Koszul Cycles

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Abstract We prove regularity bounds for Koszul cycles holding for every ideal of dimension ≤ 1 in a polynomial ring; see Theorem 3.5. In Theorem 4.7 we generalize the "$c+1$" lower bound for the Green–Lazarsfeld index of Veronese rings proved in (Bruns et al., arXiv:0902.2431) to the multihomogeneous setting. For the Koszul complex of the c-th power of the maximal ideal in a Koszul ring we prove that the cycles of homological degree t and internal degree $\geq t(c+1)$ belong to the t-th power of the module of 1-cycles; see Theorem 5.2.

1 Introduction

The Koszul complex and its homology are central objects in commutative algebra. Vanishing theorems for Koszul homology are the key to many open questions. The goal of the paper is the study of regularity bounds for Koszul cycles and Koszul homology of ideals in standard graded rings. Our original motivation comes from the study of the syzygies of Veronese varieties and, in particular, the conjecture of Ottaviani and Paoletti [12] on their Green–Lazarsfeld index, see [4].

In Sect. 2 we fix the notation and describe some canonical maps between modules of Koszul cycles. Given a standard graded ring R with maximal homogeneous ideal $\mathfrak{m}$, a homogeneous ideal I and a finitely generated graded module M, we let

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$Z_t(I,M)$ denote the module of Koszul cycles of homological degree t. Under a mild assumption, we show in 2.4 that $Z_{s+t}(I,M)$ is a direct summand of $Z_s(I,N)$ where $N = Z_t(I,M)$.

Section 3 is devoted to the description of (Castelnuovo–Mumford) regularity bounds for Koszul cycles and homology. We prove bounds of the following type:

$$\operatorname{reg}_R(Z_t(I,M)) \leq t(c+1) + \operatorname{reg}_R(M) + v \tag{1}$$

under assumptions on $\dim M/IM$. Here $\operatorname{reg}_R(N)$ denotes the (relative) Castelnuovo–Mumford regularity of a finitely generated R-module N. Note that $\operatorname{reg}_R(N)$ is the ordinary Castelnuovo–Mumford regularity if R is the polynomial ring. Furthermore it is known that $\operatorname{reg}_R(N)$ is finite if R is a Koszul algebra. If R is Koszul and $\dim M/IM = 0$, then we prove that (1) holds with $v = 0$ and where c is such that $\mathfrak{m}^c \subset I + \operatorname{Ann}(M)$ and I is generated in degrees $\leq c$, see 3.2. In 3.5 we prove that if R is a polynomial ring of characteristic 0 or big enough and $\dim M/IM \leq 1$ then (1) holds with $v = 0$ and $c = \operatorname{reg}_R(I)$. Furthermore, if R is a polynomial ring and $\dim M/IM = 0$, then we show in 3.9 that (1) holds with $c \geq$ the largest degree of a generator of I and $v = \dim[R/I]_c$.

We also give examples showing that the inequality

$$\operatorname{reg}_R Z_t(I,M) \leq t(\operatorname{reg}_R(I) + 1) + \operatorname{reg}_R(M) \tag{2}$$

cannot hold in general (i.e. without restriction on the dimension of M/IM). However (2) holds if R is a polynomial ring, $M = R/J$ and both I and J are strongly stable monomial ideals, see 3.7 and 3.8. We leave it as an open question whether (2) holds when $M = R$ and R is a polynomial ring.

In Sect. 4 we prove that, given a vector $c = (c_1,\ldots,c_d) \in \mathbb{N}_+^d$, the Segre–Veronese ring associated to c over a field of characteristic 0 or big enough, has a Green–Lazarsfeld index larger than or equal to $\min(c)+1$, see 4.7. This result was announced in [4] and improves the bound of Hering, Schenck and Smith [11] by 1.

In Sect. 5 we analyze the generators of the module $Z_t = Z_t(\mathfrak{m}^c, R)$ under the assumption that R has characteristic 0 or big enough. If R is Koszul we prove that Z_t/Z_1^t vanishes in degrees $\geq t(c+1)$, 5.2. Here Z_1^t denotes the image of the canonical map $\wedge^t Z_1 \to Z_t$. This allows us to deduce that the c-th Veronese subring of a polynomial ring S satisfies the property N_{2c} if and only if $H_1(\mathfrak{m}^c, S)^{2c} = 0$, see 5.3. Finally, we prove that the cycles given in [4] generate Z_2; see 5.5.

2 Notation and Generalities

In this section we collect notation and general facts about maps between modules of Koszul cycles. Let R be a ring, F be a free R-module of rank n, $\varphi : F \to R$ be an R-linear map and M be an R-module. All tensor products are over R. We consider the Koszul complexes $K(\varphi,R) = \bigoplus_{t=0}^n K_t(\varphi,R) = \bigwedge^\bullet F$ and $K(\varphi,M) = \bigoplus_{t=0}^n K_t(\varphi,M) = \bigwedge^\bullet F \otimes M$. The complex $K(\varphi,M)$ can be seen as a module over the exterior algebra $K(\varphi,R)$. For $a \in K(\varphi,R)$ and $f \in K(\varphi,M)$ the multiplication will be denoted by $a.f$. The differential of $K(\varphi,R)$ and $K(\varphi,M)$ will be denoted simply by φ and it satisfies

$$\varphi(a.f) = \varphi(a).f + (-1)^s a.\varphi(f)$$

for all $a \in K_s(\varphi,R)$ and $f \in K(\varphi,M)$. We let $Z_t(\varphi,M)$, $B_t(\varphi,M)$, $H_t(\varphi,M)$ denote the cycles, the boundaries and the homology in homological degree t and set $Z(\varphi,M) = \oplus Z_t(\varphi,M)$, and so on for cycles, boundaries and homology. One knows that $Z(\varphi,R)$ is a subalgebra of $K(\varphi,R)$ and that $B(\varphi,R)$ is an ideal of $Z(\varphi,R)$ so that the homology $H(\varphi,R)$ is itself an algebra. More generally, $Z(\varphi,M)$ is a $Z(\varphi,R)$-module. We let $Z_s(\varphi,R)Z_t(\varphi,M)$ denote the image of the multiplication map $Z_s(\varphi,R)\otimes Z_t(\varphi,M) \to Z_{s+t}(\varphi,M)$. Similarly, $Z_1(\varphi,R)^t$ will denote the image of the map $\bigwedge^t Z_1(\varphi,R) \to Z_t(\varphi,R)$.

In the graded setting the map φ will be assumed to be of degree 0 and F will be a direct sum of shifted copies of R. In this way the Koszul complex $K(\varphi,M)$ inherits a graded structure for the map φ and the module M. So cycles, boundaries and homology have an induced graded structure. An index on the left of a graded module always denotes the selection of the homogeneous component of that degree. If R is standard graded over a field K with maximal homogeneous ideal $\mathfrak{m}$ all the invariants we are going to study depend actually only on the image of φ and not on the map itself as long as $\ker\varphi \subseteq \mathfrak{m}F$. So, if $J = \operatorname{Im}\varphi$, we will sometimes denote $K(\varphi,R)$ simply by $K(J,R)$ and so on.

