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Gröbner bases for spaces of quadrics of codimension 3

Aldo Conca

Dipartimento di Matematica, Universitá di Genova, Via Dodecaneso 35, I-16146 Genova, Italy

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ABSTRACT

Let $R = \oplus_{i\geq 0} R_i$ be an Artinian standard graded K-algebra defined by quadrics. Assume that $\dim R_2 \leq 3$ and that K is algebraically closed, of characteristic $\neq 2$. We show that R is defined by a Gröbner basis of quadrics with, essentially, one exception. The exception is given by $K[x, y, z]/I$ where I is a complete intersection of three quadrics not containing a square of a linear form.

1. Introduction

A standard graded K-algebra R is an algebra of the form $R = K[x_1, \dots, x_n]/I$ where $K[x_1, \dots, x_n]$ is a polynomial ring over the field K and I is a homogeneous ideal with respect to the grading $\deg(x_i) = 1$. The algebra R is said to be quadratic if I is generated by quadrics (i.e. homogeneous elements of degree 2) and R is said to be Koszul if K admits a free resolution as an R-module whose maps are given by matrices of linear forms. We say that an algebra R is G-quadratic if its defining ideal has a Gröbner basis of quadrics with respect to some system of coordinates and some term order. G-quadratic algebras are Koszul and Koszul algebras are quadratic. Neither of these implications can be reversed in general; see [8].

Given a graded K-algebra R we can consider its trivial fiber extension $R' = R \circ K[y]/(y)^2$ where $y = y_1, \dots, y_m$ is a set of variables. Here $\circ$ denotes the fibre product of K-algebras. It is known that the properties of being quadratic, Koszul, G-quadratic, as well as $\dim R_i$ for $i > 1$ are unaffected by trivial fiber extensions; see [5, Lemma 4], [2, Theorem 4].

Backelin showed in [1, 4.8] that if $R = \oplus_{i\geq 0} R_i$ is a quadratic standard graded K-algebra with $\dim R_2 \leq 2$ then R is Koszul. We have shown in [5] that, under the same assumptions, R is G-quadratic with only one exception (up to trivial fiber extensions and changes of coordinates) given by the K-algebra $K[x, y, z]/(x^2, xy, y^2 - xz, yz)$.

The goal of this paper is to prove the following theorem.

Theorem 1.1. *Let K be an algebraically closed field of characteristic $\neq 2$. Let R be a standard graded K-algebra which is quadratic, Artinian and with $\dim R_2 = 3$. Then:*

(1) *R is Koszul, $R_i = 0$ for $i > 3$ and $\dim R_3 \leq 1$. Furthermore $\dim R_3 = 1$ if and only if R is a trivial fiber extension of $K[x, y, z]/I$ where I is a complete intersection of three quadrics.*
(2) *R is G-quadratic iff it is not a trivial fiber extension of $K[x, y, z]/I$ where I is a complete intersection of three quadrics not containing a square of a linear form. In particular, if $R_3 = 0$, then R is G-quadratic.*

We obtain the following corollaries.

Corollary 1.2. *Let R be a quadratic Cohen–Macaulay standard graded K-algebra. Denote by $(h_0, h_1, h_2, \dots)$ its h-vector and assume that $h_2 = 3$. Then:*

(1) *R is Koszul, $h_i = 0$ for every $i > 3$ and $h_3 \leq 1$. Furthermore $h_3 = 1$ if and only if the degree 1 component of the socle of R has dimension $h_1 - 3$.*
(2) *If $h_3 = 0$ then R is G-quadratic.*

E-mail address: conca@dima.unige.it.

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Corollary 1.3. *Let R be a quadratic Cohen–Macaulay algebra. If $e(R) \leq \operatorname{codim}(R) + 4$ then R is Koszul.*

In 1.3 $e(R)$ denotes the degree or multiplicity of R and $\operatorname{codim}(R)$ its codimension. To see that 1.2 follows form 1.1, one just considers an Artinian reduction of R. But then 1.3 follows from 1.2 in combination with [5, Corollary 9]. In particular a (non-degenerate) set of at most $n + 4$ points of $\mathbf{P}^n$ defined by quadrics is Koszul, a special case of a recent conjecture of Polishchuk [10]. Note also that there are non-Koszul quadratic Cohen–Macaulay algebras (even domains) with $e(R) = \operatorname{codim}(R) + 5$; see [6, Sect 4].

