Contemporary Mathematics
Volume **423**, 2007

Nice Initial Complexes of Some Classical Ideals

Aldo Conca, Serkan Hoşten, and Rekha R. Thomas

ABSTRACT. This is a survey article on Gorenstein initial complexes of extensively studied ideals in commutative algebra and algebraic geometry. These include defining ideals of Segre and Veronese varieties, toric deformations of flag varieties known as Hibi ideals, determinantal ideals of generic matrices of indeterminates, and ideals generated by Pfaffians of generic skew symmetric matrices. We give a summary of recent work on the construction of squarefree Gorenstein initial ideals of these ideals when the ideals are themselves Gorenstein. We also present our own independent results for the Segre, Veronese, and some determinantal cases.

1. Introduction

Let I be a homogeneous ideal in a polynomial ring R over an infinite field K and let $\beta_{ij}(I)$ be the (i,j)-th Betti number of I. Since passing to an initial ideal is a flat deformation [**15**, Chapter 15], $\beta_{ij}(I) \leq \beta_{ij}(\text{in}(I))$ for all i,j and every initial ideal $\text{in}(I)$ of I. There are many classes of toric or determinantal ideals arising from classical constructions which are known to be minimally generated by Gröbner bases in their special coordinate systems for carefully chosen term orders. These include the ideals defining Segre products and Veronese subrings of polynomial rings, the ideals of minors of generic or generic symmetric matrices of indeterminates, the ideals of Pfaffians of generic skew symmetric matrices, defining ideals of Grassmannians given by Plücker relations, etc. For any such classical ideal I there is an explicit initial ideal, $\text{in}_{\text{cla}}(I)$ (called the *classical initial ideal* of I), which is squarefree and Cohen-Macaulay, and has as many minimal generators as I in each degree, that is, $\beta_{0j}(I) = \beta_{0j}(\text{in}_{\text{cla}}(I))$ for all j (see [**5, 16, 21, 32, 33**]). However, in most cases $\beta_{ij}(I) \neq \beta_{ij}(\text{in}_{\text{cla}}(I))$ for some i and j. In fact, the breakdown usually happens already at the first syzygies; see Example 1.3 below. Therefore we are led to ask the following question.

QUESTION 1.1. Given a classical ideal I, does there exist an initial ideal $\text{in}(I)$ such that $\beta_{ij}(I) = \beta_{ij}(\text{in}(I))$ for all i,j?

EXAMPLE 1.2. Question 1.1 has a positive answer for some instances, such as when I is either the ideal of m-minors of a generic $m \times n$ matrix or the ideal of $(n-1)$-minors of a generic symmetric $n \times n$ matrix. In these cases, the classical

2000 *Mathematics Subject Classification.* Primary: 13F55; Secondary: 06A11.

initial ideal satisfies the conditions of Question 1.1. The reason follows from a general remark. Let J be a homogeneous ideal in a polynomial ring R which is generated by polynomials of degree d and higher and has codimension h. We denote the degree or multiplicity of R/J by $\deg R/J$. If R/J is Cohen-Macaulay then it is easy to see that $\deg R/J \geq \binom{h+d-1}{d-1}$. If $\deg R/J = \binom{h+d-1}{d-1}$ we say that R/J (or J) has minimal multiplicity (with respect to its initial degree). Equivalently, a Cohen-Macaulay ring R/J defined in degree d and higher has minimal multiplicity if the quotient ring of R/J by a regular sequence of $\dim R/J$ elements of degree one (its *Artinian reduction*) is isomorphic to $K[x_1, \dots, x_h]/\langle x_1, \dots, x_h\rangle^d$. It follows that the Betti numbers of J are equal to those of $\langle x_1, \dots, x_h\rangle^d$. In particular, if J is Cohen-Macaulay of minimal multiplicity and $\text{in}(J)$ is a Cohen-Macaulay initial ideal then $\beta_{ij}(J) = \beta_{ij}(\text{in}(J))$ for all i, j. The ideal of m-minors of a $m \times n$ generic matrix and the ideal of $(n-1)$-minors of a generic symmetric $n \times n$ matrix are Cohen-Macaulay of minimal multiplicity and their classical initial ideals are Cohen-Macaulay. So the remark applies in these cases.

The above examples do not represent the typical behavior. Most classical initial ideals do not have the correct Betti numbers. But one can look for other non-classical initial ideals with the property required in Question 1.1.

EXAMPLE 1.3. Let I be the ideal of 2-minors of the generic 3×3 matrix $X = (x_{ij})$ and let

$$\text{in}_{\text{cla}}(I) = \langle x_{11}x_{22}, x_{11}x_{23}, x_{11}x_{32}, x_{11}x_{33}, x_{12}x_{23}, x_{12}x_{33}, x_{21}x_{32}, x_{21}x_{33}, x_{22}x_{33}\rangle$$

be its classical initial ideal with respect to a diagonal term order. The Betti diagrams of I and $\text{in}_{\text{cla}}(I)$ are respectively:

$$\begin{matrix} 9 & 16 & 9 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \quad \text{and} \quad \begin{matrix} 9 & 16 & 10 & 2 \\ 0 & 1 & 2 & 1 \end{matrix}.$$

Now we replace $\text{in}_{\text{cla}}(I)$ with another initial ideal $\text{in}_\succ(I)$ with respect to a reverse lexicographic term order $\succ$ where the diagonal variables x_{11}, x_{22}, x_{33} are smallest. The corresponding initial ideal is

$$\langle x_{23}x_{32}, x_{21}x_{32}, x_{13}x_{32}, x_{23}x_{31}, x_{13}x_{31}, x_{12}x_{31}, x_{12}x_{23}, x_{13}x_{21}, x_{12}x_{21}\rangle.$$

One can check that the Betti diagram of $\text{in}_\succ(I)$ is identical to that of I.

In Section 8 we will generalize the phenomenon in Example 1.3 to the ideal of $(n-1)$-minors in a generic $n \times n$ matrix. However, contrary to the above examples, in many classical cases the answer to Question 1.1 is negative. In fact, the property that is asked for may fail to hold for *all* initial ideals (in the given coordinates).

EXAMPLE 1.4. Let I be the ideal of 2-minors of a generic 4×4 matrix. All initial ideals of I are squarefree, Cohen-Macaulay, and generated in degree ≤ 4, see [**33**]. With the help of the the software package CaTS [**23**] we computed all 4494288 monomial initial ideals of I. They come in 4219 distinct orbits modulo symmetries, and only 920 orbits represent quadratically generated initial ideals. Computations in CoCoA [**11**] reveal that the number of quadratic first syzygies of these initial ideals varies between 2 and 25. Since I has only linear first syzygies, that is $\beta_{1j}(I) = 0$ for $j > 2$, it follows that there is no initial ideal $\text{in}(I)$ such that $\beta_{ij}(\text{in}(I)) = \beta_{ij}(I)$ for $i = 0, 1$ and all j.

The next best thing that one could ask for is an initial ideal which has the correct number of generators and the correct Cohen-Macaulay type. We also insist on asking for squarefree initial ideals so that they can be represented by simplicial complexes. Now we can state the main question that this article addresses.

QUESTION 1.5. Given a classical ideal I which is Gorenstein does there exist a Gorenstein squarefree initial ideal of I with the same number of generators as I?

Below is the list of classical ideals for which we study Question 1.5. The first three are examples of toric ideals which we review now. A polytope $P \subset \mathbf{R}^d$ is called a *lattice polytope* if its vertices lie in $\mathbf{Z}^d$. Consider the embedding of P in $\mathbf{R}^{d+1}$ given by $P \times \{1\} := \{(p,1) \in \mathbf{R}^{d+1} : p \in P\}$ and let $C(P) \subset \mathbf{R}^{d+1}$ be the cone over $P \times \{1\}$. Then $M(P) := C(P) \cap \mathbf{Z}^{d+1}$ is a monoid whose monoid algebra is $K[M(P)] := K[x^m : m \in M(P)]$, where K is an arbitrary field and $x = (x_1, \ldots, x_{d+1})$. The algebra $K[M(P)]$ is graded by the exponent of x_{d+1}. Since $M(P)$ is finitely generated as a monoid, $K[M(P)]$ is finitely generated as a K-algebra. We say that P is *normal* if $M(P)$ is generated by the lattice points in $P \times \{1\}$ and hence $K[M(P)]$ is generated by its monomials of degree one. A sufficient condition for the normality of P is the existence of a *unimodular triangulation* of the lattice points in P.

Let $\mathcal{P}$ be the vector configuration consisting of the lattice points in $P \times \{1\}$. Then the *toric ideal* of $\mathcal{P}$ is the homogeneous ideal $I_{\mathcal{P}} = \langle y^u - y^v : \sum_{p_i \in \mathcal{P}} p_i u_i = \sum_{p_i \in \mathcal{P}} p_i v_i,\ u_i, v_i \in \mathbf{N} \rangle$ in the polynomial ring $K[y]$ where $y = (y_1, \ldots, y_s)$ and $s = |\mathcal{P}|$. When P is normal, $I_{\mathcal{P}}$ is the presentation ideal of the algebra $K[M(P)]$. See [**33**] for details on toric ideals of vector configurations.

If $K[M(P)]$ is Gorenstein, we say that P is a *Gorenstein polytope.* Let $\operatorname{int}(M(P))$ denote the lattice points in the interior of $C(P)$. It is well known that $K[M(P)]$ is Gorenstein if and only of there exists $u \in \operatorname{int}(M(P))$ such that $\operatorname{int}(M(P)) = u + M(P)$ [**7**, Chapter 6].

(1) **Segre**(m,n): Consider the Segre embedding of $\mathbf{P}^{m-1} \times \mathbf{P}^{n-1}$ in $\mathbf{P}^{mn-1}$ parametrized by the monomial map

$$K[x_{ij}] \longrightarrow K[r_1, \ldots, r_m, s_1, \ldots, s_n],\ x_{ij} \mapsto r_i s_j.$$

This is a toric variety with P equal to the product of a standard $(m-1)$-dimensional simplex and a standard $(n-1)$-dimensional simplex. The corresponding vector configuration is $\mathcal{P} = \{e_i \oplus e'_j : 1 \le i \le m,\ 1 \le j \le n\}$ where $\{e_i\}$ and $\{e'_j\}$ are the standard unit vectors of $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. Note that in this case, P is a $(m+n-1)$-dimensional polytope that lies on the hyperplane $\sum r_i + \sum s_j = 2$ in $\mathbf{R}^{m+n}$ and hence we can take $\mathcal{P}$ to be just the lattice points in P as opposed to those in $P \times \{1\}$. The toric ideal $I_{\mathcal{P}}$ is generated by the 2-minors of the $m \times n$ matrix (x_{ij}) of indeterminates. The polytope P is Gorenstein with $u = (1, 1, \ldots, 1)$ if and only if $m = n$ [**7, 20**]. In this case, we will denote the defining ideal by $I(2,n)$.

(2) **Veronese**(r,n): Consider the rth Veronese embedding of $\mathbf{P}^{n-1}$ in $\mathbf{P}^N$ where $N = \binom{r+n-1}{r-1}$ and $r \in \mathbf{N}\backslash\{0,1\}$. This defines the toric ideal $I_{\mathcal{P}}$ where P is the convex hull of all lattice points in $\mathbf{N}^n$ whose coordinates sum to r. The polytope P is $(n-1)$-dimensional and lies on the hyperplane $\sum x_i = r$ in $\mathbf{R}^n$. The ideal $I_{\mathcal{P}}$ is Gorenstein if and only if r divides n [**20**].

When $r = 2$, $I_{\mathcal{P}}$ is generated by the 2-minors of a symmetric $n \times n$ matrix of indeterminates. We will denote this ideal by $J(2,n)$ throughout the article.

(3) **Plu**(m,n): Let $X = (x_{ij})$ be a $m \times n$ matrix of indeterminates. We denote by $[a_1, \ldots, a_m]$ the m-minor of X with column indices $1 \le a_1 < \ldots < a_m \le n$. The algebra $K[[a_1, \ldots, a_m] : 1 \le a_i \le n]$ is the coordinate ring of $G(m,n)$. There are the well-known Plücker relations among these minors, see for instance [**7**, Lemma 7.2.3]. We let $K[x_\alpha : \alpha = (a_1, \ldots, a_m),\ 1 \le a_1 < \cdots < a_m \le n]$ be the polynomial ring in as many variables as the m-minors of X. We also define the K-algebra homomorphism $x_\alpha \mapsto [a_1, \ldots, a_m]$. The kernel $\mathrm{Plu}(m,n)$ of this map contains the quadratic polynomials that are preimages of the Plücker relations. Indeed, the preimages of the Plücker relations generate $\mathrm{Plu}(m,n)$. This ideal is always Gorenstein [**17**].

(4) **Hibi**(m,n): Let e_{ij} be the unit vectors in $\mathbf{N}^{m\times n}$, and let

$$\mathcal{P}_{m,n} = \{e_{1a_1} + e_{2a_2} + \cdots + e_{ma_m} : 1 \le a_1 < a_2 < \cdots < a_m \le n\}.$$

The monoid algebra $K[M(P)]$ defined by the polytope P that is the convex hull of the vectors in $\mathcal{P}_{m,n}$ is known as a Hibi ring. These rings are obtained as certain toric deformations of the coordinate ring of $G(m,n)$, the Grassmannian of m-planes in K^n. The defining toric ideal $I_{m,n}$ is always Gorenstein [**8**]. In Section 6 we will define general Hibi rings and discuss some results of Reiner and Welker [**27**]. Moreover we will describe in detail those Hibi rings obtained as Sagbi deformations of general flag varieties.

(5) **DetGen**(t,m,n), **DetSym**(t,n), and **Pfaff**(t,n): Let X be a $m \times n$ matrix of indeterminates. The ideal of all t-minors ($t > 1$) of X is called a determinantal ideal, and we will denote this ideal by $\mathrm{DetGen}(t,m,n)$. This ideal is Gorenstein if and only if $m = n$ [**8**]. Similarly, the ideal $\mathrm{DetSym}(t,n)$ will denote the ideal of t-minors of an $n \times n$ symmetric matrix X of indeterminates. This ideal is Gorenstein if and only if $n - t$ is even [**19**]. Finally, for an even integer t we let $\mathrm{Pfaff}(t,n)$ be the ideal of Pfaffians of order t of a skew symmetric $n \times n$ matrix X. The ideal of Pfaffians is always Gorenstein [**26, 2**].

This paper is organized as follows. In Section 2 we recall general facts on Stanley-Reisner rings and Gorenstein simplicial complexes. Section 3 presents very recent results of Bruns and Römer which imply that a Gorenstein toric ideal with a squarefree initial ideal possesses a squarefree initial ideal that is Gorenstein. All the toric ideals described above fall into this category. However, the construction of Bruns and Römer does not completely answer Question 1.5. This is because one does not know the degrees of the generators of the Gorenstein initial ideal that exists via their result. We will treat the toric ideals $I(2,n)$ and $J(2,n)$ more extensively in Sections 4 and 5 and answer Question 1.5 positively in these cases. In both cases we will show that the corresponding ideal has a reverse lexicographic squarefree Gorenstein initial ideal where the core of the associated simplicial complex is the boundary complex of a simplicial polytope. We will explicitly describe the facets and a two-way shelling of these simplicial complexes. In Section 6 we will examine Hibi rings more closely as the deformations of general flag manifolds. Section 8 will construct Gorenstein initial ideals of $\mathrm{DetGen}(n-1,n)$, and Section 7 will give similar constructions for $\mathrm{Pfaff}(t,n)$.

