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NEW FREE DIVISORS FROM OLD

RAGNAR-OLAF BUCHWEITZ AND ALDO CONCA

To Jürgen Herzog on his 70th birthday.

ABSTRACT. We present several methods to construct or identify families of free divisors such as those annihilated by many Euler vector fields, including binomial free divisors, or divisors with triangular discriminant matrix. We show how to create families of quasihomogeneous free divisors through the chain rule or by extending them into the tangent bundle. We also discuss whether general divisors can be extended to free ones by adding components and show that adding a normal crossing divisor to a smooth one will not succeed.

1. Introduction. The goal of this note is to describe some basic operations that allow to construct new free divisors from given ones, and to classify toric free surfaces and binomial free divisors. We mainly deal with weighted homogeneous polynomials over a field of characteristic 0, although several statements and constructions generalize to power series.

A (formal) *free divisor* is a reduced polynomial (or power series) f in variables $x_1, \dots, x_n$ over a field K such that its Jacobian ideal $J(f) = (\partial f/\partial x_1), \dots, (\partial f/\partial x_n) + (f)$ is perfect of codimension 2 in the polynomial or power series ring. For generalities about free divisors and their importance in singularity theory, we refer to, say, [**2**] and the references therein.

A determinantal characterization of free divisors is due to Saito [**10**]: a reduced polynomial f is a free divisor if and only if there exists a matrix A of size $n \times n$ with entries in the relevant polynomial or power series ring such that $\det(A) = f$ and $(\nabla f)A \equiv 0 \bmod (f)$, where

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$\nabla f = (\partial f/\partial x_1), \dots, (\partial f/\partial x_n)$ is the usual gradient of f. In that case A is called a *discriminant* (or *Saito*) *matrix* of the free divisor.

The normal crossing divisor $f = x_1 \cdots x_k$, for some $1 \leqslant k \leqslant n$, provides a simple example of a free divisor. Indeed, it is an example of a *free arrangement*, that is, a hyperplane arrangement given by linear equations $\ell_i = 0$ such that the product $f = \prod_i \ell_i$ is a free divisor, see [**9**] for more on free arrangements.

Section 2 contains generalities and notation. In Section 3 we study homogeneous polynomials that are annihilated by $n-2$ linearly independent *Euler vector fields*, that is, polynomials f such that the vector space generated by the linear derivatives $\{x_i \partial f/\partial x_i\}_{i=1,\dots,n}$ is of dimension at most 2. We show that such a polynomial is a free divisor, provided the gradient ∇f vanishes as an element of the first homology module of the associated *Buchsbaum-Rim complex*. As an application, we classify in Theorem 3.5 those *free surfaces* $\{f(x,y,z) = 0\}$ that are weighted homogeneous and annihilated by some Euler vector field.

In Section 4 we present a *composition formula* or *chain rule* for free divisors. Such a formula implies, for instance, that, if f and g are free divisors in distinct variables, then $fg(f+g)$ is also a free divisor.

In Section 5 we exhibit some *triangular* free divisors, that is, free divisors whose discriminant matrix has a triangular form. It follows, for instance, that, for natural numbers $t \geqslant 1$, $n \geqslant 2$, the polynomial $\prod_{j=2}^n (x_1^t + \cdots + x_j^t)$ is a free divisor.

In Section 6 we characterize *binomial free divisors* by showing that a binomial in $n+2$ variables $x_1, \dots, x_n, y, z$ is a free divisor if and only if it is, up to permutation and scaling of the variables, of the form

$$x_1 \cdots x_n y^u z^t \left(y^\alpha \prod x_i^{a_i} + z^\beta \prod x_i^{b_i} \right)$$

with $\min(a_i, b_i) = 0$, $\alpha, \beta > 0$, and $0 \leqslant u, t \leqslant 1$. In particular, any reduced binomial is a *factor of a free divisor*. This observation leads us to ask whether any reduced polynomial is a factor of a free divisor. We discuss this question in Section 7, where we show that the simplest approach will not work: If f is a smooth form of degree greater than 2 in more than 2 variables, then $x_1 \cdots x_n f$ is not a free divisor.

In the final Section 8, we point out that homogeneous free divisors *extend into the tangent bundle*: along with f, the polynomial

$$f\left(\frac{\partial f}{\partial x_1}y_1 + \cdots + \frac{\partial f}{\partial x_n}y_n\right)$$

in twice as many variables $x_1, \dots, x_n; y_1, \dots, y_n$ is again a free divisor. Moreover, it will again be *linear*, if this holds for f.

We want to point out that similar "extension problems" for free divisors have been considered by others as well, especially in [**4, 8, 11**].

2. Notation and generalities. Let R be the polynomial ring $K[x_1, \dots, x_n]$ or formal power series ring $K[[x_1, \dots, x_n]]$ over a field K of characteristic 0. Let $\theta := \theta_{R/K} \cong \oplus_{i=1}^n R\partial_{x_i}$ denote the module of vector fields (or K-linear derivations) of R, with ∂_{x_i} being shorthand for the corresponding partial derivative, $\partial_{x_i} := \partial/\partial x_i$. For $f \in R$, we further abbreviate $f_i := f_{x_i} := \partial_{x_i} f$, so that the *gradient* of f with respect to the chosen variables is given by the vector $\nabla f = (f_1, \dots, f_n)$.

Definition 2.1. For $a = (a_1, \dots, a_n) \in K^n$, we call the linear vector field $E_a = \sum_i a_i x_i \partial_{x_i}$ the *Euler vector field* associated to a. It is an *Euler vector field for* f if $E_a(f) = \delta f$, for some $\delta \in \mathbf{K}$.

A vector $w \in \mathbf{Z}^n$ naturally induces a $\mathbf{Z}$-grading on $K[x_1, \dots, x_n]$ by setting $\deg_w x_i = w_i$. Accordingly, one can assign to any non-zero polynomial f a degree $\deg_w(f)$, and that polynomial is w*-homogeneous*, that is, homogeneous with respect to this grading, if all the nonzero monomials in f are of degree $\deg_w(f)$. If $f \in R$ is w-homogeneous, then $E_w(f) = \deg_w(f) f$.

The Jacobian ideal $J(f)$ of f is, by definition, $(f_1, \dots, f_n) + (f) \subseteq R$. Some authors, see e.g., [**6**, page 110], call this the *Tjurina ideal* to distinguish it clearly from the ideal generated by just the partial derivatives that describes the *critical locus* of the map defined by f.

Note that $J(f) = (f_1, \dots, f_n)$ precisely when there exists a derivation $D \in \theta$ such that $D(f) = f$. This happens, for example, if f is homogeneous of non-zero degree with respect to some weight $w \in \mathbf{Z}^n$.

It is well known that, in general, $J(f)$ defines the *singular locus* of the hypersurface ring $R/(f)$, equivalently, the hypersurface $\{f = 0\}$ in affine n-space $\mathbf{A}^n_K$.

Definition 2.2. A (formal) *free divisor* is a polynomial (or power series) f, whose Jacobian ideal $J(f)$ is *perfect* of codimension 2 in R. We allow the ideal to be improper, thus, the empty set is perfect of any codimension. However, the zero ideal is, by convention, not perfect of any codimension, and we always assume $f \neq 0$.

In particular, f is then *squarefree*, equivalently, the hypersurface ring $R/(f)$ is *reduced*–and we then simply also call f *reduced*–and the singular locus of that hypersurface is a Cohen-Macaulay subscheme of codimension 2 in $\operatorname{Spec} R$.

Example 2.3. As simplest examples, any separable polynomial in $K[x]$ defines a free divisor, and so does any reduced $f \in K[x, y]$.

Saito, who introduced the notion, gave the following important criterion for f to be a free divisor:

Theorem 2.4 [**10**]. *Let $f \in R$ be reduced. Then f is a free divisor if and only if there exists a $n \times n$ matrix A with entries in R such that* $\det A = f$ *and* $(\nabla f)A \equiv 0 \bmod (f)$.

The matrix A appearing in this criterion is called a *discriminant* (or *Saito*) *matrix* of f. If the entries of A can be chosen to be linear polynomials, then f is called a *linear* free divisor. Note that f is then necessarily a homogeneous polynomial of degree n. The *normal crossing divisor* $f = x_1 \cdots x_n$ is a simple example of a linear free divisor.

Remark 2.5. It follows immediately from this criterion that a free divisor $f \in R$ remains a free divisor in any polynomial or power series ring over R. When viewed as an element of such larger ring, f is called the *suspension* of the original free divisor from R.