For standard facts on Gröbner bases we refer the reader to [9] and for standard facts on Cohen–Macaulay rings we refer the reader to [3]. The results and the examples presented were discovered by extensive computer algebra experiments performed with CoCoA [4].

2. Proof of the main result

Let K be an algebraically closed field of characteristic not 2. Let R be a standard graded K-algebra. The rank of $x \in R_1$, denoted by $\operatorname{rank} x$, is by definition $\dim xR_1$. Note that $\operatorname{rank} x = 0$ for some non-zero $x \in R_1$ iff R is a trivial fiber extension.

As we said already, the properties and the invariants under discussion, being quadratic, Koszul, G-quadratic, as well as $\dim R_i$ for $i > 1$, are unaffected by trivial fiber extensions; see [5, Lemma 4], [2, Theorem 4]. Hence in the proof of 1.1 we may assume that $\operatorname{rank} x > 0$ for every the non-zero $x \in R_1$.

The case $n = 3$ is easy: R is a complete intersection of three quadrics and 1.1 is proved in [6, Sect 6.1]. It remains to prove:

Proposition 2.1. *With the assumption of 1.1, assume further that $\operatorname{rank} x > 0$ for every non-zero $x \in R_1$ and that $n > 3$. It follows that R is G-quadratic and $R_3 = 0$.*

The main technical lemma is:

Lemma 2.2. *Under the assumption of Proposition 2.1, let $y \in R_1$ with $y^2 = 0$ and set $V = \{u \in R_1 : uy = 0\}$. If one of the following conditions holds then R is G-quadratic and $R_3 = 0$.*

(1) $\operatorname{rank} y = 3$.
(2) $\operatorname{rank} y = 2$ *and there exists $z \in V$ such that $z^2 \in yR_1$ and $zR_1 \not\subseteq yR_1$.*
(3) $\operatorname{rank} y = 2$ *and there exists $t \in R_1 \setminus V$ such that $t^2 \in yR_1$ and $tV \not\subseteq yR_1$.*
(4) $\operatorname{rank} y = 1$.

In the proofs below, the symbol L_* denotes a homogeneous polynomial of degree 1 and $*f$ means a scalar multiple of the element f.

Proof. In the four cases the proof is based on the same principle: choose a K-basis of R_1 in a suitable way, consider the associated presentation of R as a quotient of a polynomial ring, translate the assumptions into quadratic equations, check that the given quadrics already provide enough leading terms to generate all the monomials of degree 3. For simplicity of notation, we do not distinguish between the elements of R_1 and the variables of the polynomial ring that we use to present R.

Case (1) is the standard situation; see [5]. Complete y to a basis of R_1, say $y, x_2, \ldots, x_n$. Since $y^2 = 0$ and $yR_1 = R_2$, we have polynomials y^2 and $x_ix_j - yL_*$ in the defining ideal I of R. Then $(y)^2 + (x_2, \ldots, x_n)^2$ is contained in the ideal of leading terms of I with respect the revlex order associated with $x_i > y$. This is enough for concluding that R is G-quadratic and $R_3 = 0$.

Case (2). Complete y, z to a basis of R_1, say $y, z, x_3, \ldots, x_n$. We have polynomials $y^2, yz, z^2 - yL_*, x_ix_j - L_*y - L_*z$ in the defining ideal I of R. It follows that $(y, z)^2 + (x_3, \ldots, x_n)^2$ is in the ideal of leading terms of I with respect the revlex order associated with $x_i > z > y$. This is enough for concluding that R is G-quadratic and $R_3 = 0$.

Case (3). Consider a basis $y, z_2, \ldots, z_{n-2}$ of V and complete it with the given t and some other element w to a basis of R_1. Use the term order revlex $y < t < z_i < w$. Polynomials of the following form are in the defining ideal:

$$y^2, \quad yz_i, \quad z_iz_j - yL_* - tL_*, \quad t^2 - yL_*, \quad wz_i - yL_* - tL_*, \quad w^2 - yL_* - tL_*.$$

Set $W = \{u \in R_1 : ut \in yR_2\}$. Note that W is a space of dimension $n - 1$ and contains a linear form which involves w; otherwise we would have $tV \subseteq yR_1$ which contradicts the assumption of (3). Then a polynomial of the form $wt - yL_*$ is also in the defining ideal. The leading term ideal of the defining ideal of R contains $(y, z_2, \ldots, z_{n-2})^2 + (t, w)^2$. This is enough for concluding that R is G-quadratic and $R_3 = 0$.