Before we go on we like to point out that except the results in Section 4 and Section 5 which use shellings, a common theme to the results in this paper is the construction of a simplicial complex Δ such that after the cone points of Δ are removed the remaining complex is a simplicial sphere. The first appearence of this kind result in our context is the *equatorial complex* construction of Reiner and Welker [**27**] (see Section 7). Athanasiadis' result [**1**] that compressed polytopes (in particular, the Birkhoff polytopes) have regular unimodular triangulations with an equatorial complex (see Section 3) was inspired by this result. Subsequently, Bruns and Römer [**6**] generalized Athanasiadis result to all Gorenstein polytopes with a unimodular triangulation.

2. Stanley-Reisner rings and Gorenstein complexes

In this section we will recall briefly from [**7, 31**] a few important facts on Stanley-Reisner rings and Gorenstein simplicial complexes.

Let Δ be a simplicial complex and let $K[\Delta]$ be its Stanley-Reisner ring. The dimension of a face $F \in \Delta$ is $|F|-1$ and the dimension of Δ is the maximal dimension of its facets which are maximal faces with respect to inclusion. We call Δ a pure complex if all its facets have the same dimension. We denote by $\mathcal{F}(\Delta)$ the set of facets of Δ. Every simplicial complex has an (essentially unique) geometric realization. A simplicial complex Δ of dimension $d-1$ is said to be a simplicial sphere if its geometric realization is homeomorphic to the sphere $S^{d-1} \subset \mathbf{R}^d$. The Hilbert series of $K[\Delta]$ where Δ is $(d-1)$-dimensional has the form

$$\frac{h_0 + h_1 t + \cdots + h_s t^s}{(1-t)^d}$$

with $h_i \in \mathbf{Z}$, $h_s \neq 0$ and $s \leq d$. The vector $h(\Delta) := (h_0, h_1, \ldots, h_d)$ is called the *h-vector* of Δ. The a-invariant $a(K[\Delta])$ of $K[\Delta]$ is $s-d$, the degree of the Hilbert series as a rational function.

Given subsets $F_1, \ldots, F_k$ of a given set V we denote by $\langle F_1, \ldots, F_k \rangle$ the smallest simplicial complex containing the F_i. Furthermore, if Δ_1 and Δ_2 are simplicial complexes on disjoint sets of vertices V_1 and V_2, the join of Δ_1 and Δ_2 is $\Delta_1 * \Delta_2 = \{A \cup B : A \in \Delta_1, B \in \Delta_2\}$. We let $CP(\Delta) = \{v \in V : v \in F \text{ for each} F \in \mathcal{F}(\Delta)\}$ be the cone-points of Δ. We also let $\mathrm{core}(\Delta)$ be the restriction of Δ to the set of vertices not in $CP(\Delta)$. This implies that $\Delta = \mathrm{core}(\Delta) * \mathrm{Simplex}(CP(\Delta))$ where $\mathrm{Simplex}(A)$ denotes the simplex with vertices in A. Note that the elements in $CP(\Delta)$ correspond exactly to those variables which do not appear in the generators of the Stanley-Reisner ideal of Δ. So $K[\Delta]$ is just a polynomial extension of $K[\mathrm{core}(\Delta)]$.

A simplicial complex Δ is said to be Cohen-Macaulay or Gorenstein with respect to the field K if the Stanley-Reisner ring $K[\Delta]$ is Cohen-Macaulay or Gorenstein. The *link* of a face $F \in \Delta$ is $\mathrm{lk}_\Delta(F) = \{G \in \Delta : G \cup F \in \Delta, G \cap F = \emptyset\}$.

THEOREM 2.1. [**31**, Corollary 4.2], [**31**, Theorem 5.1] *A simplicial complex* Δ *is*

- *Cohen-Macaulay over* K *if and only if for all* $F \in \Delta$ *and all* $i < \dim(\mathrm{lk}_\Delta(F))$, *we have* $\tilde{H}_i(\mathrm{lk}_\Delta(F); K) = 0$, *and*

- *Gorenstein over K if and only if for all $F \in \text{core}(\Delta)$,*
$$\tilde{H}_i(\text{lk}_{\text{core}(\Delta)}(F); K) = \begin{cases} K \text{ if } i = \dim(\text{lk}_{\text{core}(\Delta)}(F)) \\ 0 \text{ if } i < \dim(\text{lk}_{\text{core}(\Delta)}(F)) \end{cases}$$

A simplicial complex Δ of dimension $d-1$ is said to be shellable if it is pure and $\mathcal{F}(\Delta)$ can be totally ordered so that for every non-minimal $F \in \mathcal{F}(\Delta)$ the simplicial complex
$$\langle F \rangle \cap \langle G \in \mathcal{F}(\Delta) : G < F \rangle \tag{1}$$
is pure of dimension $d-2$. The total order $<$ is called a shelling of Δ. Shellable simplicial complexes are Cohen-Macaulay (over any field) and their Hilbert series can be described in terms of the facets of the simplicial complex (1) as F varies. Important features of Gorenstein simplicial complexes are summarized below.

LEMMA 2.2. *Let Δ be a simplicial complex.*

(a) *If $K[\Delta]$ is Gorenstein, then $a(K[\Delta]) = -|CP(\Delta)|$. Equivalently, if $K[\Delta]$ is Gorenstein and* core(Δ) *has dimension $d-1$ then $h_d(\Delta) = 1$ and $h_i(\Delta) = 0$ for $i > d$.*

(b) *If Δ is a simplicial sphere then $K[\Delta]$ is Gorenstein.*

Furthermore assume that core(Δ) *is shellable and every face of codimension* 1 *(i.e. dimension* $\dim \text{core}(\Delta) - 1$*) is contained in exactly two facets. Then*

(c) core(Δ) *is a simplicial sphere.*

(d) *$K[\Delta]$ is Gorenstein.*

PROOF. For (a) and (b) see [**7**, Section 5.6]. Statement (c) is proved in [**4**, 4.7.22] and (d) follows from (b) and (c). □

A shelling $<$ of Δ is said to be a two-way shelling if the facets in the reversed order also give a shelling. Line shelling of simplicial polytopes are typical examples of two-way shellings. The shellings that we describe in this paper are shown to be two-way (but we do not know whether they are line-shellings).

3. Gorenstein Toric Ideals

In this section we survey recent results on Gorenstein toric ideals that are relevant to this paper. We use the notation introduced earlier on polytopes, monoid rings, and toric ideals. The following theorem was proved by Bruns and Römer [**6**] and relates to Question 1.5 addressed in this paper.

THEOREM 3.1. [**6**, Corollary 7] *Let P be a Gorenstein lattice polytope such that the set of lattice points $\mathcal{P}$ in P admits a regular unimodular triangulation. Then the toric ideal $I_{\mathcal{P}}$ has a squarefree Gorenstein initial ideal.*

We summarize the key ideas in the proof of this theorem from [**6**]. If $\text{in}_{\succ}(I)$ is a monomial initial ideal of the ideal I, then the radical ideal $\text{rad}(\text{in}_{\succ}(I))$ is squarefree and is the Stanley-Reisner ideal of a simplicial complex $\Delta(\text{in}_{\succ}(I))$. The simplicial complex $\Delta(\text{in}_{\succ}(I))$ is called the *initial complex* of I with respect to $\succ$. Theorem 8.3 in [**33**] proves that if $\text{in}_{\succ}(I_{\mathcal{P}})$ is a monomial initial ideal of the toric ideal $I_{\mathcal{P}}$, then the initial complex $\Delta(\text{in}_{\succ}(I_{\mathcal{P}}))$ is precisely the *regular triangulation* $\Delta_{\succ}(\mathcal{P})$ of $\mathcal{P}$ induced by $\succ$. Further, Corollary 8.9 in [**33**] shows that such an initial ideal is squarefree if and only if $\Delta_{\succ}(\mathcal{P})$ is unimodular. Thus when $\text{in}_{\succ}(I_{\mathcal{P}})$ is squarefree, the ring $K[y]/\text{in}_{\succ}(I_{\mathcal{P}})$ is the Stanley-Reisner ring $K[\Delta_{\succ}(\mathcal{P}]$.

The main theorem (Theorem 3) in [**6**] states that whenever M is a normal affine monoid such that the monoid algebra $K[M]$ is positively graded and Gorenstein, then there exists a simpler normal affine Gorenstein monoid algebra $K[N]$ where the monoid N is obtained as a projection of M. In the case where $M = M(P)$ and P is a normal Gorenstein lattice polytope, the monoid algebra $K[N]$ is generated in degree one in the grading inherited from $K[M]$ and hence equals $K[M(Q)]$ where Q is the polytope spanned by the exponents of the monomials in $K[N]$ of degree one. Further, Q is a Gorenstein lattice polytope with a unique interior lattice point. If we let the Hilbert series of $K[M(P)]$ be

$$\frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^{\dim(P)+1}}$$

and the h-vector $h(P) := (h_0, h_1, \ldots, h_d)$, then they show that $h(P) = h(Q) = h(\partial(Q))$ where $\partial(Q)$ is the boundary of Q (see [**6**, Corollary 4]).

In Theorem 3.1, we are given a Gorenstein lattice polytope P such that $\mathcal{P}$, the lattice points of P, admits a regular unimodular triangulation. Let $\Delta(\mathcal{P})$ be the induced regular unimodular triangulation of $\mathcal{P}$ and J the squarefree monomial initial ideal of $I_{\mathcal{P}}$ whose initial complex is $\Delta(\mathcal{P})$. Since $\mathcal{P}$ has a unimodular triangulation, P is normal and the polytope Q constructed above exists. Since the triangulation $\Delta(\mathcal{P})$ is a regular unimodular triangulation of $M(P)$, equivalently of $C(P)$, with all cones generated by elements of $\mathcal{P}$, $M(Q)$ and hence Q inherits a regular unimodular triangulation $\Delta(Q)$. Project the vertices of $\Delta(Q)$ on a sphere around the unique lattice point in Q and let P' be the simplicial polytope obtained as the convex hull of these projected vertices. Since $M(P)$ is Gorenstein there exists a unique $x \in \text{int}(M(P))$ such that $\text{int}(M(P)) = x + M(P)$. Let $p_1, \ldots, p_m$ be a subset of the minimal generating set (Hilbert basis) of $M(P)$ such that $x = p_1 + \ldots + p_m$. From the construction of Q and P' it follows that $\Delta(\mathcal{P})$ is the join of the simplicial complex $\Delta(P')$ corresponding to $\partial(P')$ and the simplex with vertices $p_1, \ldots, p_m$. This implies that the variables $y_1, \ldots, y_m$ in $K[y]$ corresponding to $p_1, \ldots, p_m$ form a regular sequence modulo J and hence $K[y]/J$ is Gorenstein since $\Delta(P')$ is the boundary of a simplicial polytope. We note that the main goal of [**6**] was to prove that if P is an integer Gorenstein polytope whose lattice points admit a unimodular triangulation then the h-vector of P is unimodal.

We now apply Theorem 3.1 to various Gorenstein lattice polytopes and their toric ideals listed in the Introduction.

(1) **Segre(n, n)**: All regular triangulations of $\mathcal{P}$ are known to be unimodular and P is Gorenstein. Hence by Theorem 3.1, $I_{\mathcal{P}}$ has a squarefree Gorenstein initial ideal. In Section 4 we will construct an explicit term order $\succ$ such that the initial ideal $\text{in}_{\succ}(I_{\mathcal{P}})$ is *quadratic*, squarefree and Gorenstein.
(2) **Veronese(r, n)**: The polytope P defining $I_{\mathcal{P}}$ is a simplex that admits a unimodular triangulation consisting of empty simplices whose facets are parallel to the facets of P. When P is Gorenstein (r divides n), Theorem 3.1 applies. In Section 5, in the case of $r = 2$ and $n = 2m$ we will exhibit an explicit initial ideal $\text{in}_{\succ}(J(2, n))$ that is quadratic, squarefree and Gorenstein.
(3) **Hibi(m, n)**: The vector configuration $\mathcal{P}$ defining the Hibi ring is affinely isomorphic to the vertices of the order polytope of the lattice of order ideals of the product of the chains $[m] \times [n-m]$, see Section 6 or [**33**,

Remark 11.11]. Order polytopes are known to have unimodular triangulations, and since $I_{\mathcal{P}}$ is a toric deformation of $\text{Plu}(m,n)$ the polytope P is also Gorenstein. Again, Theorem 3.1 applies.

There is one more polytope we have not mentioned so far which played a motivating role for both Theorem 3.1 and the earlier work of Athanasiadis [**1**].

Birkhoff(n): Recall that the nth Birkhoff polytope in $\mathbf{R}^{n\times n}$ is the convex hull of all the $n \times n$ permutation matrices. In this case, $\mathcal{P}$ equals the set of $n!$ vertices of this polytope. Birkhoff polytopes are known to be *compressed* which means that all their reverse lexicographic triangulations are unimodular [**30**]. Further, they are also Gorenstein. Hence again by Theorem 3.1, $I_{\mathcal{P}}$ has a squarefree Gorenstein initial ideal. In the rest of this section we briefly describe Athanasiadis' method.

A *special simplex* of a lattice polytope $P \subset \mathbf{R}^d$, is a collection of vertices $\Sigma = \{v_1, \ldots, v_q\}$ of P with the property that every facet of P contains all but one vertex in Σ. For the nth Birkhoff polytope, the collection of permutation matrices corresponding to the cyclic subgroup of S_n generated by the cycle $(1\,2\,3\,\cdots\,n)$ forms a special simplex. To see this note that the facets of the nth Birkhoff polytope are cut out by the hyperplanes $x_{ij} = 0$ in $\mathbf{R}^{n\times n}$ and each facet misses exactly one permutation in the above cyclic group. Note that special simplices are not contained in the boundary of P. If V is a linear subspace in $\mathbf{R}^d$, let P/V denote the quotient polytope equal to the image of P under the canonical projection $\mathbf{R}^d \to \mathbf{R}^d/V$.

LEMMA 3.2. [**1**, Proposition 2.3] *Let P be a d-dimensional polytope in $\mathbf{R}^d$ with a special simplex Σ such that $\mathcal{P}$ has a triangulation isomorphic to $\Sigma * \Delta$. Let V be the linear subspace parallel to the affine span of Σ. Then the boundary complex of the quotient polytope P/V inherits a triangulation abstractly isomorphic to Δ and its faces are precisely the faces of P that do not intersect Σ.*

LEMMA 3.3. [**1**, Lemma 3.4] *Suppose that $v_1 \prec \cdots \prec v_q \prec \cdots \prec v_{p-1} \prec v_p$ is an ordering of the vertices of a lattice polytope P such that $\Sigma = \{v_1, \ldots, v_q\}$ is a special simplex of P. Let Δ be the reverse lexicographic triangulation of $\{v_p, \cdots, v_{q+1}\}$ with respect to the order $\succ$. Then*

(1) *The reverse lexicographic triangulation $\Delta_{\succ}(P)$ is isomorphic to $\Sigma * \Delta$, and*
(2) *Δ is isomorphic to a reverse lexicographic triangulation of the boundary complex of P/V which in turn is isomorphic to the boundary complex of a simplicial polytope of the same dimension as P/V.*

We state a modified version of the main theorem in [**1**].