Fix a basis of the free module F, say $\{e_1,\dots,e_n\}$. Given $I = \{i_1,\dots,i_s\} \subset [n]$ with $i_1 < i_2 < \cdots < i_s$ we write e_I for the corresponding basis element $e_{i_1}\wedge\cdots\wedge e_{i_s}$ of $\bigwedge^s F$. If $\varphi(e_i) = u_i \in J$ we will also use the symbol $[u_{i_1},\dots,u_{i_s}]$ to denote e_I.

For disjoint subsets $A,B \subset [n]$ we set $\varepsilon(A,B) = \#\{(a,b) \in A\times B : a > b\}$ and

$$\sigma(A,B) = (-1)^{\varepsilon(A,B)}.$$

One has

$$e_A e_B = \sigma(A,B) e_{A\cup B}.$$

For further application we record the following:

Lemma 2.1. *For disjoint subsets A,B,C of $[n]$ one has*

$$\sigma(A\cup B,C)\sigma(B,A) = \sigma(B,A\cup C)\sigma(A,C).$$

Proof. Just use the fact that $\varepsilon(A\cup B,C) = \varepsilon(A,C)+\varepsilon(B,C)$ and $\varepsilon(B,A\cup C) = \varepsilon(B,A)+\varepsilon(B,C)$. □

Any element $f \in \bigwedge^s F\otimes M$ can be written uniquely as $f = \sum e_I \otimes m_I$ with $m_I \in M$ where the sum is over the subsets of cardinality s of $[n]$. If $m_I = 0$ then we will say that e_I does not appear in f. For every $f \in K_{s+t}(\varphi,M)$ and for every $I \subset [n]$ with $s = \#I$ we have a unique decomposition

$$f = a_I + e_I.b_I \tag{3}$$

with $a_I \in K_{s+t}(\varphi,M)$ and $b_I \in K_t(\varphi,M)$, and, furthermore, e_J does not appear in a_I whenever $J \supset I$ and e_S does not appear in b_I whenever $S \cap I \neq \emptyset$. With the notation above we have:

Lemma 2.2. *For every $f \in K_{s+t}(\varphi,M)$ we have:*

(a) $\sum_I e_I.b_I = \binom{t+s}{s} f$ *where $\sum_I$ stands for the sum extended to all the subsets $I \subset [n]$ with $s = \#I$.*
(b) *if $f \in Z_{s+t}(\varphi,M)$, then $b_I \in Z_t(\varphi,M)$ for every I with $s = \#I$.*

Proof. For (a) one writes $f = \sum e_J \otimes m_J$ with $J \subset [n]$ with $\#J = s+t$ and $m_J \in M$. Then one observes that $e_J.m_J$ appears in $e_I.b_I$ iff $I \subset J$. Hence $e_J.m_J$ appears in $\sum_I e_I.b_I$ exactly $\binom{t+s}{s}$ times. For (b) one applies the differential $0 = \varphi(f) = \varphi(a_I) + \varphi(e_I).b_I + (-1)^s e_I.\varphi(b_I)$ and since e_J does not appear in $\varphi(a_I) + \varphi(e_I).b_I$ whenever $J \supseteq I$ then $\varphi(b_I)$ must be 0. □

The multiplication $K_s(\varphi,R) \otimes K_t(\varphi,M) \to K_{s+t}(\varphi,M)$ can be interpreted as a map

$$K_s(\varphi,K_t(\varphi,M)) \to K_{s+t}(\varphi,M)$$

defined by $a \otimes f \to a.f$. Restricting the domain of the map to $K_s(\varphi,Z_t(\varphi,M))$ we get a map

$$K_s(\varphi,Z_t(\varphi,M)) \to K_{s+t}(\varphi,M)$$

which is indeed a map of complexes. So it induces a map

$$\alpha_t : Z_s(\varphi,Z_t(\varphi,M)) \to Z_{s+t}(\varphi,M)$$

defined by

$$\sum a \otimes f \in Z_s(\varphi,Z_t(\varphi,M)) \to \sum a.f.$$

Now we define a map

$$\gamma_t : K_{s+t}(\varphi,M) \to K_s(\varphi,K_t(\varphi,M))$$

by the formula

$$\gamma_t(f) = \sum_I e_I \otimes b_I$$

where the sum is over the $I \subset [n]$ with $\#I = s$ and b_I is determined by the decomposition (3). We claim:

Lemma 2.3. *The map $\gamma_t : K(\varphi,M) \to K(\varphi,K_t(\varphi,M))[-t]$ is a map of complexes.*

Proof. Since $K(\varphi,M) = K(\varphi,R) \otimes M$ and we have $K(\varphi,K_t(\varphi,M)) = K(\varphi,K_t(\varphi,R)) \otimes M$ it is enough to prove the statement in the case $M = R$. Then it is enough to check

$$\gamma_t \circ \varphi(e_J) = \varphi \circ \gamma_t(e_J) \quad \text{for every } J \subset [n] \text{ with } \#J = s+t.$$

Note that

$$\gamma_t(e_J) = \sum \sigma(A,B)e_A \otimes e_B$$

where the sum is over all the B such that $\#B = t$ and $A = J \setminus B$. Then

$$\varphi \circ \gamma_t(e_J) = \sum \sigma(A \cup \{p\}, B)\sigma(\{p\}, A)\varphi(e_p)e_A \otimes e_B$$

and

$$\gamma_t \circ \varphi(e_J) = \sum \sigma(\{p\}, A \cup B)\sigma(A,B)\varphi(e_p)e_A \otimes e_B$$

where in both cases the sum is over all the partitions of J into three parts $A, B, \{p\}$ with $\#B = t$. So we have to check that

$$\sigma(A \cup \{p\}, B)\sigma(\{p\}, A) = \sigma(\{p\}, A \cup B)\sigma(A,B).$$

This is a special case of 2.1. □

It follows that γ_t gives, by restriction, a map

$$Z_{s+t}(\varphi, M) \to Z_s(\varphi, K_t(\varphi, M)).$$

By virtue of 2.2, its image is indeed contained in $Z_s(\varphi, Z_t(\varphi, M))$. So we have a map

$$\beta_t : Z_{s+t}(\varphi, M) \to Z_s(\varphi, Z_t(\varphi, M))$$

and, by virtue of Lemma 2.2, we have

$$\alpha_t \circ \beta_t(f) = \binom{t+s}{s} f \qquad \text{for all } f \in Z_{s+t}(\varphi, M).$$

An immediate consequence:

Lemma 2.4. *Assume $\binom{t+s}{s}$ is invertible in R. Then $Z_{s+t}(\varphi, M)$ is a direct summand of $Z_s(\varphi, Z_t(\varphi, M))$.*

One can easily check that, in the graded setting, the maps described in this section are graded and of degree 0.

