Square-integrable imprimitivity systems

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We give a definition of square-integrability for imprimitivity systems and we prove
that the square-integrable ones share most of the properties of square-integrable
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I. INTRODUCTION

In the first part of their seminal paper,1 Duflo and Moore show the main properties of square-
integrable representations for nonunimodular locally compact topological groups. By means of
these general results, most of the properties shared by continuous wavelet transforms, generalized
Fourier operators and generalized coherent states can be easily proved.2,3

However, there are cases that require extensions of the above theory. For example, for classical
coherent states associated with the quantum harmonic oscillator, one has to consider a
representation of the Weyl–Heisenberg group that is square-integrable modulo a subgroup. To the
best of our knowledge, this notion was introduced by Borel in the case of unimodular groups and
central subgroups,4 and it includes the definition of square-integrability for projective
representations.5 In recent years, Ali extended this notion to arbitrary subgroups, considering only
finite dimensional representations of such subgroups.6

Another example is the case of the Euclidean, Poincaré, and Galilei groups, which do not have
square-integrable representations at all. To treat these groups, a definition of square-integrability
on homogeneous spaces was proposed by the use of sections, for a review, see Ref. 7; However,
in this case, due to the dependence on the section, one loses the covariance properties of the
corresponding wavelet transform as well as the existence of orthogonality relations.

Finally, the Gabor analysis,8,9 is a generalization to an arbitrary locally compact Abelian
group $G$ of the short time Fourier transform on the real line (this transform gives rise to the
classical coherent states for a suitable choice of the window function). This theory is based on the
properties of the time-frequency shift operators that define a projective square-integrable repre-
sentation of the direct product $G \times G$.

In this paper we consider imprimitivity systems for a locally compact second countable
topological group based on a locally compact second countable topological space with a continuous
action of the group. For these systems we propose a definition of square-integrability having
three main features.

First of all, given a square-integrable imprimitivity system, there exists an isometry intertwining
this system with a canonical one, playing the role of the left regular representation. Moreover,
the existence of such an isometry is a sufficient condition for square-integrability. Finally, under
some weak assumptions, the discrete part of the canonical imprimitivity system decomposes into
its irreducible components by means of the existence of orthogonality relations. We will prove
these results without the use of any theorem on the structure of imprimitivity systems.

The second feature is that, in the case of transitive imprimitivity systems, the Mackey inducing functor preserves the notion of square-integrability. Indeed, the square-integrability of such an imprimitivity system is equivalent to the square-integrability of the inducing representation of the stability subgroup.

Finally, recalling that, for groups that are semidirect products with normal Abelian factor, there is a correspondence among representations and imprimitivity systems, a representation of such groups is square-integrable if and only if the corresponding imprimitivity system is square-integrable.

At the end of the paper we apply our theory to some examples. In particular, we show that the short time Fourier transform (as well as its generalization in the framework of Gabor analysis) is the isometry associated with the imprimitivity system canonically defined by position and momentum operators for the one-dimensional quantum particle (or, in the context of signal analysis, by the time-frequency shift operators). This is an indication that our results provide the natural abstract framework for the extension of Gabor analysis to an arbitrary Abelian group. The need for this general setup is witnessed, for instance, in Ref. 9.

In this paper we focus the attention on the abstract harmonic analysis viewpoint. Nevertheless, we believe that our results can be useful in the discretisation problems and their associated numerical algorithms arising in signal process and pattern recognition, see, as a source of references, the book containing papers in Refs. 8 and 9. Moreover, they can be used, for example, in the context of quantum mechanics on phase-space,\textsuperscript{10} and in quantum tomography.\textsuperscript{11}

The notion of imprimitivity system is a central one in the theory of group representations. It was introduced by Frobenius and developed later on by Mackey in his seminal work on the classification of unitary representations for groups with normal factors. From the physical point of view, it played an important role in quantum mechanics in connection with the problem of localizability of particles, see the fundamental papers.\textsuperscript{12,13}

II. MATHEMATICAL NOTATIONS

In this section we introduce the mathematical notations we will use in the paper.

