I. INTRODUCTION

Usually, the observables in quantum mechanics are represented by self-adjoint operators that are in one-to-one correspondence with projection valued operator measures. However, in many applications (as, for example, quantum optics, quantum theory of measurement, quantization of classical dynamical systems, and localization observables of relativistic massless particles) this characterization is restrictive and one has to consider a more general description by means of positive operator valued measures (for a review of the applications in physics, see Refs. 3, 6, and 9–11).

In particular, it is of interest both in quantum mechanics and in signal analysis to describe positive operator valued measures that are covariant with respect to a unitary representation of a symmetry group $G$ (for a review see Ref. 1 and references therein). In this framework, it is well known$^{4,13}$ that, given a square-integrable irreducible representation $\pi$ of a unimodular group $G$ and a trace class, trace one positive operator $T$, the family of operators

$$Q(X) = \int_X \pi(g)T\pi(g^{-1})d\mu_G(g)$$

defines a positive operator valued measure (POVM) on $G$ covariant with respect to $\pi$ ($\mu_G$ is a Haar measure on $G$). In this article, we prove that all the covariant POVMs are of the above form for some $T$. More precisely, we show this result for non-unimodular groups and for POVMs based on the quotient space $G/Z$, where $Z$ is a central subgroup.

Let $G$ be a locally compact second countable topological group and $Z$ be a central closed subgroup. We denote by $G/Z$ the quotient group and by $g \in G/Z$ the equivalence class of $g \in G$. If $a \in G$ and $g \in G/Z$, we let $a[g] = ag$ be the natural action of $a$ on the point $g$.

Let $B(G/Z)$ be the Borel $\sigma$-algebra of $G/Z$. We fix a left Haar measure $\mu_{G/Z}$ on $G/Z$. Moreover, we denote by $\Delta$ the modular function of $G$ and of $G/Z$. 

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By representation we mean a strongly continuous unitary representation of $G$ acting on a complex and separable Hilbert space, with scalar product $\langle \cdot, \cdot \rangle$ linear in the first argument.

Let $(\pi, \mathcal{H})$ be a representation of $G$. A positive operator valued measure $Q$ defined on $G/Z$ and such that

1. $Q(G/Z) = I$;
2. for all $X \in \mathcal{B}(G/Z)$,
   $$\pi(g)Q(X)\pi(g^{-1}) = Q(g[X]) \quad \forall g \in G$$

is called $\pi$-covariant POVM on $G/Z$.

Given a representation $(\sigma, \mathcal{K})$ of $Z$, we denote by $(\lambda^\sigma, \mathcal{H}^\sigma, \mathcal{K})$ the imprimitivity system unitarily induced by $\sigma$. We recall that $\mathcal{H}^\sigma$ is the Hilbert space of $(\mu_G$-equivalence classes of) functions $f : G \to \mathcal{K}$ such that

1. $f$ is weakly measurable;
2. for all $z \in Z$,
   $$f(gz) = \sigma(z^{-1})f(g) \quad \forall g \in G;$$
3. $$\int_{G/Z} \|f(g)\|^2_{\mathcal{K}} d\mu_{G/Z}(g) < +\infty$$

with scalar product

$$\langle f_1, f_2 \rangle_{\mathcal{H}^\sigma} = \int_{G/Z} \langle f_1(g), f_2(g) \rangle_{\mathcal{K}} d\mu_{G/Z}(g).$$

The representation $\lambda^\sigma$ acts on $\mathcal{H}^\sigma$ as

$$(\lambda^\sigma(a)f)(g) := f(a^{-1}g), \quad g \in G,$$

for all $a \in G$. The projection valued measure $P^\sigma$ is given by

$$(P^\sigma(X)f)(g) := \chi_X(g)f(g), \quad g \in G,$$

for all $X \in \mathcal{B}(G/Z)$, where $\chi_X$ is the characteristic function of the set $X$.

We recall some basic properties of square integrable representations modulo a central subgroup. We refer to Ref. 2 for $G$ unimodular and $Z$ arbitrary and to Ref. 7 for $G$ non-unimodular and $Z = \{e\}$. Combining these proofs, one obtains the following result.