3 Bounds for Koszul Cycles

In this section we consider a field K and a standard graded K-algebra R with maximal homogeneous ideal $\mathfrak{m}$. In other words, R is of the form S/J where S is a polynomial ring over K with the standard grading and J is a homogeneous of S. We will consider a finitely generated graded R-module M. Let $\beta_{i,j}^R(M) = \dim_K \operatorname{Tor}_i^R(M,K)_j$ be the *graded Betti numbers* of M over R. We define the number

$$t_i^R(M) = \max\{j \in \mathbb{Z} : \beta_{i,j}^R(M) \neq 0\},$$

whenever $\operatorname{Tor}_i^R(M,K) \neq 0$ and $t_i^R(M) = -\infty$ otherwise. The *Castelnuovo–Mumford regularity* of M over R is

$$\operatorname{reg}_R(M) = \sup\{t_i^R(M) - i : i \in \mathbb{N}\}.$$

Recall that R is a *Koszul algebra* if $\operatorname{reg}_R(K) = 0$. One knows that $\operatorname{reg}_R(M)$ is finite for every finitely generated module M if R is a Koszul algebra, see Avramov and Eisenbud [2]. One says that R has the property N_p if its defining ideal J is generated by quadrics and the syzygies of the quadrics are linear for $p-1$ steps, that is, if $t_i^S(R) \leq i+1$ for $i = 1, \dots, p$. The *Green–Lazarsfeld index* of R is the largest number p such that R has the property N_p, that is,

$$\operatorname{index}(R) = \max\{p : t_i^S(R) \leq i+1 \text{ for } i = 1, \dots, p\}.$$

Conventions. Just to avoid endless repetitions, throughout this section ideals will be homogeneous, modules will be finitely generated and graded, linear maps will be graded of degree 0. Furthermore I will always denote an ideal and M a module of the current ring. The current ring will be denoted by S if it is the polynomial ring over a field K or by R if it is a standard graded K-algebra and $\mathfrak{m}$ will denote its maximal homogeneous ideal.

We start with a well-known fact that is easy to prove:

Lemma 3.1. *One has*

$$I + \operatorname{Ann}(M) \subseteq \operatorname{Ann}(M/IM) \subseteq \sqrt{I + \operatorname{Ann}(M)}.$$

We have:

Proposition 3.2. *Assume R is Koszul and $\dim M/IM = 0$. Let c be the smallest integer such that $\mathfrak{m}^c \subseteq I + \operatorname{Ann}(M)$ and I is generated in degree $\leq c$ (such a number c exists by 3.1). Set $Z_t = Z_t(I,M)$ and $H_t = H_t(I,M)$. Then, for every t,*

$$\operatorname{reg}_R(Z_t) \leq t(c+1) + \operatorname{reg}_R(M)$$

and

$$\operatorname{reg}_R(H_t) \leq t(c+1) + \operatorname{reg}_R(M) + c - 1.$$

Proof. The proof is a slight generalization of the arguments given in [4, Sect. 2]. Set $B_t = B_t(I,M)$ and $K_t = K_t(I,R)$. Note that $I + \operatorname{Ann}(M)$ annihilates H_t. Hence $\mathfrak{m}^c H_t = 0$. It follows that H_t vanishes in degrees $\geq t_0^R(Z_t) + c$ and hence $\operatorname{reg}_R(H_t) \leq t_0^R(Z_t) + c - 1 \leq \operatorname{reg}_R(Z_t) + c - 1$. So the second formula follows from the first. The short exact sequence

$$0 \to B_t \to Z_t \to H_t \to 0$$

gives

$$\operatorname{reg}_R(B_t) \leq \max\{\operatorname{reg}(Z_t), \operatorname{reg}_R(H_t) + 1\} \leq \operatorname{reg}_R(Z_t) + c$$

and

$$0 \to Z_{t+1} \to K_{t+1} \otimes M \to B_t \to 0$$

gives

$$\begin{aligned}\operatorname{reg}_R(Z_{t+1}) &\le \max\{\operatorname{reg}_R(K_{t+1}\otimes M), \operatorname{reg}_R(B_t)+1\}\\ &\le \max\{(t+1)c+\operatorname{reg}_R(M), \operatorname{reg}_R(Z_t)+c+1\}\end{aligned}$$

Now the statement can be proved by induction on t, the case $t=0$ being obvious since $Z_0 = M$. □

We single out a special case of 3.2:

Proposition 3.3. *Assume that* $\dim S/I = 0$. *Set* $Z_t = Z_t(I,M)$ *and* $H_t = Z_t(I,M)$. *Then, for every* t,

$$\operatorname{reg}_S(Z_t) \le t(\operatorname{reg}_S(I)+1)+\operatorname{reg}_S(M)$$

and

$$\operatorname{reg}_S(H_t) \le t(\operatorname{reg}_S(I)+1)+\operatorname{reg}_S(M)+\operatorname{reg}_S(I)-1.$$

Proof. The number c of 3.2 is $\le \operatorname{reg}_S(I)$. □

The following remark explains why the assumption on the dimension of S/I is necessary in 3.3.

Remark 3.4. The module $Z_1(I,M)$ sits in the exact sequence:

$$0 \to Z_1(I,M) \to F\otimes M \to IM \to 0.$$

Hence

$$\operatorname{reg}_S(IM) \le \max\{\operatorname{reg}_S(I)+\operatorname{reg}_S(M), \operatorname{reg}_S(Z_1(I,M))-1\}.$$

There are plenty of examples such that $\operatorname{reg}_S(IM) > \operatorname{reg}_S(I)+\operatorname{reg}_S(M)$ already when $M = I$, see Conca [7] or Sturmfels [13]. Therefore, in these examples, one has $\operatorname{reg}_S(Z_1(I,M)) > \operatorname{reg}_S(I)+1+\operatorname{reg}_S(M)$.

But using a result of Caviglia [5], see also Eisenbud, Huneke and Ulrich [8], we are able to show:

Theorem 3.5. *Assume that* $\dim M/IM \le 1$. *Assume also that either* $\operatorname{char} K = 0$ *or* $> t$. *Set* $Z_t = Z_t(I,M)$. *Then*

$$\operatorname{reg}_S(Z_t) \le t(\operatorname{reg}_S(I)+1)+\operatorname{reg}_S(M)$$

for every t.

