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Koszul homology and syzygies of Veronese subalgebras

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Abstract A graded K-algebra R has property N_p if it is generated in degree 1, has relations in degree 2 and the syzygies of order $\leq p$ on the relations are linear. The Green–Lazarsfeld index of R is the largest p such that it satisfies the property N_p. Our main results assert that (under a mild assumption on the base field) the cth Veronese subring of a polynomial ring has Green–Lazarsfeld index $\geq c+1$. The same conclusion also holds for an arbitrary standard graded algebra, provided $c \gg 0$.

1 Introduction

A classical problem in algebraic geometry and commutative algebra is the study of the equations defining projective varieties and of their syzygies. Green and Lazarsfeld [18,19] introduced the property N_p for a graded ring as an indication of the presence of simple syzygies. Let us recall the definition. A finitely generated $\mathbb{N}$-graded K-algebra $R = \oplus_i R_i$ over a field K satisfies property N_0 if R is generated (as a K-algebra) in degree 1. Then R can be presented as a quotient of a standard graded polynomial ring S and one says that R satisfies *property* N_p for some $p > 0$ if $\beta^S_{i,j}(R) = 0$ for

To Jürgen Herzog, friend and teacher.

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$j > i + 1$ and $1 \le i \le p$. Here $\beta^S_{i,j}(R)$ denote the graded Betti numbers of R over S. For example, property N_1 means that R is defined by quadrics, N_3 means that R is defined by quadrics and that the first and second syzygies of the quadrics are linear. We define the *Green–Lazarsfeld index* of R, denoted by $\operatorname{index}(R)$, to be the largest p such that R has N_p, with $\operatorname{index}(R) = \infty$ if R satisfies N_p for every p. It is, in general, very difficult to determine the precise value of the Green–Lazarsfeld index. Important conjectures, such as Green's conjecture on the syzygies of canonical curves [14, Chap. 9], predict the value of the Green–Lazarsfeld index for certain families of varieties.

The goal of this paper is to study the Green–Lazarsfeld index of the Veronese embeddings $v_c : \mathbb{P}^{n-1} \to \mathbb{P}^N$ of degree c of projective spaces and, more generally, of the Veronese embeddings of arbitrary varieties. Let S denote the polynomial ring in n variables over the field K. The coordinate ring of the image of v_c is the Veronese subring $S^{(c)} = \bigoplus_{i\in\mathbb{N}} S_{ic}$ of S. If $n \le 2$ or $c \le 2$ then $S^{(c)}$ is a determinantal ring whose resolution is well understood and the Green–Lazarsfeld index can be easily deduced. If $n = 2$ then $S^{(c)}$ is resolved by the Eagon–Northcott complex and so $\operatorname{index}(S^{(c)}) = \infty$. The resolution of $S^{(2)}$ in characteristic 0 is described by Jozefiak et al. in [22]; it follows that $\operatorname{index}(S^{(2)}) = 5$ if $n > 3$ and $\operatorname{index}(S^{(2)}) = \infty$ if $n \le 3$. For $n \le 6$ Andersen [1] proved that $\operatorname{index}(S^{(2)})$ is independent on char K, but for $n > 6$ and char $K = 5$ she proved that $\operatorname{index}(S^{(2)}) = 4$.

For $n > 2$ and $c > 2$ it is known that

$$c \le \operatorname{index}(S^{(c)}) \le 3c - 3. \tag{1}$$

The lower bound is due to Green [17] (and holds for any c and n). Ottaviani and Paoletti [23] established the upper bound in characteristic 0. They also showed that $\operatorname{index}(S^{(c)}) = 3c - 3$ for $n = 3$ and conjectured that $\operatorname{index}(S^{(c)}) = 3c - 3$ holds true for arbitrary $n \ge 3$; see also Weyman [28, Proposition 7.2.8]. For $n = 4$ and $c = 3$ it is indeed the case [23, Lemma 3.3]. In their interesting paper [13] Eisenbud et al. reproved some of these statements using different methods. Rubei [27] proved that $\operatorname{index}(S^{(3)}) \ge 4$ if char $K = 0$. Our main results are the following:

(i) $c + 1 \le \operatorname{index}(S^{(c)})$ if char $K = 0$ or $> c + 1$; see Corollary 4.2.
(ii) If R is a quotient of S then $\operatorname{index}(R^{(c)}) \ge \operatorname{index}(S^{(c)})$ for every $c \ge \operatorname{slope}_S(R)$; see Theorem 5.2. In particular, if R is Koszul then $\operatorname{index}(R^{(c)}) \ge \operatorname{index}(S^{(c)})$ for every $c \ge 2$

Furthermore we give characteristic free proofs of the bounds (1) and of the equality for $n = 3$; see Theorem 4.7. Our approach is based on the study of the Koszul complex associated to the cth power of the maximal ideal. Let R be a standard graded K-algebra with maximal homogeneous ideal $\mathfrak{m}$. Let $K(\mathfrak{m}^c, R)$ denote the Koszul complex associated to $\mathfrak{m}^c$, $Z_t(\mathfrak{m}^c, R)$ the module of cycles of homological degree t and $H_t(\mathfrak{m}^c, R)$ the corresponding homology module. In Sect. 2 we study the homological invariants of $Z_t(\mathfrak{m}^c, R)$. Among other facts, we prove, in a surprisingly simple way, a generalization of Green's theorem [17, Theorem 2.2] to arbitrary standard graded algebras; see Corollary 2.5. If R is a polynomial ring (or just a Koszul ring), then it follows that the regularity of $Z_t(\mathfrak{m}^c, R)$ is $\le t(c + 1)$; see Proposition 2.4.

In Sect. 3 we investigate more closely the modules $Z_t(\mathfrak{m}^c, S)$ in the case of a polynomial ring S. Lemma 3.4 describes certain cycles which then are used to prove a vanishing statement in Theorem 3.6. In Sect. 4 we improve the lower bound (1) by one, see Corollary 4.2. Proposition 4.4 states a duality of Avramov–Golod type, which is the algebraic counterpart of Serre duality. The duality is then used to establish Ottaviani and Paoletti's upper bound $\operatorname{index}(S^{(c)}) \le 3c-3$ in arbitrary characteristic (Theorem 4.7). We also show that for $n=3$ one gets $\operatorname{index}(S^{(c)}) = 3c-3$ independently of the characteristic; see Theorem 4.7.

In Sect. 5 we take R to be a quotient of a Koszul algebra D and prove that for every $c \ge \operatorname{slope}_D(R)$ we have $\operatorname{index}(R^{(c)}) \ge \operatorname{index}(D^{(c)})$; see Theorem 5.2. Here $\operatorname{slope}_D(R) = \sup\{t_i^D(R)/i : i \ge 1\}$ where $t_i^D(R)$ is the largest degree of an ith syzygy of R over D. In particular, $\operatorname{slope}_D(R) = 2$ if R is Koszul (Avramov et al. [4]) and, when $D=S$ is a polynomial ring, $\operatorname{slope}_S(R) \le a$ if the defining ideal of R has a Gröbner basis of elements of degree $\le a$. Similar results have been obtained by Park [24] under some restrictive assumptions. In the last section we discuss multigraded variants of the results presented.

