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GENERIC INITIAL IDEALS AND FIBRE PRODUCTS

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Abstract

We study the behavior of generic initial ideals with respect to fibre products. In our main result we determine the generic initial ideal of the fibre product with respect to the reverse lexicographic order. As an application we compute explicitly the generic initial ideal of a fibre product in a special case. We also prove that the fibre product of two graded ideals is componentwise linear if and only if both ideals have this property.

1. Introduction

Let K be a field of characteristic 0, $S = K[x_1, \dots, x_n]$ and I be a graded ideal of S. Unless otherwise said, initial terms are taken with respect to the degree reverse lexicographic order, rlex for short. The generic initial ideal $\mathrm{gin}(I)$ of I is an important object. It reflects many homological, geometrical and combinatorial properties of the associated rings and varieties. The ideal $\mathrm{gin}(I)$ is defined as the ideal of initial terms of $\varphi(I)$ where φ is a generic element of $\mathrm{GL}_n(K)$. Here $\mathrm{GL}_n(K)$ acts on S as the group of graded K-algebra isomorphisms. See, e.g., Eisenbud [6] for the construction, in particular for the precise definition of "generic" element, and [7, 8] for results related to the generic initial ideal.

In practice, to compute $\mathrm{gin}(I)$, one takes a random element $\varphi \in \mathrm{GL}_n(K)$ and computes, with CoCoA [1], Macaulay 2 [12] or Singular [15], the ideal of leading terms of $\varphi(I)$. The resulting ideal is, with very high probability, $\mathrm{gin}(I)$. From the theoretical point of view it is in general difficult to describe $\mathrm{gin}(I)$ explicitly. Just to give an example, important conjectures, such as the Fröberg conjecture on Hilbert functions, would be solved if we could only say what is the gin of a generic complete intersection.

Generic initial ideals can also be defined in the exterior algebra. In the exterior algebra context Kalai made in [11] conjectures about the behavior of generic initial ideals of squarefree monomial ideals with respect to standard operations on simplicial complexes. Two of these operations have purely algebraic counterparts: they corresponds to the tensor product and the fibre product over K of standard graded K-algebras. If $A = \bigoplus A_i$ and $B = \bigoplus B_i$ are standard graded K-algebras the tensor product $A \otimes_K B =$

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$\bigoplus(A_i \otimes_K B_j)$ is a (bi)graded K-algebra and the fibre product $A \circ B$ is, by definition, $A \otimes_K B/(A_1 \otimes B_1)$. Note that we have the short exact sequence of $A \otimes_K B$-modules

$$0 \to A \circ B \to A \oplus B \to K \to 0. \tag{1}$$

If we have presentations $A = K[x_1, \dots, x_n]/I$ and $B = K[x_{n+1}, \dots, x_{n+m}]/J$ then $A \otimes_K B$ has presentation $K[x_1, \dots, x_{n+m}]/I + J$ and $A \circ B$ has presentation $K[x_1, \dots, x_{n+m}]/F(I, J)$ where

$$F(I, J) = I + J + Q$$

and

$$Q = (x_i : 1 \le i \le n)(x_j : n + 1 \le j \le m + n).$$

Denote by $\operatorname{gin}_n(I)$ the gin of I in $K[x_1, \dots, x_n]$ with respect to rlex induced by $x_1 > \dots > x_n$. Similarly denote by $\operatorname{gin}_m(J)$ the gin of J in $K[x_{n+1}, \dots, x_{n+m}]$ with respect to rlex induced by $x_{n+1} > \dots > x_{n+m}$. Kalai's conjectures [11] correspond, in the symmetric algebra, to the following statements:

$$\operatorname{gin}(I + J) = \operatorname{gin}(\operatorname{gin}_n(I) + \operatorname{gin}_m(J)), \tag{2}$$

$$\operatorname{gin}(F(I, J)) = \operatorname{gin}(F(\operatorname{gin}_n(I), \operatorname{gin}_m(J))). \tag{3}$$

As observed by Nevo [13], the exterior algebra counterpart of Equality (2) does not hold. Hence it is not surprising to discover that (2) does not hold in the symmetric algebra as well. Here is a simple example:

EXAMPLE 1.1. Set $n = 2$, $m = 3$ and $I = (x_1, x_2)^2$ and $J = (x_3^2, x_3x_4, x_4x_5)$. Then $\operatorname{gin}_2(I) = I$ since I is strongly stable, $\operatorname{gin}_3(J) = (x_3, x_4)^2$ since J defines a Cohen–Macaulay ring of codimension 2. But $\operatorname{gin}(I + J)$ and $\operatorname{gin}(\operatorname{gin}_n(I) + \operatorname{gin}_m(J))$ differ already in degree 2 since x_1x_4 belongs to the second, but not to the first ideal.

On the other hand, Nevo (see [13] and [14]) proved the analog of (3) for shifted simplicial complexes with respect to exterior and symmetric algebraic shifting. Our main result asserts that (3) holds for arbitrary graded ideals:

Theorem 1.2. *Let $I \subset K[x_1, \dots, x_n]$ and $J \subset K[x_{n+1}, \dots, x_{n+m}]$ be graded ideals. Then we have* $\operatorname{gin}(F(I, J)) = \operatorname{gin}(F(\operatorname{gin}_n(I), \operatorname{gin}_m(J)))$.

Note that our methods and strategy of proofs as presented in Section 2 and Section 3 follow the ideas of Nevo and can be seen as a purely algebraic version of his results.

Using the results in Section 3 it is also possible to compute explicitly $\operatorname{gin}(F(I, J))$ in some cases. As an example we show that

$$\operatorname{gin}(Q) = (x_i x_j : i + j \leq n + m,\ i \leq j \text{ and } i \leq \min\{n, m\})$$

and further that $\operatorname{gin}(Q^k) = \operatorname{gin}(Q)^k$ for all $k \in \mathbb{N}$. The latter result is surprising, since we do not know other families of ideals satisfying it. (See 3.6 for further remarks.)

Recall the following definition of Herzog–Hibi [9]. A graded ideal L in some polynomial ring over the field K is called *componentwise linear* if for all $k \in \mathbb{N}$ the ideal $L_{\langle k\rangle}$ has a k-linear resolution (i.e. $\operatorname{Tor}_i(L_{\langle k\rangle}, K)_{i+j} = 0$ for $j \neq k$) where $L_{\langle k\rangle}$ is the ideal generated by all elements of degree k of L. In Section 4 we show that componentwise linearity behaves well with respect to fibre products. More precisely, given two graded ideals $I \subset K[x_1, \dots, x_n]$ and $J \subset K[x_{n+1}, \dots, x_{n+m}]$ such that $I \subseteq \mathfrak{m}_n^2$ and $J \subseteq \mathfrak{m}_m^2$, we prove that I and J are componentwise linear if and only if $F(I, J)$ is componentwise linear.