A different way to state the criterion, and to link it with the definition we chose, denote $\mathrm{Der}\,(-\log f) \subseteq \theta$ those vector fields D such that $D(f) \in (f)$, equivalently, $D(\log f) = D(f)/f$ is a well-defined element of R. With this notation, one has a short exact sequence of R-modules

$$0 \longrightarrow \mathrm{Der}\,(-\log f) \longrightarrow \theta \xrightarrow{df} J(f)/(f) \longrightarrow 0,$$

and a reduced f is a free divisor if, and only if, $\mathrm{Der}\,(-\log f)$ is a free R-module, necessarily of rank n. A discriminant matrix is then simply the matrix of the inclusion $\mathrm{Der}\,(-\log f) \subseteq \theta$, when bases of these free modules are chosen.

Now we turn to our results.

3. Polynomials annihilated by many Euler vector fields. In this section we assume that:

(a) $f \in R$ is a nonzero squarefree polynomial that belongs to the ideal of its derivatives, $f \in (f_1, \dots, f_n) \subseteq R$.

(b) The K-vector space of Euler vector fields annihilating f has dimension at least $n-2$. In other words, there exist $n-2$ linearly independent Euler vector fields $E_j = \sum_i a_{ij} x_i \partial_{x_i}$, for $j = 1, \dots, n-2$, such that $E_j(f) = 0$. Denote by A the $n \times (n-2)$ scalar matrix (a_{ij}) and by B the matrix $(a_{ij} x_i)$ of the same size.

Under these assumptions, the Jacobian ideal of f is equal to the ideal of its partial derivatives and has codimension at least two. To show that it defines a Cohen-Macaulay subscheme of codimension 2, it suffices thus to find a *Hilbert-Burch matrix*, necessarily of size $n \times (n-1)$, for the partial derivatives. By assumption, we have a matrix equation in R of the form

$$(\nabla f) B = (0, 0, \dots, 0).$$

We need one more syzygy! More precisely; see, for example, [**5**, 20.4] to get a Hilbert-Burch matrix for $(f_1, \dots, f_n)$; we want a column vector $w := (w_1, \dots, w_n)^T$ with entries from R such that we have an equality of sequences of elements from R of the form

$$(f_1, \dots, f_n) = I_{n-1}(C),$$

where C is obtained from B by appending the column vector w, and I_{n-1} denotes the sequence of appropriately signed maximal minors of the indicated $n \times (n-1)$ matrix.

Define an R-linear map from R^n to R^n through

$$\varepsilon(w_1, \dots, w_n) := I_{n-1}(B \mid w),$$

where $(B \mid w)$ denotes the $n \times (n-1)$-matrix obtained from B by adding column w.

Clearly, $B \circ \varepsilon = 0$, and the sequence of free (graded) R-modules

$$\mathbf{BR}(B) \equiv \Big(F_2 = R^n(n-1) \overset{\partial_2=\varepsilon}{\longrightarrow} F_1 = R^n(-1) \overset{\partial_1=B}{\longrightarrow} F_0 = R^{n-2} \to 0\Big)$$

is the beginning of the *Buchsbaum-Rim complex* for matrix B; see, for example, [**5**, Appendix A.2]. By the given setup, the vector $\nabla f \in F_1$ is a cycle in this complex, and the required vector w exists if, and only if, the class of ∇f is zero in the first homology group $H_1(\mathbf{BR}(B))$ of this Buchsbaum-Rim complex.

Now, if the ideal of the maximal minors of B has the maximal possible codimension, equal to $n-(n-2)+1 = 3$, then the entire Buchsbaum-Rim complex is exact and, so, in particular, $H_1(\mathbf{BR}(B)) = 0$.

The minor of B obtained by deleting rows i and j is the monomial $u_{ij}x_1 \cdots x_n/x_ix_j$, where u_{ij} is the minor of A obtained by deleting the rows corresponding to i and j. The ideal generated by these minors will have maximal codimension if, and only if, all the maximal minors of A are non-zero.

Summing up, we have the following result.

Proposition 3.1. *Under assumptions* (a) *and* (b)*, and with the notation as above,*

(1) *the polynomial f is a free divisor if, and only if, the class of ∇f in the first homology $H_1(\mathbf{BR}(B))$ of the Buchsbaum-Rim complex associated to B vanishes.*

(2) *If all the maximal minors of A are non-zero, then f is a free divisor.*

Example 3.2. Consider

$$f = ux^a - vx^b$$

with $u, v \in K$ nonzero and $a, b \in \mathbf{N}^n$ different exponents with $\min(a_i, b_i) \leqslant 1$, for each i, to ensure that f is reduced. The Euler vector field $\sum_{i=1}^n c_i x_i \partial/\partial x_i$ then annihilates f if, and only if, $\sum a_i c_i = 0$ and $\sum b_i c_i = 0$. Assuming $a_i b_j - a_j b_i \neq 0$ for some pair of indices $i < j$, the space of Euler vector fields annihilating f has dimension $n - 2$. The corresponding $n \times (n-2)$ coefficient matrix A then satisfies $\binom{a}{b} A = 0$, where $\binom{a}{b}$ is the obvious $2 \times n$ matrix of scalars. Linear algebra tells us that the maximal minors of A are then, up to sign and a common non-zero constant, equal to the maximal minors of $\binom{a}{b}$. By virtue of Proposition 3.1 (2), we can conclude that, if $a_i b_j - a_j b_i \neq 0$ for all pairs $i < j$, then the binomial f is a free divisor.

In Section 6 we will give a complete characterization of homogeneous binomial free divisors.

In three variables the considerations above lead to a complete characterization of free divisors that are weighted homogeneous and annihilated by an Euler vector field. To write down the corresponding Hilbert-Burch matrices in a compact form, the following tool will be useful.

Definition 3.3. Let $d > 0$ be a natural number, $R = K[x_1, \ldots, x_n]$ a polynomial ring over a field K of characteristic zero, and $y = \{y_1, \ldots, y_m\}$ a subset of the variables x. Define a K-linear endomorphism $(\deg + d)_y^{-1}$ on R through the following action on monomials:

$$(\deg + d)_y^{-1}(x^e) := \frac{1}{|e|_y + d} x^e,$$

where $|e|_y := \sum_{i, x_i \in y} e_i$ denotes the usual total degree of x^e with respect to the variables y.

In words, $(\deg + d)_y^{-1}$ has polynomials that are homogeneous of total degree a in the variables y as eigenvectors of eigenvalue $1/(a+d)$. If y is the set of all variables, then the corresponding K-linear endomorphism will simply be denoted by $(\deg + d)^{-1}$.

As is well known, the endomorphism just defined can be used to split in characteristic zero the tautological Koszul complex on the variables. Here we will use the following form.

Lemma 3.4. *Let $V = \oplus_i K x_i$ be the indicated vector space over K and $V \cong \oplus_i K\xi_i$, $x_i \mapsto \xi_i$ an isomorphic copy of it. Let $\mathbf{K}^\bullet = \mathbf{S}_K V \otimes_K \Lambda_K V \cong R \otimes_K \Lambda^\bullet_K(\xi_1, \dots, \xi_n)$ be the exterior algebra over R on variables ξ_i, the graded R-module underlying the usual Koszul complex.*

The R-linear derivation $\partial := \sum_i a_i x_i(\partial/\partial\xi_i)$ defines a differential on $\mathbf{K}$ for any choice of $a_i \in K$. Let $W \subseteq V$ denote the subspace generated by those variables y among the x, for which $a_i \neq 0$, and denote by η_j the corresponding variables among the ξ_i in the isomorphic copy of W.

If $\omega \in \mathbf{K}^m$ is a cycle *for ∂, then the class of ω in $H_i(\mathbf{K}^\bullet, \partial)$ is zero if, and only if, $\omega = 0$ in $R/(y) \otimes \Lambda^i(V/W)$. In that case, $\omega' := (\sum_j (1/a_j)\, d\eta_j \partial_{y_j}) \circ (\deg + d)_y^{-1}(\omega)$ provides a boundary, $\partial(\omega') = \omega$.*

Theorem 3.5. *Let K be a field of characteristic zero and $f \in K[x, y, z]$ a reduced polynomial in three variables such that f is contained in the ideal of its partial derivatives, $f \in (f_x, f_y, f_z)$.*

Assume further that there is a triple (a, b, c) of elements of K that are not all zero such that the Euler vector field

$$E = ax\frac{\partial}{\partial x} + by\frac{\partial}{\partial y} + cz\frac{\partial}{\partial z}$$

satisfies $E(f) = 0$.