THEOREM 3.4. [**1**, Theorem 3.5] *Suppose P is a lattice polytope and $v_1 \prec \cdots \prec v_q \prec \cdots \prec v_{p-1} \prec v_p$ is an ordering of its vertices such that (i) P is compressed and (ii) $\Sigma = \{v_1, \ldots, v_q\}$ is a special simplex of P. Then the h-vector $h(P)$ equals $h(\partial(Q))$ where Q is a simplicial polytope whose boundary is isomorphic to the reverse lexicographic triangulation of $\{v_p, \cdots, v_{q+1}\}$ with respect to the order $\succ$.*

PROOF. First we invoke the fact that if Δ is any unimodular triangulation of $\mathcal{P}$, then $h(P) = h(\Delta)$. In the situation of the theorem, since P is compressed, $\Delta_{\succ}(P)$ is unimodular and hence $h(P) = h(\Delta_{\succ}(P))$. By Lemma 3.3 (i), $\Delta_{\succ}(P) = \Sigma * \Delta$ where Δ is the reverse lexicographic triangulation of $\{v_p, \cdots, v_{q+1}\}$ with respect to the order $\succ$. Thus

$$h(P) = h(\Delta_{\succ}(P)) = h(\Sigma * \Delta) = h(\Delta)$$

where the third equality follows from standard facts about joins of simplicial complexes and the fact that the h-vector of a simplex is always 1. By Lemma 3.3 (ii), Δ is isomorphic to the boundary complex of a simplicial polytope Q whose boundary is isomorphic to the reverse lexicographic triangulation of $\{v_p, \cdots, v_{q+1}\}$ with respect to the order $\succ$ which completes the proof. □

The following is a modified version of Corollaries 4.1 and 4.2 in [**1**] adapted to this paper.

COROLLARY 3.5. Let P be a compressed Gorenstein lattice polytope. Then the toric ideal $I_{\mathcal{P}}$ has a squarefree Gorenstein initial ideal. In particular, the toric ideal of the nth Birkhoff polytope has a squarefree Gorenstein initial ideal.

PROOF. Since P is Gorenstein, there exists unique $x \in \text{int}(M(P))$ such that $\text{int}(M(P)) = x + M(P)$. Let $v_1, \ldots, v_q$ be vertices of P such that $x = v_1 + \ldots + v_q$. Athanasiadis proves that $\Sigma = \{v_1, \ldots, v_q\}$ is a special simplex of P [**1**, Corollary 4.1]. Now consider any reverse lexicographic ordering of the vertices of P such that $v_q \succ \ldots \succ v_1$ comes last in the ordering. Then the conclusion of Theorem 3.4 holds. Let J be the initial ideal $\text{in}_{\succ}(I_{\mathcal{P}})$. Since $\Delta_{\succ}(P) = \Sigma * \Delta$ (from Theorem 3.4) is the initial complex of J, J is squarefree. Further, since Δ is the boundary complex of a simplicial polytope and Σ is a simplex, $K[y]/J$ is Gorenstein. □

Note that Theorem 3.1 is a generalization of Corollary 3.5. Further examples of Gorenstein lattice polytopes that satisfy the conditions of Corollary 3.5 can be found in [**22**]. We will see in Section 4 that the polytope P of Segre(n, n) also satisfies the conditions of Corollary 3.5 providing yet another proof that its toric ideal has a squarefree Gorenstein initial ideal.

4. Gorenstein Segre products

As we indicated already, $I(2, n)$ is generated by the 2-minors of a $n \times n$ matrix $X = (x_{ij})$ of indeterminates, and it is an ideal of the polynomial ring $K[x_{ij}]$. The Hilbert series of $K[x_{ij}]/I(2, n)$ is given by

$$\sum_i \binom{n-1}{i}^2 z^i/(1-z)^{2n-1}.$$

So the a-invariant is $n - 1 - (2n - 1) = -n$, and therefore any squarefree Gorenstein initial complex of $I(2, n)$ must have exactly n cone points. The classical initial ideal of $I(2, n)$ is the one associated to a "diagonal" term order, namely a term order which selects main diagonals as initial terms of minors and it is generated by the products $x_{ij}x_{hk}$ with $i < h$ and $j < k$. The facets of this initial complex are the paths from $(n, 1)$ to $(1, n)$ in an $n \times n$ grid. Table 1 shows a typical facet of the classical initial complex of $I(2, 4)$. Since $(n, 1)$ and $(1, n)$ (corresponding to the variables x_{n1} and x_{1n}) are the only points that belong to every facet, this initial complex has only two cone points. So for $n > 2$ it is not Gorenstein.
In order to construct a Gorenstein initial complex we consider a term order where the initial term of a minor is its main diagonal unless the main diagonal of the minor involves elements of the main diagonal of the matrix. Formally, for every $i < h$ and $j < k$ the initial term of the minor $x_{ij}x_{hk} - x_{ik}x_{hj}$ is $x_{ij}x_{hk}$ unless $i = j$ or $h = k$. We can define such a term order by a reverse lexicographic order $x_{11} \prec x_{22} \prec \cdots \prec x_{nn} \prec \{x_{ij} : i \neq j\}$ where the latter set of variables are ordered

TABLE 1

			*
	*	*	*
	*		
*	*		

so that $x_{ij} \succ x_{hk}$ if $|i-j| < |h-k|$. For instance for $n=4$, we could use: $x_{12} \succ x_{21} \succ x_{23} \succ x_{32} \succ x_{34} \succ x_{43} \succ x_{13} \succ x_{24} \succ x_{31} \succ x_{42} \succ x_{14} \succ x_{41} \succ x_{44} \succ x_{33} \succ x_{22} \succ x_{11}$.

The initial terms of the 2-minors are the monomials in the variables x_{ab} with $a \neq b$ of the following form:

$$\begin{array}{llll} x_{ik}x_{hj} & \text{if} & i=j \text{ or } h=k & (1) \\ x_{ij}x_{hk} & \text{if} & i<h \text{ and } j<k & (2) \end{array} \qquad (*)$$

Note that (1) is obvious by construction while (2) follows immediately from the fact that if $i<h$ and $j<k$ then $\max(|i-j|,|h-k|) < \max(|k-i|,|h-j|)$.

PROPOSITION 4.1. The ideal $H(2,n)$ generated by the monomials described in $(*)$ is an initial ideal of $I(2,n)$ with respect to $\succ$.

It is clear that $H(2,n) \subseteq \text{in}_\succ(I(2,n))$. To prove equality we use the following well-known fact.

LEMMA 4.2. *Let J and I be homogeneous ideals in a polynomial ring R. Assume that $J \subseteq I$, $\dim R/J = \dim R/I$, $\deg R/J \geq \deg R/I$ and J is pure (i.e. all its associated primes have the same dimension). Then $J = I$.*

The proof of this fact is a simple exercise in primary decompositions. Suppose $d = \dim R/I = \dim R/J$. Let $J = Q_1 \cap \cdots \cap Q_s$ be the primary decomposition of J. By assumption $\dim R/Q_i = d$ for all i. Then $\deg R/J = \sum \deg R/Q_i$. Now, since $J \subseteq I$, each primary component of I of dimension d must contain one of the Q_i. As $\deg R/I = \deg R/J$, this forces the intersection of the primary components of I of dimension d to be exactly J. So $I \subseteq J$ and hence $I = J$.

We apply Lemma 4.2 with $I = \text{in}_\succ(I(2,n))$ and $J = H(2,n)$. Because passing to initial ideals is a flat deformation the dimension and the degree of $\text{in}_\succ(I(2,n))$ are equal to that of $I(2,n)$: $\dim K[x_{ij}]/I(2,n) = 2n-1$ and $\deg K[x_{ij}]/I(2,n) = \binom{2n-2}{n-1}$. So to prove Proposition 4.1 it suffices to show the following.

LEMMA 4.3. *The ideal $H(2,n)$ is pure of dimension $2n-1$ and degree $\binom{2n-2}{n-1}$.*

PROOF. Let Δ be the simplicial complex associated with $H(2,n)$. By construction the cone points of Δ are $CP = \{x_{11}, \ldots, x_{nn}\}$ and we may concentrate our attention on $\Delta' := \text{core}(\Delta)$. We have to show that Δ' is pure and has exactly $\binom{2n-2}{n-1}$ facets of dimension $n-2$. Let us describe the facets of Δ'. For every nonempty proper subset R of $[n]$ we define:

$$\Delta_R = \{F \in \Delta' : F \subseteq R \times ([n] \setminus R)\}.$$

The generators of $H(2,n)$ of type (1) imply that every face $F = \{(a_1,b_1), \ldots, (a_k,b_k)\}$ of Δ' has $\{a_1,\ldots,a_k\} \cap \{b_1,\ldots,b_k\} = \emptyset$. In particular F belongs to Δ_R with $R = [n] \setminus \{b_1,\ldots,b_k\}$, and hence $\Delta' = \cup \Delta_R$. The generators of type (2) imply that Δ_R is exactly the simplicial complex of the subsets of the grid $R \times ([n] \setminus R)$ which do not contain 2-diagonals. If $R = \{r_1, \ldots, r_p\}$ and $[n] \setminus R = \{c_1, \ldots, c_{n-p}\}$ with $r_1 < \cdots < r_p$ and $c_1 < \cdots < c_{n-p}$, then a facet of Δ_R is a path in the grid $R \times ([n] \setminus R)$ from (r_p, c_1) to (r_1, c_{n-p}). We deduce two important facts. First, any facet of Δ_R has $n-1$ elements and it involves all the elements of R as row indices and all the elements in $[n] \setminus R$ as column indices. Second, a facet of Δ_R cannot be a facet of Δ_S if $R \neq S$. So the set of facets of Δ' is simply the disjoint union of the facets of Δ_R as R varies. This implies that Δ' is pure of dimension $n-2$. Note that the number of facets of Δ_R is $\binom{n-2}{p-1}$ $(p = |R|)$. In general, the number of paths in a $a \times b$ grid from the bottom left to the top right is $\binom{a+b-2}{a-1}$. Then the number of facets of Δ' is:

$$\sum_{p=1}^{n-1} \binom{n}{p}\binom{n-2}{p-1} = \sum_{p=0}^{n-2} \binom{n}{n-1-p}\binom{n-2}{p} = \binom{2n-2}{n-1}.$$

□

In order to prove that $H(2,n)$ is Gorenstein, according to Lemma 2.2, it suffices to prove that every face of Δ' of codimension one is contained in exactly two facets and we need to describe a shelling. Actually we will describe a two-way shelling of Δ'. First some notation.

Given a grid of size $a \times b$ we look at paths connecting the lower left corner box S (start) to the upper right corner box E (end) consisting of horizontal steps to the right or vertical steps up. Such a path consists of 4 types of points as we go from S to E: a left turn (⤴), a right turn (↱), isolated point in a column (•), and isolated point in a row (∘). This definition is illustrated by Table 2.

TABLE 2

A path

			*	*
	*	*	*	
	*			
*	*			

and the type of its points

			↱	•
	↱	•	⤴	
	∘			
•	⤴			

We say that a subset A of the points of the grid has full support if it intersects each row and each column. Clearly a path from S to E has full support.

LEMMA 4.4. *Let P be a path in a grid and $x \in P$. We have:*

i) *x is a turn (⤴ or ↱) of P if and only if $P \setminus \{x\}$ has full support if and only if there is exactly one other path Q in the grid containing $P \setminus \{x\}$. The path Q is obtained from P and x by "flipping" x.*

ii) *x is of type • or ∘ in P if and only if $P \setminus \{x\}$ does not have full support if and only if P is the only path in the grid containing $P \setminus \{x\}$.*

To give an example, flipping the turn on the last row and second column in the path of Table 2 we get the path of Table 3.

LEMMA 4.5. *Let $P \in \Delta_R$ be a facet of Δ' and let x be a point of P. Then there are exactly two facets P and Q of Δ' containing $P \setminus \{x\}$. The path Q is described as follows:*

TABLE 3

			$*$	$*$
	$*$	$*$	$*$	
$*$	$*$			
$*$				

i) *If x is a turn of P then Q is the path of the grid $R \times ([n] \setminus R)$ (i.e. a facet of Δ_R) obtained by flipping x.*
ii) *If x is of type $\bullet$ then let c be the column index of x. Set $R' = R \cup \{c\}$. Then $P \setminus \{x\}$ is a face of $\Delta_{R'}$ contained in a unique facet Q of $\Delta_{R'}$. (See Table 4).*
iii) *If x is of type $\circ$ then let r be the row index of x. Set $R' = R \setminus \{r\}$. Then $P \setminus \{x\}$ is a face of $\Delta_{R'}$ contained in a unique facet Q of $\Delta_{R'}$.*

PROOF. i) Note that $P \setminus \{x\}$ has full support in the grid $R \times ([n] \setminus R)$. Hence any facet of Δ' containing $P \setminus \{x\}$ is a facet of Δ_R. Now we use i) of Lemma 4.4. ii) The support of $P \setminus \{x\}$ is $R \times ([n] \setminus R')$. So, among all the Δ_S, $P \setminus \{x\}$ belongs only to Δ_R and to $\Delta_{R'}$. In both Δ_R and $\Delta_{R'}$ the set $P \setminus \{x\}$ does not have full support. By ii) of Lemma 4.4 there is exactly one facet P in Δ_R and exactly one facet Q in$\Delta_{R'}$ containing $P \setminus \{x\}$. The statement iii) is dual to statement ii). □

For an illustration of the construction of Lemma 4.5 ii) see Table 4, where $n = 9$, $R = \{1, 4, 6, 9\}$, $c = 5$, $x = (4, 5)$. The first two arrays show P and $P \setminus \{x\}$ in the grid $R \times ([n] \setminus R)$ and the second two show $P \setminus \{x\}$ and Q in $R' \times ([n] \setminus R')$.