2 General bounds

In this section we consider a standard graded K-algebra R with maximal homogeneous ideal $\mathfrak{m}$, which is a quotient of a polynomial ring S, say $R = S/I$ where I is homogeneous (and may contain elements of degree 1). For a finitely generated graded R-module M let $\beta_{i,j}^R(M) = \dim_K \operatorname{Tor}_i^R(M,K)_j$ be the *graded Betti numbers* of M over R. We define the number

$$t_i^R(M) = \max\{j \in \mathbb{Z} : \beta_{i,j}^R(M) \neq 0\},$$

if $\operatorname{Tor}_i^R(M,K) \neq 0$ and $t_i^R(M) = -\infty$ otherwise. The *(relative) regularity of M over R* is given by

$$\operatorname{reg}_R(M) = \sup\{t_i^R(M) - i : i \in \mathbb{N}\}$$

and the *Castelnuovo–Mumford regularity* of M is

$$\operatorname{reg}(M) = \operatorname{reg}_S(M) = \sup\{t_i^S(M) - i : i \in \mathbb{N}\};$$

it has also the cohomological interpretation

$$\operatorname{reg}(M) = \max\{j : H_{\mathfrak{m}}^i(M)_{j-i} \neq 0 \text{ for some } i \in \mathbb{N}\}$$

where $H_{\mathfrak{m}}^i(M)$ denotes the ith local cohomology module of M. One defines the *slope* of M over R by

$$\operatorname{slope}_R(M) = \sup\left\{\frac{t_i^R(M) - t_0^R(M)}{i} : i \in \mathbb{N}, i > 0\right\},$$

and the *Backelin rate* of R by

$$\operatorname{Rate}(R) = \operatorname{slope}_R(\mathfrak{m}) = \sup\left\{\frac{t_i^R(K)-1}{i-1} : i \in \mathbb{N}, i > 1\right\}.$$

The Backelin rate measures the deviation from being Koszul: in general, $\operatorname{Rate}(R) \geq 1$, and R is Koszul if and only if $\operatorname{Rate}(R) = 1$. Finally, the *Green–Lazarsfeld index* of R is given by

$$\operatorname{index}(R) = \sup\{p \in \mathbb{N} : t_i^S(R) \leq i+1 \text{ for every } i \leq p\}.$$

It is the largest non-negative integer p such that R satisfies the property N_p. Note that we have $\operatorname{index}(R) = \infty$ if and only if $\operatorname{reg}(R) \leq 1$, that is, the defining ideal of R has a 2-linear resolution. On the other hand, $\operatorname{index}(R) \geq 1$ if and only if R is defined by quadrics. In general, $\operatorname{reg}(M)$ and $\operatorname{slope}_R(M)$ are finite (see [4]) while $\operatorname{reg}_R(M)$ can be infinite. However, $\operatorname{reg}_R(M)$ is finite if R is Koszul, see Avramov and Eisenbud [5].

Remark 2.1 The invariants $\operatorname{reg}(M)$ and $\operatorname{index}(R)$ are defined in terms of a presentation of R as a quotient of a polynomial ring but do not depend on it. The assertion is a consequence of the following formula which is obtained, for example, from the graded analogue of [3, Theorem 2.2.3]: if $x \in R_1$ and $xM = 0$, then

$$\beta_{i,j}^R(M) = \beta_{i,j}^{R/(x)}(M) + \beta_{i-1,j-1}^{R/(x)}(M).$$

We record basic properties of these invariants. The modules are graded and finitely generated and the homomorphisms are of degree 0.

Lemma 2.2 *Let R be a standard graded K-algebra, N and M_j, $j \in \mathbb{N}$, be R-modules and $i \in \mathbb{N}$.*

(a) *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence. Then

$$\begin{aligned} t_i^R(M_1) &\leq \max\{t_i^R(M_2), t_{i+1}^R(M_3)\}, \\ t_i^R(M_2) &\leq \max\{t_i^R(M_1), t_i^R(M_3)\}, \\ t_i^R(M_3) &\leq \max\{t_i^R(M_2), t_{i-1}^R(M_1)\}. \end{aligned}$$

(b) *Let*

$$\cdots \to M_{k+1} \to M_k \to M_{k-1} \to \cdots \to M_1 \to M_0 \to N \to 0$$

be an exact complex. Then

$$t_i^R(N) \leq \max\{t_{i-j}^R(M_j) : j = 0, \ldots, i\}$$

and

$$\operatorname{reg}_R(N) \leq \sup\{\operatorname{reg}_R(M_j) - j : j \geq 0\}.$$

(c) *If N vanishes in degree $> a$ then $t_i^R(N) \leq t_i^R(K) + a$.*
(d) *Let J be a homogeneous ideal of R. If $\operatorname{reg}_R(R/J) = 0$, then*

$$\operatorname{index}(R/J) \geq \operatorname{index}(R).$$

Proof To prove (a) one just considers the long exact homology sequence for $\operatorname{Tor}^R(\cdot, K)$. For (b) one uses induction on i and applies (a). Part (c) is proved by induction on $a - \min\{j : N_j \neq 0\}$. For (d) one applies (c) to the minimal free resolution of R/J as an R-module. For every i one gets $t_i^S(R/J) \leq \max\{t_{i-j}^S(R(-j)) : j = 0, \dots i\}$. But we have $t_{i-j}^S(R(-j)) = t_{i-j}^S(R) + j$. If $i \leq \operatorname{index}(R)$ then $t_{i-j}^S(R) \leq i - j + 1$. It follows that $t_i^S(R/J) \leq i+1$ for every $i \leq \operatorname{index}(R)$. Hence $\operatorname{index}(R/J) \geq \operatorname{index}(R)$. □

Let M be an R-module and let $K(\mathfrak{m}^c, M)$ be the graded Koszul complex associated to the cth power of the maximal ideal of R. Let $Z_i(\mathfrak{m}^c, M)$, $B_i(\mathfrak{m}^c, M)$, $H_i(\mathfrak{m}^c, M)$ denote the ith cycles, boundaries and homology of $K(\mathfrak{m}^c, M)$, respectively. We have:

Lemma 2.3 *Set $Z_i = Z_i(\mathfrak{m}^c, M)$. For every $a \geq 0$ and $i \geq 0$ we have:*

$$\begin{aligned} t_a^R(Z_{i+1}) \leq \max \Big\{ & t_a^R(M) + (i+1)c, \\ & t_{a+1}^R(Z_i), \\ & t_0^R(Z_i) + c + (a+1)\operatorname{Rate}(R) \Big\}. \end{aligned}$$

Proof Set $B_i = B_i(\mathfrak{m}^c, M)$ and $H_i = H_i(\mathfrak{m}^c, M)$. Recall that $\mathfrak{m}^c H_i = 0$ and hence H_i vanishes in degrees $> t_0(Z_i) + c - 1$. It follows from Lemma 2.2(c) that

$$t_a^R(H_i) \leq t_0^R(Z_i) + c - 1 + t_a^R(K).$$

The short exact sequences

$$0 \to B_i \to Z_i \to H_i \to 0$$

and

$$0 \to Z_{i+1} \to K_{i+1} \to B_i \to 0$$

now imply that

$$\begin{aligned} t_a^R(Z_{i+1}) &\leq \max\{t_a^R(M) + (i+1)c, t_{a+1}^R(B_i)\} \\ &\leq \max\{t_a^R(M) + (i+1)c, t_{a+1}^R(Z_i), t_{a+2}^R(H_i)\} \\ &\leq \max\{t_a^R(M) + (i+1)c, t_{a+1}^R(Z_i), t_0^R(Z_i) + c - 1 + t_{a+2}^R(K)\}. \end{aligned}$$

Since, by the very definition, $t^R_{a+2}(K) \leq 1 + (a+1)\,\mathrm{Rate}(R)$ the desired result follows. □

Lemma 2.3 allows us to bound $t^R_a(Z_i)$ inductively in terms of the various $t^R_j(M)$ and of $\mathrm{Rate}(R)$:

Proposition 2.4 *Set* $Z_i = Z_i(\mathfrak{m}^c, M)$.