2. Linear algebra and generic initial ideals

In this section we present some tools from linear algebra used in this paper. We follow ideas of Nevo (see [13] and [14]) who treated the analog for shifted simplicial complexes with respect to exterior and symmetric algebraic shifting.

Let K be a field with $\operatorname{char} K = 0$ and $S = K[x_1, \dots, x_n]$ be the standard graded polynomial ring. Given a monomial $x^a \in S_d$ we define a K-linear map $\tau_{x^a} : S \to S$ by

$$\tau_{x^a}(x^b) = \begin{cases} x^{b-a} & \text{if} \quad a \preceq b, \\ 0 & \text{else.} \end{cases}$$

For a polynomial $g = \sum_{a \in \mathbb{N}^n} \lambda_a x^a \in S$ we extend this definition and consider the K-linear map $\tau_g = \sum_{a \in \mathbb{N}^n} \lambda_a \tau_{x^a}$. In the following we will use the rule

$$\tau_{gh}(l) = \tau_g(\tau_h(l)) \quad \text{for homogeneous} \quad g, h, l \in S$$

which is easily verified.

Given a graded ideal $I \subset S$ and the reverse lexicographic order $<_{\text{rlex}}$ induced by $x_1 > \cdots > x_n$ on S, the algebra S/I has a K-basis $\mathscr{B}(S/I)$ consisting of those monomial x^a such that $x^a \notin \operatorname{in}(I)$ where $\operatorname{in}(I)$ always denotes the initial ideal of I with respect to rlex.

Induced by the inclusion of $\mathscr{B}(S/I)$ into $\mathscr{B}(S)$, we may consider S/I as a graded K-vector subspace of S. (So we are not using the quotient structure of S/I as an S-module in the following if not otherwise stated.) Given a homogeneous element $g \in S_d$ the map τ_g can be restricted to S/I. The main object we are studying are the various kernels

$$\operatorname{Ker}_{S/I}(\tau_g)_e = \operatorname{Ker}(\tau_g : (S/I)_e \to (S/I)_{e-d}).$$

We fix now a second generic K-basis $f_j = \sum_{j=1}^{n} a_{ij} x_i$ of S_1 where $A = (a_{ij}) \in K^{n\times n}$ is invertible and study the map τ_{f^a} where f^a is a monomial in the new K-basis $f_1, \dots, f_n$. Generic means here that with the automorphism $\varphi\colon S \to S$, $x_i \mapsto f_i$ we can compute the generic initial ideal $\operatorname{gin}(I)$ with respect to rlex induced by $x_1 > \cdots > x_n$ of a given graded ideal $I \subset S$ via $\operatorname{gin}(I) = \operatorname{in}(\varphi^{-1}(I))$.

We need the following alternative way to decide whether a monomial belongs to a generic initial ideal or not. This criterion is well-known to specialists but we include it for the sake of completeness. For a (homogeneous) element $g \in S$ we denote by $\overline{g}$ the residue class in S/I (induced by the quotient structure of S/I).

Lemma 2.1. *Let $I \subset S$ be a graded ideal and $x^a \in S$ be a monomial. Then*

$$x^a \in \operatorname{gin}(I) \iff \overline{f^a} \in \operatorname{span}_K\{\overline{f^b}\colon x^b <_{\text{rlex}} x^a,\ |b| = |a|\} \subset S/I.$$

Proof. We have that

$$\begin{aligned} & x^a \in \operatorname{gin}(I) = \operatorname{in}(\varphi^{-1}(I)) \\ \iff & x^a - \sum_{x^b <_{\text{rlex}} x^a,\ |b|=|a|} \lambda_b x^b \in \varphi^{-1}(I) \quad \text{for some} \quad \lambda_b \in K \\ \iff & \overline{x^a} \in \operatorname{span}_K\{\overline{x^b}\colon x^b <_{\text{rlex}} x^a,\ |b| = |a|\} \subset S/\varphi^{-1}(I) \\ \iff & \overline{f^a} \in \operatorname{span}_K\{\overline{f^b}\colon x^b <_{\text{rlex}} x^a,\ |b| = |a|\} \subset S/I. \end{aligned}$$

This concludes the proof. □

Let $\mathscr{B}(S/I) = \{x^c\colon x^c \notin \operatorname{in}(I)\}$ be the K-basis of S/I as discussed above. Similarly $\mathscr{B}(S/\operatorname{gin}(I))$ is the natural K-basis of $S/\operatorname{gin}(I)$, i.e. those monomials x^c with $x^c \notin \operatorname{gin}(I)$. We consider these monomials either as monomials in S or in the corresponding residue class ring of S. The generic initial ideal $\operatorname{gin}(I)$ can also be characterized via the kernels $\operatorname{Ker}_{S/I}(\tau_{f^a})_e$. More precisely, we have:

Proposition 2.2. *Let $d > 0$ be a positive integer, $x^a \in S_d$ be a monomial and $I \subset S$ be a graded ideal. Then*

$$\dim_K \bigcap_{x^b <_{\text{rlex}} x^a,\ |b|=d} \operatorname{Ker}_{S/I}(\tau_{f^b})_d = |\{x^c \in \mathscr{B}(S/\operatorname{gin}(I))\colon |c| = d,\ x^a \leq_{\text{rlex}} x^c\}|.$$

In particular, $x^a \in \mathscr{B}(S/\operatorname{gin}(I))$ if and only if

$$\dim_K \bigcap_{x^b <_{\text{rlex}} x^a,\ |b|=d} \operatorname{Ker}_{S/I}(\tau_{f^b})_d > \dim_K \bigcap_{x^b \leq_{\text{rlex}} x^a,\ |b|=d} \operatorname{Ker}_{S/I}(\tau_{f^b})_d.$$

Proof. Let $g \in (S/I)_d$ be a homogeneous element expressed as a linear combination of the elements of $\mathscr{B}(S/I)$, i.e.

$$g = \sum_{x^c \in \mathscr{B}(S/I)} \lambda_c x^c \quad \text{for some} \quad \lambda_c \in K.$$