Proof. By induction on t. For $t=1$ note that, by [5], we have $\operatorname{reg}_S(IM) \le \operatorname{reg}_S(I) + \operatorname{reg}_S(M)$ and the short exact sequence of 3.4 implies that $\operatorname{reg}_S(Z_1) \le \operatorname{reg}_S(I) + 1+\operatorname{reg}_S(M)$. For $t>1$, by virtue of 2.4 we have that Z_t is a direct summand of $Z_{t-1}(I,Z_1)$. Hence $\operatorname{reg}_S(Z_t) \le \operatorname{reg}_S(Z_{t-1}(I,Z_1))$. Since $\operatorname{Ann}(Z_1) \supseteq \operatorname{Ann}(M)$ we have $\operatorname{Ann}(Z_1)+I \supseteq \operatorname{Ann}(M)+I$ and, by 3.1, $\dim Z_1/IZ_1 \le M/IM \le 1$. Hence, by induction, we have

$$\operatorname{reg}_S(Z_{t-1}(I,Z_1)) \le (t-1)(\operatorname{reg}_S(I)+1)+\operatorname{reg}_S(Z_1).$$

Since $\operatorname{reg}_S(Z_1) \le \operatorname{reg}_S(I)+1+\operatorname{reg}_S(M)$ has been already established, the desired inequality follows. □

Question 3.6. (1) Does the inequality in 3.5 hold over a Koszul algebra R? And is the assumption on the characteristic needed?
(2) Is it true that $\operatorname{reg}_S(Z_t(I,S)) \le t(\operatorname{reg}_S(I)+1)$ holds for every homogeneous ideal $I \subset S$?

Since $Z_1(I,S)$ is the first syzygy module of I the inequality of 3.6 is actually an equality for $t=1$. An indication that the answer to 3.6(2) might be "yes" for some classes of ideals is given in 3.7 and 3.8. Recall that a monomial ideal $I \subset S = K[x_1,\dots,x_n]$ is *strongly stable* if whenever a monomial $m \in I$ is divisible by a variable x_i, then $mx_j/x_i \in I$ for every $j<i$. In characteristic 0 the strongly stable ideals are exactly the ideals of S which are fixed by the Borel group of the upper triangular matrices of $\mathrm{GL}_n(K)$ acting on S. The Eliahou–Kervaire complex [9] gives the graded minimal free resolution of strongly stable ideals. For us it is important to recall that if I is strongly stable then $\operatorname{reg}_S(I)$ is the largest degree of a minimal generator of I.

Proposition 3.7. *Let $I \subset S$ be a strongly stable ideal. Set $Z_t = Z_t(I,S)$. Then Z_t is generated by elements of degree $\le t(\operatorname{reg}_S(I)+1)$.*

Proof. Set $c = \operatorname{reg}_S(I)$. The idea of the proof follows essentially the argument given in [4, Theorem 3.3]. We note first that, as we are dealing with a monomial ideal I, the modules Z_t have a natural $\mathbb{Z}^n$-graded structure as long as we consider the free presentation $F \to I$ associated with the monomial generators of I. We do a double induction on n and on t. The case $n=1$ is obvious. The case $t=1$ is easy and follows from the description of the (first) syzygies of I given in [9]. By induction on t it is enough to verify that Z_t/Z_1Z_{t-1} is generated in degree $< t(c+1)$. Hence it suffices to show that every $\mathbb{Z}^n$-graded element $f \in Z_t$ of total degree $q \ge t(c+1)$ can be written modulo Z_1Z_{t-1} as a multiple of an element in Z_t of total degree $<q$. Let $\alpha \in \mathbb{Z}^n$ be the $\mathbb{Z}^n$-degree of f. If $\alpha_n = 0$ then we can conclude by induction on n. Therefore we may assume that $\alpha_n > 0$. Let $u \in I$ be a monomial generator of I with $x_n \mid u$ and $[u]$ the corresponding free generator of F. We have the decomposition

$$f = a + [u]b$$

with $b \in Z_{t-1}$ and $[u]$ does not appear in a. Note that b has degree $q - \deg(u) \ge q - c$. Since Z_{t-1} is generated by elements of degree $\le (t-1)(c+1)$ we may write

$$b = \sum_{j=1}^{s} \lambda_j v_j z_j \tag{4}$$

where $\lambda_j \in K$, $z_j \in Z_{t-1}$ are $\mathbb{Z}^n$ graded and the v_j are monomials of positive degree.

Let $\lambda_j v_j z_j$ be a summand in (4). If x_n does not divide v_j, then choose $i < n$ such that $x_i \mid v_j$. Since $x_i u / x_n \in I$, there exists a monomial generator of I, say u_1, such that $u_1 \mid x_i u / x_n$, say $u_1 w = x_i u / x_n$. Set $z' = x_i[u] - x_n w[u_1] \in Z_1$ and subtract the element

$$\lambda_j \frac{v_j}{x_i} z' z_j \in Z_{t-1} Z_1$$

from f. Repeating this procedure for each $\lambda_j v_j z_j$ in (4) such that x_n does not divide v_j we obtain a cycle $f_1 \in Z_t$ of degree α such that

(i) $f \equiv f_1 \bmod Z_1 Z_{t-1}$;
(ii) if $v[u_1, \ldots, u_t]$ appears in f_1 and $u \in \{u_1, \ldots, u_t\}$, then $x_n \mid v$.

We repeat the described procedure for each monomial generator $u \in I$ with $x_n \mid u$. We end up with an element $f_2 \in Z_t$ of degree α such that

(iii) $f \equiv f_2 \bmod Z_1 Z_{t-1}$;
(iv) if $v[u_1, \ldots, u_t]$ appears in f_2 and $x_n \mid u_1 \cdots u_t$, then $x_n \mid v$.

Note that if $v[u_1, \ldots, u_t]$ appears in f_2 and $x_n \nmid u_1 \cdots u_t$, then $x_n \mid v$ by degree reasons. Hence for every $v[u_1, \ldots, u_t]$ appearing in f_2 we have $x_n \mid v$. Therefore $f_2 = x_n g$, and $g \in Z_t$ has degree $< q$. This completes the proof. □

Indeed a much stronger statement holds:

Theorem 3.8. *Let I, J be strongly stable ideals of S. Then*

$$\operatorname{reg}_S(Z_t(I, S/J)) \leq t(\operatorname{reg}_S(I) + 1) + \operatorname{reg}_S(S/J)$$

for every t.

Theorem 3.8 has been proved by Satoshi Murai in collaboration with the second author and is part of an ongoing project.

The following result, whose proof is surprisingly simple, generalizes Green's theorem [10, Theorem 2.16]:

Theorem 3.9. *Let $I \subset S$ such that $\dim M/IM = 0$. Let $c \in \mathbb{N}$ be such that I is generated in degrees $\leq c$ and set $v = \dim[S/I]_c$. Set $Z_t = Z_t(I, M)$ and $H_t = H_t(I, M)$. One has*

$$\operatorname{reg}_S(Z_t) \leq t(c+1) + \operatorname{reg}_S(M) + v$$

and

$$\operatorname{reg}_S(H_t) \leq t(c+1) + \operatorname{reg}_S(M) + v + c - 1$$

for every t.