Proposition 1: Let $(\pi, \mathcal{H})$ be an irreducible representation of $G$ and $\gamma$ be the character of $Z$ such that

$$\pi(z) = \gamma(z)I_{\mathcal{H}} \quad \forall z \in Z.$$

The following facts are equivalent:

1. there exists a vector $u \in \mathcal{H}$ such that
   $$0 < \int_{G/Z} \|u, \pi(g)u\|_{\mathcal{H}}^2 d\mu_{G/Z}(g) < +\infty;$$

2. $(\pi, \mathcal{H})$ is a subrepresentation of $(\lambda^\gamma, \mathcal{H}^\gamma)$.

If either of the above conditions is satisfied, there exists a self-adjoint injective positive operator $C$ such that

$$\pi(g)C = \Delta(g)^{-1/2}C\pi(g) \quad \forall g \in G,$$
and an isometry $\Sigma: \mathcal{H} \otimes \mathcal{H}^* \rightarrow \mathcal{H}^\gamma$ such that

1. for all $u \in \mathcal{H}$ and $v \in \text{dom} \ C$
\[
\Sigma(u \otimes v^*)(g) = (u, \pi(g) Cv)_{\mathcal{H}}, \quad g \in G;
\]

2. for all $g \in G$
\[
\Sigma(\pi(g) \otimes I_{\mathcal{H}^*}) = \lambda(g) \Sigma;
\]

3. the range of $\Sigma$ is the isotypic space of $\pi$ in $\mathcal{H}^\gamma$.

If Eq. (1) is satisfied, $(\pi, \mathcal{H})$ is called square-integrable modulo $Z$. The square of $C$ is called formal degree of $\pi$ (see Ref. 7). In particular, when $G$ is unimodular, $C$ is a multiple of the identity.

II. CHARACTERIZATION OF $Q$

We fix an irreducible representation $(\pi, \mathcal{H})$ of $G$ and let $\gamma$ be the character such that $\pi|_Z = \gamma I_{\mathcal{H}}$. The following theorem characterizes all the POVM on $G/Z$ covariant with respect to $\pi$ in terms of positive trace one operators on $\mathcal{H}$.

**Theorem 2:** The irreducible representation $\pi$ admits a covariant POVM based on $G/Z$ if and only if $\pi$ is square-integrable modulo $Z$.

In this case, let $C$ be the square root of the formal degree of $\pi$. There exists a one-to-one correspondence between covariant POVMs $Q$ on $G/Z$ and positive trace one operators $T$ on $\mathcal{H}$ given by

\[
\langle Q(T(X)v, u)_{\mathcal{H}} \rangle = \int_X \langle TC \pi(g^{-1})v, C\pi(g^{-1})u \rangle_{\mathcal{H}} \, d\mu_{G/Z}(g)
\]

for all $u, v \in \text{dom} \ C$ and $X \in \mathcal{B}(G/Z)$.

**Proof:** Let $Q$ be a $\pi$-covariant POVM. According to the generalized imprimitivity theorem there exists a representation $(\sigma, \mathcal{K})$ of $Z$ and an isometry $W: \mathcal{H} \rightarrow \mathcal{H}^\sigma$ intertwining $\pi$ with $\lambda^\sigma$ such that

\[
Q(X) = W^* P^\sigma(X) W
\]

for all $X \in \mathcal{B}(G/Z)$.

Define the following closed invariant subspace of $\mathcal{K}$

\[
\mathcal{K}_\gamma = \{ v \in \mathcal{K} | \sigma(z) v = \gamma(z) v \}.
\]

Let $\sigma_1$ and $\sigma_2$ be the restrictions of $\sigma$ to $\mathcal{K}_\gamma$ and $\mathcal{K}_\gamma^\perp$, respectively. The induced imprimitivity system $(\lambda^\sigma, P^\sigma, \mathcal{H}^\sigma)$ decomposes into the orthogonal sum

\[
\mathcal{H}^\sigma = \mathcal{H}^{\sigma_1} \oplus \mathcal{H}^{\sigma_2}.
\]

If $f \in \mathcal{H}^{\sigma_1}$ and $z \in Z$, then

\[
(\lambda^\sigma(z) f)(g) = f(z^{-1}g) = f(gz^{-1}) = \sigma(z) f(g), \quad g \in G.
\]

On the other hand, if $u \in \mathcal{H}$ and $z \in Z$, we have

\[
(\lambda^\sigma(z) Wu)(g) = (W \pi(z) u)(g) = \gamma(z) (Wu)(g), \quad g \in G.
\]