Write f^b for $b \in \mathbb{N}^n$ with $|b| = d$ as

$$f^b = \sum_{a' \in \mathbb{N}^n,\ |a'|=d} A_{ba'} x^{a'}$$

where $A_{ba'}$ is a polynomial in the a_{ij} of the base change matrix $A = (a_{ij})$. We get that

$$\tau_{f^b}(g) = \sum_{x^c \in \mathscr{B}(S/I)} A_{bc}\lambda_c.$$

Hence

$$\dim_K \bigcap_{x^b <_{\mathrm{rlex}} x^a,\ |b|=d} \mathrm{Ker}_{S/I}(\tau_{f^b})_d$$

equals the dimension of the solution space $\mathscr{S}$ of the linear systems of equations $M_a z = 0$ with the matrix $M_a = (A_{bc})$ where $x^b <_{\mathrm{rlex}} x^a,\ x^c \in \mathscr{B}(S/I)$ and $|b| = |c| = d$. It follows from Lemma 2.1 that

$$\mathrm{rank}\ M_a = |\{x^c \in \mathscr{B}(S/\mathrm{gin}(I))\colon |c| = d,\ x^c <_{\mathrm{rlex}} x^a\}|.$$

We compute

$$\begin{aligned} \dim_K \bigcap_{x^b <_{\mathrm{rlex}} x^a,\ |b|=d} \mathrm{Ker}_{S/I}(\tau_{f^b})_d &= \dim_K \mathrm{Ker}(M_a\colon (S/I)_d \to (S/I)_d) \\ &= \dim_K (S/I)_d - \mathrm{rank}\ M_a \\ &= \dim_K (S/\mathrm{gin}(I))_d - \mathrm{rank}\ M_a \\ &= \{x^c \in \mathscr{B}(S/\mathrm{gin}(I))\colon |c| = d,\ x^a \leq_{\mathrm{rlex}} x^c\} \end{aligned}$$

and this is what we wanted. □

For later applications we give a second characterization of the membership in the generic initial ideals which is similar to Proposition 2.2. At first we need the following Lemma:

Lemma 2.3. *Let $a \in \mathbb{N}^n$ and $d = |a| < e$. Then*

$$\mathrm{Ker}_{S/I}(\tau_{f^a})_e = \bigcap_{j=1}^{n} \mathrm{Ker}_{S/I}(\tau_{f^{a+\varepsilon_j}})_e.$$

Proof. Let $g \in \operatorname{Ker}_{S/I}(\tau_{f^a})_e$. Then for $i = 1, \dots, n$ we have that $\tau_{f^{a+\varepsilon_j}}(g) = \tau_{f_j}(\tau_{f^a}(g)) = 0$. Thus $g \in \bigcap_{i=1}^n \operatorname{Ker}_{S/I}(\tau_{f^{a+\varepsilon_j}})_e$.

On the other hand let $g \in (S/I)_e \setminus \operatorname{Ker}_{S/I}(\tau_{f^a})_e$. Then $0 \neq \tau_{f^a}(g) \in (S/I)_{e-d}$ is a nonzero element of degree > 0. Since the set $\{f^c \colon c \in \mathbb{N}^n,\ |c| = e-d\}$ is a K-basis for S_{e-d}, there must exist an f^c such that $\tau_{f^c}(\tau_{f^a}(g)) \neq 0$. Choose an $i \in \mathbb{N}$ with $c_i > 0$. Then

$$0 \neq \tau_{f^c}(\tau_{f^a}(g)) = \tau_{f^{a+c}}(g) = \tau_{f^{c-\varepsilon_i}}(\tau_{f^{a+\varepsilon_i}}(g)) \neq 0$$

and hence $\tau_{f^{a+\varepsilon_i}}(g) \neq 0$. Thus $g \notin \bigcap_{i=1}^n \operatorname{Ker}_{S/I}(\tau_{f^{a+\varepsilon_i}})_e$. This concludes the proof. □

For a monomial $1 \neq x^b \in S$ with $b \in \mathbb{N}^n$ we set $\min(x^b) = \min(b) = \min\{i \colon b_i > 0\}$. Furthermore we define $\min(1) = 1$. Given a monomial $x^a \in S_d$ we denote by

$$\operatorname{Sh}(a) = \left\{ x^b \in S_{d+1} \colon \frac{x^b}{x_{\min(b)}} = x^a \right\}$$

the *shadow* of x^a by multiplying with x_i such that $i \leq \min(a)$. Let

$$\operatorname{maxSh}(a) = \max_{<_{\mathrm{rlex}}} \operatorname{Sh}(a)$$

and

$$\operatorname{minSh}(a) = \min_{<_{\mathrm{rlex}}} \operatorname{Sh}(a)$$

be the maximal and minimal element of $\operatorname{Sh}(a)$ with respect to $<_{\mathrm{rlex}}$ on S. Now we partition the generic initial ideal using the set $\operatorname{Sh}(a)$.

Proposition 2.4. *Let $d > 0$, $x^a \in S_d$ be a monomial and $I \subset S$ be a graded ideal. Then*

$$\begin{aligned} &|\operatorname{Sh}(a) \cap \mathscr{B}(S/\operatorname{gin}(I))| \\ &= \dim_K \bigcap_{x^b <_{\mathrm{rlex}} x^a,\ |b|=d} \operatorname{Ker}_{S/I}(\tau_{f^b})_{d+1} - \dim_K \bigcap_{x^b \leq_{\mathrm{rlex}} x^a,\ |b|=d} \operatorname{Ker}_{S/I}(\tau_{f^b})_{d+1}. \end{aligned}$$

Proof. It follows from Lemma 2.3 that

$$\dim_K \bigcap_{x^b <_{\mathrm{rlex}} x^a,\ |b|=d} \operatorname{Ker}_{S/I}(\tau_{f^b})_{d+1} = \dim_K \bigcap_{x^b <_{\mathrm{rlex}} x^a,\ |b|=d}\ \bigcap_{j=1}^n \operatorname{Ker}_{S/I}(\tau_{f^{b+\varepsilon_j}})_{d+1}.$$