(a) *Assume* $c \geq \mathrm{slope}_R(M)$. *Then for all* $a, i \in \mathbb{N}$ *we have*

$$t^R_a(Z_i) \leq t^R_0(M) + ic + \max\{a\,\mathrm{slope}_R(M), (a+i)\,\mathrm{Rate}(R)\}.$$

In particular, $\mathrm{slope}_R(Z_i) \leq \max\{\mathrm{slope}_R M, (1+i)\,\mathrm{Rate}(R)\}$.

(b) *Assume* R *is Koszul, i.e.,* $\mathrm{Rate}(R) = 1$. *Then for all* $a, i \in \mathbb{N}$ *we have*

$$t^R_a(Z_i) \leq a + i(c+1) + \mathrm{reg}_R(M).$$

In particular, $\mathrm{reg}_R(Z_i) \leq i(c+1) + \mathrm{reg}_R(M)$.

Proof To show (a) one uses that $t^R_a(M) \leq t^R_0(M) + a\,\mathrm{slope}_R(M)$ in combination with Lemma 2.3 and induction on i. For (b) one observes that $t^R_a(M) \leq a + \mathrm{reg}_R(M)$ in combination with Lemma 2.3 and induction on i. □

In particular, we have the following corollary that generalizes Green's theorem [17, Theorem 2.2] to arbitrary standard graded K-algebras.

Corollary 2.5 *Set* $Z_i = Z_i(\mathfrak{m}^c, R)$. *Then:*

(a) $t^R_0(Z_i) \leq ic + \min\{i\,\mathrm{Rate}(R), i + \mathrm{reg}(R)\}$.
(b) $H_i(\mathfrak{m}^c, R)_{ic+j} = 0$ *for every* $j \geq \min\{i\,\mathrm{Rate}(R), i + \mathrm{reg}(R)\} + c$.

Proof To prove (a) one notes that setting $M = R$ and $a = 0$ in Proposition 2.4 (a) one has $t^R_0(Z_i) \leq ic + i\,\mathrm{Rate}(R)$. Then one considers R as an S-module and sets $M = R$ and $a = 0$ in Proposition 2.4 (b). One has $t^S_0(Z_i) \leq i(c+1) + \mathrm{reg}(R)$. Since $t^S_0(Z_i) = t^R_0(Z_i)$ we are done. To prove (b) one uses (a) and the fact that $\mathfrak{m}^c H_i(\mathfrak{m}^c, R) = 0$. □

3 Koszul cycles

In this section we concentrate our attention on the Koszul complex $K(\mathfrak{m}^c) = K(\mathfrak{m}^c, S)$ where $S = K[X_1, \dots, X_n]$ is a standard graded polynomial ring over a field K and $\mathfrak{m} = (X_1, \dots, X_n)$ is its maximal homogeneous ideal. The Koszul complex $K(\mathfrak{m}^c)$ is indeed an S-algebra, namely the exterior algebra $S \otimes_K \bigwedge^{\bullet} S_c \cong \bigwedge^{\bullet} F$ where F is the free S-module of rank equal to $\dim_K S_c = \binom{n-1+c}{n-1}$. The differential of $K(\mathfrak{m}^c)$ is denoted by ∂; it is an antiderivation of degree -1. We consider the cycles $Z_t(\mathfrak{m}^c, S)$, simply denoted by $Z_t(\mathfrak{m}^c)$, of the Koszul complex $K(\mathfrak{m}^c)$, and the S-subalgebra $Z(\mathfrak{m}^c) = \bigoplus_t Z_t(\mathfrak{m}^c)$ of $K(\mathfrak{m}^c)$.

For $f_1, \dots, f_t \in S_c$ and $g \in S$ we set

$$g[f_1, \dots, f_t] = g \otimes f_1 \wedge \cdots \wedge f_t \in K_t(\mathfrak{m}^c).$$

The elements $[u_1, \dots, u_t]$ for distinct monomials $u_1, u_2, \dots, u_t$ of degree c (ordered in some way) form a basis of $K_t(\mathfrak{m}^c)$ as an S-module. We call them *monomial free generators* of $K_t(\mathfrak{m}^c)$. The elements $v[u_1, \dots, u_t]$, where $u_1, u_2, \dots, u_t$ are distinct monomials of degree c and v is a monomial of arbitrary degree, form a basis of the K-vector space $K_t(\mathfrak{m}^c)$. They are called *monomials* of $K_t(\mathfrak{m}^c)$. Evidently $K(\mathfrak{m}^c)$ is a $\mathbb{Z}$-graded complex, but it is also $\mathbb{Z}^n$-graded with the following assignment of degrees: $\deg v[u_1, \dots, u_t] = \alpha$ where $vu_1 \cdots u_t = X^\alpha$.

Every element $z \in K_t(\mathfrak{m}^c)$ can be written uniquely as a linear combination

$$z = \sum f_i[u_{i1}, \dots, u_{it}]$$

of monomial free generators $[u_{i1}, \dots, u_{it}]$ with coefficients $f_i \in S$. We call f_i the *coefficient* of $[u_{i1}, \dots, u_{it}]$ in z. Note that z is $\mathbb{Z}$-homogenous of degree $ct + j$ if every f_i is homogeneous of degree j. In this case z has *coefficients of degree* j. Note also that z is homogeneous of degree $\alpha \in \mathbb{Z}^n$ in the $\mathbb{Z}^n$-grading if for every i one has $f_i = \lambda_i v_i$ such that $\lambda_i \in K$ and v_i is a monomial with $v_i u_{i1} \cdots u_{it} = X^\alpha$. Given $z \in K(\mathfrak{m}^c)$ and a monomial $v[u_1, \dots, u_t]$ we say that $v[u_1, \dots, u_t]$ *appears in* z if it has a non-zero coefficient in the representation of z as K-linear combination of monomials of $K(\mathfrak{m}^c)$. An immediate consequence of Proposition 2.4 is:

Lemma 3.1 *We have* $\operatorname{reg}(Z_t(\mathfrak{m}^c)) \le t(c+1)$. *In particular,* $Z_t(\mathfrak{m}^c)$ *is generated by elements of degree* $\le t(c+1)$.

Remark 3.2 It is easy to see and well known that $Z_1(\mathfrak{m}^c)$ is generated by the elements $X_i[X_j b] - X_j[X_i b]$ where b is a monomial of degree $c-1$.

We write $Z_1(\mathfrak{m}^c)^t$ for $\bigwedge^t Z_1(\mathfrak{m}^c) \subset Z_t(\mathfrak{m}^c)$, and similarly for other products.