Proof. The first inequality can be deduced from the second using the standard short exact sequences relating B_t, Z_t and H_t. We prove the second inequality by induction on v. If $v = 0$ then $\mathfrak{m}^c \subset I$ and the assertion has been proved in 3.2. Now let $v > 0$.

Observe that H_t is annihilated by $I + \mathrm{Ann}(M)$. Hence (by 3.1) $\dim H_t = 0$ and its regularity is the largest degree in which H_t does not vanish. Take $f \in S_c \setminus I$ and set $J = I + (f)$. Note that the minimal generators of I are minimal generators of J and that $\dim[S/J]_c = v - 1$. We have a short exact sequence of Koszul homology [3, 1.6.13]:

$$H_{t+1}(J,M) \to H_t(-c) \to H_t.$$

By construction $H_t(-c)$ does not vanish in degree $\mathrm{reg}_S(H_t) + c$ while H_t vanishes in that degree. It follows that $H_{t+1}(J,M)$ does not vanish in degree $\mathrm{reg}_S(H_t) + c$ and hence $\mathrm{reg}_S(H_{t+1}(J,M)) \geq \mathrm{reg}_S(H_t) + c$. By induction we know that $\mathrm{reg}_S(H_{t+1}(J, M)) \leq (t+1)(c+1) + \mathrm{reg}_S(M) + (v-1) + c - 1$. It follows that

$$\mathrm{reg}_S(H_t) + c \leq (t+1)(c+1) + \mathrm{reg}_S(M) + (v-1) + c - 1,$$

that is,

$$\mathrm{reg}_S(H_t) \leq t(c+1) + \mathrm{reg}_S(M) + v + c - 1. \qquad \square$$

Remark 3.10. (a) Let $I \subset S$ be the ideal generated by a proper subspace V of forms of degree c such that $I\mathfrak{m} = \mathfrak{m}^{c+1}$. Then $\mathrm{reg}_S(I) = c + 1$. Set $Z_t = Z_t(I,S)$ and $v = \dim S_c/V$. By virtue of 3.3 we have $\mathrm{reg}_S(Z_t) \leq t(c+2)$ while 3.9 gives $\mathrm{reg}_S(Z_t) \leq t(c+1) + v$. So for small t the first bound is better than the second and the other way round for large t.

(b) Since $H_0 = M/IM$, for $t = 0$ the bound of 3.9 takes the form $\mathrm{reg}_S(M/IM) \leq \mathrm{reg}_S(M) + v + c - 1$. Even the case $M = S$ is interesting: it says that if $\sqrt{I} = \mathfrak{m}$, I is generated in degree $\leq c$ and $v = \dim[S/I]_c$ then $\mathfrak{m}^{c+v} \subset I$.

4 Green–Lazarsfeld Index for Segre–Veronese Rings

The goal of this section is to prove a result 4.7 about the Green–Lazarsfeld index of Segre–Veronese rings which was announced in [4]. We first need to generalize some results of [4] to the multihomogeneous setting.

Let $d \in \mathbb{N}$ and $m = (m_1, \dots, m_d) \in \mathbb{N}^d$ and $\mathrm{c} = (c_1, \dots, c_d) \in \mathbb{N}^d$. We consider the polynomial ring $S = K[x_{ij} : 1 \leq i \leq d,\ \ 1 \leq j \leq m_i]$ with the $\mathbb{Z}^d$ graded structure induced by assigning $\deg x_{ij} = e_i \in \mathbb{Z}^d$. Consider the ideals $\mathfrak{m}_i = (x_{ij} | j = 1, \dots, m_i)$ and

$$\mathfrak{m}^{\mathrm{c}} = \prod_{i=1}^{d} \mathfrak{m}_i^{c_i}.$$

Then the module of Koszul cycles $Z_t(\mathfrak{m}^{\mathrm{c}}, S)$ has a $\mathbb{Z}^d$-graded structure and also a finer $\mathbb{Z}^m = \mathbb{Z}^{m_1} \times \cdots \mathbb{Z}^{m_d}$-graded structure. We have:

Lemma 4.1. *The module $Z_t(\mathfrak{m}^{\mathrm{c}}, S)$ is generated by elements that either have $\mathbb{Z}^d$-degree bounded above by the vector $t\mathrm{c} + (t-1)\sum e_i$ or belong to U_i^t for some i in $\{1, \dots, d\}$ where $U_i = Z_1(\mathfrak{m}^{\mathrm{c}}, S)_{\mathrm{c}+e_i}$.*

Proof. Set $Z_t = Z_t(\mathfrak{m}^{\mathrm{c}}, S)$ and give it the natural $\mathbb{Z}^m$ graded structure. The proof is a multigraded variant of the argument used above in 3.7. First note that given a monomial generator u of $\mathfrak{m}^{\mathrm{c}}$, a variable $x_{ij}|u$ and k such that $1 \leq k \leq m_i$ and $k \neq j$, the monomial $v = ux_{ik}/x_{ij}$ belongs to $\mathfrak{m}^{\mathrm{c}}$ and the element $x_{ik}[u] - x_{ij}[v]$ belongs to U_i. It is well known that these syzygies generate Z_1 and that $\mathfrak{m}^{\mathrm{c}}$ has a linear resolution. Now assume that $f \in Z_t$ is a $\mathbb{Z}^m$-homogeneous element of degree $(\alpha_1, \dots, \alpha_d)$, $\alpha_i = (\alpha_{i1}, \dots, \alpha_{im_i}) \in \mathbb{Z}^{m_i}$. Assume that $|\alpha_i| \geq t(c_i + 1)$ for some i, say for $i = 1$. We may also assume that $\alpha_{11} \neq 0$. Using induction on t, the rewriting procedure described in the proof of 3.7 and the linear syzygies described above, we may write $f = x_{11} g \bmod U_1 Z_{t-1}$. Since $g \in Z_t$, the conclusion follows by induction on t. □

Next we note that [4, Lemma 3.4] can be extended to the present setting:

Lemma 4.2. *Let $\alpha \in \mathbb{N}^d$ be a vector such that $\alpha \leq \mathrm{c}$ componentwise. Let $a_1, a_2 \dots, a_{t+1}$ be monomials of $\mathbb{Z}^d$-degree equal to α and $b_1, b_2 \dots, b_t \in S$ monomials of degree $\mathrm{c} - \alpha$. Then*

$$\sum_{\sigma \in \mathscr{S}_{t+1}} (-1)^{\sigma} a_{\sigma(t+1)} [b_1 a_{\sigma(1)}, b_2 a_{\sigma(2)}, \dots, b_t a_{\sigma(t)}] \qquad (5)$$

belongs to $Z_t(\mathfrak{m}^{\mathrm{c}}, S)$.