If $X$ is a locally compact second countable (lcsc) topological space (in the following all the topological spaces are assumed to be Hausdorff), we let $\mathcal{C}_c(X)$ be the vector space of the compactly supported continuous functions on $X$ and $C_0(X)$ be the Banach space of the continuous functions vanishing at infinity with the sup norm $\|\cdot\|_{\text{sup}}$. We consider $C_0(X)$ as a commutative $C^*$-algebra with respect to the pointwise multiplication $(f_1,f_2)\mapsto f_1\cdot f_2$ and the complex conjugation $f\mapsto \bar{f}$. We denote by $\mathcal{B}(X)$ the $\sigma$-algebra of the Borel subsets of $X$. A \textit{(positive) measure} is a positive linear form on $C_c(X)$, and a \textit{(complex) bounded measure} is a continuous linear form on $C_0(X)$. As usual, we use the same symbol for the extensions of the above linear forms to $\mathcal{L}^1(X)$.

Given a lcsc topological group $G$ with identity $e$, we denote by $dg$ a left invariant Haar measure and by $\Delta_G$ its modular function. By $G$-space, we mean a lcsc topological space endowed with a continuous action of $G$. If $X$ is such a space, we denote by $g[x]$ the action of $g\in G$ on the point $x\in X$, and by $G[x]$ the orbit at $x$. Given a map $f$ defined on $X$, we let, for all $g\in G$,

$$f^g(x)=f(g^{-1}[x]) \quad x\in X,$$

and, fixed $g\in G$, we define the operator $l_g$ acting in $C_0(X)$ as $l_g f=f^g$. Obviously, $l_g$ is a well-defined isometric $\ast$-homomorphism of $C_0(X)$ and

$$l_{g_1g_2}=l_{g_1}l_{g_2}, \quad l_{g^{-1}}=l_g^{-1}, \quad g_1,g_2,g\in G.$$

Moreover, given a measure $\nu$ on $X$, we denote by $\nu^g$ the measure on $X$,

$$\int_X f(x)\,d\nu^g(x) = \int_X f^g(x)\,d\nu(x) \quad f\in C_c(X).$$
A positive nonzero measure $\nu$ on $X$ is said to be \textit{quasi-invariant} if, for all $g \in G$, $\nu^g$ is equivalent to $\nu$, \textit{relatively invariant} if $\nu^g$ is proportional to $\nu$, and \textit{invariant} if $\nu^g = \nu$.

By \textit{Hilbert space} we mean a complex separable Hilbert space with scalar product $(\cdot, \cdot)$ linear in the first argument. We denote by $\| \cdot \|$ the corresponding norm. We use the word \textit{representation} (of a lcsc group $G$ acting in a Hilbert space) to mean a unitary representation of $G$ that is continuous with respect to the strong operator topology.

Let $G$ be a lcsc group and $X$ a $G$-space. We consider the topological product $G \times X$ as a $G$-space with respect to the following action of $G$:

$$g[(h,x)] = (gh, g[h,x]), \quad g \in G, (h,x) \in G \times X.$$ 

We introduce the following notation. If $\phi_1$ and $\phi_2$ are two complex functions such that, for all $(g,x) \in G \times X$, the function

$$G \ni h \mapsto \phi_1(h,x)\phi_2(h^{-1}[(g,x)]) \in \mathbb{C}$$

is integrable with respect to $dg$, then we define $\phi_1 \ast \phi_2$ as the function on $G \times X$ given by

$$(\phi_1 \ast \phi_2)(g,x) = \int_G \phi_1(h,x)\phi_2(h^{-1}[(g,x)])dh.$$ 

Moreover, a couple $(M, U)$ is said to be an \textit{imprimitivity system} (for $G$ based on $X$) acting in a Hilbert space $\mathcal{H}$ if