It follows that $(Wu)(g) \in \mathcal{K}_\gamma$ for $\mu_G$—almost every $g \in G$, that is, $Wu \in \mathcal{H}^{\sigma_1}$. So it is not restrictive to assume that
\( \sigma = \gamma I_K \)

for some Hilbert space \( K \). Clearly, we have

\[
\mathcal{H}^\sigma = \mathcal{H}^\gamma \otimes K, \quad \lambda^\sigma = \lambda^\gamma \otimes I_K.
\]

In particular, \( \pi \) is a subrepresentation of \( \lambda^\gamma \), hence it is square-integrable modulo \( \mathbb{Z} \).

Due to Proposition 1, the operator \( W' = (\Sigma^* \otimes I_K)W \) is an isometry from \( \mathcal{H} \) to \( \mathcal{H} \otimes \mathcal{H}^* \otimes \mathcal{K} \) such that

\[
W' \pi(g) = (\pi(g) \otimes I_{\mathcal{H}^* \otimes \mathcal{K}})W' \quad \forall g \in G.
\]

Since \( \pi \) is irreducible, given Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), a standard result asserts that \( \mathcal{C}(\pi \otimes I_{\mathcal{H}_1}, \pi \otimes I_{\mathcal{H}_2}) = I_{\mathcal{H}} \otimes L(\mathcal{H}_1, \mathcal{H}_2) \). In the present case, this means that

\[
W'u = u \otimes B \quad \forall u \in \mathcal{H}
\]

for some \( B \in \mathcal{H}^* \otimes \mathcal{K} \). Since \( W' \) is isometric, \( B \) has Hilbert–Schmidt norm 1.

Let \( (e_i)_{i \geq 1} \) be an orthonormal basis of \( \mathcal{H} \) such that \( e_i \in \text{dom } C \). Then

\[
B = \sum_i e_i^* \otimes k_i,
\]

where \( k_i \in \mathcal{K} \) and \( \sum_i \|k_i\|_{\mathcal{K}}^2 = 1 \).

If \( u \in \text{dom } C \), one has that

\[
(Wu)(g) = [(\Sigma \otimes I_K)(u \otimes B)](g)
\]

\[
= \sum_i \langle \Sigma(u \otimes e_i^*), k_i \rangle
\]

\[
= \sum_i \langle u, \pi(g)Ce_i \rangle \otimes k_i
\]

\[
= \sum_i \langle Ce_i, \pi(g)^{-1}u \rangle \otimes k_i
\]

\[
= \sum_i (e_i^* \otimes k_i)(C \pi(g^{-1})u),
\]

where the series converges in \( \mathcal{H}^\sigma \). On the other hand, for all \( g \in G \) the series \( \sum_i (e_i^* \otimes k_i) \times (C \pi(g^{-1})u) \) converges to \( BC \pi(g^{-1})u \), where we identify \( \mathcal{H}^* \otimes \mathcal{K} \) with the space of Hilbert–Schmidt operators. By uniqueness of the limit,

\[
(Wu)(g) = BC \pi(g^{-1})u, \quad g \in G.
\]

If \( u, v \in \text{dom } C \), the corresponding covariant POVM is given by

\[
\langle Q(X)v, u \rangle = \langle P^\sigma(X)Wv, Wu \rangle_{\mathcal{H}^\sigma}
\]

\[
= \int_{G/\mathbb{Z}} \Delta_X(\hat{g}) (BC \pi(g^{-1})v, BC \pi(g^{-1})u)_{\mathcal{H}} d\mu_{G/\mathbb{Z}}(\hat{g})
\]

\[
= \int_X \langle TC \pi(g^{-1})v, C \pi(g^{-1})u \rangle_{\mathcal{H}} d\mu_{G/\mathbb{Z}}(\hat{g}),
\]
where

$$T := B^* B$$

is a positive trace class trace one operator on $\mathcal{H}$.