We then have the following possibilities, up to renaming the variables:

(1) *If $abc \neq 0$, then f is a free divisor with Hilbert-Burch matrix*

$$(f_x, f_y, f_z) = I_2 \begin{pmatrix} ax & \left(\frac{1}{c} - \frac{1}{b}\right)(\deg + 2)^{-1}(f_{yz}) \\ \\ by & \left(\frac{1}{a} - \frac{1}{c}\right)(\deg + 2)^{-1}(f_{xz}) \\ \\ cz & \left(\frac{1}{b} - \frac{1}{a}\right)(\deg + 2)^{-1}(f_{xy}) \end{pmatrix}$$

where f_{**} denotes the corresponding second order derivative of f.

(2) *If $a = 0$, but $bc \neq 0$, then f is a free divisor if, and only if, $f_x \in (y, z)$. If that condition is verified and $f_x = yg + zh$, then $f_y/cz = -f_z/by$ is an element of R, and a Hilbert-Burch matrix is given by*

$$(f_x, f_y, f_z) = I_2 \begin{pmatrix} 0 & f_y/cz = -f_z/by \\ by & -h/c \\ cz & g/b \end{pmatrix}.$$

(3) *If* $a = b = 0$, *then* f *is independent of* z *and, so, as the suspension of a reduced plane curve, is a free divisor.*

Proof. We simply need to verify that the Hilbert-Burch matrix is correct. One may now either use the preceding lemma, or calculate directly, as we will do. We just verify that, in case (1), the minor obtained when deleting the first row is correct, leaving the remaining calculations to the interested reader. It suffices to check the case when $f = x^{e_1}y^{e_2}z^{e_3}$ is a monomial with $ae_1+be_2+ce_3 = 0$ and $e_i \geqslant 0, |e| > 0$. Then,

$$\begin{aligned}
&by(1/b - 1/a)(\deg + 2)^{-1}(f_{xy}) - cz(1/a - 1/c)(\deg + 2)^{-1}(f_{xz}) \\
&\qquad = by(1/b - 1/a)(\deg + 2)^{-1}(e_1e_2x^{e_1-1}y^{e_2-1}z^{e_3}) \\
&\qquad\quad - cz(1/a - 1/c)(\deg + 2)^{-1}(e_1e_3x^{e_1-1}y^{e_2}z^{e_3-1}) \\
&\qquad = \frac{e_1e_2}{|e|}(1 - b/a)x^{e_1-1}y^{e_2}z^{e_3} - \frac{e_1e_3}{|e|}(c/a - 1)x^{e_1-1}y^{e_2}z^{e_3} \\
&\qquad = f_x\left(e_2(a - b) - e_3(c - a)\right)/a|e| \\
&\qquad = f_x\left((e_2 + e_3)a - e_2b - e_3c\right)/a|e| \\
&\qquad = f_x,
\end{aligned}$$

as required. □

To apply this result, we need to detect Euler vector fields annihilating given polynomials, and the following remark is useful for this purpose.

Remark 3.6. Assume that f is a polynomial that is homogeneous with respect to two weights, $w, v \in \mathbf{Z}^n$. For every $a, b \in \mathbf{Z}$, the polynomial f is then homogeneous with respect to $aw + bv$, of degree $a\deg_w(f) + b\deg_v(f)$. Taking $a = \deg_v(f)$ and $b = -\deg_w(f)$, we conclude that f is homogeneous of degree 0 with respect to $\deg_v(f)w - \deg_w(f)v$, and so the corresponding Euler vector field annihilates f. If, further, some degree $a\deg_w(f) + b\deg_v(f)$ is not zero, then f satisfies the assumption (a) from the beginning.

This remark can be applied as follows.

Example 3.7. Set

$$f(x,y,z) = x^{\gamma_1} y^{\gamma_2} z^{\gamma_3} \Pi_{i=1}^{k}(x^a - \alpha_i y^b z^c)$$

with $a, b, c, k \in \mathbf{N} \setminus \{0\}$, $\gamma_j \in \{0,1\}$ and $\alpha_i \in K$. Assume that the α_i are non-zero and distinct so that f is reduced. Then f is a free divisor if, and only if, not both γ_2 and γ_3 equal 0, equivalently, $\gamma_2 + \gamma_3 > 0$. To prove the statement, take $v = (0, c, -b)$ and $w = (b, a, 0)$, so that f becomes homogeneous with respect to both v and w, satisfying

$$\deg_v(f) = c\gamma_2 - b\gamma_3 \quad \text{and} \quad \deg_w(f) = b\gamma_1 + a\gamma_2 + kab \neq 0.$$

Hence, by the remark above, $f \in (f_x, f_y, f_z)$, and the Euler vector field associated to

$$\begin{aligned} \deg_v(f)w - \deg_w(f)v &= (c\gamma_2 - b\gamma_3)(b, a, 0) \\ &\quad - (b\gamma_1 + a\gamma_2 + kab)(0, c, -b) \\ &= -b(-c\gamma_2 + b\gamma_3, a\gamma_3 \\ &\qquad + c\gamma_1 + kac, -b\gamma_1 - a\gamma_2 - kab) \end{aligned}$$

annihilates f. Clearly, the second and the third coordinates of this vector are non-zero, while the first one equals $b(c\gamma_2 - b\gamma_3)$. Now, if γ_2 or γ_3 is non-zero, then $f_x \in (y, z)$, and we conclude by Theorem 3.5, either part (1) or (2), that f is a free divisor.

On the other hand, if $\gamma_2 = \gamma_3 = 0$, then f contains a pure power of x and so $f_x \notin (y, z)$. We may then conclude by Theorem 3.5 (2) that f is not a free divisor.

Remark 3.8. Some isolated members of this family of examples have been identified as free divisors before:

$$f = y(x^2 - yz) \quad \text{or} \quad f = xy(x^2 - yz),$$

the quadratic cone with, respectively, one or two planes, of which one is tangent, or

$$f = y(x^2 - y^2 z),$$

the Whitney umbrella with an adjoint plane, see [**8**].

FIGURE 1. The free divisors defined by $h = yz(x^2 - 5yz)(x^2 - \frac{1}{2}yz)(x^2 + yz)$ (left) and $h = yz(x^2 - \frac{1}{2}y^2z)(x^2 + 5y^2z)$ (right).

A remarkable feature of this example is that it exhibits free surfaces with arbitrarily many irreducible components that are not suspended, in that we can, for example, extend the family of examples involving quadratic cones to

$$f = x^{\gamma_1} y^{\gamma_2} z^{\gamma_3} \prod_{i=1}^{k} (x^2 - \alpha_i yz)$$

for $k \geqslant 1$, $\gamma_j \in \{0, 1\}$ with $\gamma_2 + \gamma_3 \neq 0$ and scalars $\alpha_i \in K$ satisfying $\prod_{i=1}^{k} \alpha_i \prod_{i<j} (\alpha_i - \alpha_j) \neq 0$. Such f will clearly have $\gamma_1 + \gamma_2 + \gamma_3 + k$ many irreducible components, $1 \leqslant \gamma_1 + \gamma_2 + \gamma_3 \leqslant 3$ among them planes.

4. A chain rule for quasihomogeneous free divisors. We start with a simple observation: if $f \in K[x] = K[x_1, \ldots, x_n]$ and $g \in K[y] = K[y_1, \ldots, y_m]$ are free divisors, then $fg \in K[x, y]$ is a free divisor. To see this, one just takes the discriminant matrices A, B associated to f and g, and notes that the block matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

is a discriminant matrix for fg that one can think of as the pullback of the planar normal crossing divisor along the map with components (f, g). Such free divisors have been called "product-unions" by Damon [**3**] or "splayed" divisors by Aluffi and Faber [**1**].

If $f = f_1 \cdots f_k$ is square free, then a vector field D is logarithmic for f if, and only if, D is logarithmic for each f_i, as

$$D(\log f) = \sum_i D(\log f_i) = \sum_i \frac{D(f_i)}{f_i}$$

can only be an element of R if that holds for the summands.

We now use these observations to establish a *chain rule* for free divisors. In this form, the result and its proof are due to Mond and Schulze [**8**, Theorem 4.1], while we originally had obtained a weaker result. We include an algebraic version of the proof, and strengthen their result by removing the hypothesis that no f_i be a smooth divisor.

Theorem 4.1. *Let $k \geqslant 1$ be an integer, K a field of characteristic zero. Assume, given a free divisor $f = f_1 \cdots f_k \in R = K[x_1, \ldots, x_n]$ that admits vector fields E_j, for $j = 1, \ldots, k$, satisfying $E_j(f_i) = \delta_{ij} f_i$, where δ_{ij} is the Kronecker delta.*

If $H = y_1 \cdots y_k H_1 \in Q := K[y_1, \ldots, y_k]$ is a free divisor such that f and $H_1(f_1, \ldots, f_k)$ are without common factor, then the polynomial $\widetilde{H} := H(f_1, \ldots, f_k) \in R$ is a free divisor.