1. $M$ is a nondegenerate $\ast$-representation of the $C^\ast$-algebra $C_0(X)$ in $\mathcal{H}$;
2. $U$ is a representation of $G$ in $\mathcal{H}$;
3. for all $f \in C_0(X)$ and $g \in G$

$$U_gM(f)U_g^{-1} = M(I_gf).$$

We denote by $P^M$ the unique projection valued measure in $\mathcal{H}$ such that

$$M(f) = \int_X f(x)dP^M(x) \quad f \in C_0(X),$$

where the integral is in the weak operator topology. Given an imprimitivity system $(M, U)$ acting in a Hilbert space $\mathcal{H}$, according to Ref. 14, we define, for all $\phi \in C_\infty(G \times X)$, the operator on $\mathcal{H}$,

$$U(\phi) = \int_G M(\phi(g, \cdot))U_gdg,$$ \hspace{1cm} (1)

where the integral is in the strong operator topology.

\textbf{Lemma 1:} The operator $U(\phi)$ defined by Eq. (1) is well defined, bounded and $\|U(\phi)\| \leq \int_G \|\phi(g, \cdot)\|_{\sup}dg$.

\textbf{Proof:} Let $u \in \mathcal{H}$. Clearly, the map $g \mapsto \phi(g, \cdot) = \Phi(g)$ is continuous from $G$ to $C_0(X)$ and it has compact support. Since $f \mapsto M(f)$ is continuous from $C_0(X)$ to $L(\mathcal{H})$ (with the operator norm), then $g \mapsto M(\Phi(g))$ is continuous from $G$ to $L(\mathcal{H})$. Moreover, the map $g \mapsto U_gu$ is continuous from $G$ to $\mathcal{H}$, hence $g \mapsto M(\Phi(g))U_gu$ is continuous from $G$ to $\mathcal{H}$ and

$$\|M(\Phi(g))U_gu\| \leq \|u\|\|\Phi(g)\|_{\sup}.$$ 

Then, the map $g \mapsto M(\Phi(g))U_gu$ is $dg$-integrable, the operator $U(\phi)$ is well defined and

$$\|U(\phi)u\| \leq \|u\|\int_G \|\Phi(g)\|_{\sup}dg.$$ 

From the above equation it follows that $U(\phi)$ is bounded and $\|U(\phi)\| \leq \int_G \|\phi(g, \cdot)\|_{\sup}dg$. \hfill \square
Remark 1: Denote by \( \mathcal{L} \) the space \( C_c(G \times X) \) with the norm
\[
\| \phi \| = \int_G \| \phi(g, \cdot) \|_{\sup} dg.
\]
If \( \phi_1, \phi_2 \in \mathcal{L} \), then one can easily check that \( \phi_1 \star \phi_2 \) is well defined and belongs to \( \mathcal{L} \). Moreover, given \( \phi \in \mathcal{L} \), the function
\[
\phi^\#(g, x) = \Delta_G(g^{-1}) \phi(g^{-1}, g^{-1}[x]) \ (g, x) \in G \times X,
\]
is in \( \mathcal{L} \). One can show, see, for example, Chap. 6 of Ref. 14, that, with respect to the above operations, \( \mathcal{L} \) is an involutive normed algebra and the map \( \phi \mapsto U(\phi) \) is a \( * \)-representation of \( \mathcal{L} \) in \( \mathcal{H} \). This representation characterizes completely the imprimitivity system \( (M, U) \). The completion \( L^1(G, dg, C_0(X)) \) of \( \mathcal{L} \), with respect to the above norm, is an involutive Banach algebra and it is the analogous of the group algebra \( L^1(G) \) for groups.