Theorem 3.3 *For every t the module $Z_t(\mathfrak{m}^c)/Z_1(\mathfrak{m}^c)^t$ is generated in degree $< t(c+1)$.*

Proof The assertion is proved by induction on t. For $t = 1$ there is nothing to do. By induction it is enough to verify that $Z_t(\mathfrak{m}^c)/Z_1(\mathfrak{m}^c)Z_{t-1}(\mathfrak{m}^c)$ is generated in degree $< t(c+1)$. Since $Z_t(\mathfrak{m}^c)$ is $\mathbb{Z}^n$-graded and generated in degree $\le t(c+1)$, it suffices to show that every $\mathbb{Z}^n$-graded element $f \in Z_t(\mathfrak{m}^c)$ of total degree $t(c+1)$ can be written modulo $Z_1(\mathfrak{m}^c)Z_{t-1}(\mathfrak{m}^c)$ as a multiple of an element in $Z_t(\mathfrak{m}^c)$ of total degree $< t(c+1)$. Let $\alpha \in \mathbb{Z}^n$ be the $\mathbb{Z}^n$-degree of f. Permuting the coordinates if necessary, we may assume $\alpha_n > 0$.

Let $u \in S$ be a monomial of degree c with $X_n \mid u$. We write $f = a + b[u]$ with $a \in K_t(\mathfrak{m}^c)$ and $b \in K_{t-1}(\mathfrak{m}^c)$ such that a, b involve only free generators $[u_1, \dots, u_s]$ $(s = t, t-1)$ with $u_i \ne u$ for all i. Since

$$0 = \partial(f) = \partial(a) + \partial(b)[u] \pm bu$$

it follows that $\partial(b) = 0$, i.e., $b \in Z_{t-1}(\mathfrak{m}^c)$. Note that b has coefficients of degree t. Since $Z_{t-1}(\mathfrak{m}^c)$ is generated by elements with coefficients of degree $\le t-1$ we may write

$$b = \sum_{j=1}^{s} \lambda_j v_j z_j \tag{2}$$

where $\lambda_j \in K$, $z_j \in Z_{t-1}(\mathfrak{m}^c)$ and the v_j are monomials of positive degree.

Let $\lambda_j v_j z_j$ be a summand in (2). If X_n does not divide v_j, then choose $i < n$ such that $X_i \mid v_j$. We set $z' = X_i[u] - X_n[u'] \in Z_1(\mathfrak{m}^c)$ where $u' = uX_i/X_n$, and subtract from f the element

$$\lambda_j \frac{v_j}{X_i} z_j z' \in Z_{t-1}(\mathfrak{m}^c) Z_1(\mathfrak{m}^c).$$

Repeating this procedure for each $\lambda_j v_j z_j$ in (2) such that X_n does not divide v_j we obtain a cycle $f_1 \in Z_t(\mathfrak{m}^c)$ of degree α such that

(i) $f = f_1 \mod Z_1(\mathfrak{m}^c) Z_{t-1}(\mathfrak{m}^c)$;
(ii) if a monomial $v[u_1, \dots, u_t]$ appears in f_1 and $u \in \{u_1, \dots, u_t\}$, then $X_n \mid v$.

We repeat the described procedure for each monomial u of degree c with $X_n | u$. We end up with an element $f_2 \in Z_t(\mathfrak{m}^c)$ of degree α such that

(iii) $f = f_2 \mod Z_1(\mathfrak{m}^c) Z_{t-1}(\mathfrak{m}^c)$;
(iv) if a monomial $v[u_1, \dots, u_t]$ appears in f_2 and $X_n \mid u_1 \cdots u_t$, then $X_n \mid v$.

Note that if $v[u_1, \dots, u_t]$ appears in f_2 and $X_n \nmid u_1 \cdots u_t$, then $X_n \mid v$ by degree reasons. Hence for every monomial $v[u_1, \dots, u_t]$ appearing in f_2 we have $X_n \mid v$. Therefore $f_2 = X_n g$, and $g \in Z_t(\mathfrak{m}^c)$ has degree $< t(c+1)$. This completes the proof. □

Next we describe some cycles which are needed in the following. For $t \in \mathbb{N}$, $t \ge 1$ let $\mathcal{S}_t$ be the group of permutations of $\{1, \dots, t\}$.

Lemma 3.4 *Let s, t be integers such that $1 \le s \le c$ and $t > 0$. Let $a_1, a_2 \dots, a_{t+1} \in S$ be monomials of degree s and $b_1, b_2 \dots, b_t \in S$ monomials of degree $c - s$. Then*

$$\sum_{\sigma \in \mathcal{S}_{t+1}} (-1)^{\sigma} a_{\sigma(t+1)} [b_1 a_{\sigma(1)}, b_2 a_{\sigma(2)}, \dots, b_t a_{\sigma(t)}] \tag{3}$$

belongs to $Z_t(\mathfrak{m}^c)$.

Proof We apply the differential of $K(\mathfrak{m}^c)$ to (3) and observe that for distinct integers $j_1, j_2, \dots, j_{i-1}, j_{i+1}, \dots, j_t$ in the range of 1 to $t+1$ the free generator

$$[b_1 a_{j_1}, b_2 a_{j_2}, \dots, b_{i-1} a_{j_{i-1}}, b_{i+1} a_{j_{i+1}} \dots, b_t a_{i_t}]$$

appears twice in the image. The coefficients differ just by -1 because the corresponding permutations differ by a transposition. Thus the element in (3) is indeed a cycle. □

Remark 3.5 (a) Of course, it may happen that a cycle described in Lemma 3.4 is identically 0. But for $t = 1$ and $s = 1$ these cycles take the form

$$X_i[bX_j] - X_j[bX_i],$$

and, as said already in Remark 3.2, they generate $Z_1(\mathfrak{m}^c)$. For $s = c$ the cycles in Lemma 3.4 are the boundaries of $K_t(\mathfrak{m}^c)$ (multiplied by $t!$). Hence for $c = 1$ the cycles in 3.4 generate the algebra $Z(\mathfrak{m})$. So there is some evidence that the cycles in Lemma 3.4 might generate $Z(\mathfrak{m}^c)$ in general.

(b) For $n = 3, c = 2, t = 2$ and $s = 1$ with $a_i = X_i$ for $i = 1, 2, 3$ and $b_i = X_i$ for $i = 1, 2$ the cycle in (3) is

$$\begin{array}{llll} + & X_3[X_1^2, X_2^2] & - \; X_2[X_1^2, X_2X_3] & - \overbrace{X_3[X_1X_2, X_1X_2]}^{0} \\ + & X_1[X_1X_2, X_2X_3] & + \; X_2[X_1X_3, X_1X_2] & - \; X_1[X_1X_3, X_2^2], \end{array}$$

a non-zero element in $Z_2(\mathfrak{m}^2)$.

Let $B_i(\mathfrak{m}^c) \subset Z_i(\mathfrak{m}^c)$ denote the S-module of boundaries in $K_i(\mathfrak{m}^c)$.

Theorem 3.6 *We have*

$$(c+1)!\, \mathfrak{m}^{c-1} Z_1(\mathfrak{m}^c)^c \subset B_c(\mathfrak{m}^c).$$

Proof For a monomial $b \in S$ of degree $c - 1$ and variables X_i, X_j we set

$$z_b(X_i, X_j) = X_i[bX_j] - X_j[bX_i].$$

As observed in Remark 3.2, the elements $z_b(X_i, X_j)$ generate $Z_1(\mathfrak{m}^c)$. Let $a, b \in S$ be monomials of degree $c - 1$. We note that

$$az_b(X_i, X_j) + bz_a(X_i, X_j) = \partial\left([aX_i, bX_j] + [bX_i, aX_j]\right) \in B_1(\mathfrak{m}^c),$$

that is,

$$az_b(X_i, X_j) = -bz_a(X_i, X_j) \mod B_1(\mathfrak{m}^c). \tag{4}$$

Let $b_1, \dots, b_{c+1} \in S$ be monomials of degree $c - 1$, and let $X_{ij} \in \{X_1, \dots, X_n\}$ for $i = 1, \dots, c$ and $j = 0, 1$ be variables. By construction, the elements

$$f = b_{c+1} \prod_{i=1}^{c} z_{b_i}(X_{i0}, X_{i1}) \in Z_c(\mathfrak{m}^c)$$

generate $\mathfrak{m}^{c-1} Z_1(\mathfrak{m}^c)^c$. We have to show that $(c+1)!\, f \in B_c(\mathfrak{m}^c)$.