Proof. Because f is a free divisor, its R-module of logarithmic vector fields $\operatorname{Der}(-\log f)$ is free. It contains the vector fields E_i, because $E_i(f) = f$ by the product rule. Further, the E_i are linearly independent over R, as $0 = \sum_{i=1}^k g_i E_i \in \theta$ implies $0 = \sum_{i=1}^k g_i E_i(f_j) = g_j f_j$, and so $g_j = 0$ for each j. In this way, $\oplus_{i=1}^k R E_i$ becomes a free submodule of $\operatorname{Der}(-\log f)$.

Now any $D \in \operatorname{Der}(-\log f)$ is logarithmic for each f_i as those elements of R are relatively prime, f being squarefree. Therefore, $D \mapsto \sum_{i=1}^k D(\log f_i) E_i$ provides an R-linear map $\operatorname{Der}(-\log f) \to \oplus_{i=1}^k R E_i$ that splits the inclusion, and whose kernel consists of those derivations D that satisfy $D(f_i) = 0$ for each i.

Therefore, we can extend the E_i to a basis $(E_1, \ldots, E_k, D_1, \ldots, D_{n-k})$ of $\operatorname{Der}(-\log f)$ as R-module, with $D_j(f_i) = 0$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n-k$.

Let C be the $n \times n$ matrix over R that expresses the just chosen basis of $\operatorname{Der}(-\log f)$ in terms of the partial derivatives $\partial/\partial x_j$, for $j = 1, \ldots, n$, so that

$$(E_1, \dots, E_k, D_1, \dots, D_{n-k}) = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) C.$$

The matrix C is then a discriminant matrix for f, and, in particular, $\det C = f$.

Now we turn to $H \in Q$ and observe that any $D \in \mathrm{Der}_Q(-\log H)$, a logarithmic derivation for H over Q, is necessarily of the form $D = \sum_{r=1}^{k} y_r b_r(\partial/\partial y_r)$ for suitable elements $b_r \in Q$, as H contains by assumption $y_1 \cdots y_k$ as a factor, whence $D(\log y_r) = b_r$ must be in Q. In matrix form, a discriminant matrix for H can be factored as

$$A := \mathrm{diag}\,(y_1, \dots, y_k) B,$$

where the first factor is the diagonal matrix with entries y_r and $B = (b_{rs})$ is a $k \times k$ matrix over Q so that the vector fields $\sum_r y_r b_{rs}(\partial/\partial y_r)$ form a Q-basis of $\mathrm{Der}_Q(-\log H)$. Because $\det A = H$ by Saito's criterion in Theorem 2.4, it follows that $\det B = H_1 \in Q$.

Next note that the given f_i define a substitution homomorphism $Q \to R$ that sends $y_i \mapsto f_i$. For any $b \in Q$, we denote $\widetilde{b} = b(f_1, \dots, f_k)$ its image in R. We claim that a derivation $\widetilde{D} := \sum_r \widetilde{b}_r E_r$ is logarithmic for $\widetilde{H} \in R$, if $D := \sum_r y_r b_r(\partial/\partial y_r)$ is logarithmic for $H \in Q$. In fact, the usual chain rule for derivations first yields

$$\begin{aligned}
\widetilde{D}(\widetilde{H}) &= \sum_{r=1}^{k} \widetilde{b}_r E_r(\widetilde{H}) \\
&= \sum_{r=1}^{k} \widetilde{b}_r \sum_{s=1}^{k} \widetilde{\frac{\partial H}{\partial y_s}} E_r(f_s) \\
&= \sum_{r=1}^{k} f_r \widetilde{b}_r \widetilde{\frac{\partial H}{\partial y_r}}
\end{aligned}$$

as $E_r(f_s) = \delta_{rs} f_r$ by assumption. Now the last term equals $\widetilde{D(H)}$, the image of $D(H)$ under substitution. Thus, if $D(H)$ is in $(H) \subseteq Q$, its image is in $(\widetilde{H}) \subseteq R$, and so $\widetilde{D}$ is indeed logarithmic for $\widetilde{H}$.

On the other hand, if D is a derivation on R that vanishes on each f_i, then applying the chain rule yet again shows

$$D(\widetilde{H}) = \sum_{r=1}^{k} \widetilde{\left(\frac{\partial H}{\partial y_r}\right)} D(f_r) = 0,$$

whence such D is in particular logarithmic for $\widetilde{H}$. Putting everything together,

$$\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) C \begin{pmatrix} \widetilde{B} & 0 \\ 0 & I_{n-k} \end{pmatrix},$$

with I_{n-k} the identity matrix of indicated size, represents n logarithmic vector fields for $\widetilde{H}$. Taking determinants, we get

$$\begin{aligned} \det\left(C \begin{pmatrix} \widetilde{B} & 0 \\ 0 & I_{n-k} \end{pmatrix}\right) &= \det C \det \widetilde{B} \\ &= \det C \widetilde{\det B} = f_1 \cdots f_k \widetilde{H_1} = \widetilde{H}. \end{aligned}$$

Thus, the proof will be completed by Saito's criterion Theorem 2.4, once we show that $\widetilde{H_1}$ is squarefree, as by assumption f is already squarefree and relatively prime to $\widetilde{H}_1$. To this end, we use the Jacobi criterion, see e.g., [**7**, 30.3]. The rank of the Jacobi matrix

$$\left(\frac{\partial f_i}{\partial x_j}\right)_{j=1,\dots,n}^{i=1,\dots,k}$$

is k outside of $\{f = 0\}$, as $E_1(f_1) \cdots E_k(f_k) = f$ is in the ideal of maximal minors of that matrix. Therefore, R is smooth over Q outside of $\{f = 0\}$, and the inverse image $\{\widetilde{H}_1 = 0\}$ of $\{H_1 = 0\}$ remains thus reduced. □

We mention the following special case of Theorem 4.1 as an example.

Corollary 4.2. *If $f \in K[x] = K[x_1, \dots, x_n]$ and $g \in K[y] = K[y_1, \dots, y_m]$ are free divisors that are weighted homogeneous, then $fg(f + g) \in K[x, y]$ is a free divisor.*

Remark 4.3. In the original treatment of Theorem 4.1 in [**8**], the hypothesis that f and $H_1(f_1, \dots, f_k)$ are without common factors is missing. That hypothesis is, however, necessary, as is shown by the following example that Eleonore Faber kindly provided.

Take $f_1 = (1+u)(x^2-y^3)$, $f_2 = (1+v)(y^2-x^3)$ and $f_3 = (1+w)(f_1^3+f_2^2)$ in $R = K[x,y,u,v,w]$. A calculation in SINGULAR shows readily that $f = f_1f_2f_3$ is a free divisor. The vector fields $E_1 = (1+u)\partial/\partial u, E_2 = (1+v)\partial/\partial v$, and $E_3 = (1+w)\partial/\partial w$ certainly satisfy $E_i(f_j) = \delta_{ij}f_i$.

Now take $H(y_1,y_2,y_3) = y_1y_2y_3(y_1^3+y_2^2)$, a binomial free divisor according to Theorem 6.1, and observe that

$$H(f_1,f_2,f_3) = f_1f_2f_3(f_1^3+f_2^2) = f_1f_2(1+w)(f_1^3+f_2^2)^2$$

is not reduced, thus, is not a free divisor, as f and $H_1(f_1,f_2,f_3)$ have the factor $f_1^3+f_2^2$ in common.

5. Triangular free divisors. Let K be a field of characteristic zero. Assume, given a "seed" $F_0 \in R := K[y_1,\dots,y_n]$, and define inductively for $i>0$ polynomials

$$F_i := \alpha_i x_i^{a_i} + \beta_i F_{i-1}^{b_i} \in Q := R[x_1,\dots,x_i]$$

for natural numbers $a_i, b_i > 0$ and $\alpha_i, \beta_i \in K$ with $\alpha_i \neq 0$.

Proposition 5.1. *Assume F_0 is a free divisor in R with discriminant $(n\times n)$-matrix A over R. If $F := F_iF_{i-1}\cdots F_0$ is reduced, then it is a free divisor over Q with "triangular" discriminant matrix of the form*

$$B = \begin{pmatrix} A & 0 & 0 & \cdots & 0 \\ * & F_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ * & * & * & F_{i-1} & 0 \\ * & * & * & * & F_i \end{pmatrix}$$

where the entries marked "$*$" represent elements of Q that can be calculated explicitly.

Proof. First observe that the determinant of the displayed matrix certainly equals F. It thus remains to prove that we can choose the columns to represent logarithmic vector fields for it.