The following result is a version of Schur lemma for imprimitivity systems:

Lemma 2: Let \( (M, U) \) and \( (N, V) \) be two imprimitivity systems acting in \( \mathcal{H} \) and \( \mathcal{K} \), respectively. Let \( T \) be a closed operator from \( \mathcal{H} \) to \( \mathcal{K} \) with a dense domain such that, for all \( f \in C_0(X) \) and \( g \in G \),
\[
N(f) V_g T \subset TM(f) U_g.
\]
Suppose that \( (M, U) \) is irreducible, then \( T \) is a multiple of an isometry.

Proof: The proof is standard. Consider the selfadjoint operator \( T^* T \). Fix \( f \in C_0(X) \) and \( g \in G \), then
\[
M(f) U_g T^* T = (M(1_g \overline{\varphi}^f) U_g^{-1})^* T^* T \subset T^* T = T^* N(f) V_g T \subset TM(f) U_g.
\]
Moreover, let \( E \rightarrow Q(E) \) be the spectral measure of \( T^* T \), then, see, for example, Theorem 4.11 of Ch. X of Ref. 15, for all \( E \in B([1]) \),
\[
M(f) U_g Q(E) \subset Q(E) M(f) U_g.
\]
Since \( Q(E) \) is bounded the above relation is an equality. Since \( (M, U) \) is irreducible, it follows that \( Q(E) \) is proportional to the identity. Hence \( |T| \) is a scalar and one concludes using the polar decomposition of \( T \).

III. SQUARE INTEGRABLE IMPRIMITIVITY SYSTEMS

In the following we give the definition of square-integrability for imprimitivity systems and we prove the main properties of this class of systems.

In this section, \( G \) is a lcsc topological group with a fixed left Haar measure \( dg \) and \( X \) is a \( G \)-space with a fixed quasi-invariant measure \( dx \). We observe that quasi-invariant measures always exist, but, in general, they are not canonical, since two of them need not be proportional or equivalent.

First of all, we define a canonical measure on \( G \times X \) in terms of the measures \( dg \) and \( dx \). We denote by \( dg dx \) the product measure on \( G \times X \) and we let \( \mu \) be the measure on \( G \times X \) given by
\[
\int_{G \times X} \phi(g, x) d\mu(g, x) = \int_{G \times X} \phi(g, g[x]) dgdx \quad \phi \in C_c(G \times X).
\]

Lemma 3: With the above notations, the measure \( \mu \) is well-defined, invariant and equivalent to \( dg dx \).

Moreover, there is a measurable function \( \lambda : G \times X \rightarrow ]0, \infty[ \), called the cocycle of \( dx \), such that
(1) for $dg$-almost all $g \in G$,

$$\int_X f(g^{-1}[x])dx = \int_X f(x)\lambda(g,x)dx \quad f \in C_c(X);$$

(2) for $(dgdx)$-almost all $(g,h,x) \in G \times G \times X$

$$\lambda(g,h,x) = \lambda(g,h[x])\lambda(h,x);$$

(3) the function $(g,x) \mapsto \lambda(g^{-1},x)$ is the density of $\mu$ with respect to $dgdx$.

Finally, if $dx'$ is another quasi-invariant measure on $X$ having density $\alpha$ with respect to $dx$, then, for all $\phi \in C_c(G \times X)$,

$$\int_{G \times X} \phi(g,x)d\mu'(g,x) = \int_{G \times X} \phi(g,x)\alpha(g^{-1}[x])d\mu(g,x),$$

where $\mu'$ is given by Eq. (2), replacing $dx$ with $dx'$.

Proof: Define $\Omega$ as the map from $G \times X$ into itself given by $\Omega(g,x) = (g,g[x])$, then $\Omega$ is clearly a homeomorphism and $\mu$ is the image measure of the measure $dgdx$ with respect to $\Omega$, so $\mu$ is well defined. We prove that it is invariant.