TABLE 4

$P =$

	2	3	5	7	8
1				$*$	$*$
4		$*$	$\bullet$	$*$	
6	$*$	$*$			
9	$*$				

$\longrightarrow$ $P \setminus \{x\} =$

	2	3	5	7	8
1				$*$	$*$
4		$*$		$*$	
6	$*$	$*$			
9	$*$				

$\longrightarrow$

$P \setminus \{x\} =$

	2	3	7	8
1			$*$	$*$
4		$*$	$*$	
5				
6	$*$	$*$		
9	$*$			

$\longrightarrow$ $Q =$

	2	3	7	8
1			$*$	$*$
4		$*$	$*$	
5		$\circ$		
6	$*$	$*$		
9	$*$			

Now we describe the shelling. First we order the set of nonempty proper subsets of $[n]$. Such a subset is represented as a strictly increasing sequence of integers.

$$S = \{a_1, \ldots, a_s\} < R = \{b_1, \ldots, b_t\} \iff \begin{cases} a_j < b_j \text{ for the smallest } j \text{ such that} \\ a_j \neq b_j \\ \text{or} \\ s < t \text{ and } a_i = b_i \text{ for all } i = 1, \ldots, s \end{cases}$$

DEFINITION 4.6. Let F and G be facets of Δ', say F is a facet of Δ_R and G is a facet of Δ_S. We set:

$$F < G \iff \begin{cases} R < S \\ \text{or} \\ R = S \text{ and } F < G \text{ in the standard shelling of } \Delta_R \end{cases}$$

The standard shelling of Δ_R is defined as follows: let F, G be facets (paths) in the corresponding grid. Then we set $F < G$ if the first step in which they differ (always going from bottom-left to top-right) is vertical for F and (hence) horizontal for G. See Table 5 for the standard shelling in the 3×3 grid.

TABLE 5

*	*	*
*		
*		

<

	*	*
*	*	
*		

<

		*
*	*	*
*		

<

	*	*
	*	
*	*	

<

		*
	*	*
*	*	

<

		*
		*
*	*	*

For every facet F of Δ_R we define:

$$F^- = \begin{array}{l} \{x \in F : x \text{ is a left turn }\} \cup \\ \{x \in F : x \text{ is of type } \bullet \text{ and its column index is } < \max(R)\} \cup \\ \{x \in F : x \text{ is of type } \circ \text{ and its row index is} \max(R)\} \end{array}$$

and

$$F^+ = F \setminus F^- = \begin{array}{l} \{x \in F : x \text{ is a right turn }\} \cup \\ \{x \in F : x \text{ is of type } \bullet \text{ and its column index is } > \max(R)\} \cup \\ \{x \in F : x \text{ is of type } \circ \text{ and its row index is} < \max(R)\} \end{array}$$

In Table 6 the symbols $+$ or $-$ mark whether that point is in F^+ or F^-.

TABLE 6

	2	3	5	7	10
1				+	+
4		+	−	−	
6	+	−			
8	+				
9	−				

PROPOSITION 4.7. The total order of the facets of Δ' in Definition 4.6 is a two-way shelling of Δ'. Precisely, for every facet F of Δ' one has:

$$\langle F \rangle \cap \langle G : G < F \rangle = \langle F \setminus \{x\} : x \in F^- \rangle \quad (1)$$

and

$$\langle F \rangle \cap \langle G : G > F \rangle = \langle F \setminus \{x\} : x \in F^+ \rangle. \quad (2)$$

where in (1) it is assumed that F is not the minimal facet of Δ' and in (2) it is not the maximal.

In order to prove Proposition 4.7 we will show the two inclusions $\supseteq$ and $\subseteq$ in (1) and (2) separately. The first inclusion is equivalent to the following

CLAIM 4.8. For every facet F of Δ' and for $x \in F$ let G be the unique facet other than F containing $F \setminus \{x\}$. Then we have $G > F$ if $x \in F^+$ and $G < F$ if $x \in F^-$.

PROOF. Suppose F is a facet of Δ_R. The statement is clear if x is a turn where G is obtained by flipping the turn x. In this case, $G < F$ if x is a left turn and $G > F$ if x is a right turn. If x is of type $\bullet$ then $G \in \Delta_{R'}$ where $R' = R \cup \{c\}$ and c is the column index of x. We conclude that $R' < R$ if and only if $c < \max(R)$. Finally if x is of type $\circ$ then $G \in \Delta_{R'}$ where $R' = R \setminus \{r\}$ and r is the row of x. And this time we conclude that $R' < R$ if and only if $r = \max(R)$. □

The reverse inclusions in (1) and (2) translate to two more claims.

CLAIM 4.9. If F, H are facets of Δ' and $H < F$ then there exists $x \in F^-$ such that $x \notin H$.

CLAIM 4.10. If F, G are facets of Δ' and $F < G$ then there exists $y \in F^+$ such that $y \notin G$.

PROOF OF CLAIM 4.9. Suppose F is a facet of Δ_R. If H also belongs to Δ_R, then the desired x is a left turn of F. If instead H is in Δ_S for some $S \neq R$ then $S < R$ because $H < F$. We let $S = \{a_1 < \cdots < a_s\}$ and $R = \{b_1 < \cdots < b_t\}$. There are two cases.

Case 1: $a_j < b_j$ for some j and $a_i = b_i$ for every $i < j$. Then $a_j \notin R$ and $a_j < b_j \leq \max(R)$. The points of F in column a_j are not in H since a_j is a row index for H. So it is enough to show that column a_j intersects F^-. If F has an isolated point x in this column, then we are done since we know that $a_j < \max(R)$. If F has a left turn in column a_j then we are also done. Otherwise a_j is the first column of the grid of F and the first step of the path is vertical. But the starting point of the path is of type $\circ$ in the row with index $\max(R)$ and column a_j. This concludes the proof in this case.

Case 2: $s < t$ and $a_i = b_i$ for $i = 1, \ldots, s$. The points of F in row b_t are not in H since b_t is a column index for H. So it is enough to show that row b_t of F intersects F^-. This is clear because in the last row we have either a left turn or an element of type $\circ$ (note that F has at least two rows). □

PROOF OF CLAIM 4.10. As above if G is a facet of Δ_R such that $F \in \Delta_R$ then the desired y is a right turn of F. If instead G is in Δ_S for some $S \neq R$ then $S > R$ because $G > F$. We let $S = \{a_1 < \cdots < a_s\}$ and $R = \{b_1 < \cdots < b_t\}$ and study two cases.

Case 1: $a_j > b_j$ for some j and $a_i = b_i$ for every $i < j$. The points of F in row b_j are not in G since b_j is a column index for G. So we are done if row b_j intersects F^+. This is the case if F has a right turn in row b_j. It is also the case if F has an isolated point in row b_j and $b_j < \max(R)$. So we may assume that $b_j = \max(R)$ and F has no right turn in that row. Now either $j = 1$ (i.e. $R = \{b_1\}$) or $j > 1$ and the first step of F is vertical. If $j = 1$ then $(1, a_1)$ is of type $\bullet$ for F and we are done. If $j > 1$ and the first step of F in vertical then a_j is a column index for F and is $> \max(R) = b_j$. If in column a_j for F we have an isolated point or a right turn then we are done. There is just one possibility left: a_j is the last column index for F and there is no point of type $\bullet$ in that column. So the ending point of F is reached with a vertical step. But then the ending point (b_1, a_j) is of type $\circ$ and $b_1 < b_j$. So we are done.

Case 2: $t < s$ and $a_i = b_i$ for $i = 1, \ldots, t$. All the points of F in column a_s are not in G since a_s is a row index for G. So we are done if column a_s intersects F^+. This is the case if F has a point of type $\bullet$ in column a_s or a right turn in that column. Otherwise a_s must be the largest column index for F and the last step of F is vertical. But then $t > 1$ (because there is a vertical step) and the last point of F, namely (b_1, a_s) is of type $\circ$ for F with row index $b_1 < \max(R) = b_t$. So (b_1, a_s) is in F^+. This concludes the proof of Claim 4.10. □

Thus we have shown that $H(2,n)$ gives a Gorenstein initial complex of $I(2,n)$. The goal of the rest of this section is to prove that for many reverse lexicographic initial ideals similar to $H(2,n)$ the core of the initial complex is the boundary of a simplicial polytope. To this end we construct a reverse lexicographic triangulation of the point configuration $\mathcal{P}$ whose toric ideal is $I_{\mathcal{P}} = I(2,n)$. We first review some facts about these triangulations and $\mathcal{P}$.

Let $\mathcal{P} = \{a_1, \ldots, a_n\} \subset \mathbf{Z}^d$ be a point configuration. The reverse lexicographic triangulation of $\mathcal{P}$ (as well as the corresponding polytope P) with respect to the ordering $a_1 \succ a_2 \succ \cdots \succ a_n$ is obtained as follows (see [**33**, Chapter 8]): let $F_1, \ldots, F_k$ be the facets of P that do not contain a_n. Then

$$\Delta_{\succ}(\mathcal{A}) \;=\; \bigcup_{i=1}^{k} \bigcup_{G \in \Delta_{\succ}(F_i)} G \cup \{a_n\}$$

where G runs over the facets of $\Delta_{\succ}(F_i)$. Observe that the definition implies that a_n is a cone point of $\Delta_{\succ}(P)$.

Denote by $I(2,m,n)$ the toric ideal of Segre(m,n) generated by the 2-minors of a generic $m \times n$ matrix. In this case, $\mathcal{P}$ is $\Sigma_{m-1} \times \Sigma_{n-1}$ where Σ_k is the standard simplex in $\mathbf{R}^{k+1}$ of dimension k. In this case the point configuration is

$$\mathcal{P}(m,n) \;:=\; \{e_i \oplus f_j \;:\; 1 \le i \le m, 1 \le j \le n\}$$

where e_i and f_j are the standard unit vectors in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively. We will identify the columns of $\mathcal{P}(m,n)$ with the variables in the polynomial ring $K[x_{ij}]$. The convex hull of $\mathcal{P}(m,n)$ which we denote by $P(m,n)$ has dimension $m+n-1$. Each face of $P(m,n)$ is $F \times G$ where F and G are faces of Σ_{m-1} and Σ_{n-1}, respectively. In other words, facets of $P(m,n)$ are $F \times \Sigma_{n-1}$ and $\Sigma_{m-1} \times G$ where F and G run over the facets of Σ_{m-1} and Σ_{n-1}, respectively. This implies the following.

Proposition 4.11. Let u_i and v_j be the coordinate functions of $\mathbf{R}^m$ and $\mathbf{R}^n$ $(m, n \ge 2)$. Then the facets of $P(m,n)$ in $\mathbf{R}^m \oplus \mathbf{R}^n$ are precisely the $m+n$ faces supported by $u_i = 0$ for $i = 1, \ldots, m$ and $v_j = 0$ for $j = 1, \ldots, n$.

We will also need the following lemma.

Lemma 4.12. *Assume that* $0 \le i \le m \le n$. *Let* $x_{11} \prec x_{22} \prec \cdots \prec x_{ii} \prec \{x_{ij} : i \ne j\}$ *be a reverse lexicographic order where the variables in the latter set are ordered arbitrarily. Then* $x_{11}, \ldots, x_{ii}$ *are cone points of the triangulation* $\Delta_{\succ}$. *Moreover, if* $i = m > 1$, *the simplicial complex obtained by removing* $x_{11}, \ldots, x_{mm}$ *is* core $\Delta_{\succ}$.

PROOF. We induct on $m+n$. The first non-trivial cases are $m+n=3$ and $m+n=4$, and the statements are easy to check. The case $i=0$ is vacuous for any $m+n$. So suppose $i \geq 1$. By the definition of $\Delta_{\succ}$ we know x_{11} is a cone point. There are exactly two facets of $P(m,n)$ that do not contain x_{11}, namely the facets defined by $u_1 = 0$ and $v_1 = 0$. These facets are isomorphic to $P(m-1,n)$ and $P(m,n-1)$ respectively. They go with $I(2,m-1,n)$ and $I(2,m,n-1)$ corresponding to generic matrices obtained by deleting the first row and deleting the first column (respectively) of an $m \times n$ matrix. In the first case, by cyclically permuting the columns, and in the second case by cyclically permuting the rows, we will be in the case $m+n-1 < m+n$ and $x_{22} \prec \cdots \prec x_{ii}$ are the variables that are smallest. By induction they are cone points on both facets, and hence cone points of $\Delta_{\succ}$. For the last statement, observe that after removing $x_{11}, \ldots, x_{mm}$ the remaining faces that need to be triangulated are defined by $u_i = 0$ for $i \in I \subset [m]$ together with $v_j = 0$ for $j \in [m] \setminus I$. If there were another cone point x_{ij}, the corresponding $e_i \oplus f_j$ had to be in every one of these faces. But clearly that cannot happen. □

THEOREM 4.13. *Assume that* $0 \leq i \leq m \leq n$. *Let* $x_{11} \prec x_{22} \prec \cdots \prec x_{ii} \prec \{x_{ij} : i \neq j\}$ *be a reverse lexicographic order. After removing the cone points* $x_{11}, \ldots, x_{ii}$ *from the triangulation* $\Delta_{\succ}$, *the remaining simplicial complex is a* $m+n-i-2$ *dimensional ball if* $m < n$ *or if* $m = n$ *and* $i < m$, *and it is a* $n-2$ *dimensional sphere if* $i = m = n$.

PROOF. Again, the proof is by induction on $m+n$. One more time the cases $m+n=3$ and $m+n=4$ are easy to check. As in the proof of the above lemma, after removing the cone point x_{11}, the rest of the triangulation is the union of the reverse lex triangulations of the two facets of $P(m,n)$ defined by $u_1 = 0$ and $v_1 = 0$ respectively. These facets were isomorphic to $P(m-1,n)$ and $P(m,n-1)$, and we we will use our induction hypothesis on them. Assume that we remove the cone points $x_{22}, \ldots, x_{ii}$ from these two facets. We get the following statements by induction: if $m < n$ or if $i < m = n$, the two simplicial complexes are $m+n-i-2$ dimensional balls. They are glued along the simplicial complex obtained by triangulating the unique face at the intersection of the two facets, namely the face defined by $u_1 = v_1 = 0$, and removing the cone points $x_{22}, \ldots, x_{ii}$. This face is isomorphic to $P(m-1,n-1)$, and hence after the removal we get a $m+n-i-3$-dimensional ball. But this one-lower-dimensional ball is on the boundary of the two balls. So the gluing gives again an $m+n-i-2$-dimensional ball. When $i = m = n$, we obtain two $n-2$-dimensional balls, glued by an $n-3$-dimensional sphere. If we can show that this $n-3$-sphere is exactly the boundary of the two balls, then after gluing we will get a $n-2$-dimensional sphere. Let us concentrate on one ball B, obtained from the facet $u_1 = 0$. After removing the cone points, this simplicial complex is the union of simplicial complexes obtained by triangulating the faces F_I defined by $u_1 = 0$ and $u_i = 0$ for $i \in I \subset \{2, \ldots, m\}$ and $v_j = 0 \in J = \{2, \ldots, m\} \setminus I$. So the simplices that will make up the boundary of B are precisely the simplices on the facets of F_I which belong to a unique F_I. The facets of F_I are obtained by either setting $u_s = 0$ where $s \in J$ or setting $v_t = 0$ where $t \in I \cup \{1\}$. In the first case, this facet of F_I is also a facet of $F_{I \cup \{s\}}$ defined by $v_s = 0$. In the second case, if $t \neq 1$, it is the facet of $F_{I \setminus \{t\}}$ defined by $u_t = 0$. Only when $t = 1$, this facet of F_I belongs to the $n-3$-dimensional sphere which is on the boundary of this B. Symmetric arguments hold for the second facet defined by $v_1 = 0$, and we are done. □

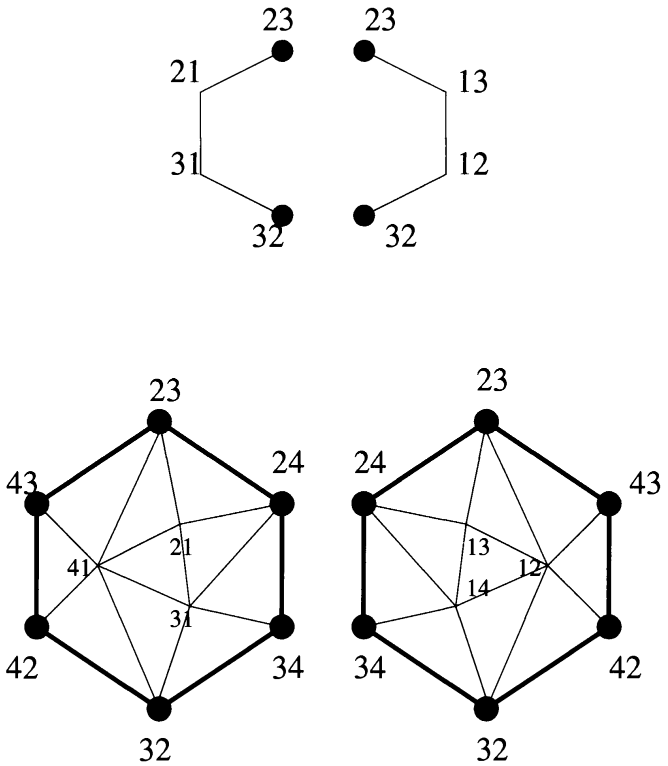


FIGURE 1. How the balls glue to a sphere

COROLLARY 4.14. Any initial ideal of $I(2,n)$ with respect to the reverse lex order $x_{11} \prec x_{22} \cdots \prec x_{nn} \prec \{x_{ij} : i \neq j\}$ is Gorenstein and quadratic.