Case (4). We have that $R/(y)$ is Artinian with Hilbert series $1 + (n - 1)x + 2x^2 + \cdots$. By [5] we know that there exists $t \in R_1$ such that $t^2 \in yR_1$ and $R_2 = yR_1 + tR_1$. Complete y and t to a basis of R_1 with elements $x_3, \ldots, x_n$ and use the revlex order associated with $y < x_i < t$. In the defining ideal of R we have polynomials

$$y^2, \quad x_ix_j - yL_* - tL_*, \quad t^2 - yL_*$$

and therefore its initial ideal contains the ideal initial terms $(x_3, \ldots, x_n)^2 + (y^2, t^2)$. Furthermore, since $yV = 0$ in R and V has dimension $n - 1$, we have initial terms Wy where W is a set of variables of cardinality $n - 1$ containing y. So either all the x_i are in W or t is in W. In the first case the initial term ideal contains $(x_3, \ldots, x_n, y)^2 + (t^2)$, in the second $(x_3, \ldots, x_n)^2 + (y, t)^2$. In both cases we are done. $\square$

Another auxiliary fact:

Lemma 2.3. *Assume S is a quadratic standard graded K-algebra with Hilbert series $1+3x+x^2$ and $z \in S_1$ such that $z^2 \neq 0$. Then there exists $u \in S_1$ such that $u^2 = 0$ and $uz \neq 0$.*

Proof. We argue by contradiction. Let $v \in S_1$ not a multiple of z. We have equations $v^2 = az^2$ and $vz = bz^2$ with $a, b \in K$. By contradiction, there is no $\alpha \in K$ such that $(v+\alpha z)^2 = 0$ and $z(v+\alpha z) \neq 0$, that is, no $\alpha \in K$ such that $a+2\alpha b+\alpha^2 = 0$ and $b+\alpha \neq 0$. In other words, $\alpha = -b$ is the only solution of $a+2\alpha b+\alpha^2 = 0$, that is, $a = b^2$. Complete z to a basis of S_1 with elements t, w. By the argument above we have the equations

$$t^2 = b^2z^2, \qquad tz = bz^2, \qquad w^2 = c^2z^2, \qquad wz = cz^2.$$

We have also an equation $wt = dz^2$. Since $(t+w)^2 = (b^2+c^2+d)z^2$ and $(t+w)z = (b+c)z^2$ the argument above applied to $t+w$ yields

$$(b^2+c^2+2d) = (b+c)^2,$$

that is $d = bc$. So the polynomials defining S are

$$t^2 - b^2z^2, \qquad tz - bz^2, \qquad w^2 - c^2z^2, \qquad wz - cz^2, \qquad wt - bcz^2.$$

These polynomials are contained in the ideal $(t-bz, w-cz)$, contradicting the fact that S is Artinian. $\square$

Now we are ready to prove 2.1:

Proof. Fix K-bases of R_1 and R_2. The condition $y^2 = 0$ for an element $y \in R_1$ is expressed by $\dim R_2$ quadratic equations in the $\dim R_1$ coefficients of y. Since $\dim R_1 = n > \dim R_2 = 3$ and K is algebraically closed, there exists $y \in R_1$, non-zero, such that $y^2 = 0$. Further, $\operatorname{rank} y > 0$ by assumption. If $\operatorname{rank} y = 3$ or 1 we conclude by 2.2 Case (1) and Case (4). So we may assume that $\operatorname{rank} y = 2$. Let $V = \{u \in R_1 : uy = 0\}$; V is a $n-2$-dimensional subspace of R_1. We discuss three cases:

Case 1: $V^2 \not\subseteq yR_1$.

Case 2: $V^2 \subseteq yR_1$ and $VR_1 \not\subseteq yR_1$.

Case 3: $VR_1 \subseteq yR_1$.

For Case 1 we argue as follows: Let $z \in V$ be such that $z^2 \notin yR_1$ (here we use that the characteristic of K is not 2). Complete y, z to a K-basis of V with elements z_i. Since R_2/yR_1 is one-dimensional, generated by z^2, we may replace z_i with $z_i - *z$ and assume that $z_i^2 \in yR_1$. Now, if for some i, $z_iR_1 \not\subseteq yR_1$, we end up in case (2) of Lemma 2.2. Hence we have to discuss the case in which $z_iR_1 \subseteq yR_1$. In other words, the z_i are in the socle of $R/(y)$. Modding out these socle elements we get an algebra S with Hilbert series $1+3x+x^2$ and the residue class of z in S satisfies $z^2 \neq 0$. So by 2.3 there exists $w \in R_1$ such that $wz \notin yR_1$ and $w^2 \in yR_1$. This is case (3) of 2.2.