Let $\phi \in C_c(G \times X)$ and $g \in G$, then

$$\int_{G \times X} \phi^\prime(h,x)d\mu(h,x) = \int_{G \times X} \phi(g^{-1}h,g^{-1}h[x])dhdx$$

$$= \int_X \int_G \phi(h,h[x])dhdx = \int_{G \times X} \phi(h,x)d\mu(h,x).$$

We prove that $\mu$ is equivalent to $dgdx$. To this aim, it is sufficient to prove that, given $E \in \mathcal{B}(G \times X)$, then $E$ is $dgdx$-negligible if and only if $\Omega(E)$ is $dgdx$-negligible. The set $E$ is negligible if and only if, for $dg$-almost all $g \in G$, the section $E_g$ is $dx$-negligible. Since $dx$ is equivalent to $dx^g$, this last condition is equivalent to the fact that, for $dg$-almost all $g \in G$, $g[E_g] = \Omega(E)_g$ is $dx$-negligible and, hence, to the fact that $\Omega(E)$ is $dgdx$-negligible.

We now prove the second claim. By a standard result on $G$-spaces, see, for example, Theorem 5.10 of Ref. 16, there is a positive and measurable function $\lambda$ such that conditions in items 1.-2. above are satisfied. Let now $\xi$ be the density of $\mu$ with respect to $dgdx$, which is a $dgdx$-locally integrable positive function, and choose $\phi \in C_c(G \times X)$ positive, then

$$\int_G \int_X \lambda(g^{-1},x)\phi(g,x)dxdg = \int_G \int_X \phi(g,g[x])dxdg$$

$$= \int_{G \times X} \phi(g,g[x])dgdx$$

$$= \int_{G \times X} \phi(g,x)d\mu(g,x)$$

$$= \int_{G \times X} \phi(g,x)\xi(g,x)dxdg < \infty.$$

From Fubini theorem, it follows that $(g,x) \mapsto \lambda(g^{-1},x)$ is $dgdx$-locally integrable and, for $dgdx$-almost all $(g,x) \in G \times X$,

$$\lambda(g^{-1},x) = \xi(g,x).$$
so condition in item (3) holds.

Let now $dx'$ be a measure on $X$ having density $\alpha$ with respect to $dx$, then, for all $\phi \in C_c(G \times X)$,

$$\int_{G \times X} \phi(g,x) d\mu'(g,x) = \int_{G \times X} \phi(g,g[x]) dg dx'$$

$$= \int_{G \times X} \phi(g,g[x]) \alpha(x) dg dx$$

$$= \int_{G \times X} \phi(g,g[x]) \alpha(g^{-1}[x]) d\mu(g,x).$$

So the last claim follows. \qed

The following definition of square-integrability for imprimitivity systems is the cornerstone of the paper.

**Definition:** Let $(M, U)$ be an irreducible imprimitivity system acting in a Hilbert space $\mathcal{H}$. The system $(M, U)$ is said to be *square-integrable* with respect to $dx$ if there are $u,v \in \mathcal{H}$, $u,v \neq 0$ and a function $c_{u,v}$ in $L^2(G \times X, \mu)$ such that, for all $\phi \in C_c(G \times X)$,

$$\langle u, U(\tilde{\phi}) v \rangle = \int_{G \times X} c_{u,v}(g,x) \phi(g,x) d\mu(g,x). \quad (4)$$

For all $u,v \in \mathcal{H}$ we denote by $c_{u,v}$ the linear form on $C_c(G \times X)$,

$$\phi \mapsto \langle u, U(\tilde{\phi}) v \rangle.$$

If there exists a locally $\mu$-integrable function on $G \times X$ such that Eq. (4) holds we say that $c_{u,v}$ is a *locally $\mu$-integrable function* and we use the same symbol $c_{u,v}$ to denote this function. We say that a vector $v \in \mathcal{H}$ is *admissible* if there is a nonzero vector $u \in \mathcal{H}$ such that $c_{u,v}$ is a function square-integrable with respect to $\mu$. Clearly, $(M, U)$ is square-integrable if and only if there is a nonzero admissible vector.