Now we prove a multigraded version of [4, Theorem 3.6]:

Lemma 4.3. *For every $i = 1, \dots, d$ let $\mathrm{b} = \mathrm{c} - e_i$ and set $U_i = Z_1(\mathfrak{m}^{\mathrm{c}}, S)_{\mathrm{c}+e_i}$. Then*

$$(c_i + 1)!\, \mathfrak{m}^{\mathrm{b}} U_i^{c_i} \subset \mathfrak{m}_i^{c_i} Z_{c_i}(\mathfrak{m}^{\mathrm{c}}, S) + B_{c_i}(\mathfrak{m}^{\mathrm{c}}, S).$$

Proof. Set $u = c_i$ and $Z_u = Z_{c_i}(\mathfrak{m}^{\mathrm{c}}, S)$ and $B_u = B_{c_i}(\mathfrak{m}^{\mathrm{c}}, S)$. The generators of U_i are of the form

$$z_a(y_0, y_1) = y_0[ay_1] - y_1[ay_0]$$

where a is a monomial of $\mathbb{Z}^d$-degree equal to b and $y_0, y_1 \in \{x_{i1}, \dots, x_{im_i}\}$. So we have to take u such elements, say $z_{a_j}(y_{0j}, y_{1j})$ with $j = 1, \dots, u$, another monomial of degree b, say a_{u+1}, and we have to prove that

$$(u+1)! a_{u+1} \prod_{j=1}^{u} z_{a_j}(y_{0j}, y_{1j}) \in \mathfrak{m}_i^u Z_u + B_u. \qquad (6)$$

The symmetrization argument given in the proof of [4, Theorem 3.6] works in this case as well to prove that the left hand side of (6) can be rewritten, modulo boundaries, as

$$\sum y_{i_1 1} \cdots y_{i_u, u} W_i$$

where $i = (i_1, \dots, i_u) \in \{0, 1\}^u$ and W_i are cycles of the type described in 4.2. □

An $\mathbb{N}^d$-graded K-algebra $R = \bigoplus_{\alpha \in \mathbb{N}^d} R_\alpha$ is called standard if $R_0 = K$ and R is generated by R_{e_i} with $i = 1, \dots, d$. Clearly R can be presented as a quotient of an

$\mathbb{N}^d$-graded polynomial ring $S = K[x_{ij} : 1 \le i \le d,\ \ 1 \le j \le m_i]$ with the $\mathbb{N}^d$-graded structure induced by assigning $\deg x_{ij} = e_i \in \mathbb{Z}^d$. Given a vector

$$\mathrm{c} = (c_1, \dots, c_d) \in \mathbb{N}^d,$$

we can consider the *Segre–Veronese subring* of R associated to it, namely

$$R^{(\mathrm{c})} = \bigoplus_{j \in \mathbb{N}} R_{j\mathrm{c}}.$$

Our goal is to study the Green–Lazarsfeld index of $R^{(\mathrm{c})}$. We note that $R^{(\mathrm{c})}$ is a quotient ring of $S^{(\mathrm{c})}$. Furthermore one has:

Lemma 4.4. *(a)* $\operatorname{reg}_{S^{(\mathrm{c})}}(R^{(\mathrm{c})}) = 0$ *for* $\mathrm{c} \gg 0$.
(b) $\operatorname{index}(R^{(\mathrm{c})}) \ge \operatorname{index}(S^{(\mathrm{c})})$ *for* $\mathrm{c} \gg 0$.

More precisely, both statements hold provided one has $i\mathrm{c} \ge \alpha$ *componentwise for every* $i \in \mathbb{Z}$ *and* $\alpha \in \mathbb{Z}^d$ *such that* $\beta^S_{i,\alpha}(R) \ne 0$.

Proof. A detailed proof of (a) is given in the bigraded setting by Conca, Herzog, Trung and Valla [6]. The same argument works as well for multigradings. Then (b) follows from (a) and [4, Lemma 2.2]. □

Consider the symmetric algebra T of the K-vector space S_c (i.e. a polynomial ring of Krull dimension $\dim_K S_\mathrm{c}$), and the natural surjection $T \to S^{(\mathrm{c})}$. The Betti numbers of $S^{(\mathrm{c})}$ as a T-module can be computed via Koszul homology.

Lemma 4.5. *We have*

$$\beta^T_{i,j}(S^{(\mathrm{c})}) = H_i(\mathfrak{m}^\mathrm{c}, S)_{j\mathrm{c}}.$$

Proof. One notes that $S^{(\mathrm{c})}$ is a direct summand of S and then proceeds as in [4, Lemma 4.1] □

So we may reinterpret 4.1 in terms of syzygies of $S^{(\mathrm{c})}$, obtaining:

Corollary 4.6. *One has* $\beta^T_{i,j}(S^{(\mathrm{c})}) = 0$ *provided* $(j-i-1)\min(\mathrm{c}) \ge i$.

Proof. Since $\mathfrak{m}^\mathrm{c}$ annihilates $H_i(\mathfrak{m}^\mathrm{c}, S)$, it follows from 4.1 that $H_i(\mathfrak{m}^\mathrm{c}, S)_\alpha = 0$ for every $\alpha \ge i(\mathrm{c} + \sum e_i) + \mathrm{c}$ componentwise. Replacing α with $j\mathrm{c}$ we have that $\beta^T_{i,j}(S^{(\mathrm{c})}) = 0$ if $j\mathrm{c} \ge i(\mathrm{c} + \sum e_i) + \mathrm{c}$ which is equivalent to $(j-i-1)\min(\mathrm{c}) \ge i$. □

In [11] Hering, Schenck and Smith proved that $\operatorname{index}(S^{(c)}) \ge \min(c)$. We improve the bound by one:

Theorem 4.7. *One has* $\min(\mathrm{c}) \le \operatorname{index}(S^{(\mathrm{c})})$. *Moreover,* $\min(\mathrm{c}) + 1 \le \operatorname{index}(S^{(\mathrm{c})})$ *if* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > 1 + \min(c)$.

Proof. The first statement is an immediate consequence of 4.6. In fact, if $i \le \min(\mathrm{c})$ then $(j-i-1)\min(\mathrm{c}) \ge i$ for every $j > i+1$ and hence, by 4.6, $t^T_i(S^{(\mathrm{c})}) = i+1$. Set $u = \min(\mathrm{c})$. For the second statement, we have to show that $H_{u+1}(\mathfrak{m}^\mathrm{c}, S)_{j\mathrm{c}} = 0$ for every $j > u+2$. By virtue of 4.1 we know that $Z_{u+1}(\mathfrak{m}^\mathrm{c}, S)$ is generated by:

(1) elements of degree $\le (u+1)\,\mathrm{c} + u\sum_s e_s$ and
(2) elements of U_i^{u+1} where $U_i = Z_1(\mathfrak{m}^{\mathrm{c}}, S)_{\mathrm{c}+e_i}$ and $i = 1,\dots,d$; they have degree $(u+1)\,\mathrm{c} + (u+1)e_i$.