The proof proceeds by induction on $i \geqslant 0$, the case $i = 0$ being true by assumption. For $i \geqslant 1$, set $G = F/F_i$, and assume that the result is correct for G. The last column in B represents the vector field $D = F_i\partial/\partial x_i$, and we now show that it is a logarithmic vector field for F, that is, F divides $D(F)$:

$$D(F) = D(F_i)G = F_i\frac{\partial F_i}{\partial x_i}G = \left(\frac{\partial F_i}{\partial x_i}\right)F,$$

the first equality due to the fact that G is independent of x_i.

To finish the proof, it suffices now to establish the following:

Lemma 5.2. *Let D be a logarithmic vector field for G as an element of $R[x_1,\dots,x_{i-1}]$.*

(1) *D is a logarithmic vector field for each factor $F_0,\dots,F_{i-1}$ of G, so that $c_{F_j} := D(F_j)/F_j \in R[x_1,\dots,x_{i-1}]$ for each $j = 0,\dots,i-1$.*

(2) *The vector field*

$$\widetilde{D} = \frac{b_i c_{F_{i-1}}}{\alpha_i a_i}x_i\frac{\partial}{\partial x_i} + D$$

is the unique extension of D to a logarithmic vector field for F in Q. It satisfies

$$\widetilde{D}(F) = \Big((b_i+1)c_{F_{i-1}} + \sum_{j=0}^{i-2} c_{F_j}\Big)F.$$

Proof. The first part was already pointed out above: if D is any logarithmic vector field for a product fg of coprime factors, then it is necessarily a logarithmic vector field for each factor.

Now we turn to the derivation D given in the statement. Assume there is an extension $\widetilde{D} = u(\partial/\partial x_i) + D$ of D to a logarithmic vector field for F. We then get first from the product rule

$$\widetilde{D}(F) = \widetilde{D}(F_i)G + F_i\widetilde{D}(G),$$

and by definition of $\widetilde{D}$ and F_i, this evaluates to

$$= \Big(u\alpha_i a_i x_i^{a_i-1} + \beta_i b_i F_{i-1}^{b_i-1}D(F_{i-1})\Big)\,G + F_iD(G)$$

as $\widetilde{D}(H) = D(H)$ for H equal to either F_{i-1} or G,

$$= \left(u\alpha_i a_i x_i^{a_i-1} + \beta_i b_i c_{F_{i-1}} F_{i-1}^{b_i}\right) G + c_G F_i G,$$

as D is, respectively, logarithmic for F_{i-1} and for G with the indicated multipliers.

Due to $F = F_i G$, we see that $\widetilde{D}(F)$ will be a multiple of F if, and only if, $F_i = \alpha_i x_i^{a_i} + \beta_i F_{i-1}^{b_i}$ divides $u\alpha_i a_i x_i^{a_i-1} + \beta_i b_i c_{F_{i-1}} F_{i-1}^{b_i}$, if, and only if,

$$u = b_i c_{F_{i-1}} x_i / a_i,$$

and in that case

$$\widetilde{D}(F) = (b_i c_{F_{i-1}} + c_G)F.$$

It follows that

$$\widetilde{D} := \frac{b_i c_{F_{i-1}}}{a_i} x_i \frac{\partial}{\partial x_i} + D$$

is the unique extension of D to a logarithmic vector field for F, as claimed. Finally, observe that the multiplier in question is

$$\begin{aligned} c := \frac{\widetilde{D}(F)}{F} &= b_i c_{F_{i-1}} + c_G \\ &= b_i c_{F_{i-1}} + \sum_{j=0}^{i-1} c_{F_j} \\ &= (b_i + 1) c_{F_{i-1}} + \sum_{j=0}^{i-2} c_{F_j}, \end{aligned}$$

and that finishes the proof. □

To end the proof of Proposition 5.1, if the result holds for $i-1$, we extend the column that represents the logarithmic vector field D for $G = F_{i-1} \cdots F_0$ in the displayed discriminant matrix by adding the corresponding coefficient $[(b_i c_{F_{i-1}})/a_i] x_i$ of $\partial/\partial x_i$ in $\widetilde{D}$ as the entry in the last row of the discriminant matrix for F. □

Note that, in Proposition 5.1, we may take as seed F_0 any reduced polynomial in two variables.

FIGURE 2. The union of a cylinder over an A_1-curve and an A_2-surface given by $h = (x^2 - y^2)(x^2 - y^2 + z^3)$ (left) and the union of a cylinder over an A_2-curve and an E_8-surface given by $h = (x^2 + y^3)(x^2 + y^3 - z^5)$ (right).

Example 5.3. Given positive integers $t_1, \dots, t_i$, for $j = 2, \dots, i$, set $G_j = x_1^{t_1} + \cdots + x_j^{t_j}$. Take $F_0 = G_2$ as a seed, and set $a_j = t_{j+2}, b_j = \alpha_j = \beta_j = 1$ to obtain $F_j = G_{j+2}$ for $j = 0, \dots, i-2$. The resulting product $G = G_2 \cdots G_i$ of Brieskorn-Pham polynomials is a free divisor by Proposition 5.1.

One can easily calculate the entries of the discriminant matrix. To illustrate, we treat the case where each exponent is $t = 2$, so that $G_j = x_1^2 + \cdots + x_j^2$.

The first column can be taken as representing the usual Euler vector field that is the unique extension of the Euler vector field for G_2. The second column can be taken to correspond to the vector field $D = -x_2\partial/\partial x_1 + x_1\partial/\partial x_2$ that in turn corresponds to the automorphism interchanging x_1 and x_2. As for this D, one has $D(G_2) = 0$, and Lemma 5.2 shows that the corresponding matrix entries below the second row will be zero as well.

Now we indicate how to obtain the entries of columns 3 through i. Counting from the top, start with $D = G_j\partial/\partial x_j$, thus, putting G_j as the entry in the jth row as first nonzero entry in column $j \geqslant 3$, and note that $D(G_j) = 2x_jG_j$, so that $c_{G_j} = 2x_j$. By Lemma 5.2, the entry below it will be

$$a_{j+1,j} = \frac{b_{j+1}c_{G_j}}{a_{j+1}}x_{j+1} = \frac{c_{G_j}}{2}x_{j+1} = x_jx_{j+1}.$$

Now $c_{G_{j+1}} = 2x_j$ again, and induction shows that a relevant discriminant matrix can be taken in the form

$$B = \begin{pmatrix} x_1 & -x_2 & 0 & 0 & \cdots & 0 \\ x_2 & x_1 & 0 & 0 & \cdots & 0 \\ x_3 & 0 & G_3 & 0 & \cdots & 0 \\ x_4 & 0 & x_3x_4 & G_4 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ x_i & 0 & x_3x_i & x_4x_i & \cdots & G_i \end{pmatrix}.$$

6. Binomial free divisors. The goal of this section is to investigate binomials $(ux^a + vx^b)x^c$, with $u, v \in K, uv \neq 0$, and exponent vectors a, b, c with $|a|, |b| \geqslant 1$, $\min(a_i, b_i) = 0$, that are free divisors. This forces each entry of c to be in $\{0, 1\}$, and we can absorb the constants u, v into the variables to reduce to the form $F = L(M + N)$, where L is a product of distinct variables and M, N are coprime monomials.

We further assume $R = K[x_1, \dots, x_{n+2}]$, with K as usual a field of characteristic 0, and we may suppose that F involves all the variables, as otherwise it is just a suspension of a divisor that satisfies this requirement.

With these preparations we show the following result.

Theorem 6.1. *The binomial $F = L(M + N)$ as above is a free divisor if*

(a) *at most one of the variables appearing in M does not appear in L, and*

(b) *at most one of the variables appearing in N does not appear in L.*

Note that, if F is required to involve all variables, then these conditions imply $\deg L \geqslant n$.