We will explain and motivate our definition after the next lemma that characterizes the square-integrability in terms of properties of $M$ and $U$.

**Lemma 4:** Let $(M, U)$ be an imprimitivity system acting in $\mathcal{H}$. Given $u,v \in \mathcal{H}$, the following conditions are equivalent:

1. the linear form $c_{u,v}$ is a locally $\mu$-integrable function;
2. for $dg$-almost all $g \in G$, the bounded measure on $X$,

$$f \mapsto \langle u, M(f) U_v \rangle$$

has a (complex) density $\eta_{u,v,g}$ with respect to $dx$.

If any of the above conditions is satisfied, then, for $dg$-almost all $g \in G$, there is a $dx$-negligible set $A_g$ such that

$$\eta_{u,v,g}(x) = \lambda(g^{-1}, x) c_{u,v}(g,x) \quad x \in X, x \notin A_g,$$

and $c_{u,v}$ belongs to $L^2(G \times X, \mu)$ if and only if

$$\int_G \int_X \lambda(g^{-1},x)^{-1} |\eta_{u,v,g}(x)|^2 dxdg < \infty.$$
Proof: We prove that the first condition implies the second one. Since \( c_{u,v} \) is locally \( \mu \)-integrable and \( G \) second countable, then, for almost all \( g \in G \), the function \( x \mapsto c_{u,v}(g,g[x]) \) is locally \( dx \)-integrable and, for all \( f \in C_c(X) \), the function \( g \mapsto \int_X f(g[x])c_{u,v}(g,g[x])\,dx \) is locally \( dg \)-integrable.

Then, fixed \( f \in C_c(X) \), for all \( \varphi \in C_c(G) \), by Fubini theorem,

\[
\int_G \varphi(g) \int_X f(g[x])c_{u,v}(g,g[x])\,dx
g = \int_{G \times X} \varphi(g)f(x)c_{u,v}(g,x)\,d\mu(g,x)
\]

\[
\quad = \langle u, U(f \otimes \varphi) \rangle
\]

\[
\quad = \int_G \varphi(g) \langle u, M(f)U_g \rangle v \,dg.
\]

Hence, there is a \( dg \)-negligible set \( A_f \subset G \), such that for all \( g \notin A_f \),

\[
\langle u, M(f)U_g \rangle = \int_X f(g[x])c_{u,v}(g,g[x])\,dx.
\]

Since \( X \) is second countable, it follows that, for \( dg \)-almost all \( g \in G \), the bounded measure \( f \mapsto \langle u, M(f)U_g \rangle \) has density \( c_{u,v}(g,\cdot) \) with respect to \( dx^{s-1} \) and, taking into account that, by Lemma 3, for \( dg \)-almost all \( g \in G \), \( \lambda(g^{-1},\cdot) \) is the density of \( dx^{s-1} \) with respect to \( dx \), the above bounded measure has density

\[
\eta_{u,v} := \lambda(g^{-1},\cdot)c_{u,v}(g,\cdot)
\]

with respect to \( dx \). The claim is proven and, in particular, Eq. (5) holds.

Conversely, assume condition 2. By definition, for all \( \phi \in C_c(G \times X) \),

\[
\langle u, U(\phi) \rangle = \int_G \int_X \phi(g,x) \eta_{u,v,g}(x)\,dx\,dg.
\]

(6)

We claim that there is a \( \mu \)-locally integrable function \( c_{u,v} \) defined on \( G \times X \) such that,

\[
\langle u, U(\phi) \rangle = \int_{G \times X} \phi(g,x)c_{u,v}(g,x)\,d\mu(g,x).
\]