Conversely, assume that $\pi$ is square integrable and let $T$ be a positive trace class trace one operator on $\mathcal{H}$. Then

$$B := \sqrt{T}$$

is a (positive) operator belonging to $\mathcal{H}^* \otimes \mathcal{H}$ such that $B^* B = T$ and $\|B\|_{\mathcal{H}^* \otimes \mathcal{H}} = 1$. The operator $W$ defined by

$$Wv := (\Sigma \otimes I_\mathcal{H})(v \otimes B) \quad \forall v \in \mathcal{H}$$

is an isometry intertwining $(\pi, \mathcal{H})$ with the representation $(\lambda^\sigma, \mathcal{H}^\sigma)$, where

$$\sigma = \gamma I_\mathcal{H}.$$ 

Define $Q_T$ by

$$Q_T(X) = W^* P^\sigma(X) W, \quad X \in \mathcal{B}(G/Z).$$

With the same computation as above, one has that

$$\langle Q_T(X)u, v \rangle_{\mathcal{H}} = \int_X \langle TC\pi(g^{-1})u, C\pi(g^{-1})v \rangle_{\mathcal{H}} d\mu_{G/Z}(g)$$

for all $u, v \in \text{dom} \ C$.

Finally, we show that the correspondence $T \mapsto Q_T$ is injective. Let $T_1$ and $T_2$ be positive trace one operators on $\mathcal{H}$, with $Q_{T_1} = Q_{T_2}$. Set $T = T_1 - T_2$. Since $\pi$ is strongly continuous, for all $u, v \in \text{dom} \ C$ the map

$$G/Z \ni g \mapsto \langle TC\pi(g^{-1})v, C\pi(g^{-1})u \rangle_{\mathcal{H}} = \Delta(g)^{-1} \langle T \pi(g^{-1})Cu, \pi(g^{-1})Cu \rangle_{\mathcal{H}} \in \mathbb{C}$$

is continuous. Since

$$\int_X \langle TC\pi(g^{-1})v, C\pi(g^{-1})u \rangle_{\mathcal{H}} d\mu_{G/Z}(g) = \langle [Q_{T_1}(X) - Q_{T_2}(X)]v, u \rangle_{\mathcal{H}} = 0$$

for all $X \in \mathcal{B}(G/Z)$, we have

$$\langle TC\pi(g^{-1})v, C\pi(g^{-1})u \rangle_{\mathcal{H}} = 0 \quad \forall g \in G/Z.$$ 

In particular,

$$\langle TCu, Cu \rangle_{\mathcal{H}} = 0,$$

so that, since $C$ has dense range, $T = 0$. 

Remark 3: Scutaru shows in Ref. 13 that there exists a one-to-one correspondence between positive trace one operators on $\mathcal{H}$ and covariant POVMs $Q$ based on $G/Z$ with the property

$$\text{tr}Q(K) < +\infty$$

for all compact sets $K \subset G/Z$. Theorem 2 shows that every covariant POVM $Q$ based on $G/Z$ shares property (3).
Remark 4: If \( G \) is unimodular, then \( C = \lambda I \), with \( \lambda > 0 \), and one can normalize \( \mu_{G/Z} \) so that \( \lambda = 1 \). Hence,

\[
Q_T(X) = \int_X \pi(g) T \pi(g^{-1}) \, \text{d}\mu_{G/Z}(g) \quad \forall X \in \mathcal{B}(G/Z),
\]

the integral being understood in the weak sense.

Remark 5: If \( T = \eta^* \otimes \eta \), with \( \eta \in \text{dom } C \) and \( \| \eta \|_{H^1} = 1 \), we observe that

\[
\langle Q_T(X)v, u \rangle_{H^1} = \int_X \langle C \pi(g^{-1})v, \eta \rangle_{H^1} \langle \eta, C \pi(g^{-1})u \rangle_{H^1} \, \text{d}\mu_{G/Z}(g)
\]

\[
= \int_X \langle v, \pi(g) C \eta \rangle_{H^1} \langle \pi(g) C \eta, u \rangle_{H^1} \, \text{d}\mu_{G/Z}(g)
\]

\[
= \int_X (W_{C \eta}v)(g)(W_{C \eta}u)(g) \, \text{d}\mu_{G/Z}(g)
\]

for all \( u, v \in \text{dom } C \), where \( W_{C \eta} : H \to H^1 \) is the wavelet operator associated to the vector \( C \eta \). In particular,

\[Q_T(X) = W_{C \eta}^* P^\gamma(X) W_{C \eta}.\]

III. TWO EXAMPLES

A. The Heisenberg group

In quantum mechanics, the study of positive operator valued measures covariant with respect to suitable representations of the Heisenberg group is motivated by two problems. They appear as a natural tool in the construction of coherent states associated with the quantum harmonic oscillator (see, for example, Ref. 1). Moreover, they describe the possible localization observables on the phase space of a one dimensional classical particle (for an account, see Ref. 12).