Let $\sigma \in \mathcal{S}_{c+1}$ be an arbitrary permutation. From Eq. (4) and from the inclusion $B_1(\mathfrak{m}^c)Z_{c-1}(\mathfrak{m}^c) \subset B_c(\mathfrak{m}^c)$ it follows that

$$f = (-1)^\sigma b_{\sigma(c+1)} \prod_{i=1}^{c} z_{b_{\sigma(i)}}(X_{i0}, X_{i1}) \mod B_c(\mathfrak{m}^c).$$

Hence

$$(c+1)!f = \sum_{\sigma \in \mathcal{S}_{c+1}} (-1)^\sigma b_{\sigma(c+1)} \prod_{i=1}^{c} z_{b_{\sigma(i)}}(X_{i0}, X_{i1}) \mod B_c(\mathfrak{m}^c). \tag{5}$$

In the right-hand side of (5) we replace $z_{b_{\sigma(i)}}(X_{i0}, X_{i1})$ with $X_{i0}[b_{\sigma(i)}X_{i1}] - X_{i1}[b_{\sigma(i)} X_{i0}]$, then expand the product and collect the multiples of $X_{1j_1} \cdots X_{cj_c}$ for $j = (j_1, \ldots, j_c) \in \{0, 1\}^c$. We can rewrite Eq. (5) as

$$(c+1)!f = \sum_{j \in \{0,1\}^c} (-1)^{j_1+\cdots+j_c} X_{1j_1} \cdots X_{cj_c} W_j \mod B_c(\mathfrak{m}^c), \tag{6}$$

where

$$W_j = \sum_{\sigma \in \mathcal{S}_{c+1}} (-1)^\sigma b_{\sigma(c+1)}[X_{1i_1} b_{\sigma(1)}, \ldots, X_{ci_c} b_{\sigma(c)}]$$

with $i_k = 1 - j_k$ for $k = 1, \ldots, c$. Lemma 3.4 yields $W_j \in Z_c(\mathfrak{m}^c)$. Since $\mathfrak{m}^c Z_c(\mathfrak{m}^c) \subset B_c(\mathfrak{m}^c)$ we get

$$X_{1j_1} \cdots X_{cj_c} W_j = 0 \mod B_c(\mathfrak{m}^c).$$

Thus Eq. (6) implies $(c+1)!f \in B_c(\mathfrak{m}^c)$ as desired. □

As a consequence we obtain:

Corollary 3.7 *The homology $H_t(\mathfrak{m}^c)_{tc+j}$ vanishes if $j \geq t + c$. If $t \geq c$ and the characteristic of K is either 0 or $> c+1$, then $H_t(\mathfrak{m}^c)_{tc+j} = 0$ for $j \geq t + c - 1$.*

Proof The first statement is a special case of Corollary 2.5. For the second, set $j = t + c - 1$. We have to prove that $H_t(\mathfrak{m}^c)_{tc+j} = 0$. Theorem 3.3 implies that $Z_t(\mathfrak{m}^c)$ is generated by some elements z_i of degree $< t(c+1)$ and by some elements w_i of $Z_1(\mathfrak{m}^c)^t$ of degree $t(c+1)$. Hence an element $f \in Z_t(\mathfrak{m}^c)_{tc+j}$ can be written as $f = \sum a_i z_i + \sum b_i w_i$ with $a_i \in \mathfrak{m}^c$ and $b_i \in \mathfrak{m}^{c-1}$ by degree reasons. Now $\sum a_i z_i \in \mathfrak{m}^c Z_t(\mathfrak{m}^c) \subset B_t(\mathfrak{m}^c)$. In view of Theorem 3.6 we furthermore have

$$\sum b_i w_i \in \mathfrak{m}^{c-1} Z_1(\mathfrak{m}^c)^t = \mathfrak{m}^{c-1} Z_1(\mathfrak{m}^c)^c Z_1(\mathfrak{m}^c)^{t-c} \subset B_c(\mathfrak{m}^c) Z_1(\mathfrak{m}^c)^{t-c} \subset B_t(\mathfrak{m}^c).$$

Summing up, $f \in B_t(\mathfrak{m}^c)$ and hence $H_t(\mathfrak{m}^c)_{tc+j} = 0$. □

Remark 3.8 The coefficient $(c+1)!$ in Theorem 3.6 and the assumption on the characteristic in Corollary 3.7 are necessary. For $n = 7, c = 2$ and $\operatorname{char} K = 3$ we have checked that $\mathfrak{m}Z_1(\mathfrak{m}^2)^2 \not\subset B_2(\mathfrak{m}^2)$ and that $\dim H_2(\mathfrak{m}^2)_7 = 1$. More precisely, $H_2(\mathfrak{m}^2)$ has dimension 1 in the multidegree $(1, 1, 1, 1, 1, 1, 1)$ if $\operatorname{char} K = 3$.

Another consequence of Theorem 3.6 is the following:

Corollary 3.9 *Assume* $\operatorname{char} K$ *is* 0 *or* $> c+1$. *Then* $\operatorname{reg} Z_{t+1}(\mathfrak{m}^c) \leq (t+1)(c+1)-1$ *for every* $t \geq c$. *In particular,* $Z_1(\mathfrak{m}^c)^{c+1} \subset \mathfrak{m}Z_{c+1}(\mathfrak{m}^c)$.

Proof To prove the first assertion, let us denote by Z_t the module $Z_t(\mathfrak{m}^c)$ and similarly for B_t, H_t and K_t. The short exact sequences

$$0 \to B_t \to Z_t \to H_t \to 0 \quad \text{and} \quad 0 \to Z_{t+1} \to K_{t+1} \to B_t \to 0$$

imply that $\operatorname{reg}(Z_{t+1}) \leq \max\{\operatorname{reg}(Z_t) + 1, \operatorname{reg}(H_t) + 2\}$. Using Lemma 3.1 and Corollary 3.7 one obtains $\operatorname{reg}(Z_{t+1}) \leq (t+1)(c+1) - 1$ for every $t \geq c$. The second assertion follows immediately from the first. $\square$

4 The Green–Lazarsfeld index of Veronese subrings of polynomial rings

Again we consider a standard graded K-algebra R of the form $R = S/I$ where K is a field, $S = K[X_1, \dots, X_n]$ is a polynomial ring over K graded by $\deg(X_i) = 1$ and $I \subset S$ is a graded ideal.

Given $c \in \mathbb{N}$, $c \geq 1$ and $0 \leq k < c$, we set

$$V_R(c, k) = \bigoplus_{i \in \mathbb{N}} R_{k+ic}.$$

Observe that $R^{(c)} = V_R(c, 0)$ is the usual cth Veronese subring of R, and that the $V_R(c, k)$ are $R^{(c)}$-modules known as the *Veronese modules* of R. For a finitely generated graded R-module M we similarly define

$$M^{(c)} = \bigoplus_{i \in \mathbb{Z}} M_{ic}.$$

We consider $R^{(c)}$ as a standard graded K-algebra with homogeneous component of degree i equal to R_{ic}, and $M^{(c)}$ as a graded $R^{(c)}$-module with homogeneous components M_{ic}. The grading of the $R^{(c)}$-module $V_R(c, k)$ is given by $V_R(c, k)_i = R_{k+ic}$. In particular, the latter modules are all generated in degree 0 with respect to this grading.