Clearly, we can assume that \( \eta_{u,v,g} \) is positive, so that \( \phi \mapsto \langle u, U(\phi) \rangle \) is a positive measure on \( G \times X \). We will prove that this measure has density with respect to \( \mu \). Let \( (\phi_n) \) be a decreasing sequence of positive functions in \( C_c(G \times X) \) such that \( \lim_n \int_{G \times X} \phi_n \,d\mu = 0 \), then, by monotone convergence theorem and since \( \phi_n \geq 0 \), we have \( \lim_n \phi_n = 0 \), \( \mu \)-almost everywhere. Since \( \mu \) is equivalent to \( dgdx \), for \( dg \)-almost all \( g \in G \),

\[
\lim_n \phi_n(g,\cdot) = 0 \quad dx\text{-a.e.},
\]

and, hence,

\[
\lim(n \eta_{u,v,g} \phi_n(g,\cdot)) = 0 \quad dx\text{-a.e.}
\]

Moreover, \( \eta_{u,v,g} \) is in \( L^1(X,dx) \), being the density of a bounded measure; then \( (\eta_{u,v,g} \phi_n \times (g,\cdot))_n \) is a positive decreasing sequence in \( L^1(X,dx) \), so, by monotone convergence theorem, for \( dg \)-almost all \( g \in G \),

\[
\lim_n \eta_{u,v,g} \phi_n = 0 \quad dgdx\text{-a.e.}
\]
\[
\lim_{n} \int_{X} \eta_{n,v,g}(x) \phi_{n}(g,x) dx = 0
\]
and, by the same arguments,
\[
\lim_{n} \int_{G} \int_{X} \eta_{n,v,g}(x) \phi_{n}(g,x) dx dg = 0,
\]
where, for all \( n \in \mathbb{N} \), the function \( g \mapsto \int_{X} \eta_{n,v,g}(x) \phi_{n}(g,x) dx \) is \( dg \)-measurable, since it is equal \( dg \)-almost everywhere to the continuous function
\[
g \mapsto \langle u,M(\phi_{n}(g,\cdot))U_{g}v \rangle.
\]
Hence, by Eq. (6), \( \lim_{n} \langle u, U(\phi_{n})v \rangle = 0 \). This shows that the measure \( \phi \mapsto \langle u, U(\phi)v \rangle \) has density with respect to \( \mu \), see, for example, Theorem 6.5.3 of Ref. 17.

Let \( c_{u,v} \) be such a density, which is a \( \mu \)-locally integrable function, then, by definition, for all \( \phi \in C_{c}(G \times X) \),
\[
\langle u, U(\phi)v \rangle = \int_{G \times X} \phi(g,x)c_{u,v}(g,x) d\mu(g,x).
\]
Finally, assume that one of the two conditions holds, then, by Eq. (5) and Fubini theorem,
\[
\int_{G} \int_{X} \lambda(g^{-1},x)^{-1} |\eta_{u,v,g}(x)|^{2} dx dg = \int_{G \times X} |c_{u,v}(g,x)|^{2} d\mu(g,x),
\]
and the last claim is clear. \( \square \)

The above definition of square-integrability depends only on the equivalence class of \((M, U)\) and is motivated by the following observations.

\textbf{Remark 2:} Let us show that the definition given by Eq. (4) is the natural generalization of the notion of square-integrable representation to an imprimitivity system. Consider the case of \( X \) being a discrete denumerable set \( X = \{x_{k}\} \) and choose the measure \( dx \) to be the atomic measure \( dx = \sum_{k} \delta_{x_{k}} \), where \( \delta_{x_{k}} \) is the Dirac delta at point \( x_{k} \). This measure is \( G \)-invariant, so that \( \mu = dg dx \), and one has, for all \( u,v \in \mathcal{H} \),
\[
\langle M(f)u,v \rangle = \sum_{k} f(x_{k}) (P^{M}(\{x_{k}\})u,v) \quad f \in C_{0}(X),
\]
\[
\langle U(\phi)u,v \rangle = \sum_{k} \int_{G} \phi(g,x_{k}) (P^{M}(\{x_{k}\})U_{g}u,v) dg \quad \phi \in C_{c}(G \times X).
\]
Moreover, given \( u,v \in \mathcal{H} \), define the map \( \hat{c}_{u,v} \) from \( G \times X \) to \( \mathbb{C} \),
\[
\hat{c}_{u,v}(g,x) = \langle u,P^{M}(\{x\})U_{g}v \rangle.
\]
The fact that \((M, U)\) is square-integrable with respect to \( dx \) is clearly equivalent to the fact that there exist two non zero vectors \( u,v \in \mathcal{H} \) such that the map \( \hat{c}_{u,v} \) is square-integrable with respect to \( dg dx \). On the other hand, if \( X \) not discrete, one has to take care of the fact that the measures \( dx \) and \( P^{M} \) are in general diffuse. The definition we give solves this technical problem.