Since $\{x^{b'} \colon x^{b'} <_{\mathrm{rlex}} \operatorname{minSh}(a),\ |b'| = d+1\} = \{x^b x_i \colon x^b <_{\mathrm{rlex}} x^a,\ 1 \leq i \leq n,\ |b| = d\}$, the latter dimension equals

$$\dim_K \bigcap_{x^{b'} <_{\mathrm{rlex}} \operatorname{minSh}(a),\ |b'|=d+1} \operatorname{Ker}_{S/I}(\tau_{f^{b'}})_{d+1}$$

which is by Proposition 2.2 equal to

$$|\{x^c \in \mathscr{B}(S/\mathrm{gin}(I))\colon |c| = d+1,\ \ \mathrm{minSh}(a) \leq_{\mathrm{rlex}} x^c\}|.$$

Analogously we compute

$$\begin{aligned} &\dim_K \bigcap_{x^b \leq_{\mathrm{rlex}} x^a,\ |b|=d} \mathrm{Ker}_{S/I}(\tau_{f^b})_{d+1} \\ &= |\{x^c \in \mathscr{B}(S/\mathrm{gin}(I))\colon |c| = d+1,\ \ \mathrm{maxSh}(a) <_{\mathrm{rlex}} x^c\}|. \end{aligned}$$

Taking the difference in question gives rise to the desired formula. □

Given a graded ideal $I \subset S$ and a monomial $x^a \in S$ we define

$$\begin{aligned} d_I(a) &= \left|\mathrm{Sh}\left(\frac{x^a}{x_{\min(a)}}\right) \cap \mathscr{B}\left(\frac{S}{\mathrm{gin}(I)}\right)\right| \\ &= \left|\left\{x^b \in S\colon x^b \in \mathscr{B}\left(\frac{S}{\mathrm{gin}(I)}\right),\ |b| = |a| \text{ and } \frac{x^b}{x_{\min(b)}} = \frac{x^a}{x_{\min(a)}}\right\}\right|. \end{aligned}$$

Using $d_I(a)$ we can decide whether x^a belongs to $\mathrm{gin}(I)$ or not. Recall that a monomial ideal $I \subseteq S$ is called *strongly stable*, if $x_j x^a/x_i \in I$ for every $x^a \in I$ such that $x_i \mid x^a$ and $j \leq i$. Since $\mathrm{char}\, K = 0$ it is well-known that $\mathrm{gin}(I)$ is strongly stable (see, e.g., Eisenbud [6]).

Lemma 2.5. *Let $I \subset S$ be a graded ideal and $x^a \in S_d$ be a monomial with $d > 1$. Then*

$$x^a \notin \mathrm{gin}(I) \Longleftrightarrow \min\left(\frac{x^a}{x_{\min(a)}}\right) - \min(x^a) + 1 \leq d_I(a).$$

Proof. Let $x^{a'} = x^a/x_{\min(a)}$. Then clearly $\min(a') \geq \min(a)$ and $x^a \in \mathrm{Sh}(x^{a'})$. If $x^a \notin \mathrm{gin}(I)$, then $x^b \notin \mathrm{gin}(I)$ for $x^b = x_j x^{a'}$ and $\min(a) \leq j \leq \min(a')$ because $\mathrm{gin}(I)$ is strongly stable. Hence $\min(x^a/x_{\min(a)}) - \min(x^a) + 1 \leq d_I(a)$ because the considered monomials x^b are elements of $\mathrm{Sh}(x^{a'})$. Similarly we see that $\min(x^a/x_{\min(a)}) - \min(x^a) + 1 \leq d_I(a)$ implies that $x^a \notin \mathrm{gin}(I)$. □

Since $\mathrm{gin}(\mathrm{gin}(I)) = \mathrm{gin}(I)$ (see, e.g., [2] for a proof) it follows right from the definition that:

Lemma 2.6. *Let $I \subset S$ be a graded ideal and $x^a \in S_d$ be a monomial with $d > 1$. Then*

$$d_I(a) = d_{\mathrm{gin}(I)}(a).$$

3. Fibre products

We recall some notation. Let $\operatorname{gin}_n(I)$ be the generic initial ideal of a graded ideal $I \subset K[x_1, \dots, x_n]$ with respect to rlex induced by $x_1 > \dots > x_n$, let $\operatorname{gin}_m(J)$ be the generic initial ideal of a graded ideal $J \subset K[x_{n+1}, \dots, x_{n+m}]$ with respect to rlex induced by $x_{n+1} > \dots > x_{n+m}$, and let $\operatorname{gin}(L)$ be the generic initial ideal of a graded ideal $L \subset S = K[x_1, \dots, x_{n+m}]$ with respect to rlex induced by $x_1 > \dots > x_n > x_{n+1} > \dots > x_{n+m}$. Furthermore we define the ideals $\mathfrak{m}_n = (x_1, \dots, x_n) \subset K[x_1, \dots, x_n]$, $\mathfrak{m}_m = (x_{n+1}, \dots, x_{n+m}) \subset K[x_{n+1}, \dots, x_{n+m}]$ and $Q = \mathfrak{m}_n\mathfrak{m}_m \subset S$. We prove first a special version of our main result:

Proposition 3.1. *Let $I \subset K[x_1, \dots, x_n]$ and $J \subset K[x_{n+1}, \dots, x_{n+m}]$ be graded ideals. Then we have the following formulas for ideals in S:*
(i) $\operatorname{gin}(I + \mathfrak{m}_m) = \operatorname{gin}(\operatorname{gin}_n(I) + \mathfrak{m}_m)$.
(ii) $\operatorname{gin}(J + \mathfrak{m}_n) = \operatorname{gin}(\operatorname{gin}_m(J) + \mathfrak{m}_n)$.

Proof. (i): Let $f_1, \dots, f_{n+m} \subset S_1$ be a K-basis such that for the S-automorphism $\varphi\colon S \to S$, $x_i \mapsto f_i$ we can compute $\operatorname{gin}(I + \mathfrak{m}_m)$ and $\operatorname{gin}(\operatorname{gin}_n(I) + \mathfrak{m}_m)$ by using φ^{-1}. We denote the residue class of f_i in $K[x_1, \dots, x_n] = S/\mathfrak{m}_m$ by g_i.

Let $\psi\colon K[x_1, \dots, x_n] \to K[x_1, \dots, x_n]$ be an $K[x_1, \dots, x_n]$-automorphism such that the ideals $\operatorname{gin}_n(I)$ and $\operatorname{gin}_n(\operatorname{gin}_n(I))$ can be computed by using ψ^{-1}.