If F is a homogeneous *binomial, that is,* $\deg M = \deg N$, *then the preceding sufficient conditions are also necessary.*

Proof. For the first claim, we can write, up to a permutation of the variables and setting $y = x_{n+1}$ and $z = x_{n+2}$,

$$F = x_1 \cdots x_n y^u z^t G$$

where

$$G = x^a y^\alpha + x^b z^\beta$$

and $a, b \in \mathbf{N}^n$ with $\min(a_i, b_i) = 0$, $\alpha, \beta > 0$ and $u, t \in \{0, 1\}$. Let V be the K vector space generated by the monomials $x_1 \cdots x_n x^a y^{u+\alpha} z^t$ and $x_1 \cdots x_n x^a y^u z^{t+\beta}$ involved in F. Obviously, V is two-dimensional, the elements F, zF_z form a basis, and V contains $x_i F_{x_i}$ for each $i = 1, \dots, n+2$. So we get the relations

$$x_i F_{x_i} + v_i z F_z \equiv 0 \pmod{F}, \tag{1}$$

with some $v_i \in K$, for $i = 1, \dots n$. Now note that

$$F_y = x_1 \cdots x_n z^t (uG + \alpha x^a y^{\alpha-1+u}) \tag{2}$$

and

$$F_z = x_1 \cdots x_n y^u (tG + \beta x^b z^{\beta-1+t}) \tag{3}$$

whence we also get the relations

$$\beta y F_y + \alpha z F_z \equiv 0 \pmod{F} \tag{4}$$

and

$$-y^u (tG + \beta x^b z^{\beta-1+t}) F_y + z^t (uG + \alpha x^a y^{\alpha-1+u}) F_z = 0. \tag{5}$$

Collecting this information in the $(n+2) \times (n+2)$ matrix,

$$A = \begin{pmatrix} x_1 & 0 & & \cdots & 0 & 0 \\ 0 & x_2 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & \cdots & x_n & 0 & 0 \\ 0 & 0 & \cdots & 0 & \beta y & -y^u (tG + \beta x^b z^{\beta-1+t}) \\ v_1 z & v_2 z & \cdots & v_n z & \alpha z & z^t (uG + \alpha x^a y^{\alpha-1+u}) \end{pmatrix}$$

it follows from (1) and (4) that the first $n+1$ entries of $(\nabla F)A$ are congruent to 0 modulo F, while (5) implies that the last entry of $(\nabla F)A$ already equals 0 in R. Finally, it is straightforward that

$$\det A = (\beta\alpha + u\beta + t\alpha)F \quad \text{and} \quad \beta\alpha + u\beta + t\alpha \neq 0,$$

whence we conclude from Saito's criterion in Theorem 2.4 that F is a free divisor.

Next we show that if F is a *homogeneous* free divisor then conditions (a) and (b) are satisfied. We argue by contradiction. Suppose that F is a free divisor that involves all variables, but fails one of the conditions (a) or (b). By symmetry, and after permutating the variables, we may assume that F is of the form:

$$F = x^a y^\alpha z^\beta + x^b,$$

where we set $y = x_{n+1}$ and $z = x_{n+2}$ as before, and $a, b \in \mathbf{N}^n$, $\alpha > 0$, $\beta > 0$. With J again the Jacobian ideal of F, note that $(y, z) \subseteq (J : x^a y^{\alpha-1} z^{\beta-1})$. Since J is perfect of codimension 2, either (y, z) is a minimal prime of J or $x^a y^{\alpha-1} z^{\beta-1} \in J$. In the former case, $F \in J \subset (y, z)$ implies $x^b \in (y, z)$, and that is impossible. In the latter case,

$$x^a y^{\alpha-1} z^{\beta-1} \in J \subseteq (y^{\alpha-1} z^\beta, y^\alpha z^{\beta-1}) + (\partial x^b/\partial x_i;\ i = 1, \ldots, n),$$

and so $x^a y^{\alpha-1} z^{\beta-1}$ must be divisible by $\partial x^b/\partial x_i$ for some i. This contradicts the homogeneity of F. ◻

Example 6.2. A particular case of Theorem 6.1 has recently been presented independently by Simis and Tohaneanu [**11**, Proposition 2.11]: In our notation from the proof above, they take a homogeneous binomial of the form $G = x^a y^\alpha + z^\beta$, with $\alpha > 0$, $|a| + \alpha = \beta$, and $a_i \neq 0$ for $i = 2, \ldots, n$ in $x^a = x_1^{a_1} \cdots x_n^{a_n}$, so that G is homogeneous of degree β and the only potentially missing variable in the first summand is x_1. The authors then affirm that

$$\begin{aligned} &F = x_1 \cdots x_n (x^a y^\alpha + z^\beta) \quad \text{and} \\ &F = \frac{x_1 \cdots x_n}{x_i} y (x^a y^\alpha + z^\beta) \quad \text{for some } i = 1, \ldots, n, \end{aligned}$$

are homogeneous free divisors. Theorem 6.1 shows that, in each case, zF is a homogeneous free divisor as well.

7. "Divisors" of free divisors. The results of the previous sections show that:

(1) Any reduced homogeneous binomial has a multiple that is a free divisor by Theorem 6.1.

(2) If K is algebraically closed, then any quadric Q can be put in standard form $x_1^2 + \cdots + x_i^2$. Hence, it has a multiple that is a free divisor by Example 5.3.

(3) If f, g are free divisors in distinct sets of variables, then $f + g$ divides the free divisor $fg(f + g)$ by Corollary 4.2.

So we are led to ask:

Question 7.1. Let f be a (homogeneous) reduced polynomial. Does there exist a free divisor g such that f divides g?

This question is also raised and addressed in [**4, 8, 11**].

In light of the discussion above, the first case to look at is that of cubics in three variables. Again, by Example 5.3, we know that the Fermat cubic $x^3 + y^3 + z^3$ divides the free divisor $(x^3 + y^3)(x^3 + y^3 + z^3)$. So, what about other smooth cubics or smooth hypersurfaces in general? What we can prove is a negative result: it asserts that a smooth form, in $n > 2$ variables of degree larger than 2, times a product of n linearly independent linear forms is never a free divisor.

Theorem 7.2. *Let f be a smooth form of degree $k = \deg f > 2$ in $n > 2$ variables and $\ell_1, \ell_2, \dots, \ell_n$ linearly independent linear forms. Set $g = \ell_1 \cdots \ell_n f$, and denote $J(g) \subseteq R = K[x_1, \dots, x_n]$ the Jacobian ideal of g. Then one has:*

(1) *g is not a free divisor, instead*

(2) $\operatorname{depth} R/J(g) \leq \min(\max(0, n - k), n/2) < n - 2$.

In particular, if $k \geqslant n$ then $\operatorname{depth} R/J(g) = 0$.

Since $k > 2$ and $n > 2$ imply $\max(0, n - k) < n - 2$, assertion (1) indeed follows from (2) as claimed. To prove (2) in Theorem 7.2, we need to set up some notation. To avoid confusion, $\langle a_1, \dots, a_n \rangle$ will denote the vector with coordinates a_i, while $(a_1, \dots, a_n)$ denotes the ideal or module generated by the a_i. For a form f, we set $\widehat{f_i} = x_i f_i + f$, with $f_i = \partial f / \partial x_i$ as before.

Lemma 7.3. *Let f be a form in $K[x_1, \dots, x_n]$. If $g = x_1 \cdots x_n f$ is reduced, then the ideals $J(g)$ and $(x_i f_i;\ i = 1, \dots, n)$ of R have the same projective dimension. In particular, g is a free divisor if, and only if, $(x_i f_i;\ i = 1, \dots, n)$ is perfect of codimension 2.*

Proof. Set $y_i = x_1 \cdots x_n / x_i$, and note that $g_i = y_i \widehat{f_i}$. If $\langle \alpha_1, \dots, \alpha_n \rangle$ is a syzygy of ∇g, then $\langle \alpha_1 \widehat{f_1}, \dots, \alpha_n \widehat{f_n} \rangle$ is thus a syzygy of $\langle y_1, \dots, y_n \rangle$. By the Hilbert-Burch theorem, the syzygy module of $\langle y_1, \dots, y_n \rangle$ is generated by $x_1 e_1 - x_i e_i$ with $i = 2, \dots, n$, whence there exist polynomials $a_2, \dots, a_n$ such that

$$\begin{aligned} \alpha_1 \widehat{f_1} &= (a_2 + \cdots + a_n) x_1 \quad \text{and} \\ \alpha_i \widehat{f_i} &= -a_i x_i \quad \text{for } i = 2, \dots, n. \end{aligned}$$

Since g is squarefree, x_i does not divide f, whence that variable must divide α_i for each i. In other words, $\alpha_i = x_i \beta_i$ for suitable $\beta_i \in R$, and then $\langle \beta_1, \dots, \beta_n \rangle$ is a syzygy of $\langle \widehat{f_1}, \dots, \widehat{f_n} \rangle$.

Therefore, the R-linear map $\psi : R^n \to R^n$ sending e_i to $x_i e_i$ induces an isomorphism between the syzygy module of $\langle \widehat{f_1}, \dots, \widehat{f_n} \rangle$ and the syzygy module of ∇g.

Because f is homogeneous, one has the Euler relation $f = (1/k) \sum_i x_i f_i$, whence

$$(\widehat{f_i};\ i = 1, \dots, n) \subseteq (x_i f_i; i = 1, \dots, n).$$

Using the Euler relation once more, one obtains as well $\sum_{i=1}^n \widehat{f_i} = (\deg f + n) f$; thus, $f \in (\widehat{f_i};\ i = 1, \dots, n)$, and then also

$$(x_i f_i; i = 1, \dots, n) \subseteq (\widehat{f_i};\ i = 1, \dots, n).$$

Accordingly, these ideals agree.