PROOF. Any initial ideal of $I(2,m,n)$ is squarefree, since $\mathcal{P}(m,n)$ is a totally unimodular configuration. So it is enough to show that core $\Delta_{\succ}$ is a simplicial sphere. And this follows from Lemma 4.12 and Theorem 4.13. The fact that any such initial ideal is quadratic is a consequence of the following remark. □

REMARK 4.15. A set of polynomials is a universal Gröbner basis for the ideal they generate if it is a Gröbner basis of the ideal with respect to all term orders, and it is a reverse lex universal Gröbner baisis if it is a Gröbner basis with respect to all reverse lex term orders. The 2-minors of a generic $m \times n$ matrix do not form a universal Gröbner unless $\min(m,n) = 2$. A universal Gröbner basis for the ideal $I(2,m,n)$ can be described in terms of the cycles of the complete bipartite graph $K_{m,n}$, see [**33**, 4.11 and 8.11] or [**35**, 8.1.10]. Nevertheless, the 2-minors of a generic $m \times n$ matrix do form a universal reverse lex Gröbner basis of $I(2,m,n)$. This can be checked by using the Buchberger criterion.

Let us illustrate the theorem for $m = n = 2$, $m = n = 3$ and $m = n = 4$. In the first case core $\Delta_{\succ}$ consists of the two isolated vertices x_{12} and x_{21}, and this is a 0-sphere. For the other cases Figure 1 shows the two balls and how they glue along their boundary. In the last case, we order the variables so that the initial terms of the minors that do not touch the variables $x_{11}, \ldots, x_{44}$ are the main diagonals.

COROLLARY 4.16. The simplicial sphere constructed in Theorem 4.13 is the boundary of a simplicial polytope.

PROOF. The simplex spanned by $e_i \oplus f_i$ corresponding to $x_{11}, \cdots, x_{nn}$ is a special simplex as in Section 3, and the reverse lexicographic term order we used is the kind in Lemma 3.3. Now the result follows from this lemma and Theorem 3.4. □

5. Gorenstein Veronese varieties

In Section 3 we have indicated that the ideal defining the Veronese variety $\text{Ver}(r,n)$ is Gorenstein if and only if r divides n. Theorem 3.1 guarantees that this ideal has a squarefree Gorenstein initial ideal. In this section we look at the case $r = 2$, and give two independent proofs of the same result. These results are in the same spirit as in Section 4: the first constructs a squarefree initial ideal that corresponds to a simplicial complex with a two-way shelling, and the second constructs an initial complex which is a polytopal sphere.

Recall that the ideal $J(2,n)$ generated by the 2-minors of an $n \times n$ generic symmetric matrix $X = (x_{ij})$ is the defining ideal of $\text{Ver}(2,n)$. It is an ideal of the polynomial ring $K[x_{ij}] = K[x_{ij} : 1 \le i \le j \le n]$. The Hilbert series of $K[x_{ij}]/J(2,n)$ is

$$\sum_i \binom{n}{2i} z^i/(1-z)^n.$$

We assume that $n = 2m$. Then the degree of the h-vector is m and the a-invariant is $n - m = m$. Therefore any Gorenstein initial complex of $J(2,n)$ must have exactly m cone points.

The classical initial complex (associated to diagonal orders) is described as follows: its facets are the paths in the "upper triangle" in an $n \times n$ grid

$$T = \{(i,j) \in \mathbf{N}^2 : 1 \le i \le j \le n\}$$

with starting point $(1,n)$ and end point (i,i) for some i, $1 \le i \le n$.

Table 7 shows T and a typical facet of the classical initial complex of $J(2,4)$, where we put ∘ in those positions which are in the triangle but not in the path.

TABLE 7

∘	∘	∘	∘
	∘	∘	∘
		∘	∘
			∘

∘	∘	∘	*
	∘	*	*
		*	∘
			∘

The only cone point of the classical initial complex of $J(2,n)$ is $(1,n)$, and hence it is not Gorenstein if $n > 2$. In order to describe a Gorenstein initial complex we consider a term order such that the initial term of a 2-minor of (x_{ij}) is its main diagonal unless the main diagonal involves elements from the set $CP = \{(1,n),(2,n-1),\ldots,(m,m+1)\}$. An example of such a term order is a reverse lexicographic order where the variables corresponding to CP (namely $x_{1n}, x_{2,n-1}, \ldots, x_{m,m+1}$) are followed by the rest of the variables which are totally ordered so that $x_{ij} \succ x_{hk}$ if $|i-j| < |h-k|$. For instance, for $n = 4$ this term order can be taken as the reverse lexicographic order with $x_{11} \succ x_{22} \succ x_{33} \succ x_{44} \succ x_{12} \succ x_{34} \succ x_{13} \succ x_{24} \succ x_{23} \succ x_{14}$.

By construction, the initial terms of the 2-minors are the monomials not involving variables in CP of the following two kinds:

$$\left.\begin{array}{lcr} x_{ij}x_{hk} & \text{if a+b=n+1 for some } a \in \{i,j\} \text{ and } b \in \{h,k\} & (1) \\ x_{ij}x_{hk} & \text{with } i \leq j, h \leq k, i < h, j < k & (2) \end{array}\right. \quad (**)$$

Let $K(2,n)$ be the ideal generated by these monomials. We want to show that $K(2,n) = \mathrm{in}_{\succ}(J(2,n))$. According to Lemma 4.2, it suffices to show that $K(2,n)$ and $\mathrm{in}_{\succ}(J(2,n))$ have the same dimension and degree and that $K(2,n)$ is pure. The dimension and the degree of $J(2,n)$ are $\dim K[x_{ij}]/J(2,n) = n$ and $\deg K[x_{ij}]/J(2,n) = 2^{n-1}$.

LEMMA 5.1. *Let $\Delta' = \mathrm{core}(\Delta)$ be the core of the simplicial complex Δ associated with $K(2,n)$. Then Δ' is pure and has 2^{n-1} facets with m vertices.*

PROOF. Consider the family $\mathcal{A}$ of subsets A of $[n]$ of cardinality m and such that $i+j \neq n+1$ for every $i,j \in A$. Note that any $A \in \mathcal{A}$ is completely determined by its intersection with $[m]$. In other words, the cardinality of $\mathcal{A}$ is 2^m. For any $A \in \mathcal{A}$ we set

$$T_A = \{(i,j) : i \leq j \text{ and } i,j \in A\} \qquad \text{and} \qquad \Delta_A = \{F \in \Delta' : F \subseteq T_A\}$$

The monomials of type (1) imply $\Delta' = \cup_A \Delta_A$ as A varies in $\mathcal{A}$. The monomials of type (1) do not have any effect on Δ_A while those of type (2) imply that Δ_A is exactly the simplicial complex of the subsets of the small triangle T_A which do not contain any 2-diagonal. In other words any Δ_A is the classical initial complex of $J(2,m)$. Each Δ_A has 2^{m-1} facets each of cardinality m. Each facet of Δ_A involves (either as a row or column index) all the indices of A. Then the set of the facets of Δ is the disjoint union of the set of the facets of Δ_A with $A \in \mathcal{A}$. It follows that Δ is pure and has $2^m 2^{m-1} = 2^{n-1}$ facets. $\square$

Our next goal is to prove that Δ is Gorenstein. Given $A = \{a_1, \dots, a_m\}$ with $a_1 < \cdots < a_m$, we consider paths in T_A starting with the box $S = (a_1, a_m)$ and ending with a box (a_i, a_i) on the diagonal. Each step is either a horizontal step to the left or a vertical step downwards. Such a path consists of 3 types of points as we travel from S to a diagonal box: a left turn ($\curvearrowleft$) with the convention that the last point is a left turn if the last step to a diagonal box is horizontal; a right turn ($\hookleftarrow$) with the convention that the last point is a right turn if the last step is vertical; and an isolated point ($\bullet$) if this point is the only one on the path on a row or column a_j. In the latter case we say that the point is isolated with index a_j. For an illustration of this definition see Table 8. As in Lemma 4.5 we prove:

TABLE 8

Path:

0	0	0	0	0	*
	0	0	*	*	*
		0	*	0	0
			*	0	0
				0	0
					0

Types:

0	0	0	0	0	$\bullet$
	0	0	$\curvearrowleft$	$\bullet$	$\hookleftarrow$
		0	$\bullet$	0	0
			$\hookleftarrow$	0	0
				0	0
					0

LEMMA 5.2. *Let $P \in \Delta_A$ be a facet of Δ' and let x be a point of P. Then there are exactly two facets P and Q of Δ' containing $P \setminus \{x\}$. The path Q is described as follows:*

i) *If x is a turn of P then Q is the path of the triangle T_A (i.e. a facet of Δ_A) obtained by flipping x.*
ii) *If x is of type • suppose that it is the only point of the path involving the index $i \in A$. We set $A' = A \setminus \{i\} \cup \{n+1-i\}$, and then $P \setminus \{x\}$ is a face of $\Delta_{A'}$ contained in a unique facet Q of $\Delta_{A'}$.*

We order the A's lexicographically, i.e. if $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_m\}$ then

$$A < B \iff a_j < b_j \text{ for the smallest } j \text{ such that } a_j \neq b_j.$$

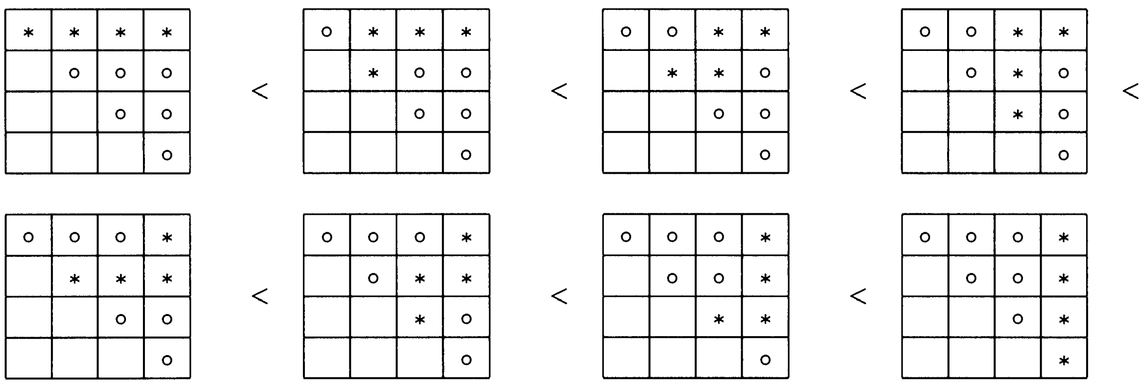
And also we define a total order on the set of facets of Δ' which will turn out to be a shelling.

DEFINITION 5.3. Let F and G be facets of Δ', say F is a facet of Δ_A and G is a facet of Δ_B. We set:

$$F < G \iff \begin{cases} A < B \\ \text{or} \\ A = B \text{ and } F < G \text{ in the standard shelling of } \Delta_A \end{cases}$$

The standard shelling of Δ_A is defined as follows: let F, G be facets (paths) in the corresponding T_A. Then $F < G$ if the first step in which the paths differ going from top-right to bottom-left is horizontal for F and (hence) vertical for G. Table 9 shows the standard shelling Δ_A when $m = 4$.

TABLE 9



For every facet F of Δ_A we define:

$F^- = \{x \in F : x \text{ is a right turn }\} \bigcup \{x \in F : x \text{ is of type } \bullet \text{ with index } > m\}$ and

$F^+ = F \backslash F^- = \{x \in F : x \text{ is a left turn }\} \bigcup \{x \in F : x \text{ is of type } \bullet \text{ with index } \leq m\}$

The proof of the next proposition is similar to that of Proposition 4.7 but easier since here everything is fully symmetric.

PROPOSITION 5.4. The total order described above is a two-way shelling of Δ'. Precisely, for every facet F of Δ' one has:

$$\langle F \rangle \cap \langle G : G < F \rangle = \langle F \setminus \{x\} : x \in F^- \rangle \quad (1)$$

and

$$\langle F \rangle \cap \langle G : G > F \rangle = \langle F \setminus \{x\} : x \in F^+ \rangle \quad (2)$$

By Lemmas 5.1, 5.2, Proposition 5.4 and applying Lemma 2.2 we have proved that $K(2,n)$ is a squarefree Gorenstein initial ideal of $J(2,n)$. A stronger statement would be to claim that Δ' is the boundary complex of a simplicial polytope. The rest of the section will prove this result.

The point configuration defining $\mathrm{Ver}(2,n)$ is

$$\mathcal{P}(2,n) = \{e_i + e_j \,: 1 \le i \le j \le n\} \subset \mathbf{R}^n,$$

and the convex hull of these points is the polytope $P(2,n)$ with vertices $2e_1, \dots, 2e_n$. The variable x_{ij} corresponds to the point $e_i + e_j$. The facets of $P(2,n)$ are defined by the coordinate hyperplanes $y_i = 0$ for $i = 1, \dots, n$. To construct the desired initial complex we will use the same reverse lexicographic term order we introduced earlier. We order the anti-diagonal variables $AD = \{x_{m,m+1} \succ \cdots \succ x_{1n}\}$, and then we order the rest of the variables $X^c = X \setminus AD$ in such a way, so that the faces of $P(2,n)$ not containing $e_1 + e_n, \dots, e_m + e_{m+1}$ are triangulated by unimodular simplices. Because these faces are isomorphic to $P(2,m)$ such a coherent ordering of X^c are possible.