For a monomial $x^a \in S$ we have that

$$\begin{aligned}
& x^a \in \operatorname{gin}(I + \mathfrak{m}_m) \\
\iff & \overline{f^a} \in \operatorname{span}_K\{\overline{f^b}\colon x^b <_{\text{rlex}} x^a,\ |b| = |a|\} \subset S/(I + \mathfrak{m}_m) \\
\iff & \overline{g^a} \in \operatorname{span}_K\{\overline{g^b}\colon x^b <_{\text{rlex}} x^a,\ |b| = |a|\} \subset K[x_1, \dots, x_n]/I \\
\iff & \overline{\psi^{-1}(g^a)} \in \operatorname{span}_K\{\overline{\psi^{-1}(g^b)}\colon x^b <_{\text{rlex}} x^a,\ |b| = |a|\} \subset K[x_1, \dots, x_n]/\psi^{-1}(I) \\
\iff & \overline{\psi^{-1}(g^a)} \in \operatorname{span}_K\{\overline{\psi^{-1}(g^b)}\colon x^b <_{\text{rlex}} x^a,\ |b| = |a|\} \subset K[x_1, \dots, x_n]/\operatorname{gin}_n(I).
\end{aligned}$$

Here the first equality follows from Lemma 2.1. The second one is trivial. The third one follows since ψ^{-1} is an K-automorphism and the last equality follows since $S/\psi^{-1}(I)$ and $K[x_1, \dots, x_n]/\operatorname{gin}_n(I) = K[x_1, \dots, x_n]/\operatorname{in}_{<_{\text{rlex}}}(\psi^{-1}(I))$ share a common K-vector space basis. Similarly we compute

$$\begin{aligned}
& x^a \in \operatorname{gin}(\operatorname{gin}_n(I) + \mathfrak{m}_m) \\
\iff & \overline{f^a} \in \operatorname{span}_K\{\overline{f^b}\colon x^b <_{\text{rlex}} x^a,\ |b| = |a|\} \subset S/(\operatorname{gin}_n(I) + \mathfrak{m}_m) \\
\iff & \overline{g^a} \in \operatorname{span}_K\{\overline{g^b}\colon x^b <_{\text{rlex}} x^a,\ |b| = |a|\} \subset K[x_1, \dots, x_n]/\operatorname{gin}_n(I) \\
\iff & \overline{\psi^{-1}(g^a)} \in \operatorname{span}_K\{\overline{\psi^{-1}(g^b)}\colon x^b <_{\text{rlex}} x^a,\ |b| = |a|\} \subset K[x_1, \dots, x_n]/\psi^{-1}(\operatorname{gin}_n(I)) \\
\iff & \overline{\psi^{-1}(g^a)} \in \operatorname{span}_K\{\overline{\psi^{-1}(g^b)}\colon x^b <_{\text{rlex}} x^a,\ |b| = |a|\} \subset K[x_1, \dots, x_n]/\operatorname{gin}_n(\operatorname{gin}_n(I)) \\
\iff & \overline{\psi^{-1}(g^a)} \in \operatorname{span}_K\{\overline{\psi^{-1}(g^b)}\colon x^b <_{\text{rlex}} x^a,\ |b| = |a|\} \subset K[x_1, \dots, x_n]/\operatorname{gin}_n(I).
\end{aligned}$$

The last equality is new and it follows from the fact that $\operatorname{gin}_n(\operatorname{gin}_n(I)) = \operatorname{gin}_n(I)$. Thus we see that

$$\operatorname{gin}(I + \mathfrak{m}_m) = \operatorname{gin}(\operatorname{gin}_n(I) + \mathfrak{m}_m)$$

and this shows (i). Analogously one proves (ii). $\square$

We need the following result which relates the numbers $d_I(a)$ introduced in Section 2 of various ideals.

Proposition 3.2. *Let $I \subset K[x_1, \dots, x_n]$ and $J \subset K[x_{n+1}, \dots, x_{n+m}]$ be graded ideals and $x^a \in S_d$ be a monomial with $d > 1$. We have that*

$$d_{I+J+Q}(a) = d_{I+\mathfrak{m}_m}(a) + d_{J+\mathfrak{m}_n}(a).$$

Proof. As a K-vector space we may see $S/(I+\mathfrak{m}_m)$ and $S/(J+\mathfrak{m}_n)$ as subspaces of $S/(I + J + Q)$ using the standard K-bases $\mathscr{B}(S/(I + \mathfrak{m}_m))$, $\mathscr{B}(S/(J + \mathfrak{m}_n))$ and $\mathscr{B}(S/(I + J + Q))$ respectively. Using the equality $(I + \mathfrak{m}_m) \cap (J + \mathfrak{m}_n) = I + J + Q$ it follows from a direct computation that the homomorphism of K-vector spaces

$$\rho_d \colon S/(I + \mathfrak{m}_m)_d \oplus S/(J + \mathfrak{m}_n)_d \to S/(I + J + Q)_d, \quad (g, h) \mapsto g + h,$$

is surjective with trivial kernel for $d > 0$. Thus it is an isomorphism for $d > 0$.

Let $x^{a'} = x^a/x_{\min(a)}$. Then $|a'| = d - 1 > 0$. The following diagram

$$\begin{tikzcd}
S/(I + \mathfrak{m}_m)_d \oplus S/(J + \mathfrak{m}_n)_d \arrow[r, "\rho_d"] \arrow[d, "\bigoplus_{x^b <_{\mathrm{rlex}} x^{a'}} \tau_{f^b}"', "\bigoplus_{x^b <_{\mathrm{rlex}} x^{a'}} \tau_{f^b}"] & S/(I + J + Q)_d \arrow[d, "\bigoplus_{x^b <_{\mathrm{rlex}} x^{a'}} \tau_{f^b}"'] \\
\bigoplus_{x^b <_{\mathrm{rlex}} x^{a'}} S/(I + \mathfrak{m}_m)_1 \oplus \bigoplus_{x^b <_{\mathrm{rlex}} x^{a'}} S/(J + \mathfrak{m}_n)_1 \arrow[r, "\bigoplus_{x^b <_{\mathrm{rlex}} x^{a'}} \rho_1"] & \bigoplus_{x^b <_{\mathrm{rlex}} x^{a'}} S/(I + J + Q)_1
\end{tikzcd}$$

is commutative where the direct sums are take over monomials of degree $d - 1$. It follows that

$$\begin{aligned}
&\dim_K \bigcap_{x^b <_{\mathrm{rlex}} x^{a'},\ |b|=d-1} \operatorname{Ker}_{S/(I+J+Q)}(\tau_{f^b})_d \\
&= \dim_K \bigcap_{x^b <_{\mathrm{rlex}} x^{a'},\ |b|=d-1} \operatorname{Ker}_{S/(I+\mathfrak{m}_m)}(\tau_{f^b})_d + \dim_K \bigcap_{x^b <_{\mathrm{rlex}} x^{a'},\ |b|=d-1} \operatorname{Ker}_{S/(J+\mathfrak{m}_m)}(\tau_{f^b})_d.
\end{aligned}$$