Let $T = \operatorname{Sym}(R_c)$ be the symmetric algebra on vector space R_c, that is,

$$T = K[Y_u : u \in B_c]$$

where B_c is any K-basis of R_c. When $R = S$ the basis B_c can be taken as the set of monomials of degree c. The canonical map $T \to R^{(c)}$ is surjective, and the modules $V_R(c, k)$ are also finitely generated graded T-modules (generated in degree 0).

With the notation of the preceding sections we have:

Lemma 4.1 *For $i \in \mathbb{N}$, $j \in \mathbb{Z}$ and $0 \le k < c$ we have*

$$\beta^T_{i,j}(V_R(c,k)) = \dim_K H_i(\mathfrak{m}^c, R)_{jc+k}.$$

Proof Let $K(T_1)$ be the Koszul complex (of T-modules) associated to the elements Y_u with $u \in B_c$. We observe that

$$\beta^T_{i,j}(V_R(c,k)) = \operatorname{Tor}^T_i(K, V_R(c,k))_j = \dim_K H_i(K(T_1)) \otimes_T V_R(c,k))_j.$$

But the last homology is $H_i(\mathfrak{m}^c)_{jc+k}$, the ith homology of the complex $K(\mathfrak{m}^c)_{jc+k}$. □

Lemma 4.1 and Corollary 3.7 imply:

Corollary 4.2 *For all integers $i \ge 0$ and $k = 0, \dots, c-1$ we have*

$$t^T_i(V_S(c,k)) < 1 + i + \frac{i-k}{c}.$$

If K has characteristic 0 or $> c+1$ and $i \ge c$, then

$$t^T_i(V_S(c,k)) < 1 + i + \frac{i-k-1}{c}.$$

Remark 4.3 Let $S = K[X_1, \dots, X_n]$. Andersen [1] proved that the graded Betti numbers $\beta^T_{ij}(S^{(2)})$ do not depend on the characteristic of K if $i \le 4$ or if $i = 5$ and $n \le 6$. She also proved that, for $n \ge 7$, one has $\beta^T_{5,7}(S^{(2)}) \ne 0$ in characteristic 5 while $\beta^T_{5,7}(S^{(2)}) = 0$ in characteristic 0. Thus, for $n \ge 7$ one has

$$\operatorname{index}(S^{(2)}) = \begin{cases} 5, & \operatorname{char} K = 0, \\ 4, & \operatorname{char} K = 5. \end{cases}$$

Also note that already $\beta_{2,3}(V_S(2,1))$ depends on the characteristic if $n \ge 7$, as follows from the data in Remark 3.8.

We now record a duality on $H(\mathfrak{m}^c)$. It can be seen as an Avramov–Golod type duality (see [8, Theorem 3.4.5]) or as an algebraic version of Serre duality.

Proposition 4.4 *Let $N = \binom{n+c-1}{c}$. Then*

$$\dim_K H_i(\mathfrak{m}^c)_j = \dim_K H_{N-n-i}(\mathfrak{m}^c)_{Nc-n-j}, \qquad i, j \in \mathbb{Z},\ i, j \ge 0.$$

Proof For this proof (and only here) we consider the grading on the polynomial ring $T = K[Y_u : u \in S \text{ monomial}, \deg u = c]$ in which Y_u has degree c. The polynomial ring S in its standard grading is a finitely generated graded T-module as usual.

Note that the canonical module of S is $\omega_S = S(-n)$, and that the canonical module of T is $\omega_T = T(-Nc)$. Recall that

$$\operatorname{Ext}^j_T(S, T(-Nc)) = \begin{cases} 0 & \text{if } j < N-n, \\ S(-n) & \text{if } j = N-n. \end{cases}$$

(See, e.g., [8, Theorem 3.3.7 and Theorem 3.3.10].) Let F be a minimal graded free T-resolution of S. Computing $\operatorname{Ext}^i_T(S, T(-Nc))$ via $\operatorname{Hom}_T(F, T(-Nc))$, the minimal graded free T-resolution of $S(-n)$, we see immediately that $\beta^T_{i,j}(S) = \beta^T_{N-n-i,Nc-j}(S(-n))$. Then

$$\begin{aligned} \dim_K H_i(\mathfrak{m}^c)_j &= \beta^T_{i,j}(S) \\ &= \beta^T_{N-n-i,Nc-n-j}(S) = \dim_K H_{N-n-i}(\mathfrak{m}^c)_{Nc-n-j}. \end{aligned}$$

□

Example 4.5 Let char $K = 0$. Computer algebra systems as CoCoA [10], Macaulay 2 [16] or Singular [20] can easily compute the following diagram for $\dim_K H(\mathfrak{m}^3)$ in the case $n = 3$:

	0	1	2	3	4	5	6	7	
0	1	–	–	–	–	–	–	–	←
1	3	15	21	–	–	–	–	–	
2	6	49	105	147	105	21	–	–	
3	**0**	27	105	189	189	105	27	–	←
4	–	**0**	21	105	147	105	49	6	
5	–	–	**0**	0	–	21	15	3	
6	–	–	–	**0**	0	–	–	1	←

The (i, j)-entry of the table is $\dim_K H_i(\mathfrak{m}^c)_{ic+j}$ and "–" indicates that entry is 0. By selecting the rows whose indices are multiples of $c = 3$ (those marked by the arrows in the diagram) one gets the Betti diagram of $S^{(3)}$. Green's theorem [17, Theorem 2.2] implies the vanishing in the positions of the boldface zeros and below. Our result implies the vanishing in the positions of the non-bold zeros and below. (Also see Weyman [28, Example 7.2.7] for the case $n = c = 3$.)

Using the duality we prove the upper bound for $\operatorname{index}(S^{(c)})$ due to Ottaviani and Paoletti [23] (in arbitrary characteristic). To this end we need a variation of [25, Corollary 2.10].

Proposition 4.6 *Let $(e_i : i = 1, \ldots, m)$ be a basis of the vector space $\bigwedge^t S_c$ (thus $m = \binom{N}{t}$ with $N = \binom{n-1+c}{n-1}$). Let*

$$z = \sum_{i=1}^{m} f_i e_i$$

be a non-zero element in $Z_t(\mathfrak{m}^c)$. *Then the* K*-vector space generated by the coefficients* f_i *of* z *has dimension* $\geq t+1$.

Proof Since the K-vector space dimension of the space of coefficients does not depend on the basis, it is enough to prove the assertion for the monomial basis (e_i). We use induction on t.

The case $t = 0$ is obvious. So assume $t > 0$. Fix a term order, for example the lexicographic term order, on S. Let $C(z)$ denote the vector space generated by the coefficients of z. As already discussed in the proof of Theorem 3.3, for every monomial u of degree c we may write $z = a + b[u]$ with $b \in Z_{t-1}(\mathfrak{m}^c)$. Choose u to be the largest monomial (with respect to the term order) such that the corresponding b is non-zero. By induction $\dim_K C(b) \geq t$ and $C(b) \subset C(z)$.