It follows that the first syzygy module of the ideal $J(g)$ and that of the ideal $(x_1 f_1, \dots, x_n f_n)$ differ only by a free summand–whose rank is in fact the K-dimension of the vector space of Euler vector fields annihilating f. So the statement follows. □

Example 7.4. Let us illustrate the preceding result.

(a) Consider $f = \sum_{i=1}^k u_i M_i$ with $0 \neq u_i \in K$, with M_i pairwise coprime monomials of same degree, and set $g = x_1 \cdots x_n f$. Then

$\operatorname{depth} R/J(g) = n - k$, because here the ideal $(x_i f_i)_{i=1,\dots,n}$ is the complete intersection ideal $(M_1, \dots, M_k)$.

(b) Let f be the *Cayley form* in n variables, the elementary symmetric polynomial of degree $n-1$, that can be written

$$f = x_1 \cdots x_n (x_1^{-1} + \cdots + x_n^{-1}),$$

and consider $g = x_1 \cdots x_n f$.

Denoting J_k the ideal generated by all square-free monomials of degree k, it is well known that J_k is perfect of codimension $n-k+1$. The radical of the Jacobian ideal of f is easily seen to be J_{n-2}. So f is irreducible and, for $n \geqslant 3$, singular with singular locus of codimension 3.

On the other hand, one checks that $(x_i f_i;\ i = 1, \dots, n) = J_{n-1}$ and Lemma 7.3 therefore verifies that g is a free divisor, as was also observed in [**8**], where further a discriminant matrix is given.

(c) For a given form f, smooth and in generic coordinates, the elements $(x_i f_i)_i$ tend to form a regular sequence. In that case, the resolution of the first syzygy module of $J(g)$ is thus given by the corresponding tail of the Koszul complex on $(x_i f_i)_i$, shifted in degree and, therefore, $R/(x_i f_i)_i$ embeds as the nonzero Artinian submodule $H^0_{(x_i;\ i=1,\dots,n)}(R/J(g))$ into $R/J(g)$, forcing $\operatorname{depth} R/J(g) = 0$. As a concrete example, take a Fermat hypersurface $f = \sum_{i=1}^n x_i^k$, with $k \geqslant 1$, $n \geqslant 3$.

(d) For a subset A of $\{1, \dots, n\}$, set $x_A = \Pi_{i \in A} x_i$. With notation as in Lemma 7.3, one obviously has

$$(f_i;\ i \in A) \subseteq (x_i f_i;\ i = 1, \dots, n) : (x_A).$$

Accordingly, either $x_A \in (x_i f_i)_i$ or the projective dimension of $R/(x_i f_i)_i$ is at least the codimension of $R/(f_i;\ i \in A)$. In particular, if $\deg f > n$, then no such monomial is in $(x_i f_i)_i$, and we see again that $\operatorname{depth} R/J(g) = 0$.

The last example leads to the following result.

Proposition 7.5. *Assume $f \in R = K[x_1, \dots, x_n]$ with $n > 2$ is smooth of degree $k > 2$, and let $\ell_1, \dots, \ell_n$ be linearly independent* *linear forms. With $g = \ell_1 \cdots \ell_n f$, one then has*

$$\operatorname{depth} R/J(g) \leqslant \max(0, n-k).$$

Proof. Changing coordinates, we may assume that $\ell_i = x_i$. Set $v = \min(k, n)$. In view of Example 7.4 (d) to Lemma 7.3, it is enough to show that $x_1 \cdots x_v \notin (x_1 f_1, \dots, x_n f_n)$. If $k > n$ this is obvious. If $k \leq n$, then $v = k$, and we argue as follows. Suppose by contradiction that

$$(*) \qquad x_1 \cdots x_k = \sum_i \lambda_i x_i f_i$$

with $\lambda_i \in K$. Let $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be a monomial in the support of f that is different from $x_1 \cdots x_k$. From $(*)$, it follows that $\sum_{i=1}^n \lambda_i \alpha_i = 0$. If we show that the support of f contains at least n monomials different from $x_1 \cdots x_k$ whose exponents are linearly independent, we can conclude that $\lambda_i = 0$ for all i, thus, contradicting $(*)$. Since f is smooth, for each i, there exists some $j = j(i)$, such that the monomial $x_i^{k-1} x_j$ is in the support of f.

We claim that the exponents of $x_i^{k-1} x_{j(i)}$, for $i = 1, \dots, n$, are indeed linearly independent. To prove this, consider the linear map $h : \mathbf{C}^n \to \mathbf{C}^n$ defined as $h(e_i) = e_{j(i)}$. Any such map is easily seen to satisfy $(h^{n!} - 1)h^n = 0$, whence the eigenvalues of h are either 0 or roots of unity. In particular, no integer m with $|m| > 1$ is a root of the characteristic polynomial $\det(-tI + h)$ of h. Therefore, we have that $\det(-tI + h) \neq 0$ at $t = -k + 1$, and this proves the claim. □

As for a last ingredient, note the following.

Lemma 7.6. *If $f \in R = K[x_1, \dots, x_n]$ is smooth, then the codimension of $(x_i f_i)_{i=1,\dots,n}$ is at least $n/2$.*

Proof. Let P be a minimal prime of $I = (x_i f_i)_{i=1,\dots,n}$ in R. If c is the number of variables x_i contained in P, then that prime ideal contains at least $n - c$ of the f_i. Hence, P contains two regular sequences: one of length c and the other of length $n - c$. So the codimension of I is at least $n/2$. □

The *Proof of Theorem* 7.2 is now obtained by combining Lemma 7.3, Proposition 7.5 and Lemma 7.6. $\square$

Remark 7.7. As far as we know, in Example 7.4 (d), it might even be true that, for *any* smooth f in *any* system of coordinates, $x_1 \cdots x_n \notin (x_1 f_1, \ldots, x_n f_n)$, so that then, in particular, depth $R/J(g) = 0$ always.

However, for a smooth f, the ideal $(x_i f_i)_{i=1,\ldots,n}$ can be of codimension $n/2$, but, of course, only for n even. For example,

$$f = (x_1^{k-1} + x_2^{k-1})x_2 + (x_3^{k-1} + x_4^{k-1})x_4$$

is smooth, and the codimension of $(x_i f_i)_{i=1,\ldots,4}$ is 2. Nevertheless, in this case, $R/(x_i f_i)_{i=1,\ldots,n}$ still has depth 0 since $x_1x_2x_3x_4 \notin (x_i f_i)_{i=1,\ldots,n}$.

8. Extending free divisors into the tangent bundle. Let $R = K[x_1, \ldots, x_n]$ be as before, and set $R' = R[y_1, \ldots, y_n]$. Define a map $^* : R \to R'$ by

$$f^* = \sum_{i=1}^{n} y_i \frac{\partial f}{\partial x_i}$$

for every $f \in R$. Clearly, * is a K-linear derivation. For a matrix $C = (c_{ij})$ with entries in R, we set $C^* = (c_{ij}^*)$.

Theorem 8.1. *Let $f \in R$ be a homogeneous free divisor of degree $k > 0$. Then ff^* is a free divisor in R', in $2n$ variables and of total degree $2k$, that is linear if f is so.*

Proof. First note that ff^* is reduced because f^* is irreducible. By contradiction, if f^* were reducible, then, since f^* is homogeneous of degree 1 in the y's, the partial derivatives of f had a non-trivial common factor contradicting the fact that f is reduced.

Secondly, we identify a discriminant matrix for ff^*. Since f is homogeneous, a discriminant matrix for f can be constructed as follows. Because $J(f)$ is a perfect ideal of codimension 2, we can find a Hilbert-Burch matrix $B = (b_{ij})$ for $J(f)$, of size $n \times (n-1)$, such that the $(n-1)$-minor of B obtained by removing the i-th row is $(-1)^{i+1}\partial f/\partial x_i$.