Case 2: Take $z \in V$ such that $zR_1 \not\subseteq yR_1$ and note that this is case (2) of 2.2.

Case 3: In $R/(y)$ the space $V/(y)$ belongs to the socle. So since R is quadratic and Artinian, the algebra $R/(V)$ has Hilbert series $1+2x+x^2$. For such an algebra it is easy to see that there exist independent linear forms t, w such that $t^2 = 0$ and $w^2 = 0$. Lifting back to R, we have that a basis of R_2 is given by ty, wy, wt and for every $z \in V$ not a multiple of y we get the equations

$$y^2 = 0, \qquad yz = 0, \qquad z^2 = L_1y, \qquad t^2 = L_2y, \qquad w^2 = L_3y, \qquad zw = L_4y, \qquad zt = L_5y \tag{2.1}$$

where the L_i are linear forms in t and w, say

$$L_i = \lambda_{i,1}t + \lambda_{i,2}w.$$

Now we look for linear forms of type $\ell = t + az + by$ such that $\ell^2 = 0$. The condition $\ell^2 = 0$ translates into the polynomial system

$$\begin{cases} \lambda_{2,1} + 2a\lambda_{5,1} + 2b + a^2\lambda_{1,1} = 0 \\ \lambda_{2,2} + 2a\lambda_{5,2} + a^2\lambda_{1,2} = 0. \end{cases}$$

Now, assume that

$$\lambda_{1,2} \neq 0 \quad \text{or} \quad \lambda_{5,2} \neq 0. \tag{2.2}$$

Then we can solve the second equation to obtain the value of a and, substituting in the first, we get the value of b. In other words, assuming (2.2), there exists $\ell = t + az + by$ such that $\ell^2 = 0$ in R. We evaluate now the rank of such an ℓ. We have

$$\ell y = ty \quad \text{and} \quad \ell w = tw + aL_4y + bwy \tag{2.3}$$

and since these two elements of R_2 are linearly independent, we can conclude that $\operatorname{rank} \ell \geq 2$. If $\operatorname{rank} \ell = 3$ then we are done by 2.2(1). Hence we may assume that $\operatorname{rank} \ell = 2$. This implies that ℓz and ℓt are linear combinations of ℓy and ℓw. Now $\ell z = (\lambda_{5,2} + a\lambda_{1,2})wy + *ty$ and $\ell t = (\lambda_{2,2} + a\lambda_{5,2})wy + *ty$. Summing up, if $\operatorname{rank} \ell = 2$ then

$$\lambda_{5,2} + a\lambda_{1,2} = 0 \quad \text{and} \quad \lambda_{2,2} + a\lambda_{5,2} = 0. \tag{2.4}$$

In this case, the space $V(\ell) := \{u \in R_1 : u\ell = 0\}$ contains $z + \gamma y$ with $\gamma = -\lambda_{5,1} - a\lambda_{1,1}$. Note that $(z + \gamma y)^2 = \lambda_{1,1}ty + \lambda_{1,2}wy$ and we claim that $(z + \gamma y)^2 \notin \ell R_1$. If, otherwise, $(z + \gamma y)^2 \in \ell R_1$ then, since the rank $\ell = 2$ and hence the elements (2.3) are a basis of R_1, we must have $\lambda_{1,2} = 0$. By (2.4) it follows that $\lambda_{5,2} = 0$ contradicting the assumption (2.2). We can conclude that $V(\ell)^2 \not\subseteq \ell R_1$. This is Case (1) with ℓ playing the role of y and we are done. Summing up, if (2.2) holds then we are done because we find an element ℓ with $\ell^2 = 0$ which has either rank 3 or rank 2 and $\{u \in R_1 : u\ell = 0\}^2 \not\subseteq \ell R_1$.