So an element $f \in Z_{u+1}(\mathfrak{m}^{\mathrm{c}}, S)_{j\,\mathrm{c}}$ can come from a generator of type (1) by multiplication of elements of degree $\alpha \in \mathbb{Z}^d$ such that

$$\alpha \ge j\,\mathrm{c} - (u+1)\,\mathrm{c} + u\sum_s e_s.$$

Since $j > u+2$, we have

$$\alpha \ge 2\,\mathrm{c} + u\sum_s e_s \ge \mathrm{c}\,.$$

So $f \in \mathfrak{m}^{\mathrm{c}} Z_{u+1}(\mathfrak{m}^{\mathrm{c}}, S)$ and $f = 0$ in homology. Alternatively, $f \in Z_{u+1}(\mathfrak{m}^{\mathrm{c}}, S)_{j\,\mathrm{c}}$ can come from a generator of type (2) by multiplication of elements of degree $\alpha \in \mathbb{Z}^d$ such that

$$\alpha = j\,\mathrm{c} - (u+1)\,\mathrm{c} - (u+1)e_i \ge 2\,\mathrm{c} - (u+1)e_i.$$

If $c_i > u$ then $\alpha \ge \mathrm{c}$, and we conclude as above that $f = 0$ in homology. If, instead, $c_i = u$, then $\alpha \ge 2\,\mathrm{c} - (c_i+1)e_i$. We have that

$$f \in \mathfrak{m}^{2\,\mathrm{c} - (c_i+1)e_i} U_i^{c_i+1} = \mathfrak{m}^{\mathrm{c} - c_i e_i} \mathfrak{m}^{\mathrm{c} - e_i} U_i^{c_i} U_i.$$

But, assuming K has either characteristic 0 or $> u+1$, 4.3 implies:

$$\mathfrak{m}^{\mathrm{c} - e_i} U_i^{c_i} \subset \mathfrak{m}_i^{c_i} Z_{c_i}(\mathfrak{m}^{\mathrm{c}}, S) + B_{c_i}(\mathfrak{m}^{\mathrm{c}}, S).$$

Hence

$$f \in \mathfrak{m}^{\mathrm{c}} Z_u(\mathfrak{m}^{\mathrm{c}}, S) U_i + B_{c_i}(\mathfrak{m}^{\mathrm{c}}, S) U_i \subset B_{c_i+1}(\mathfrak{m}^{\mathrm{c}}, S)$$

and we conclude that $f = 0$ in homology. □

5 Generating Koszul Cycles

In this section we consider the Koszul cycles $Z_t(I,R)$ where R is standard graded and I is a homogeneous ideal. For simplicity, in this section we let Z_t denote the cycles $Z_t(I,R)$, and similarly write B_t, H_t and K_t for boundary, homology and components of the Koszul complex $K(I,R)$. We consider the multiplication map

$$Z_s \otimes Z_t \to Z_{s+t} \tag{7}$$

and we want to understand in which degrees it is surjective. Note that the map (7) has a factorization

$$Z_s \otimes Z_t \xrightarrow{u_{s,t}} Z_s(I, Z_t) \xrightarrow{\alpha_t} Z_{s+t}$$

where the first map $u_{s,t}$ is the canonical one and the second is the map α_t described in Sect. 2.

Proposition 5.1. *Suppose R has characteristic 0 or larger than $s+t$. Then:*

(1) The multiplication map $Z_s \otimes Z_t \to Z_{s+t}$ is surjective in degree j if the module $\operatorname{Tor}_1^R(K_{s-1}/B_{s-1}, Z_t)$ vanishes in degree j.
(2) If $R = K[x_1, \dots, x_n]$ and $\dim R/I = 0$ then the multiplication map $Z_s \otimes Z_t \to Z_{s+t}$ is surjective in degree j for every $j \geq \operatorname{reg}_R Z_s + \operatorname{reg}_R Z_t$. In particular, the map $Z_s \otimes Z_t \to Z_{s+t}$ is surjective in degree j for every $j \geq (s+t)(\operatorname{reg}_R(I)+1)$.

Proof. To prove (1) we note that, as α_t is surjective, we may as well consider the map $u_{s,t} : Z_s \otimes Z_t \to Z_s(\varphi, Z_t)$. Tensoring

$$0 \to Z_s \to K_s \to B_{s-1} \to 0$$

and

$$0 \to B_{s-1} \to K_{s-1} \to K_{s-1}/B_{s-1} \to 0$$

with Z_t, we have exact sequences

$$Z_s \otimes Z_t \to K_s \otimes Z_t \xrightarrow{f} B_{s-1} \otimes Z_t \to 0$$

and

$$\operatorname{Tor}_1^R(K_{s-1}/B_{s-1}, Z_t) \to B_{s-1} \otimes Z_t \xrightarrow{g} K_{s-1} \otimes Z_t.$$

The composition $g \circ f$ is the map of the Koszul complex $K_s \otimes Z_t \to K_{s-1} \otimes Z_t$. So $Z_s(\varphi, Z_t) = \ker(g \circ f)$ and the image of $u_{s,t}$ is $\ker f$. It follows that $u_{s,t}$ is surjective in degree j iff g is injective in degree j, that is $\operatorname{Tor}_1^R(K_{s-1}/B_{s-1}, Z_t)$ vanishes in degree j.

To prove (2) we first observe, since $\sqrt{I} = \mathfrak{m}$, one has that $(Z_t)_P$ is free for every prime ideal $P \neq \mathfrak{m}$. Hence $\operatorname{Tor}_i^R(M, Z_t)$ has Krull dimension 0 for every finitely generated R-module M and every $t \geq 0$ and $i > 0$.

Then we may apply [8, Corollary 3.1] and have that

$$\operatorname{reg}_R \operatorname{Tor}_i^R(M, Z_t) \leq \operatorname{reg}_R M + \operatorname{reg}_R Z_t + i$$

and in particular

$$\operatorname{reg}_R \operatorname{Tor}_1^R(K_{s-1}/B_{s-1}, Z_t) \leq \operatorname{reg}_R K_{s-1}/B_{s-1} + \operatorname{reg}_R Z_t + 1.$$

But $\operatorname{reg}_R K_{s-1}/B_{s-1} = \operatorname{reg}_R Z_s - 2$ and hence

$$\operatorname{reg}_R \operatorname{Tor}_1^R(K_{s-1}/B_{s-1}, Z_t) \leq \operatorname{reg}_R Z_s + \operatorname{reg}_R Z_t - 1$$

In other words, $\operatorname{Tor}_1^R(K_{s-1}/B_{s-1}, Z_t)$ vanishes in degrees $\geq \operatorname{reg}_R Z_s + \operatorname{reg}_R Z_t$. Together with (1) this concludes the proof of (2). □

Theorem 5.2. *Assume that R is Koszul and K has characteristic 0 or $> t$ and take $I = \mathfrak{m}^c$. Then for every t the module Z_t/Z_1^t vanishes in degree $\geq t(c+1)$ and Z_1^t has an R-linear resolution.*