If $C(b) \neq C(z)$ then clearly $\dim_K C(z) \geq t+1$. If instead $C(b) = C(z)$, then $C(a) \subset C(b)$. Let v be the largest monomial appearing in the elements of $C(b)$. The inclusion $C(a) \subset C(b)$ implies that every monomial appearing in the elements of $C(a)$ is $\leq v$. But $\partial(a) \pm bu = 0$ and hence $C(\partial(a)) = C(bu) = uC(b)$. The monomial vu appears in $C(bu)$. Every monomial in $C(\partial(a))$ is of the form wu_1 where w is a monomial appearing in $C(a)$ and u_1 is a monomial of degree c which is an "exterior" factor of some free generator appearing in z. By construction $w \leq v$ and $u_1 < u$. It follows that $wu_1 \neq vu$, a contradiction with $C(\partial(a)) = uC(b)$. □

Theorem 4.7 *For* $n \geq 3$ *and* $c \geq 3$ *one has* $\operatorname{index}(S^{(c)}) \leq 3c-3$, *and equality holds for* $n = 3$.

Proof We first consider the case $n = 3$. By an inspection of the Hilbert function of (the Cohen–Macaulay ring) $S^{(c)}$ one sees immediately that $\operatorname{reg} S^{(c)} \leq 2$, that is, $t_i^T(S^{(c)}) \leq i+2$ for every $i \geq 0$. From Theorem 4.1 and Proposition 4.4 we have

$$\beta_{i,j}^T(S^{(c)}) = \dim_K H_i(\mathfrak{m}^c)_{jc} = \dim_K H_{N-3-i}(\mathfrak{m}^c)_{Nc-3-jc}.$$

Therefore $t_i^T(S^{(c)}) \leq i+1$ if and only if

$$H_{N-3-i}(\mathfrak{m}^c)_{(N-3-i)c+c-3} = 0,$$

and, since the boundaries have coefficients of degree $\geq c$, this is equivalent to

$$Z_{N-3-i}(\mathfrak{m}^c)_{(N-3-i)c+c-3} = 0.$$

So we have to analyze the cycles in $Z_{N-3-i}(\mathfrak{m}^c)$ with coefficients of degree $c-3$.

It follows from Proposition 4.6 that

$$N-3-i+1 \leq \dim_K S_{c-3} = \binom{c-1}{2}$$

if there exists a non-zero cycle $z \in Z_{N-3-i}(\mathfrak{m}^c)$ with coefficients of degree $c-3$. Thus there are no cycles in that degree if $N-3-i \geq \binom{c-1}{2}$. Hence

$$t_i^T(S^{(c)}) \leq i+1 \quad \text{for } 0 \leq i \leq 3c-3,$$

that is, $\operatorname{index}(S^{(c)}) \geq 3c-3$. It remains to show that $S^{(c)}$ does not satisfy the property N_{3c-2}. We have to find a non-zero cycle in $Z_j(\mathfrak{m}^c)$ with coefficients of degree $c-3$ where $j = N-3-i$. Note that $j = \binom{c-1}{2} - 1$ and so $j+1 = \dim S_{c-3}$. Take the monomials $u'_1, \ldots, u'_{j+1}$ of degree $c-3$ and set $u_k = u'_k X_1 X_2 X_3$ for $k = 1, \ldots, j+1$. Then

$$w = \partial([u_1, \ldots, u_{j+1}]) \in Z_j(\mathfrak{m}^c)$$

is non-zero boundary with coefficients of degree c. But we can divide each coefficient of w by $X_1X_2X_3$ to obtain a non-zero cycle $z \in Z_j(\mathfrak{m}^c)$ with coefficients of degree $c-3$. It follows that $S^{(c)}$ does not satisfy the property N_{3c-2}. This concludes the proof for $n = 3$.

Now let $n > 3$. Recall that $H_i(\mathfrak{m}^c)$ is multigraded. For a vector $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$ with $a_i = 0$ for $i > 3$ let $b = (a_1, a_2, a_3)$. We may identify

$$H_i(\mathfrak{m}^c)_a = H_i(\mathfrak{m}_3^c)_b$$

where $H_i(\mathfrak{m}_3^c)$ is the corresponding Koszul homology in 3 variables. Since for $n = 3$ the cth Veronese does not satisfy N_{3c-2} it follows that the same is true for all $n \geq 3$, proving that $\operatorname{index}(S^{(c)}) \leq 3c-3$. □

Remark 4.8 It is well-known that $\operatorname{reg} S^{(c)} \leq n-1$ in general, i.e., $t_i(S^{(c)}) \leq i+n-1$. Analogously to the proof of Theorem 4.7 one can determine the largest i such that $t_i(S^{(c)}) < i+n-1$. Again this is determined by elements in $Z_i(\mathfrak{m}^c)$ with coefficients of degree $c-n$. It remains to count the monomials of S in that degree. For example, for $c \geq n = 4$ one obtains $t_i(S^{(c)}) < i+3$ if and only if $i \leq 2c^2-2$.

5 The Green–Lazarsfeld index of Veronese subrings of standard graded rings

Let D be a Koszul K-algebra and I be a homogeneous ideal of D. Set $R = D/I$. We want to relate the Green–Lazarsfeld index of $R^{(c)}$ to that of $D^{(c)}$. For a polynomial ring S Aramova et al. proved in [2, Theorem 2.1] that the Veronese modules $V_S(c,k)$ have a linear resolution over the Veronese ring $S^{(c)}$. We show that this property holds for Koszul algebras in general.

Lemma 5.1 *Assume D is a Koszul algebra and, for a given c, let $T = \operatorname{Sym}(D_c)$ be the symmetric algebra of D_c.*

(a) *The Veronese module $V_D(c,k)$ has a linear resolution as a $D^{(c)}$-module.*
(b) *For every $k = 0, \ldots, c-1$ we have*

$$t_i^T(V_D(c,k)) \leq t_i^T(D^{(c)}).$$

Proof (a) Let $\mathfrak{m}$ denote the homogeneous maximal ideal of D, and set $A = D^{(c)}$ and $V_k = V_D(c,k)$. We prove by induction on i that $t_i^A(V_k) \leq i$ for all i and k. For $i = 0$ the assertion is obvious and it is so for $k = 0$ and $i \geq 0$, too. Assume that $i > 0$.

The ideal $\mathfrak{m}^k$ is generated in degree k and, since D is Koszul, it has a linear resolution over D. Shifting that resolution by k, we obtain a complex

$$\cdots \to F_i \to F_{i-1} \to \cdots \to F_1 \to F_0 \to 0$$

resolving $\mathfrak{m}^k(k)$ and such that $F_i = D(-i)^{\beta_i}$. Applying the exact functor $(\ )^{(c)}$ to it we get an exact complex of A-modules

$$\cdots \to F_i^{(c)} \to F_{i-1}^{(c)} \to \cdots \to F_1^{(c)} \to A^{\beta_0} \to V_k \to 0.$$

Note that $D(-j)^{(c)} = V_e(-\lceil j/c \rceil)$ where $e = c\lceil j/c \rceil - j$. Therefore $F_j^{(c)} = V_{e_j}(-\lceil j/c \rceil)^{\beta_j}$ where $e_j = c\lceil j/c \rceil - j$. Applying Lemma 2.2 (b) to the complex above we have

$$t_i^A(V_k) \le \max\{t_{i-j}^A(V_{e_j}) + \lceil j/c \rceil : j = 0, \ldots, i\}.$$

Obviously $t_i^A(V_{e_0}) = t_i^A(A) = -\infty$ and, by induction, $t_{i-j}^A(V_{e_j}) \le i - j$ for $j = 1, \ldots, i$. Therefore

$$t_i^A(V_k) \le \max\{i - j + \lceil j/c \rceil : j = 1, \ldots, i\} = i$$

and this concludes the proof of (a). For (b) we may apply Lemma 2.2(b) to the minimal A-free resolution of V_k and to get the desired inequality. □

Now we prove the main result of this section.