Analogously we see that

$$\begin{aligned}&\dim_K \bigcap_{x^b \le_{\mathrm{rlex}} x^{a'},\ |b|=d-1} \mathrm{Ker}_{S/(I+J+Q)}(\tau_{f^b})_d \\ &= \dim_K \bigcap_{x^b \le_{\mathrm{rlex}} x^{a'},\ |b|=d-1} \mathrm{Ker}_{S/(I+\mathfrak{m}_m)}(\tau_{f^b})_d + \dim_K \bigcap_{x^b \le_{\mathrm{rlex}} x^{a'},\ |b|=d-1} \mathrm{Ker}_{S/(J+\mathfrak{m}_m)}(\tau_{f^b})_d.\end{aligned}$$

Now Proposition 2.4 implies that

$$\begin{aligned}d_{I+J+Q}(a) &= |\mathrm{Sh}(a') \cap \mathscr{B}(S/\mathrm{gin}(I+J+Q))| \\ &= |\mathrm{Sh}(a') \cap \mathscr{B}(S/\mathrm{gin}(I+\mathfrak{m}_m))| + |\mathrm{Sh}(a') \cap \mathscr{B}(S/\mathrm{gin}(J+\mathfrak{m}_n))| \\ &= d_{I+\mathfrak{m}_m}(a) + d_{J+\mathfrak{m}_n}(a).\end{aligned}$$

This concludes the proof. □

We are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Let $x^a \in S_d$ be a monomial for an integer $d > 0$. If $d = 1$ then $\mathrm{gin}(I + J + Q)_1 = \mathrm{gin}(\mathrm{gin}_n(I) + \mathrm{gin}_m(J) + Q)_1$ for trivial reasons. Assume that $d > 1$. Applying Lemma 2.6, Proposition 3.1 and Proposition 3.2 we see that

$$\begin{aligned}d_{I+J+Q}(a) &= d_{I+\mathfrak{m}_m}(a) + d_{J+\mathfrak{m}_n}(a) = d_{\mathrm{gin}(I+\mathfrak{m}_m)}(a) + d_{\mathrm{gin}(J+\mathfrak{m}_n)}(a) \\ &= d_{\mathrm{gin}(\mathrm{gin}_n(I)+\mathfrak{m}_m)}(a) + d_{\mathrm{gin}(\mathrm{gin}_m(J)+\mathfrak{m}_n)}(a) = d_{\mathrm{gin}_n(I)+\mathfrak{m}_m}(a) + d_{\mathrm{gin}_m(J)+\mathfrak{m}_n}(a) \\ &= d_{\mathrm{gin}_n(I)+\mathrm{gin}_m(J)+Q}(a).\end{aligned}$$

Thus it follows from Lemma 2.5 that

$$x^a \notin \mathrm{gin}(I+J+Q) \Longleftrightarrow x^a \notin \mathrm{gin}(\mathrm{gin}_n(I) + \mathrm{gin}_m(J) + Q).$$ □

REMARK 3.3. It is clear that rlex plays an important role in the arguments we have used to prove Theorem 1.2. For other term orders the statement of Theorem 1.2 fails to be true. For instance, for the lex order, $m = n = 3$ the ideals $I = (x_1^2, x_1x_2)$ and $J = (x_4^2, x_5x_6)$ do not fulfill Theorem 1.2.

REMARK 3.4. Let Γ be a simplicial complex on the vertex set $\{x_1, \dots, x_n\}$. We denote the *symmetric algebraic shifted complex* of Γ by $\Delta^s(\Gamma)$. In general a shifting operation Δ in the sense of Kalai [11] replaces a complex Γ with a combinatorially simpler complex $\Delta(\Gamma)$ that still detects some combinatorial properties of Γ. Symmetric algebraic shifting Δ^s is an example of such an operation and an be realized by using a kind of polarization of the generic initial ideal with respect to the reverse lexicographic order of the Stanley–Reisner ideal of Γ in the polynomial ring $K[x_1, \dots, x_n]$. (See Herzog [8] or Kalai [11] for details.)

For two simplicial complexes Γ_1 on the vertex set $\{x_1,\dots,x_n\}$ and Γ_2 on the vertex set $\{x_{n+1},\dots,x_{n+m}\}$ we denote by $\Gamma_1 \uplus \Gamma_2$ the *disjoint union* of Γ_1 and Γ_2 on the vertex set $\{x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}\}$. Using Theorem 1.2 it is not difficult to prove that

$$\Delta^s(\Gamma_1 \uplus \Gamma_2) = \Delta^s(\Delta^s(\Gamma_1) \uplus \Delta^s(\Gamma_2)).$$

This result was conjectured by Kalai in [11] and it was proved for exterior and symmetric shifting by Nevo. (See [13] and [14].)

In view of Theorem 1.2, the ideal $\operatorname{gin}(F(I, J))$ can be computed by first computing $\operatorname{gin}_n(I)$, $\operatorname{gin}_m(J)$ and then $\operatorname{gin}(F(\operatorname{gin}_n(I), \operatorname{gin}_m(J)))$. One would like to describe the generators of the ideal $\operatorname{gin}(F(I, J))$ in terms of those of $\operatorname{gin}_n(I)$ and $\operatorname{gin}_m(J)$. The first step in this direction is the following:

Proposition 3.5. *Let $Q = (x_i : 1 \le i \le n)(x_j : n+1 \le j \le m+n) \subset S$. Then:*
(i) $\operatorname{gin}(Q) = (x_i x_j : i + j \le n + m,\ i \le j$ *and* $i \le \min\{n, m\})$.
(ii) $\operatorname{gin}(Q^k) = \operatorname{gin}(Q)^k$ *for all* k.

