \leq s$. From Lemma 5.1 we know that $\mathrm{in}_\tau I$ does not have generators in degree $\geq s$. Then $\mathrm{in}_\tau I_d = \mathrm{in}_\tau I_s S_{d-s}$ for all $d \geq s$. On the other hand, it follows from Lemma 5.5 that $\mathrm{Seg}_\tau(I)_d = \mathrm{Seg}_\tau(I)_s S_{d-s}$ for all $d \geq s$. We have seen already that $\mathrm{in}_\tau I_s = \mathrm{Seg}_\tau(I)_s$. Therefore we may conclude that $\mathrm{in}_\tau I_d = \mathrm{Seg}_\tau(I)_d$ also for all $d \geq s$. We have shown that $\mathrm{in}_\tau I = \mathrm{Seg}_\tau(I)$. From this it follows that $\mathrm{gin}_\tau I = \mathrm{Seg}_\tau(I)$ (see the construction/definition of gin given in Eisenbud (1995, Theorem 15.18)). $\square$

We can now prove the main result of this section:

Proof (Theorem 1.2). Let $P_1, \dots, P_s$ be generic points in $\mathbb{P}^r$. Fix a coordinate system on $\mathbb{P}^r$ and let $(a_{i0}, a_{i1}, \dots, a_{ir})$ be the coordinates of P_i. It is enough to show that the assumption of Proposition 5.6 holds (in the given coordinates) for a generic choice of the a_{ij}. For any $d \leq s$ consider the $s \times \binom{r+d}{r}$ matrix X_d whose rows are indexed by the points, the columns by the monomials of degree d and whose ij-th entry is obtained by evaluating the j-th monomial at the i-th point. The assumption of Proposition 5.6 is equivalent to the fact that any maximal minor of X_d is non-zero for $d \leq s$. If we consider the a_{ij} as variables over some base field then every minor of X_d is a non-zero polynomial in the a_{ij} since no cancellation can occur in the expansion. So these are finitely many non-trivial polynomial conditions on the coordinates of the points. $\square$

As we have already said, the genericity condition required in Theorem 1.2 implies that the Hilbert function of the ideal I of s points of $\mathbb{P}^r$ is given is the expected one:

$$\dim[S/I]_j = \min\left(s, \binom{r+j}{r}\right).$$

One may wonder whether it is enough to assume that the Hilbert function is generic to conclude that $\mathrm{gin}_\tau\, I$ is $\mathrm{Seg}_\tau(I)$ for an ideal of points. The next example answers this question.

Example 5.7. (a) Consider the ideal I of 7 points of $\mathbf{P}^3$ with generic Hilbert function. The ideal I contains 3 quadrics. If the 3 quadrics have a common linear factor, then $\mathrm{gin}(I_2)$ is $x_0(x_0, x_1, x_2)$ no matter what the term order is. So in particular, $\mathrm{gin}_{\mathrm{revlex}}\, I$ is not $\mathrm{Seg}_{\mathrm{revlex}}(I)$ in degree 2. Explicitly, one can take the seven points with coordinates $(0, 0, 0, 1), (0, 0, 1, 1), (0, 0, 2, 1), (0, 1, 0, 1), (0, 1, 1, 1), (0, 2, 0, 1), (1, 0, 0, 1)$.

(b) Consider the 10 points of $\mathbf{P}^3$ with coordinates $(a, b, c, 1)$ where a, b, c are non-negative integers with $a + b + c \leq 2$ and let I be the corresponding ideal. One can check with (and even without) the help of a computer algebra system that the 10 points have the generic Hilbert function and that any generic projection to $\mathbf{P}^2$ gives 10 points on a cubic. This, in turn, implies that $\mathrm{gin}_{\mathrm{lex}}\, I$ contains x_2^3 while $\mathrm{Seg}_{\mathrm{lex}}(I)$ does not contain it.

The next example shows that, even for Hilbert functions of generic points in $\mathbb{P}^2$, the segment ideals are special among the Borel-fixed ideals.

Example 5.8. Consider the ideal I of seven generic points in $\mathbb{P}^2$. The Hilbert function of S/I is $(1, 3, 6, 7, 7, 7, \ldots)$. There are exactly eight Borel-fixed ideals with this Hilbert function, they are:

(1)	$(x^3, x^2y, x^2z, xy^3, xy^2z, xyz^3, xz^5, y^7)$,	lex
(2)	$(x^3, x^2y, x^2z, xy^3, xy^2z, xyz^3, y^6)$,	(6, 2, 1)
(3)	$(x^3, x^2y, x^2z, xy^3, xy^2z, y^5)$,	(4, 2, 1)
(4)	$(x^3, x^2y, x^2z, xy^3, y^4)$,	
(5)	$(x^3, x^2y, xy^2, x^2z^2, xyz^3, xz^5, y^7)$,	
(6)	$(x^3, x^2y, xy^2, x^2z^2, xyz^3, y^6)$,	
(7)	$(x^3, x^2y, xy^2, x^2z^2, y^5)$,	
(8)	(x^3, x^2y, xy^2, y^4)	revlex .

The ideals (1)–(3) and (8) are segments (with respect to the term order or weight indicated on the right) while the remaining four are non-segments. Let us check, for instance, that (4) is not a segment. Suppose, by contradiction, it is a segment with respect to a term order τ. Then since x^2z is in and xy^2 is out, we have $x^2z >_\tau xy^2$ and hence $xz >_\tau y^2$. We deduce that $xy^2z >_\tau y^4$. But since y^4 is in then also xy^2z must be in and this is a contradiction. Summing up, among the eight Borel-fixed ideals only (1)–(3) and (8) are gins of I.

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