The Heisenberg group \( H \) is \( \mathbb{R}^3 \) with composition law

\[
(p, q, t)(p', q', t') = \left( p + p', q + q', t + t' + \frac{pq' - qp'}{2} \right).
\]

The center of \( H \) is

\[Z = \{(0,0,t) | t \in \mathbb{R}\},\]

and the quotient group \( G/Z \) is isomorphic to the Abelian group \( \mathbb{R}^2 \), with projection

\[q(p, q, t) = (p, q).\]

The Heisenberg group is unimodular with Haar measure

\[\text{d}\mu_{G/Z}(p, q) = \frac{1}{2\pi} \, \text{d}p \text{d}q.\]

Given an infinite dimensional Hilbert space \( \mathcal{H} \) and an orthonormal basis \( (e_n)_{n \geq 1} \), let \( a, a^* \) be the corresponding ladder operators. Define
$Q = \text{closure of } \frac{1}{\sqrt{2}}(a + a^*)$, \\
$P = \text{closure of } \frac{1}{\sqrt{2i}}(a - a^*)$.

It is known\textsuperscript{4,8} that the representation

$$\pi(p,q,t) = e^{i(t+pQ+qP)}$$

is square-integrable modulo $\mathbb{Z}$ and $C=1$.

It follows from Theorem 2 that any $\pi$-covariant POVM $Q$ based on $\mathbb{R}^2$ is of the form

$$Q(X) = \frac{1}{2\pi} \int e^{i(pQ+qP)} T e^{-i(pQ+qP)} dp dq, \quad X \in \mathcal{B}(\mathbb{R}^2),$$

for some positive trace one operator on $\mathcal{H}$. Up to our knowledge, the complete classification of the POVMs on $\mathbb{R}^2$ covariant with respect to the Heisenberg group has been an open problem till now.

### B. The $ax + b$ group

The $ax + b$ group is the semidirect product $G = \mathbb{R} \ltimes \mathbb{R}_+$, where we regard $\mathbb{R}$ as additive group and $\mathbb{R}_+$ as multiplicative group. The composition law is

$$(b,a)(b',a') = (b + ab', aa').$$

The group $G$ is nonunimodular with left Haar measure

$$d\mu_G(b,a) = a^{-2} db da$$

and modular function

$$\Delta(b,a) = \frac{1}{a}.$$ 

Let $\mathcal{H} = L^2((0,\infty), dx)$ and $(\pi^+, \mathcal{H})$ be the representation of $G$ given by

$$[\pi^+(b,a)f](x) = a^{1/2} e^{2\pi ibx} f(ax), \quad x \in (0,\infty).$$

It is known\textsuperscript{8} that $\pi$ is square-integrable, and the square root of its formal degree is

$$(Cf)(x) = \Delta(0,x)^{1/2} f(x) = x^{-1/2} f(x), \quad x \in (0,\infty),$$

acting on its natural domain.

By means of Theorem 2 every POVM based on $G$ and covariant with respect to $\pi^+$ is described by a positive trace one operator $T$ according to Eq. (2). Explicitly, let $(\epsilon_i)_{i\geq 1}$ be an orthonormal basis of eigenvectors of $T$ and $\lambda_i \geq 0$ be the corresponding eigenvalues. If $u \in L^2((0,\infty), dx)$ is such that $x^{-1/2} u \in L^2((0,\infty), dx)$, the $\pi^+$-covariant POVM corresponding to $T$ is given by
\[ \langle Q_T(X)u,u \rangle_H = \int_X \langle TC \pi^+(g^{-1})u,C \pi^+(g^{-1})u \rangle_H d\mu_G(g) \]
\[ = \int_X \sum_i \lambda_i \langle (C \pi^+(g^{-1})u,e_i) \rangle_H^2 d\mu_G(g) \]
\[ = \sum_i \lambda_i \int_X \left| \int_{\mathbb{R}^+} x^{-\frac{1}{2}} a^{-\frac{1}{2}} e^{-2\pi ibx/a} u \left( \frac{x}{a} \right) e_j(x) dx \right|^2 a^{-2} db da. \]