Proof. Every ideal which is a product of ideals of linear forms has a linear resolution, see [4]. The gin-rlex of an ideal with linear resolution has linear resolution [6]. Hence we know that $\operatorname{gin}(Q^k)$ is generated in degree $2k$ for all k. Therefore it is enough to prove equality (i) in degree 2 and to prove equality (ii) in degree $2k$. To check that (i) holds in degree 2, it is enough to prove the inclusion $\supseteq$ since the two vector spaces involved have vector space dimension equal to mn. It follows from Lemma 2.5 that for a monomial $x_i x_j$ with $i \le j$ we have

$$x_i x_j \in \operatorname{gin}(Q) \iff j - i + 1 > d_Q(\varepsilon_i + \varepsilon_j)$$

where for $k \in \{1, \dots, n+m\}$ we denote by ε_k the k-th standard basis vector of $\mathbb{Z}^{n+m}$. By Lemma 2.6 and Proposition 3.2 we know

$$\begin{aligned} d_Q(\varepsilon_i + \varepsilon_j) &= d_{\mathfrak{m}_m}(\varepsilon_i + \varepsilon_j) + d_{\mathfrak{m}_n}(\varepsilon_i + \varepsilon_j) = d_{\operatorname{gin}(\mathfrak{m}_m)}(\varepsilon_i + \varepsilon_j) + d_{\operatorname{gin}(\mathfrak{m}_n)}(\varepsilon_i + \varepsilon_j) \\ &= d_{(x_1,\dots,x_m)}(\varepsilon_i + \varepsilon_j) + d_{(x_1,\dots,x_n)}(\varepsilon_i + \varepsilon_j) \end{aligned} \tag{4}$$

where we used the obvious facts $\operatorname{gin}(\mathfrak{m}_m) = (x_1, \dots, x_m)$ and $\operatorname{gin}(\mathfrak{m}_n) = (x_1, \dots, x_n)$.

Thus we have to compute $d_{(x_1,\dots,x_m)}(\varepsilon_i + \varepsilon_j)$ and $d_{(x_1,\dots,x_n)}(\varepsilon_i + \varepsilon_j)$ respectively. We treat the case $n \le m$; if $n > m$ one argues analogously. At first we consider

$$1 \le i,\ j \le n + m \quad \text{with} \quad i + j \le n + m,\ \ i \le j \quad \text{and} \quad i \le n.$$

It follows from the definition that

$$d_{(x_1,\dots,x_m)}(\varepsilon_i + \varepsilon_j) = |\{x_l x_j : m + 1 \le l \le j \le m + n\}|.$$

This number is only nonzero if $j \geq m+1$ and in this case we have $d_{(x_1,\ldots,x_m)}(\varepsilon_i+\varepsilon_j) = j-m$. Similarly, we have that $d_{(x_1,\ldots,x_n)}(\varepsilon_i+\varepsilon_j)$ is only nonzero if $j \geq n+1$ and in this case we have $d_{(x_1,\ldots,x_n)}(\varepsilon_i+\varepsilon_j) = j-n$. In any case, if $i+j \leq m+n$, then it follow from Equation (4) that

$$d_Q(\varepsilon_i+\varepsilon_j) \leq (j-n)+(j-m) \leq (m-i)+(j-m) < j-i+1.$$

Hence we get that $x_i x_j \in \operatorname{gin}(Q)$ as desired. As explained above, this proves (i).

As observed above, to prove (ii) it is enough to prove the asserted equality in degree $2k$. Consider the K-algebra $K[\varphi(Q_2)]$ generated by $\varphi(Q_2)$ for a generic change of coordinates φ. The assertion we have to prove is equivalent to the assertion that the initial algebra of $K[\varphi(Q_2)]$ is generated as a K-algebra by the elements $x_i x_j$ described in (i) which generate $\operatorname{gin}(Q)$ as an ideal.

We claim that:

(5) The algebra relations among those $x_i x_j$ are of degree 2.

It follows from results in [3] or [16] that the second Veronese ring of S can be presented as P/L where $P = K[y_{l,k}\colon 1 \leq l \leq k \leq n+m]$ is a polynomial ring and $L \subseteq P$ is an ideal which is generated in degree 2. Indeed there exists even a Gröbner basis of L of degree 2. We obtain a presentation of the K-algebra generated by the $x_i x_j$ of (5) by eliminating all those $y_{l,k}$ where l, k do not satisfy the conditions in (i). In particular, this shows claim (5).

Hence, by the Buchberger-like criterion for being a Sagbi basis [5], it suffices to prove that $\operatorname{gin}(Q^2) = \operatorname{gin}(Q)^2$ in degree 4. This can be done by checking that the two vector spaces involved have the same dimension. The dimension of $\operatorname{gin}(Q^2)_4$ is equal to that of Q_4^2 which is $\binom{n+1}{2}\binom{m+1}{2}$. It remains to compute the dimension of $\operatorname{gin}(Q)_4^2$. Observe that a basis of $\operatorname{gin}(Q)_4^2$ is given by the monomials $x_i x_j x_h x_k$ such that the following conditions hold:

$$\text{(6)} \qquad \begin{cases} 1 \leq i \leq j \leq h \leq k \leq m+n, \\ j \leq \min(n, m), \\ i+k \leq n+m, \\ j+h \leq n+m. \end{cases}$$

It would be nice to have a bijection showing that the number, call it $W(n, m)$, of (i, j, h, k) satisfying (6) is equal to $\binom{n+1}{2}\binom{m+1}{2}$. We content ourself with a non-bijective proof. Assuming that $n \leq m$ and using the identities $\sum_{l=s}^{t} l = (t-s+1)(t+s)/2$ and $\sum_{l=s}^{t} 1 = (t-s+1)$, a straight forward (but tedious) evaluation of the formula

$$\sum_{i=1}^{n}\sum_{j=i}^{n}\sum_{h=j}^{m+n-j}\sum_{k=h}^{m+n-i} 1$$

allows us to write $W(n, m)$ as

$$\sum_{i=1}^{n} \binom{n+m-2i+3}{3} \tag{7}$$

plus

$$-\binom{m+2}{4} - \binom{n+2}{4} + \binom{m-n+2}{4}. \tag{8}$$

The expression (7) can be evaluated, depending on whether $m + n$ is odd or even, using the formulas:

$$\sum_{i=0}^{p} \binom{2i+u}{3} = \begin{cases} \dfrac{p(p+1)(2p^2+2p-1)}{6} & \text{if} \quad u = 1, \\ \dfrac{(p^2-1)p^2}{3} & \text{if} \quad u = 0. \end{cases} \tag{9}$$

It follows that the expression of (7) is indeed equal to

$$\frac{1}{6n^3m} + \frac{1}{6nm^3} + \frac{1}{6n^3} + \frac{1}{2nm^2} + \frac{1}{6nm} - \frac{1}{6n}. \tag{10}$$