Proof. We prove the first assertion by induction on t. It is enough to prove that the multiplication map $Z_1 \otimes Z_{t-1} \to Z_t$ is surjective in degrees $j \geq tc+t$. By virtue of 5.1(1), it is enough to prove that $\operatorname{Tor}_1^R(R/\mathfrak{m}^c, Z_{t-1})$ vanishes for $j \geq tc+t$. But since $R/\mathfrak{m}^c$ vanishes in degree $\geq c$, it is easy to see that $\operatorname{Tor}_1^R(R/\mathfrak{m}^c, Z_{t-1})$ vanishes in degrees $\geq t_1^R(Z_{t-1})+c$. We know [4, Proposition 2.4] that $\operatorname{reg}_R(Z_i) \leq ic+i$ for every i (here we use the fact that R is Koszul). So we have $t_1^R(Z_{t-1})-1 \leq (t-1)c+(t-1)$, i.e. $t_1^R(Z_{t-1}) \leq (t-1)c+t$. Then we have $t_1(Z_{t-1})+c \leq (t-1)c+t+c = tc+t$. This proves the first assertion. For the second, one just notes that $\operatorname{reg}_R(Z_t) \leq tc+t$ and that Z_1^t coincides with Z_t truncated in degree $t(c+1)$. Therefore Z_1^t must have an R-linear resolution. □

We have the following consequence:

Corollary 5.3. *Let $S = K[x_1,\ldots,x_n]$ with $\operatorname{char} K = 0$ or $> 2c$. One has $H_1(\mathfrak{m}^c, S)^{2c} = 0$ iff $\operatorname{index}(S^{(c)}) \geq 2c$, i.e. $S^{(c)}$ has the N_{2c}-property.*

Proof. By virtue of 5.2 $Z_{2c}(\mathfrak{m}^c, S)$ coincides with $Z_1(\mathfrak{m}^c, S)^{2c}$ in degrees $\geq 2c(c+1)$. Hence, by assumption, $H_{2c}(\mathfrak{m}^c, S)$ vanishes in degrees $\geq 2c(c+1)$. This implies that $\beta^T_{2c,j}(S^{(c)}) = 0$ if $jc \geq 2c(c+1)$, that is, $j \geq 2c+2$. In other words, $t^T_{2c}(S^{(c)}) \leq 2c+1$. Since $S^{(c)}$ is Cohen–Macaulay, one can conclude that $t_i^T(S^{(c)}) \leq i+1$ for $i = 1,\ldots,2c$, that is, $\operatorname{index}(S^{(c)}) \geq 2c$. □

Remark 5.4. The interesting aspect of Corollary 5.3 is that we know explicitly the generators of $Z_1(\mathfrak{m}^c, S)$ and hence the inclusion $Z_1(\mathfrak{m}^c, S)^{2c} \subset B_{2c}$ boils down to a quite concrete statement. Unfortunately we have not been able to settle it. Note also that Ottaviani and Paoletti conjectured that $\operatorname{index}(S^{(c)}) = 3c-3$ apart from few known exceptions and at least in characteristic 0, see [12] or [4] for the precise statements. In [4] we have proved that $\operatorname{index}(S^{(c)}) \geq c+1$.

As we mentioned in [4] there are computational evidences that the cycles of [4, Lemma 3.4] generate $Z_t(\mathfrak{m}^c, S)$. We show below that this is the case for $t = 2$ and any c. To this end we recall that for every monomial b of degree $c-1$ and for variables x_j, x_k we have an element $z_b(x_j, x_k) = x_j[bx_k] - x_k[bx_j] \in Z_1(\mathfrak{m}^c, S)$. It is well-know and easy to see that the elements $z_b(x_j, x_k)$ generate $Z_1(\mathfrak{m}^c, S)$. For a monomial a we set $\max(a) = \max\{i : x_i | a\}$ and $\min(a) = \min\{i : x_i | a\}$. More precisely, the elements $z_b(x_j, x_k)$ with $j < k$ and $\max(b) \leq k$ form a Gröbner basis of $Z_1(\mathfrak{m}^c, S)$ with respect to any term order selecting $x_j[bx_k]$ as leading term of $z_b(x_j, x_k)$. We have:

Proposition 5.5. *If K has characteristic $\neq 2$ then the module $Z_2(\mathfrak{m}^c, S)$ is generated by two types of elements:*

(1) The elements of [4, Lemma 3.4] of degree $2c+1$,
(2) and by the elements of $Z_1(\mathfrak{m}^c, S)^2$ of degree $2c+2$, that is, the elements of the form $z_a(x_i, x_j) z_b(x_h, x_k)$.

Proof. Consider the map

$$\alpha_1 : Z_1(\mathfrak{m}^c, Z_1(\mathfrak{m}^c, S)) \to Z_2(\mathfrak{m}^c, S)$$

of Sect. 2. We know that $Z_2(\mathfrak{m}^c, S)$ has regularity $\leq 2c+2$ and the only generators of degree $2c+2$ are the elements of (2). So we only need to deal with the elements of degree $2c+1$. To this end we look at the component of degree $2c+1$ of α_1. Let a,b be monomials of degree $c-1$. The element

$$[ax_i] \otimes z_b(x_j, x_k) + [ax_k] \otimes z_b(x_i, x_j) + [ax_j] \otimes z_b(x_k, x_i) \tag{8}$$

belong to $Z_1(\mathfrak{m}^c, Z_1(\mathfrak{m}^c, S))$ and has degree $2c+1$. The image under α_1 of the elements in (8) are exactly the cycles of [4, Lemma 3.4] in $Z_2(\mathfrak{m}^c, S)$. Since α_1 is surjective, to complete the proof it is enough to prove the following statement:

Claim. The cycles described in (8) generate $Z_1(\mathfrak{m}^c, Z_1(\mathfrak{m}^c, S))$ in degree $2c+1$.

Let $F \in Z_1(\mathfrak{m}^c, Z_1(\mathfrak{m}^c, S))$ be an element of degree $2c+1$. So F is a sum of elements of the form $[u] \otimes f$ with u a monomial of degree c and $f \in Z_1(\mathfrak{m}^c, S)$ with $\deg(f) = c+1$. Choose $[u]$ to be the largest in the lexicographic order induced by $x_1 > \cdots > x_n$ and look at the coefficient f of $[u]$ in F, i.e.

$$F = [u] \otimes f + \text{ sum of terms } [v] \otimes g \text{ with } v < u.$$

Let $x_j[bx_k]$ be the leading term of f with $j < k$ and $\max(b) \leq k$. If $\min(u) < j$ then we may add a suitable scalar multiple of (8) to "kill" the leading term of F and we are done. If instead $\min(u) \geq j$, then, since

$$0 = \varphi(F) = uf + \text{ sum of terms } vg \text{ with } v < u$$

we have that $x_j u[bx_k]$ must cancel, and so $x_j u = x_s v$ for some $v < u$ in the lex-order. But this is impossible. □

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