We can also look for elements of the form $\ell = w + az + by$ satisfying $\ell^2 = 0$. The situation is completely symmetric. Therefore we get the desired conclusion unless

$$\lambda_{1,2} = 0 \quad \text{and} \quad \lambda_{5,2} = 0 \quad \text{and} \quad \lambda_{1,1} = 0 \quad \text{and} \quad \lambda_{4,1} = 0, \tag{2.5}$$

that is, Eq. (2.1) takes the form

$$y^2 = 0, \qquad yz = 0, \qquad z^2 = 0, \qquad t^2 = L_2y, \qquad w^2 = L_3y, \qquad zw = \lambda_{4,2}wy, \qquad zt = \lambda_{5,1}ty$$

for every $z \in V$ which is not multiple of y. It follows that $z^2 = 0$ for all the $z \in V$. This implies $V^2 = 0$ (here we use again that K has characteristic not 2). In particular we see that the element $y_1 = z - \lambda_{4,2}y$ of V has $y_1^2 = 0$ and $y_1V = 0$ and $y_1w = 0$. So rank $y_1 = 1$ and we are done by 2.2(4). □

The following examples show that the cases described in 2.2 do indeed arise in an essential way. By this we mean:

Example 2.4. There are examples where all the elements with $\ell^2 = 0$ have rank 3. This is the generic situation. In four variables, an ideal generated by the squares of seven general linear forms has this property. Explicitly, in $K[t, z, w, y]$ the algebra defined by $(y^2, z^2, w^2, t^2, (t + z + w + y)^2, (t + 2z + 4w + 8y)^2, (t + 3z + 9w + 27y)^2)$ has this property.

Example 2.5. There are examples where all the elements with $\ell^2 = 0$ have rank 2 and (2) does apply. For example, in $K[t, z, w, y]$ the ideal $(y^2, yz, z^2 - wy, t^2, tw, w^2 - tz, wz)$ defines an algebra R with only two elements with $\ell^2 = 0$, namely y and t, and both have rank 2. For both y and t one can apply 2.2(2).

Example 2.6. There are examples where all the elements with $\ell^2 = 0$ have rank 2 and (2) does not apply while 2.2(3) does apply. The ideal $(y^2, yz, w^2, wz, t^2, tz, z^2 + ty + wy)$ defines an algebra with three elements with $\ell^2 = 0$, namely y, w, t. They all have rank 2. While y does not fit into 2.2(2) or (3), both t and w satisfy 2.2(3) but not (2).

Example 2.7. There are examples where the elements with $\ell^2 = 0$ all have rank ≤ 2 and for those of rank 2 case (2) or (3) do not apply. The ideal $(y^2, zy, z^2, t^2 - ty - 2wy, w^2 - 3ty - 4wy, tz - ty, wz - 2wy)$ defines an algebra where the elements with $\ell^2 = 0$ are the elements of type $ay + bz$. They all have rank ≤ 2. Those of rank exactly 2 do not fit into 2.2(2) or (3). Among the elements with the property $\ell^2 = 0$ there are exactly two elements of rank 1, namely $y - z$ and $2y - z$.

3. Nets of conics

Our main result asserts that quadratic Artinian algebras with $\dim R_2 = 3$ are Koszul and most of them are G-quadratic. What about dropping the assumption that they are Artinian? We will discuss in this section the case of quadratic (not necessarily Artinian) algebras with Hilbert series $1 + 3x + 3x^2 + \cdots$. In [2] the authors make a detailed study of the Koszul property of the quadratic quotients of $K[x, y, z]$. The most difficult case is that of a quotient defined by three quadrics, that is, an algebra with Hilbert series $1 + 3x + 3x^2 + \cdots$. It turns out that there exist exactly (up to change of coordinates) two quotients of $K[x, y, z]$ defined by three quadrics that are not Koszul. They are the algebras defined by the ideal number (12) and number (14) in the list below.

To proceed with the discussion let us recall a few facts. Vector spaces of quadrics of dimension 3 in three variables are classically called nets of conics. The main ingredient for the proof of [2, Theorem 1] is a classification result for nets of conics up to the action of $GL_3(K)$. This classification can be found in full detail in the paper of Wall [11] or in an old preprint of Emsalem and Iarrobino [7]. Over the complex numbers, there are 15 types of nets of conics; fourteen of them are just one point and one type is one-dimensional. With respect to [11], we have chosen slightly different normal forms to minimize the total number of terms involved or (as in case 15) to maximize the symmetry. The list of nets of conics is