Now we construct the corresponding reverse lexicographic triangulation of $P(2,n)$. Because $K(2,n)$ is a squarefree initial ideal, this triangulation is unimodular. In fact we can describe its pieces as we have done before: after "pulling" the points corresponding to the anti-diagonal variables we are left with the faces of $P(2,n)$ defined by setting m coordinates $y_i = 0$ where $i \in A$ as in the proof of Lemma 5.1. These faces are isomorphic to $P(2,m)$, and as Veronese polytopes they are triangulated further using the usual diagonal term order into unimodular simplices. We note two facts. First, after pulling the point corresponding to x_{1n} there are exactly two facets F_1 and F_n of $P(2,n)$, defined by $y_1 = 0$ and $y_n = 0$ respectively, which do not contain this point. Second, the intersection of these two facets defined by $y_1 = y_n = 0$ is the polytope $P(2,n-2)$.

THEOREM 5.5. *The core of the simplicial complex obtained by triangulating $P(2,n)$ using the above term order is an $(m-1)$-dimensional sphere.*

PROOF. We use induction on m. The case for $m = 1$ is clear since $P(2,2)$ is the convex hull of $2e_1, e_1 + e_2, 2e_2$ in $\mathbf{R}^2$. Suppose the statement is true for $k \le m-1$. As we observed above the facet F_1 and F_n of $P(2,2m)$ are triangulated polytopes, and their intersection is $P(2,2m-2)$. Since these two facets contain the remaining $m-1$ cone points in their intersection, the core of (the triangulation of) $F_1 \cap F_2$ is the intersection of the cores of F_1 and F_n. By induction, the former is an $(m-2)$-dimensional sphere. The core of F_1 is supported on smaller Veronese polytopes obtained by setting m coordinates $y_i = 0$ where $i \in A$ as in the proof of Lemma 5.1 and $n \notin A$, and similarly the core of F_n is supported on those where $1 \notin A$. All of the former contains the point $2e_n$ and all of the latter contains the point $2e_1$. Now since the core of $F_1 \cap F_2$ is supported by faces obtained by setting both $y_1 = y_n = 0$, we conclude that the core of the triangulation of F_1 is supported on cones with apex $2e_n$ and that of F_n is supported on cones with apex $2e_1$. Since the core of $F_1 \cap F_n$ is an $m-2$-dimensional sphere we conclude that the core of F_1 and F_2 are $(m-1)$-dimensional balls, and their boundary is precisely the core of $F_1 \cap F_n$. This shows that the triangulation of $P(2,n)$ is an $(m-1)$-dimensional sphere. □

Figure 2 illustrates the construction in the above proof for $m = 1, 2, 3$.

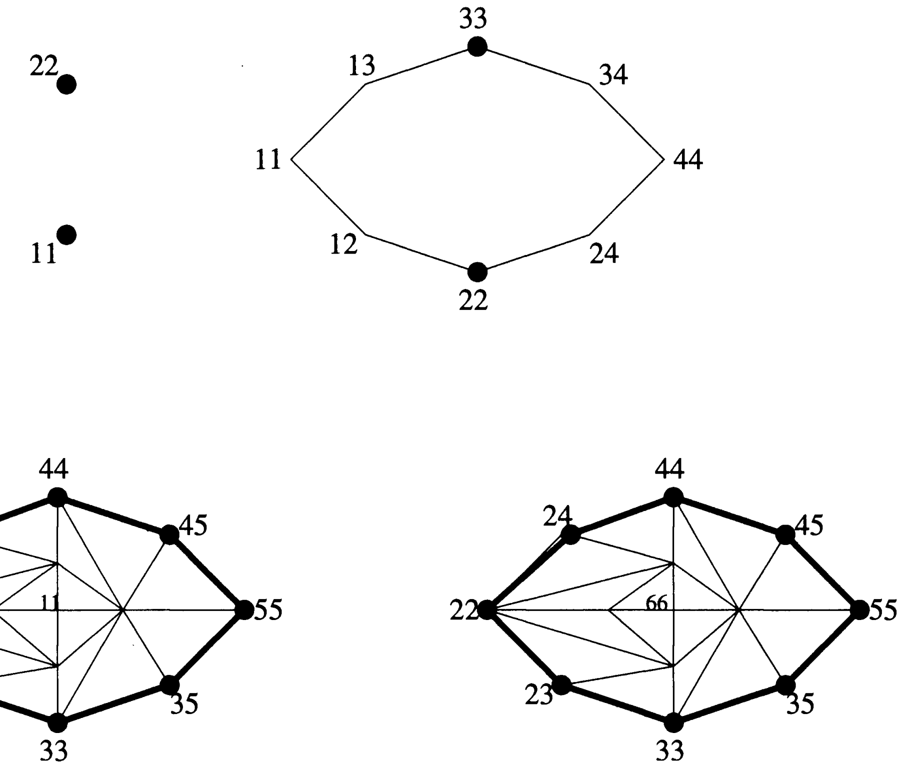


FIGURE 2. How the balls glue to a sphere

COROLLARY 5.6. The sphere constructed in Theorem 5.5 is a polytopal sphere.

PROOF. We note that the simplex that is the convex hull of the cone points $e_1 + e_n, \cdots, e_m + e_{m+1}$ is a special simplex of $P(2, n)$ as defined in Section 3. The triangulation we get is a reverse lexicographic one used in Theorem 3.4. $\square$

6. Hibi rings and flag varieties

In this section, first we recall the definition and main properties of Hibi rings associated with distributive lattices. Then we describe a result of Reiner and Welker [**27**] that implies the existence of Gorenstein initial complexes for ideals defining Gorenstein Hibi rings. Finally we will illustrate Sagbi deformations of the coordinate rings of flag varieties to certain Hibi rings. For general facts on Sagbi bases and Sagbi deformations we refer the reader to [**5, 10**] and [**33**, Chapter 11].

6.1. Hibi rings. Let $(P, \leq)$ be a finite poset. If there is no danger of confusion we will denote the poset only by the underlying set P. A (possibly empty) subset I of P is an order ideal if $x \leq y \in I$ implies $x \in I$. The set $J(P)$ of the order ideals of P is a poset under set inclusion. This poset is a distributive lattice where join and meet operations correspond to taking unions and intersections. The celebrated Birkhoff's theorem [**3**] asserts that any finite distributive lattice L is lattice-isomorphic to $J(P)$ for some poset P. Indeed one can take P to be the set of *join-irreducible* elements of L with the poset structure induced by L. An element $x \in L$ is called join-irreducible if it is not the minimum of L and cannot be written as $y \vee z$ for $z, y < x$. More precisely, $J(P) \simeq L$ as lattices under the map sending any order ideal $I = \{x_1, \ldots, x_k\}$ of P to $x_1 \vee x_2 \vee \cdots \vee x_k$ where, by convention, the image of $\emptyset$ is the $\hat{0}$ of L.

For any distributive lattice L let R_L be the polynomial ring over the field K whose variables are the elements of L. For each pair of incomparable elements $x, y \in L$ one defines the Hibi relation $xy - (x \wedge y)(x \vee y)$. The Hibi ideal I_L is the ideal of R_L generated by all the Hibi relations and the Hibi ring of L is the K-algebra defined by I_L:

$$I_L = (xy - (x \wedge y)(x \vee y) : x, y \text{ incomparable in } L) \quad \text{and} \quad H(L) = R_L/I_L.$$

Hibi proved in [**21**] that $H(L)$ is a normal Cohen-Macaulay domain and is a homogeneous algebra with straightening law (ASL). The main point of Hibi's proof is to describe $H(L)$ as a toric ring. Let Q be the *order polytope* of P, i.e., the convex hull of $\{\chi_I : I \in J(P)\}$ where χ_I is the 0/1 characteristic vector of I: $\chi_I(p) = 1$ if $p \in I$ and $\chi_I(p) = 0$ otherwise. Birkhoff's theorem induces a K-algebra isomorphism $K[M(Q)] \simeq H(L)$. Here $K[M(Q)]$ is the monoid algebra associated to Q as in Section 1. An important consequence of the ASL property is that $\text{in}_\succ(I_L) = (xy : x \text{ and } y \text{ incomparable})$ with respect to any reverse lexicographic order where $x \prec y$ whenever $x <_L y$. Hibi proved also that $H(L)$ is Gorenstein if and only if P is graded i.e. all the maximal chains of P have the same cardinality.

THEOREM 6.1 (Hibi). *If L is a distributive lattice then*

(1) *the Hibi ring $H(L)$ is a toric, normal, Cohen-Macaulay ASL,*
(2) *the ideal I_L has a quadratic squarefree initial ideal whose associated simplicial complex is the chain complex of L, and*
(3) *$H(L)$ is Gorenstein if and only if the poset of join-irreducible elements in L is graded.*

Then Theorem 3.1 implies the following result.

THEOREM 6.2. *Let L be a distributive lattice and assume that $H(L)$ is Gorenstein. Then I_L has a squarefree initial ideal which is Gorenstein and whose associated simplicial complex is a cone over a simplicial polytope.*

A proof of Theorem 6.2 is given by Reiner and Welker in [**27**]. We give a few details of their approach. Let P be the graded poset such that $L = J(P)$. We may assume that P is a poset on $[n]$ and has rank r. A chain of order ideals $I_1 \subset I_2 \subset \cdots \subset I_t$ is called *equatorial* if $f := \chi_{I_1} + \cdots + \chi_{I_t}$ has the property that $\min_{p \in P} f(p) = 0$ and for every $j \in [2, r]$, there exists a covering relation $p_{j-1} < p_j$ with p_{j-1} of rank $j-1$ and p_j of rank j such that $f(p_{j-1}) = f(p_j)$. On the other hand, $I_1 \subset I_2 \subset \cdots \subset I_t$ is *rank-constant* if f is constant along ranks of P, i.e., $f(p) = f(q)$ whenever p and q are elements of the same rank in P.

DEFINITION 6.3. [**27**, Definition 3.7] The equatorial complex $\Delta_{eq}(P)$ is the subcomplex of the order complex $\Delta(J(P))$ whose faces are indexed by the equatorial chains of non-empty order ideals in P.

Theorem 3.6 in [**27**] proves that the collection of simplices $\{\text{conv}(\chi_I : I \in \mathcal{R} \cup \mathcal{E})\}$ where $\mathcal{R}$ (respectively $\mathcal{E}$) is a chain of non-empty rank constant (equatorial) order ideals in P, gives a unimodular triangulation of the order polytope $\mathcal{O}(P)$ called the equatorial triangulation of $\mathcal{O}(P)$. Let $\mathcal{O}_{eq}(P)$ be the quotient polytope $\mathcal{O}(P)/V$ where V is the linear subspace spanned by the characteristic vectors of the rank-constant ideals in P. This quotient polytope can be identified with the orthogonal projection of $\mathcal{O}(P)$ onto $V^\perp$. Reiner and Welker show that the equatorial complex $\Delta_{eq}(P)$ can be realized as the boundary complex of the simplicial polytope Q that is obtained by a reverse lexicographic triangulation of $\mathcal{O}_{eq}(P)$ where the vertices of $\mathcal{O}_{eq}(P)$ corresponding to order ideals I with smaller cardinality come first. This means the following: if we order the vertices of $\mathcal{O}(P)$ reverse lexicographically where those corresponding to the rank-constant order ideals come first and then the rest is ordered according to the cardinality of the order ideals, the unimodular triangulation we obtain is the simplicial join of the simplex given by the rank-constant ideals and $\Delta_{eq}(P)$. Now the initial ideal $\text{in}_{eq}(I_L)$ of I_L with respect to the above reverse lexicographic term order is squarefree. Moreover, the core of the corresponding initial complex is $\Delta_{eq}(P)$ which is the boundary complex of a simplical polytope.

Furthermore, Reiner and Welker show that $\text{in}_{eq}(I_L)$ is quadratic if the width (i.e. the largest size of an antichain) of P is at most 2 and need not be so if the width is larger than 2. In particular, they give a positive answer to Question 1.5 for Hibi ideals associated to graded posets P of width at most 2.

6.2. Flag varieties and their deformation to Hibi rings. Let V be a vector space of dimension n over an algebraically closed field K. The Grassmann variety $G(m,n)$ is the set of m-dimensional subspaces of V. It is a projective variety embedded in the projective space $\mathbf{P}^{N-1}$ where $N = \binom{n}{m}$ via the Plücker map. The coordinate ring $\text{Grass}(m,n)$ of $G(m,n)$ in this embedding is the K-subalgebra of $K[x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$ generated by the m-minors of the $m \times n$ matrix $X = (x_{ij})$. The algebra $\text{Grass}(m,n)$ has a toric deformation to the Hibi ring associated to the poset of maximal minors (see [**33**, Chapter 11]). More generally, a similar statement holds also for flag varieties. As we explain below, these toric deformations are simple consequences of the straightening law for generic minors. We first recall the definition of flag varieties and their multi-homogeneous coordinate rings and then describe the toric deformation in the language of Sagbi bases.

Consider a sequence $1 \leq m_1 < m_2 < \cdots < m_k < n$ and set $M = \{m_1, \ldots, m_k\}$. Define $F(M,n) = \{V_1 \subset V_2 \subset \cdots \subset V_k \subset V : V_i \text{ a vector space of dimension } m_i\}$. Let $X = (x_{ij})$ be a $m_k \times n$ matrix of variables. For $p \leq m_k$ and $a_1 < \cdots < a_p \leq n$ we denote by $[a_1, \ldots, a_p]$ the p-minor of X with row indices $1, 2, \ldots, p$ and column indices $a_1, \ldots, a_p$. If we set $L(M,n) = \{[a_1, \ldots, a_p] : a_1 < \cdots < a_p \leq n,\ p \in M\}$, the multi-homogeneous coordinate ring $\text{Flag}(M,n)$ of $F(M,n)$ is

$$\text{Flag}(M,n) = K[[a_1, \ldots, a_p] : [a_1, \ldots, a_p] \in L(M,n)].$$

With the partial order $[a_1, \ldots, a_p] \leq [b_1, \ldots, b_q]$ if $p \geq q$ and $a_i \leq b_i$ for $i = 1, \ldots, q$, the set of minors $L(M,n)$ becomes a distributive lattice. The straightening law for generic minors (see [**12**] or [**8**]) asserts that the polynomial ring $K[x_{ij}]$ has a K-basis whose elements are products of minors of X of various order. It implies immediately that $\text{Flag}(M,n)$ has a K-basis $B(M,n)$ whose elements are the products $\delta_1 \ldots \delta_v$ with $\delta_i \in L(M,n)$ and $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_v$.