Theorem 5.2 *Assume D is a Koszul algebra and $R = D/I$. Let $c \ge \operatorname{slope}_D(R)$. Then* $\operatorname{index}(R^{(c)}) \ge \operatorname{index}(D^{(c)})$.

Proof To prove the statement we set $A = D^{(c)}$ and $B = R^{(c)}$. By virtue of Lemma 2.2 (d) it is enough to show that $\operatorname{reg}_A(B) = 0$. Let

$$\cdots \to F_p \to \cdots \to F_1 \to F_0 \to R \to 0$$

be the minimal graded free resolution of R over D. Since $(\ )^{(c)}$ is an exact functor, we obtain an exact complex of finitely generated graded A-modules

$$\cdots \to F_p^{(c)} \to \cdots \to F_1^{(c)} \to F_0^{(c)} \to B \to 0. \tag{7}$$

Hence by virtue of Lemma 2.2 (b) we have

$$\operatorname{reg}_A(B) \le \max\{\operatorname{reg}_A(F_i^{(c)}) - i : i \ge 0\}.$$

Note that $D(-k)^{(c)} = V_D(c, e)(-\lceil k/c \rceil)$ where $e = c\lceil k/c \rceil - k$. Hence, by virtue of Lemma 5.1, $\operatorname{reg}_A(D(-k)^{(c)}) = \lceil k/c \rceil$. Therefore, since $F_i = \bigoplus_{k \in \mathbb{Z}} S(-k)^{\beta_{ik}^D(R)}$ we get $\operatorname{reg}_A(F_i^{(c)}) = \lceil t_i^D(R)/c \rceil$. Summing up,

$$\operatorname{reg}_A(B) \leq \max\{\lceil t_i^D(R)/c \rceil - i : i \geq 0\}.$$

If $c \geq \operatorname{slope}_D(R)$, then $t_i^D(R) \leq ci$ and hence $\operatorname{reg}_A(B) = 0$. This concludes the proof. □

As a corollary we have:

Corollary 5.3 *Let S be a polynomial ring and $R = S/I$ a standard graded algebra quotient of it and let $c \geq \operatorname{slope}_S(R)$. Then* $\operatorname{index}(R^{(c)}) \geq \operatorname{index}(S^{(c)})$. *In particular,*

(a) $\operatorname{index}(R^{(c)}) \geq c$. *Furthermore, if K has characteristic* 0 *or* $> c+1$, *then we have* $\operatorname{index}(R^{(c)}) \geq c+1$.
(b) *If* $\dim R_1 = 3$, *then* $\operatorname{index}(R^{(c)}) \geq 3c-3$.

Note that $\operatorname{slope}_S(R) = 2$ if R is Koszul; see [4]. Furthermore $\operatorname{slope}_S(R) \leq a$ if R is defined by either a complete intersection of elements of degree $\leq a$ or by a Gröbner basis of elements of degree $\leq a$.

Remark 5.4 Sometimes the bound in Theorem 5.2 can be improved by a more careful argumentation. Let $R = S/I$ and let T be the symmetric algebra of S_c. For instance, using the argument of the proof of Theorem 5.2 one shows that

$$t_i^T(R^{(c)}) \leq \max\{t_{i-j}^T(S^{(c)}) + \lceil t_j^S(R)/c \rceil\} : j = 0, 1, \ldots, i\}.$$

It follows that $\operatorname{index}(R^{(c)}) \geq p$ if $c \geq p$ and $\operatorname{index}(R) \geq p$, a result proved by Rubei in [26]. It is very easy to show that $R^{(c)}$ is defined by quadrics, i.e. $\operatorname{index}(R^{(c)}) \geq 1$ provided $c \geq t_1^S(R)/2$. Similarly, one can prove that $\operatorname{index}(R^{(c)}) \geq p$ if

$$c \geq \max\{p,\ \ \max\{t_j^S(R)/j : j = 1, \ldots, p-1\},\ \ t_p^S(R)/(p+1)\}.$$

Remark 5.5 (a) Let us say that a positively graded K-algebra is *almost standard* if R is Noetherian and a finitely generated module over $K[R_1]$. If K is infinite, then this property is equivalent to the existence of a Noether normalization generated by elements of degree 1. Galliego and Purnaprajna [15, Theorem 1.3] proved a general result on the property N_p of Veronese subalgebras of almost standard K-algebras R of depth ≥ 2 over fields of characteristic 0: $R^{(c)}$ has N_p for all $c \geq \max\{\operatorname{reg}(R) + p - 1, \operatorname{reg}(R), p\}$. If $\operatorname{reg}(R) \geq 1$ and $p \geq 1$, this amounts to the property $N_{c-\operatorname{reg}(R)+1}$ of $R^{(c)}$ for all $c \geq \operatorname{reg}(R)$. Thus Theorem 5.2 gives a stronger result for standard graded algebras.
(b) Eisenbud et al. [12] proved that the Veronese subalgebras $R^{(c)}$ of standard graded K-algebra R are defined by an ideals with Gröbner bases of degree 2 for all $c \geq (\operatorname{reg}(R) + 1)/2$. It follows that these algebras are Koszul.
(c) If R is almost standard and Cohen–Macaulay, then $R^{(c)}$ is defined by an ideal with a Gröbner bases of degree 2 for every $c \geq \operatorname{reg}(R)$. See Bruns et al. [7, Theorem 1.4.1] or Bruns and Gubeladze [6, Theorem 7.41].

6 The multigraded case

The results presented in this paper have natural extensions to the multigraded case. Here we just formulate the main statements. Detailed proofs will be given in the forthcoming article [9]. Suppose $S = K[X^{(1)}, \dots, X^{(m)}]$ is a $\mathbb{Z}^m$-graded polynomial ring in which each $X^{(i)}$ is the set of variables of degree $e_i \in \mathbb{Z}^m$. For a vector $c \in (c_1, \dots, c_m) \in \mathbb{N}^m_+$ consider the cth diagonal subring $S^{(c)} = \oplus_{i \in \mathbb{N}} S_{ic}$, the coordinate ring of the corresponding Segre–Veronese embedding. The following result improves the bound of Hering et al. [21] by one:

Theorem 6.1 *With the notation above one has:* $\min(c) \le \operatorname{index}(S^{(c)})$. *Moreover, we have* $\min(c) + 1 \le \operatorname{index}(S^{(c)})$ *if* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > 1 + \min(c)$.

Similarly one has the multigraded analog of Theorem 5.2. Here one uses the fact, proved in [11], given any $\mathbb{Z}^m$-graded standard graded algebra quotient of S then if the c_j's are big enough (in terms of the multigraded Betti numbers of R over S) then $\operatorname{reg}_{S^{(c)}}(R^{(c)}) = 0$.

Proposition 6.2 *Assume that for all* $j = 1 \dots, m$ *one has* $c_j \ge \max\{\alpha_j / i : i > 0,$ $\alpha \in \mathbb{Z}^m$ *and* $\beta^S_{i,\alpha}(R) \neq 0, \}$ *then* $\operatorname{index}(R^{(c)}) \ge \operatorname{index}(S^{(c)})$.

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