(1)	(x^2, xy, y^2)	(2)	(x^2, xy, xz)	(3)	(x^2, y^2, z^2)
(4)	(xy, xz, yz)	(5)	$(x^2, y^2, z^2 + xy)$	(6)	$(xz, yz, z^2 + xy)$
(7)	(x^2, y^2, xz)	(8)	(xy, z^2, yz)	(9)	$(x^2, y^2, xz + yz)$
(10)	$(x^2, xz, y^2 + yz)$	(11)	$(x^2, xy + z^2, xz)$	(12)	$(x^2, xy + z^2, yz)$
(13)	$(x^2, yz, y^2 + z^2 + xy)$	(14)	$(xz, y^2 + yx, z^2 + xy)$		

and

(15) $(x^2 + 2jyz, y^2 + 2jxz, z^2 + 2jxy)$ with $j \in \mathbf{C}$ and $j^3 \neq 0, 1, -1/8$.

One can then study the G-quadratic property of the quadratic quotients of $K[x, y, z]$, say over $\mathbf{C}$, using the classification. We skip the uninteresting details. Below we summarize the final result.

There are five possible Hilbert series:

(a) $1+3x+3x^2+x^3$ (c.i.) (b) $(1+2x)/(1-x)$
(c) $(1+x-2x^2+x^3)/(1-x)^2$ (d) $(1+2x-x^3)/(1-x)$
(e) $(1+2x-2x^3)/(1-x)$.

Every net of conics V has a dual net of conics V^* (the orthogonal space with respect to partial differentiation). In this duality, a point $(a, b, c) \in \mathbf{P}^2$ belongs to the locus defined by V if and only the square of the linear form $ax+by+cz$ belongs to V^*.

Another interesting aspect of the story is the following. If a net V is generated by the partial derivatives of a cubic f we say that V is of gradient type. It turns out that "almost all" nets of conics are of gradient type and the corresponding cubic is also uniquely determined. For instance, the net (15) corresponds to the smooth cubic form in Hesse form $f = x^3+y^3+z^3+6jxyz$ and the conditions on j guarantee that f is smooth and not in the orbit of the Fermat cubic $x^3+y^3+z^3$.

In the following table we show, for every type, its Hilbert series (column head H-series), the number of linear forms whose squares are in the net (column head q), the number of points of the variety defined by the net (column head p), whether it is Koszul or not (column head Kos), whether it is G-quadratic or not (column head G-quad), the name given by Wall to that type of net (column head Wall), and whether it is of gradient type (column head ∇).

	H-series	q	p	Kos	G-quad	Wall	∇
(1)	(b)	∞	1	Yes	Yes	I	No
(2)	(c)	1	∞	Yes	Yes	I*	No
(3)	(a)	3	0	Yes	Yes	E	Yes
(4)	(b)	0	3	Yes	Yes	E*	Yes
(5)	(a)	2	0	Yes	Yes	D	No
(6)	(d)	0	2	Yes	Yes	D*	Yes
(7)	(d)	2	1	Yes	Yes	G	Yes
(8)	(b)	1	2	Yes	Yes	G*	No
(9)	(d)	2	1	Yes	Yes	F	No
(10)	(d)	1	2	Yes	Yes	F*	No
(11)	(b)	1	1	Yes	Yes	H	Yes
(12)	(e)	1	1	No	No	C	No
(13)	(a)	1	0	Yes	Yes	B	No
(14)	(e)	0	1	No	No	B*	Yes
(15)	(a)	0	0	Yes	No	A	Yes

The star $*$ in Wall's notation refers to the duality. The types that are G-quadratic are so in the given coordinates with the exception of (6). Applying the change of coordinates $x \to x, y \to x-z, z \to x-y$, the net (6) becomes generated by $xz-yz, x^2-xy, y^2-yz$ which is a G-basis. That (12) and (14) are not G-quadratic follows from the fact, proved in [2], that they are not Koszul. But it follows also from the simple observation that there is no quadratic monomial ideal with Hilbert series (e).

4. Final remarks

We list some questions which arise from the results presented. Let R be a quadratic standard graded K-algebra which is not a trivial fiber extension.

(1) Assume $\dim R_1 > \dim R_2 = 3$. Is R G-quadratic?
(2) Assume $\dim R_1 > \dim R_2$ and $R_3 = 0$ (or just that R is Artinian). Is R G-quadratic?

We have seen that the answer to (1) is positive for Artinian algebras. Also, the answer to (2) is positive for "generic" algebras; see [5].

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