Now comes the crucial (and easy) observation: if $\succ$ is a diagonal term order and $\delta \neq \gamma \in B(M,n)$ then $\text{in}_\succ(\delta) \neq \text{in}_\succ(\gamma)$. Recall that the *initial algebra* of a K-algebra R with respect to a term order $\succ$, denoted as $\text{in}_\succ(R)$, is the K-vector space generated by $\{\text{in}_\succ(f) : f \in R\}$. A subset L of R is a *Sagbi basis* of R with respect to $\succ$ if $\text{in}_\succ(R) = K[\text{in}_\succ(\alpha) : \alpha \in L]$. The following result is part of the folklore in this subject.

PROPOSITION 6.4. With the notation introduced above we have:

(1) the elements of $L(M,n)$ form a Sagbi basis of $\operatorname{Flag}(M,n)$, that is, the initial algebra $\operatorname{in}_{\succ}(\operatorname{Flag}(M,n)) = K[\operatorname{in}_{\succ}(f) : f \in L(M,n)]$,
(2) the elements $\operatorname{in}_{\succ}(g)$ with $g \in B(M,n)$ form a K-basis of $\operatorname{in}_{\succ}(\operatorname{Flag}(M,n))$, and
(3) $\operatorname{in}_{\succ}(\operatorname{Flag}(M,n))$ is the Hibi ring of $L(M,n)$.

PROOF. Let $f \neq 0 \in \operatorname{Flag}(M,n)$. Then f can be written as a linear combination of elements in $B(M,n)$. Since distinct elements of $B(M,n)$ have distinct initial terms, $\operatorname{in}_{\succ}(f)$ is the initial term of some element of $B(M,n)$. So any monomial in the initial algebra $\operatorname{in}_{\succ}(\operatorname{Flag}(M,n))$ is of the form $\operatorname{in}_{\succ}(g)$ for a unique $g \in B(M,n)$. This proves (1) and (2). To prove (3) note that for $\delta, \gamma \in L(M,n)$ one has $\operatorname{in}_{\succ}(\delta)\operatorname{in}_{\succ}(\gamma) = \operatorname{in}_{\succ}(\delta \wedge \gamma)\operatorname{in}_{\succ}(\delta \vee \gamma)$. This gives a surjective K-algebra homomorphism from $H(L(M,n))$ to $\operatorname{in}_{\succ}(\operatorname{Flag}(M,n))$. It must be also an isomorphism because the two rings have the same Hilbert function. □

So as we have seen, $\operatorname{Flag}(M,n)$ gets deformed to the Hibi ring $H(L(M,n))$. One knows that $\operatorname{Flag}(M,n)$ is Gorenstein (even factorial), see [**17**, Chap.9]. This implies that the Hibi ring $H(L(M,n))$ must be Gorenstein as well since it is a Cohen-Macaulay graded domain with the Hilbert function of a Gorenstein ring [**7**, Cor.4.4.6]. One can also argue the other way around: check that the poset of join-irreducible elements of $L(M,n)$ is graded (indeed, it is a distributive lattice) and deduce that the Hibi ring $H(L(M,n))$ is Gorenstein. Then, by Sagbi deformation, one has that $\operatorname{Flag}(M,n)$ is Gorenstein. Below we present some examples describing the poset of join -irreducible elements for some specific values of M and n.

Now we associate indeterminates t_α with $\alpha \in L(M,n)$ and we obtain a presentation of $\operatorname{Flag}(M,n)$ as a quotient of $K[t_\alpha : \alpha \in L(M,n)]$ via the map sending t_α to α. The kernel of this map is the Plücker ideal $\operatorname{Plu}(M,n)$:

$$\operatorname{Flag}(M,n) = K[t_\alpha : \alpha \in L(M,n)]/\operatorname{Plu}(M,n).$$

The generators of $\operatorname{Plu}(M,n)$ are quadrics which can be described in terms of multilinear algebra, see [**17**, Chap.9]. In their reduced form (in the sense of ASL theory or Sagbi basis theory) they are of the form

$$t_\alpha t_\beta - t_{\alpha\vee\beta}t_{\alpha\wedge\beta} + \dots \text{ other terms } \lambda t_\gamma t_\delta$$

where

(1) the p-minor α and q-minor β are incomparable in $L(M,n)$, and
(2) $\lambda \in \mathbf{Z}$ and in each term $\lambda t_\gamma t_\delta$ with $\lambda \neq 0$, γ is a p-minor and δ is a q-minor with $\delta < \alpha \wedge \beta$ and $\gamma > \alpha \vee \beta$ and $\operatorname{rank}\alpha + \operatorname{rank}\beta = \operatorname{rank}\gamma + \operatorname{rank}\delta$.

THEOREM 6.5. *The ideal of Plücker relations* $\operatorname{Plu}(M,n)$ *defining the flag variety* $F(M,n)$ *has an initial ideal which is squarefree and Gorenstein.*

PROOF. By Sagbi theory, any initial ideal of the ideal defining the initial algebra $H(L(M,n))$ is also an initial ideal of the ideal defining $\operatorname{Flag}(M,n)$. So it is enough to show that the toric ideal $I_{L(M,n)}$ has a Gorenstein squarefree initial ideal. But this follows from Theorem 6.2. □

EXAMPLE 6.6. Consider the Grassmannian $\operatorname{Grass}(m,n) = \operatorname{Flag}(M,n)$ with $M = \{m\}$. The join-irreducible elements of $L(M,n)$ are:

$$\delta(a,b) = [1,2,\dots,a-1,a+b,a+1+b,\dots,m+b]$$

with $a = 1, \dots, m$ and $b = 1, \dots, n-m$. Note that $\delta(a,b) \leq \delta(c,d)$ if and only if $c \leq a$ and $b \leq d$ Hence the poset of join-irreducible elements P of $L(M,n)$ is a $m \times (n-m)$ grid, i.e. the cartesian product of $[m]$ and $[n-m]$. Therefore the width of P is $\min(m, n-m)$. It follows that Question 1.5 for Grass(m,n) has a positive answer if $\min(m, n-m) \leq 2$. This is essentially the case $m = 2$. We analyze the case $m = 2$ and $n = 5$ in more detail. In this case $P = \{p < q\} \times \{1,2,3\}$ and including the empty order ideal there are ten order ideals of P which we list using their maximal elements:

$$\emptyset, \{p1\}, \{p2\}, \{p3\}, \{q1\}, \{q1,p2\}, \{q1,p3\}, \{q2\}, \{q2,p3\}, \{q3\}.$$

We label these order ideals by $[12], [13], \dots, [45]$ respectively. This ordering is consistent with the description of the join-irreducible elements $[13], [14], [15], [23], [34], [45]$. The order polytope is a six-dimensional polytope that is the convex hull of the columns of the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where the columns correspond to the order ideals in the above order and the rows correspond to $p1, p2, p3, q1, q2, q3$. The Hibi ideal I_L is generated by five relations:

$$[14][23]-[13][24], [15][23]-[13][25], [15][24]-[14][25], [15][34]-[14][35], [25][34]-[24][35].$$

According to Theorem 6.1 the first terms of these binomials are the minimal generators of the classical initial ideal. The chain $[12] \subset [13] \subset [24] \subset [35] \subset [45]$ is the maximal chain of rank constant order ideals. And according to Reiner-Welker construction we can take the following reverse lexicographic term order:

$$[12] \prec [13] \prec [24] \prec [35] \prec [45] \prec [14] \prec [23] \prec [15] \prec [25] \prec [34].$$

Then $\text{in}_{\succ}(I_L) = \langle [14][23], [14][25], [15][23], [15][34], [25][34] \rangle$, and the cone points of the corresponding simplicial complex are precisely those in the maximal chain of rank constant order ideals. Moreover, the core of this complex is the boundary complex of a pentagon whose vertices are (in cyclic order) $[14], [34], [23], [25], [15]$.

EXAMPLE 6.7. Consider Flag$(\{1,3,4,6\},7)$. The poset of join-irreducible elements of $L(M,n)$ is:

					8
234567	345678	4567	5678	678	7
134567	145678	1567	1678	178	1
124567	125678	1267	1278	128	
123567	123678	1237	1238	123	
123467	123478	1234			
123457	123458				

Here 128 stands for $[1,2,8]$ and so on.

7. Pfaffians

In this section we let X be an $n \times n$ skew symmetric matrix of indeterminates: the diagonal entries of X are zero, and $x_{ji} = -x_{ij}$ for $i < j$. For $t = 2r \leq n$ and $J = \{1 \leq j_1 < \cdots < j_t \leq n\}$ the t-minor of X obtained from the columns and rows indexed by J is of the form $p_J(x)^2$ where $p_J(x)$ is a polynomial of degree r. The polynomials $p_J(x)$ are called the Pfaffians of order $t = 2r$, and we let $\mathrm{Pfaff}(t, n)$ be the ideal generated by all Pfaffians of order t. Squarefree initial ideals of $\mathrm{Pfaff}(t, n)$ have been constructed [**16**]. However, even though $\mathrm{Pfaff}(t, n)$ is always Gorenstein [**2**] these initial ideals are not. Here we give a sketch of the construction of Gorenstein initial ideals from [**25**]. We thank Jakob Jonsson and Volkmar Welker for generously sharing their manuscript in progress. We refer the reader to this manuscript [**25**] for all the details.

The dimension and the degree of any initial ideal of $\mathrm{Pfaff}(2(r+1), n)$ is equal to that of $\mathrm{Pfaff}(2(r+1), n)$: $\dim K[X]/\mathrm{Pfaff}(2(r+1), n) = r(2n - 2r - 1)$ and

$$\deg K[X]/\mathrm{Pfaff}(2(r+1), n) = \prod_{1 \leq i \leq j \leq n-2r-1} \frac{2r+i+j}{i+j}.$$

The determinantal formula for the Hilbert series of $K[X]/\mathrm{Pfaff}(2(r+1), n)$ (see [**18**]) implies that the a-invariant is $-rn$.

Suppose $p_J(x)$ is a Pfaffian of order $2r$ associated to the row and column indices $J = \{1 \leq j_1 < \cdots < j_{2r} \leq n\}$. Then the terms of $p_J(x)$ are in bijection with the *perfect matchings* of the complete graph on $2r$ vertices labeled by the elements of J. For instance, if we take $n = 6$, $r = 2$, and $J = \{1, 2, 3, 4\}$, the corresponding Pfaffian is $x_{14}x_{23} - x_{13}x_{24} + x_{12}x_{34}$. The terms of this Pfaffian correspond to the matchings $\{1-4,\ 2-3\}$, $\{1-3,\ 2-4\}$, and $\{1-2,\ 3-4\}$, respectively. We will introduce a term order used in [**25**] that picks as initial term that term of $p_J(x)$ corresponding to the matching $\{j_1 - j_{r+1},\ j_2 - j_{r+2}, \ldots, j_r - j_{2r}\}$. For this we let $d_{ij} = \min(j - i, n + i - j)$ for $i < j$. Now we totally order the indeterminates so that $x_{ij} \prec x_{kl}$ whenever $d_{ij} < d_{kl}$, and then we use a reverse lexicographic term order induced by this ordering

EXAMPLE 7.1. Let $n = 6$ and $r = 2$. We can use the following reverse lexicographic order:

$$\begin{gathered} x_{12} \prec x_{23} \prec x_{34} \prec x_{45} \prec x_{56} \prec x_{16} \prec \\ x_{13} \prec x_{24} \prec x_{35} \prec x_{46} \prec x_{15} \prec x_{26} \prec \\ x_{14} \prec x_{25} \prec x_{36} \end{gathered}$$

The set of all Pfaffians is a Gröbner basis of $\mathrm{Pfaff}(4, 6)$ where the underlined terms are the initial terms:

$$\begin{gathered} \underline{x_{36}x_{25}} - x_{26}x_{35} - x_{56}x_{23},\ \underline{x_{36}x_{14}} - x_{46}x_{13} - x_{16}x_{34},\ \underline{x_{25}x_{14}} - x_{15}x_{24} - x_{45}x_{12}, \\ \underline{x_{14}x_{26}} - x_{24}x_{16} - x_{46}x_{12},\ \underline{x_{36}x_{15}} - x_{35}x_{16} - x_{13}x_{56},\ \underline{x_{26}x_{15}} - x_{25}x_{16} - x_{56}x_{12}, \\ \underline{x_{25}x_{46}} - x_{24}x_{56} - x_{26}x_{45},\ \underline{x_{15}x_{46}} - x_{14}x_{56} - x_{16}x_{45},\ \underline{x_{14}x_{35}} - x_{13}x_{45} - x_{15}x_{34}, \\ \underline{x_{46}x_{35}} - x_{36}x_{45} - x_{56}x_{34},\ \underline{x_{36}x_{24}} - x_{26}x_{34} - x_{46}x_{23},\ \underline{x_{35}x_{24}} - x_{25}x_{34} - x_{45}x_{23}, \\ \underline{x_{25}x_{13}} - x_{15}x_{23} - x_{35}x_{12},\ \underline{x_{26}x_{13}} - x_{16}x_{23} - x_{36}x_{12},\ \underline{x_{24}x_{13}} - x_{14}x_{23} - x_{34}x_{12} \end{gathered}$$

PROPOSITION 7.2 (cf. [**25**]). *The initial term of the Pfaffian $p_J(x)$ where $J = \{j_1 < \cdots < j_{2r}\}$ is $x_{j_1 j_{r+1}} x_{j_2 j_{r+2}} \cdots x_{j_r j_{2r}}$.*

One consequence of this proposition is the following. Let $I(r, n)$ be the squarefree ideal generated by the initial terms of the order $2r$ Pfaffians in $\mathrm{Pfaff}(2r, n)$.

This corresponds to a simplicial complex $\Delta_{n,r-1}$ and the cone points of $\Delta_{n,r-1}$ correspond to the variables which do not appear in the generators of $I(r,n)$. These variables are of the form x_{ij} where either $j-i \leq r-1$ or $n+i-j \leq r-1$. An easy counting argument shows that there are $(r-1)n$ such cone points. This is precisely $-a(\mathrm{Pfaff}(2r,n))$. Therefore it is natural to ask whether $I(r,n)$ is equal to $\mathrm{in}_{\succ}(\mathrm{Pfaff}(2r,n))$. The positive answer is the content of the next result.

PROPOSITION 7.3. [**25**, Theorem 2.1] The ideal $I(r,n)$ generated by the initial terms of the Pfaffians in the ideal $\mathrm{Pfaff}(2r,n)$ is equal to $\mathrm{in}_{\succ}(\mathrm{Pfaff}(2r,n))$.

PROOF. Clearly $I(r,n) \subseteq \mathrm{in}_{\succ}(\mathrm{Pfaff}(2r,n))$. For the other inclusion we use Lemma 4.2. The simplicial complex $\Delta_{n,r-1}$ corresponding to $I(r,n)$ can be described as follows: Let $\Omega_n = \{(i,j) : 1 \leq i < j \leq n\}$ which we will think of as the edges and diagonals of a convex n-gon. For $j \geq 1$, a j-crossing is a subset of j elements of Ω_n which mutually intersect and where all $2j$ endpoints are distinct. Then $\Delta_{n,r-1}$ is the simplicial complex of all subsets of Ω_n which do not contain an r-crossing. Observe that the minimal nonfaces of $\Delta_{n,r-1}$ are precisely r-crossings, and they correspond to the minimal generators of $I(r,n)$. By the results in [**14**] and [**24**] the simplicial complex $\Delta_{n,r}$ is a pure complex of dimension $r(2n-2r-1)-1$ and has $\prod_{1\leq i\leq j\leq n-2r-1} \frac{2r+i+j}{i+j}$ facets. Now Lemma 4.2 implies that $I(r+1,n) = \mathrm{in}_{\succ}(\mathrm{Pfaff}(2(r+1),n))$. □

Further results in [**13**] show that the core $\Delta'_{n,r}$ of $\Delta_{n,r}$ is a simplicial sphere. With this result we get the main theorem of this section.