That summing (10) with (8) one gets $\binom{m+1}{2}\binom{n+1}{2}$ is a straightforward computation. $\square$

REMARK 3.6. (a) Refining the arguments in Proposition 3.5 one can show that statement (i) and (ii) hold for every term order satisfying $x_1 > \cdots > x_{m+n}$.
(b) The equality $\operatorname{gin}(Q^k) = \operatorname{gin}(Q)^k$ is quite remarkable. We do not know other families of ideals satisfying it. It is difficult to guess a generalization of it. For instance that equality does not hold for a product of 3 or more ideals of linear forms, e.g. $(x_1, x_2)(x_3, x_4)(x_5, x_6)$. And it does not hold for the ideal defining the fibre product of 3 of more polynomial rings, e.g. $(x_1, x_2)(x_3, x_4) + (x_1, x_2)(x_5, x_6) + (x_3, x_4)(x_5, x_6)$.
(c) The statements (i) and (ii) of Proposition 3.5, can be rephrased as follows: The (n, m)-th Segre product in generic coordinates has a toric degeneration to a ladder of the second Veronese ring of the polynomial ring in $m + n - 1$ variables and this holds independently of the term order.

4. Algebraic properties of fibre products

It is easy to see how homological properties transfer from A and B to $A \circ B$. The main point is that the short exact sequence (1) allows to control local cohomology. In this section we show that the componentwise linearity behaves well with respect to fibre products. Recall that a graded ideal L in some polynomial ring over the field K

is called *componentwise linear* if for all $k \in \mathbb{N}$ the ideal $L_{\langle k \rangle}$ has a k-linear resolution (i.e. $\operatorname{Tor}_i(L_{\langle k \rangle}, K)_{i+j} = 0$ for $j \neq k$) where $L_{\langle k \rangle}$ is the ideal generated by all elements of degree k of L. (See Herzog and Hibi [9] for more details.)

In the proof we will use the following criterion of Herzog–Reiner–Welker [10, Theorem 17] to check whether an ideal is componentwise linear: Let L be a graded ideal in a polynomial ring T. Then L is componentwise linear if and only if for all $k \in \mathbb{Z}$ we have $\operatorname{reg} T/L_{\leq k} \leq k - 1$ where $L_{\leq k}$ is the ideal generated by all elements of L of degree smaller or equal to k and $\operatorname{reg} M$ denotes the Castelnuovo–Mumford regularity of a f.g. graded T-module M. Note that in [10] this result is only stated for monomial ideals, but the proof works also, more generally, for graded ideals. Recall that the Castelnuovo–Mumford regularity can be computed via local cohomology as $\operatorname{reg} M = \max\{j \in \mathbb{Z}\colon H^i_{\mathfrak{m}}(M)_{j-i} \neq 0 \text{ for some } i\}$ where $\mathfrak{m}$ denotes also the graded maximal ideal of T.

Theorem 4.1. *Let $I \subset K[x_1, \ldots, x_n]$ and $J \subset K[x_{n+1}, \ldots, x_{n+m}]$ be graded ideals such that $I \subseteq \mathfrak{m}_n^2$ and $J \subseteq \mathfrak{m}_m^2$. Then I and J are componentwise linear if and only if $F(I, J)$ is componentwise linear.*

Proof. At first we note that $F(I, J)_{\leq k} = 0$ for $k \leq 1$, and for $k \geq 2$ we have

$$F(I, J)_{\leq k} = (I + J + Q)_{\leq k} = I_{\leq k} + J_{\leq k} + Q = F(I_{\leq k}, J_{\leq k}).$$

The short exact sequence (1) with the current notation is:

$$0 \to S/F(I, J)_{\leq k} \to S/(I_{\leq k} + \mathfrak{m}_m) \oplus S/(J_{\leq k} + \mathfrak{m}_n) \to S/\mathfrak{m} \to 0.$$

It induces the long exact sequence of local cohomology modules in degree $j - i$

$$\begin{aligned} \cdots \to H^{i-1}_{\mathfrak{m}}(S/\mathfrak{m})_{j-1-(i-1)} &\to H^i_{\mathfrak{m}}(S/F(I, J)_{\leq k})_{j-i} \\ &\to H^i_{\mathfrak{m}}(S/(I_{\leq k} + \mathfrak{m}_m))_{j-i} \oplus H^i_{\mathfrak{m}}(S/(J_{\leq k} + \mathfrak{m}_n))_{j-i} \to H^i_{\mathfrak{m}}(S/\mathfrak{m})_{j-i} \to \cdots. \end{aligned}$$

Observe that for $j \geq 2$ we have that $H^{i-1}_{\mathfrak{m}}(S/\mathfrak{m})_{j-1-(i-1)} = H^i_{\mathfrak{m}}(S/\mathfrak{m})_{j-i} = 0$ because $S/\mathfrak{m}$ has regularity 0. The ring change isomorphism for local cohomology modules yields

$$H^i_{\mathfrak{m}}(S/(I_{\leq k} + \mathfrak{m}_m)) \cong H^i_{\mathfrak{m}_n}(K[x_1, \ldots, x_n]/I_{\leq k})$$

and

$$H^i_{\mathfrak{m}}(S/(J_{\leq k} + \mathfrak{m}_n)) \cong H^i_{\mathfrak{m}_m}(K[x_{n+1}, \ldots, x_{n+m}]/J_{\leq k})$$

as graded K-vector spaces. Thus we get for $j \geq 2$ the isomorphisms

$$\begin{aligned} &H^i_{\mathfrak{m}}(S/F(I, J)_{\leq k})_{j-i} \\ &\cong H^i_{\mathfrak{m}_n}(K[x_1, \ldots, x_n]/I_{\leq k})_{j-i} \oplus H^i_{\mathfrak{m}_m}(K[x_{n+1}, \ldots, x_{n+m}]/J_{\leq k})_{j-i} \end{aligned}$$

as K-vector spaces. Since the regularities of all modules in questions are greater or equal to 2, we obtain that $\operatorname{reg} S/F(I,J)_{\leq k} \leq k$ if and only if $\operatorname{reg} K[x_1,\dots,x_n]/I_{\leq k} \leq k$ and $\operatorname{reg} K[x_{n+1},\dots,x_{n+m}]/J_{\leq k} \leq k$. As noted above, this concludes the proof. $\square$

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