Adjoining $x^T = (x_1, \dots, x_n)^T$ as a column to matrix B, we obtain the matrix

$$A = (B \mid x^T)$$

that is, by construction, a discriminant matrix for f. We now claim that the following $2n \times 2n$ block matrix

$$A' = \left(\begin{array}{cc|cc} B & x^T & 0 & 0 \\ \hline B^* & 0 & B & y^T \end{array}\right)$$

is a discriminant matrix for ff^*. Its determinant is clearly ff^*, by definition of A, B and f^*. The product rule yields

$$\nabla(ff^*) = f^*(\nabla_x(f), 0) + f(\nabla_x(f^*), \nabla_x(f)),$$

and hence,

$$\nabla(ff^*)A' = f^*(\nabla_x(f), 0)A' + f(\nabla_x(f^*), \nabla_x(f))A'.$$

Now $(\nabla_x(f), 0)A' = (\nabla_x(f)A, 0) \equiv 0 \bmod (f)$, and so it remains to show that

$$(\dagger) \qquad (\nabla_x(f^*), \nabla_x(f))A' \equiv 0 \bmod (f^*).$$

Expanding returns the vector

$$\begin{aligned}&(\nabla_x(f^*), \nabla_x(f))A' \\ &\qquad = (\nabla_x(f^*)B + \nabla_x(f)B^*, \nabla_x(f^*)x^T, \nabla_x(f)B, \nabla_x(f)y^T).\end{aligned}$$

Concerning its first part, note that $\nabla_x(f^*) = \nabla_x(f)^*$, whence

$$\begin{aligned}\nabla_x(f^*)B + \nabla_x(f)B^* &= (\nabla_x(f)B)^* \quad \text{because }^* \text{ is a derivation,} \\ &= 0^* = 0 \quad \text{as } \nabla_x(f)B = 0 \text{ by construction.}\end{aligned}$$

Regarding the second component,

$$\nabla_x(f^*)x^T = (k-1)f^* \equiv 0 \bmod (f^*),$$

because f^* is homogeneous of degree $k-1$ with respect to the variables x. Finally,

$$\begin{aligned}\nabla_x(f)B &= 0 && \text{by choice of } B\text{, and} \\ \nabla_x(f)y^T &= f^* && \text{by definition.}\end{aligned}$$

Therefore, (†) holds and ff^* is confirmed as a free divisor. The assertions on degree and number of variables are obvious from the construction.

A free divisor is linear if all entries in a discriminant matrix are linear, and this property is clearly inherited by A' from A. ◻

Remark 8.2. The geometric interpretation of the hypersurface defined by ff^* is as follows.

Viewing $f \in R$ as the function $f\colon \operatorname{Spec} R = \mathbb{A}^n_K \to \mathbb{A}^1_K = \operatorname{Spec} K[t]$, its differential fits into the exact Zariski-Jacobi sequence of Kähler differential forms

$$0 \leftarrow \Omega^1_{R/K[t]} \leftarrow \Omega^1_{R/K} \cong \oplus_i R dx_i \stackrel{df\partial/\partial t}{\longleftarrow} \Omega^1_{K[t]/K} \otimes_{K[t]} R \cong R\,dt,$$

and one may interpret $R' \cong \operatorname{Sym}_R \Omega^1_R$ as the ring of regular functions on the tangent bundle $T_X \cong \operatorname{Spec} R' \cong \mathbb{A}^{2n}_K$ over $X = \operatorname{Spec} R \cong \mathbb{A}^n_K$.

This identifies $R'/(f^*)$ with the regular functions on the total space of the affine relative tangent "subbundle" $T_{X/S} \subseteq T_X$, the kernel of the Jacobian map $df : T_X \to T_S$ that consists of the vector fields vertical with respect to (the fibers of) f over the affine line $S = \operatorname{Spec} K[t]$.

Accordingly, the hypersurface H defined by ff^* is the union of that affine "bundle" with $\operatorname{Spec} R'/(f)$, the restriction of the total tangent bundle T_X to $\operatorname{Spec} R/(f)$, in turn the fibre over 0 of the function f. Equivalently, $\operatorname{Spec} R'/(f)$ is the suspended free divisor obtained as the inverse image of $\operatorname{Spec} R/(f)$ along the structure morphism $p : T_X \to X$. Thus, $H = T_{X/S} \cup \operatorname{Spec} R'/(f) = df^{-1}(0) \cup (fp)^{-1}(0) \subseteq T_X$.

$$\begin{tikzcd}
 & T_{X/S} \arrow[rr] \arrow[dl, hook] \arrow[dr, hook] & & \{0\} \arrow[dr, hook] & & \\
H \arrow[rr, hook] & & T_X \arrow[rr, "df"] \arrow[dr] & & T_S \arrow[dr] & \\
 & (fp)^{-1}(0) \arrow[ul, hook] \arrow[ur, hook] \arrow[dr] & & X \arrow[rr, "f"] & & S \\
 & & f^{-1}(0) \arrow[rr] \arrow[ur, hook] & & \{0\} \arrow[ur, hook] &
\end{tikzcd}$$

Interesting examples are hard to visualize as they will live in four or more dimensions. However, the intersection of the two (unions of) components, $T_{X/S} \cap \operatorname{Spec} R'/(f) \subseteq \operatorname{Sing} H$ is easy to understand: Geometrically, over X it fibers into the union of the hyperplanes perpendicular to $\nabla f(x)$ for some $x \in X$ on $\{f = 0\}$, that is,

$$T_{X/S} \cap \operatorname{Spec} R'/(f) = \bigcup_{x, f(x)=0} \{(x, y) \in \mathbb{A}^n \times \mathbb{A}^n \mid \nabla f(x) y = 0\}\,.$$

Example 8.3. Applying Theorem 8.1 to the normal crossing divisor $x_1 \cdots x_n$, we find that

$$(x_1 \cdots x_n)^2 \sum_{i=1}^{n} \frac{y_i}{x_i}$$

is a linear free divisor.

Remarks 8.4. Various generalizations are possible:

(1) Given a homogeneous free divisor f in a polynomial ring of dimension n, one can iterate the use of Theorem 8.1 to get an infinite family $\{F_i\}_{i \in \mathbf{N}}$ of homogeneous free divisors, defined by $F_0 = f$ and $F_{i+1} = F_i F_i^*$, where * is, of course, to be understood relative to the polynomial ring containing F_i. By construction, F_i belongs to a polynomial ring of dimension $2^i n$, its degree equals $2^i \deg f$, and it is a linear free divisor if, and only if, f is linear.

Taking $F_0 = x$ as a seed, we obtain the sequence of linear free divisors

$$\begin{aligned} &x, xy, xy(xz_1 + yz_2),\ xy(xz_1 + yz_2) \\ &\qquad (2xyz_1u_1 + y^2z_2u_1 + x^2z_1u_2 + 2xyz_2u_2 + x^2yu_3 + xy^2u_4), \ldots \end{aligned}$$

in $K[x, y, z_1, z_2, u_1, \ldots, u_4, \ldots]$.

(2) Theorem 8.1 holds also for free divisors that are weighted homogeneous of degree $d \neq 0$ with respect to some weight vector $w = (w_1, \ldots, w_n) \in \mathbf{Z}^n$. In the proof one simply replaces the column vector x^T in the discriminant matrix with $(w_1x_1, \ldots, w_nx_n)^T$. Again, linearity is preserved.

One can further generalize Theorem 8.1, as well as Remark 8.4 (1), also as follows, incorporating right away the weighted homogeneous version as in Remark 8.4 (2).

Theorem 8.5. *With notation as before, assume f weighted homogeneous of degree $d \neq 0$ with respect to some weight vector $w = (w_1, \dots, w_n) \in \mathbf{Z}^n$.*

With $m \geqslant 1$, let $R' = R[y_{ij} : 1 \leq i \leq n, 1 \leq j \leq m]$, assign weights $|y_{ij}| = w_i$, and set $f^{\{*_j\}} = \sum_i y_{ij}\partial f/\partial x_i$. Then $f \prod_{j=1}^m f^{\{*_j\}}$ is a free divisor in $(m+1)n$ variables of weighted homogeneous degree $(m+1)d$ that will be linear along with f.

Proof. The proof is a simple variation of the one given for $m = 1$. For instance, if $m = 2$, the discriminant matrix can be taken as

$$\left(\begin{array}{cc|cc|cc} B & wx^T & 0 & 0 & 0 & 0 \\ \hline B^{\{*_1\}} & 0 & B & wy_1^T & 0 & 0 \\ \hline B^{\{*_2\}} & 0 & 0 & 0 & B & wy_2^T \end{array}\right)$$

where $wx = (w_1x_1, \dots, w_nx_n)$, with wy_1, wy_2 analogous abbreviations. $\square$

In this way, one may obtain any normal crossing divisor $x_0 \cdots x_m$, starting from $f = x_0$ and using $f^{\{*_j\}} = x_j\partial f/\partial x_0 = x_j$ for $j = 1, \dots, m$.

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Department of Computer and Mathematical Sciences, University of Toronto Scarborough, Toronto, Ontario M1A 1C4, Canada
Email address: ragnar@utsc.utoronto.ca

Dipartimento di Matematica, Universitá di Genova, Via Dodecaneso 35, I-16146 Genova, Italy
Email address: conca@dima.unige.it