THEOREM 7.4. [**25**, Theorem 2.1] *The ideal* $\mathrm{in}_{\succ}(\mathrm{Pfaff}(2r,n))$ *is a squarefree Gorenstein initial ideal.*

We finish this section by pointing out that $\Delta'_{n,1}$ is the boundary complex of the polytope dual to the n-associahedron, and hence it is a polytopal sphere [**13**]. It remains open whether $\Delta'_{n,r}$ is a polytopal sphere in general.

8. Minors

In this section, we return to Question 1.1 and illustrate a family of determinantal ideals such that for each I in the family there is an initial ideal $\mathrm{in}_{\succ}(I)$ with the same Betti numbers as I.

THEOREM 8.1. *Let* I *be the ideal of* $(n-1)$*-minors of the generic* $n \times n$ *matrix* $X = (x_{ij})$ *with* $n > 2$. *Set* $V = \{x_{ij} : 0 \leq j-i \leq 1\} \cup \{x_{n1}\}$ *and* $W = \{x_{ij} : x_{ij} \notin V\}$. *Let* Y *be the matrix obtained from* X *by replacing* x_{ij} *with* 0 *if* $x_{ij} \in W$. *Let* $\succ$ *be any reverse lexicographic order on the* x_{ij} *such that* $x_{ij} > x_{hk}$ *if* $x_{ij} \in V$ *and* $x_{hk} \in W$. *Then* $\mathrm{in}_{\succ}(I)$ *is a square-free monomial ideal with Betti numbers equal to those of* I *and the core of the associated initial complex is the cyclic polytope with* $2n$ *vertices in* $\mathbf{R}^{2n-4}$. *More precisely,* $\mathrm{in}_{\succ}(I)$ *is the specialization of* I *by the regular sequence* W *or in other words,* $\mathrm{in}_{\tau}(I)$ *is the ideal of* $(n-1)$*-minors of* Y.

We note that part of Daniel Soll's thesis [**28**] has results about the initial complexes of determinantal ideals, and in a recent manuscript [**29**] Soll and Welker identify a polytopal sphere (namely the core of the simplicial complex of the type-B k-triangulations of a $2n$-gon) which is conjecturally the initial complex of the k-minors of an $n \times n$ generic matrix with respect to a special term order. Theorem 8.1 is Proposition 7 in [**29**] where they prove this conjectural equality for $k = n-1$.

Given a matrix Z we define a graph $G(Z)$ as follows. The vertices of $G(Z)$ are the elements z_{ij} such that $z_{ij} \neq 0$ and the edges are the pairs $\{z_{ij}, z_{hk}\}$ such that $i = h$ or $j = k$. Note that $G(Y)$ is a cycle of length $2n$. The statement of Theorem 8.1 remains true whenever V is a subset of the x_{ij}'s such that the corresponding matrix Y has the property that the graph $G(Y)$ is a cycle of length $2n$. The main ingredient needed in the proof of Theorem 8.1 is the following lemma.

LEMMA 8.2. *Suppose Y is a $n \times n$ matrix such that $G(Y)$ is a cycle of length $2n$. Let J_k be the ideal of k-minors of Y. Then for all $k < n$,*

$$J_k = (\prod_{v \in A} v : A \text{ is an independent set of } G(Y) \text{ with } |A| = k).$$

PROOF OF THEOREM 8.1. With the notation of Theorem 8.1, let J be the ideal generated by the $(n-1)$-minors of Y. From Lemma 8.2, J is a square-free monomial ideal. Let Δ be the initial complex of J. Lemma 8.2 implies that the facets of core(Δ) are the sets obtained as unions of $n-2$ disjoint edges of G. Since the size of any such facet is $2n-4$ and Y has $2n$ nonzero entries, the codimension of J is 4. Now using the facts that I defines a Cohen-Macaulay ring and that $I + (W) = J + (W)$, we conclude that W is a regular sequence modulo I. This in turn shows that J is a specialization of I by the regular sequence W and hence J and I have the same Betti numbers and same Hilbert function. Since $J \subset \text{in}_{\succ}(I)$ holds by construction it must then be that $J = \text{in}_{\succ}(I)$. That core(Δ) is the cyclic polytope with $2n$ vertices in $\mathbf{R}^{2n-4}$ follows from the facet description given above and Gale's evenness characterization of the facets of the cyclic polytope; see [**36**, Chapter 0]. □

It remains to prove Lemma 8.2. To this end, let us introduce some notation. Let Z be a matrix such that each row and column of Z contains at most two non-zero entries. Then each vertex of $G(Z)$ is contained in at most two edges. Therefore the connected components of $G(Z)$ are either paths or cycles. The decomposition of $G(Z)$ into connected components correspond to a block decomposition of Z as follows: if $G(Z)$ has connected components $G_1, \ldots, G_r$ then, up to row and column permutations and after eliminating zero rows and columns from Z, Z has a block decomposition of the form:

$$\begin{matrix} Z_1 & 0 & \ldots & 0 \\ 0 & Z_2 & 0 & \ldots \\ & \ddots & \ddots & \ddots \\ 0 & \ldots & 0 & Z_r \end{matrix}$$

(1) If G_i is a cycle, then it is a cycle of even length with vertices $y_1, \ldots, y_{2k}$, and Z_i is the $k \times k$ matrix

$$\begin{matrix} y_1 & y_2 & 0 & \ldots & 0 \\ 0 & y_3 & y_4 & 0 & \ldots \\ & & \ddots & \ddots & \\ 0 & \ldots & 0 & y_{2k-3} & y_{2k-2} \\ y_{2k} & 0 & \ldots & 0 & y_{2k-1} \end{matrix}$$

(2) If G_i is a path of odd length with vertices $y_1, \dots, y_{2k-1}$, then Z_i is the $k \times k$ matrix

$$\begin{matrix} y_1 & y_2 & 0 & \dots & 0 \\ 0 & y_3 & y_4 & 0 & \dots \\ & & \ddots & \ddots & \\ 0 & \dots & 0 & y_{2k-3} & y_{2k-2} \\ 0 & 0 & \dots & 0 & y_{2k-1} \end{matrix}$$

or its transpose.

(3) If G_i is a path of even length with vertices $y_1, \dots, y_{2k}$ then Z_i is the $k \times (k+1)$ matrix

$$\begin{matrix} y_1 & y_2 & 0 & \dots & & 0 \\ 0 & y_3 & y_4 & 0 & \dots & \\ & & \ddots & \ddots & & \\ 0 & \dots & 0 & y_{2k-3} & y_{2k-2} & 0 \\ 0 & 0 & \dots & 0 & y_{2k-1} & y_{2k} \end{matrix}$$

or its transpose.

It follows that if Z is a square matrix containing no zero rows or columns, then $\det Z = 0$ if one of the G_i is a path of even length. Otherwise, $\det Z$ is (up to sign) the product of the determinants $\det Z_i$ associated to the blocks. Furthermore, $\det Z_i = y_1 y_3 \dots y_{2k-1} - y_2 y_4 \dots y_{2k}$ if G_i is a cycle (case (1) above) and $\det Z_i = y_1 y_3 \dots y_{2k-1}$ if G_i is a path of odd length (case (2) above).

Proof of 8.2. Set $G = G(Y)$. Denote by V the set of vertices of G. For simplicity, we identify square-free monomials in the variables in V with subsets of V. Denote by U_k the ideal generated by the independent subsets of G of cardinality k. We have to show that $J_k = U_k$ for all $k < n$.

The inclusion $J_k \subseteq U_k$ follows from the very definition of determinant. For the other inclusion note that if Z is the $k \times k$ sub-matrix of Y with row indices $R = \{r_1, \dots, r_k\}$ and column indices $C = \{c_1, \dots, c_k\}$ then $G(Z)$ is the subgraph of G whose vertices y_{ij} satisfy $i \in R$ and $j \in C$. In particular, if $k < n$ then $G(Z)$ is not a cycle and so its connected components are lines. It follow that if $k < n$ then $\det Z$ is either 0 or a monomial in the variables of V. Consider now an independent set of cardinality $k < n$ of G, say $y_{i_1 j_1}, \dots, y_{i_k j_k}$. By construction, $i_a \neq i_b$ and $j_a \neq j_b$ if $a \neq b$. Consider the sub-matrix Z of Y with row indices $i_1, \dots, i_k$ and column indices $j_1, \dots, j_k$. By construction $y_{i_1 j_1} \cdots y_{i_k j_k}$ appears in $\det Z$ and, since we know that $\det Z$ is either 0 or a monomial, we may conclude that $\det Z$ is $y_{i_1 j_1} \cdots y_{i_k j_k}$ up to sign. This implies that $U_k \subseteq J_k$ and concludes the proof. □

References

[1] C. A. Athanasiadis. Ehrhart polynomials, simplicial polytopes, magic squares and a conjecture of Stanley, *J. Reine Angew. Math.* **583** (2005), 163-174.

[2] L. Avramov. A class of factorial domains, *Serdica* **5** (1979), no. 4, 378–379.

[3] G. Birkhoff. *Lattice theory*, American Mathematical Society Colloquium Publications, Vol. XXV, American Mathematical Society, Providence, R.I. 1967.

[4] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. Ziegler. *Oriented matroids.* Second edition. Encyclopedia of Mathematics and its Applications, 46. Cambridge University Press, Cambridge, 1999.

[5] W. Bruns and A. Conca. Gröbner bases and determinantal ideals, *Commutative algebra, singularities and computer algebra (Sinaia, 2002)*, 9–66, NATO Sci. Ser. II Math. Phys. Chem., 115, Kluwer Acad. Publ., Dordrecht, 2003.

[6] W. Bruns and T. Römer. h-vectors of Gorenstein polytopes, math.AC/0508392.

[7] W. Bruns and J. Herzog. *Cohen-Macaulay Rings*, Cambridge University Press, Cambridge, 1996.

[8] W. Bruns and U. Vetter. *Determinantal Rings*, Lecture Notes in Mathematics **1327**, Springer-Verlag, Berlin, 1988.

[9] A. Conca. Gröbner bases of ideals of minors of a symmetric matrix, *J. Algebra* **166** (1994), 406-421.

[10] A. Conca, J. Herzog and G. Valla. Sagbi bases with applications to blow-up algebras, *J. Reine Angew. Math.* **474** (1996), 113–138.

[11] CoCoA Team. *CoCoA: a system for doing Computations in Commutative Algebra*, Available at http://cocoa.dima.unige.it.

[12] P. Doubilet, G. C. Rota and J. Stein. *On the foundations of combinatorial theory IX*, Combinatorial methods in Invariant Theory. Studies in Appl. Math. 53 (1974), 185–216.

[13] A. Dress, S. Grünewald, J. Jonsson and V. Moulton. Paper in preparation.

[14] A. Dress, J. Koolen and V. Moulton. $4n - 10$, *Annals of Combinatorics* **8** (2005) 463–471.

[15] D. Eisenbud. *Commutative Algebra with a view towards Algebraic Geometry*, Graduate Texts in Mathematics **150**, Springer-Verlag, New York, 1995.

[16] J. Herzog and N. V. Trung. Gröbner bases and multiplicity of determinantal and Pfaffian ideals, *Adv. Math.* **96** (1992), no. 1, 1–37.

[17] W. Fulton. *Young tableaux, with applications to representation theory and geometry*, London Mathematical Society Student Texts **35**, Cambridge University Press, Cambridge, 1997.

[18] S. R. Ghorpade and C. Krattenthaler. The Hilbert Series of Pfaffian rings, *Algebra, arithmetic and geometry with applications* (West Lafayette, IN, 2000), 337–356, Springer, Berlin, 2004.

[19] S. Goto. On the Gorensteinness of determinantal loci, *J. Math. Kyoto Univ.* **19** (1979), no. 2, 371–374.

[20] S. Goto and K. Watanabe. On graded rings. *I. J. Math. Soc. Japan* **30** (1978), no. 2, 179–213.

[21] T. Hibi. Distributive lattices, affine semigroup rings and algebras with straightening laws, *Commutative Algebra and Combinatorics* (Kyoto, 1985), 93–109, Adv. Stud. Pure Math., **11**, North-Holland, Amsterdam, 1987.

[22] T. Hibi and H. Ohsugi. Special simplices and Gorenstein toric rings, math.AC/0503666.

[23] A. Jensen. *Gfan, a software system for Gröbner fans*, available at `http://home.imf.au.dk/ajensen/software/gfan/gfan.html`.

[24] J. Jonsson. Generalized triangulations and diagonal-free subsets of stack polyominoes, *J. Combin. Theory Ser. A* **112** (2005), 117–142.

[25] J. Jonsson and V. Welker. A spherical initial ideal of Pfaffians, Manuscript (2005).

[26] H. Kleppe and D. Laksov. The algebraic structure and deformation of Pfaffian schemes, *J. Algebra* **64** (1980), no. 1, 167–189.

[27] V. Reiner and V. Welker. On the Charney-Davis and Neggers-Stanley conjectures, *J. Combin. Theory Ser. A* **109** (2005), 247–280.

[28] D. Soll. Diploma Thesis, in preparation.

[29] D. Soll and V. Welker. Type-B generalized triangulations and determinantal ideals, Manuscript (2006).

[30] R. Stanley. Decompositions of rational convex polytopes, *Annals of Discrete Math.* **6** (1980), 333-342.

[31] R. Stanley. *Combinatorics and Commutative Algebra*, Progress in Mathematics **41**, Birkhäuser, Boston 1996.

[32] B. Sturmfels. Gröbner bases and Stanley decompositions of determinantal rings, *Math.Z.* **205** (1990), 137-144.

[33] B. Sturmfels. *Gröbner Bases and Convex Polytopes*, American Mathematical Society, Providence, RI, 1995.

[34] B. Sturmfels, *Algorithms in Invariant Theory*, Springer Verlag, Vienna, 1993.

[35] R. Villarreal. *Monomial algebras*, Monographs and Textbooks in Pure and Applied Mathematics, 238. Marcel Dekker Inc., New York, 2001.

[36] G. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics, Springer-Verlag, Berlin, 1995.

Dipartimento di Matematica, Universite degli Studi di Genova, Via Dodecaneso, 35 16146 Genova, Italy

E-mail address: `conca@dima.unige.it`

Mathematics Department, San Francisco State University, 1600 Holloway Avenue, San Francisco, CA 94132, USA

E-mail address: `serkan@math.sfsu.edu`

Department of Mathematics, University of Washington, Seattle, WA 98195,USA

E-mail address: `thomas@math.washington.edu`