\textbf{Remark 3:} With the notation of Remark 1, since \( \phi \mapsto U(\phi) \) is a *-representation of the involutive normed algebra \( \mathbb{L} \), given \( u \in \mathcal{H} \), the linear form \( \Omega_{u} \) on \( \mathbb{L} \),
\[
\phi \mapsto \langle U(\phi)u,u \rangle
\]
is of positive type, i.e., it satisfies $\langle \Omega_x, \phi^{\#} \phi \rangle \geq 0$. It is well known, see, for example, Theorem 6.28 of Ref. 14, that there is a one-to-one correspondence between cyclic imprimitivity systems and linear forms of positive type. It is natural to express the square-integrability in terms of such linear forms.

Remark 4: If $X$ reduces to a single point, so that the imprimitivity system collapses into a representation of the group $G$, our definition is precisely the one given for groups. The main difference between the two notions is that, in the case of groups, the Haar measure is canonical since it is the unique, up to a constant, invariant measure on $G$; whereas, in the case of imprimitivity systems, the notion of square-integrability depends on the choice of the measure $dx$ and, in the following, we will discuss carefully this point. In particular, given an imprimitivity system $(M, U)$ square-integrable with respect to $dx$, Corollary 1 will prove that $(M, U)$ is square-integrable with respect to any measure equivalent to $dx$. Moreover, Corollary 2 will show that there exists a minimal (with respect to the natural partial order among measures) quasi-invariant measure class where the measure $dx$ can be chosen to test the square-integrability of $(M, U)$. However, this measure class depends on the imprimitivity system, so that it is not canonical for $X$. Finally, there are many cases where there exists on $X$ a canonical quasi-invariant measure or, at least, a canonical quasi-invariant measure class. For example,

1. If $X$ is transitive, there is only one quasi-invariant measure class, see, for example Ref. 14.
2. If $G$ is a Lie group, $X$ a manifold and the action on $X$ smooth, all the Lebesgue measures on $X$ are quasi-invariant and equivalent among them, see, for example Ref. 18.
3. If $X$ has, by itself, a group structure compatible with its topology (so that $X$ is a lcsc topological group) and the action of $G$ preserves this group structure, then any left invariant Haar measure $dx$ (unique up to a constant) of $X$ is relatively invariant with respect to the action of $G$. Indeed, given $g \in G$, since $x \mapsto g[x]$ is a topological group isomorphism, for all $y \in X$ and $f \in C_c(X)$,

$$\int_X f(x)d\gamma = \int_X f(g^{-1}[g(y^{-1})x])dx = \int_X f(g^{-1}[x])dx,$$

then $d\gamma$ is a Haar measure of $X$ and, hence, there is a strictly positive constant $\lambda(g)$ such that $d\gamma = \lambda(g)dx$, i.e., $dx$ is relatively invariant.

Remark 5: Suppose that $X$ is the dual group of an Abelian group $A$ such that the action of $G$ preserves the composition law of $X$. It is well known that there is a one-to-one correspondence between the imprimitivity systems $(M, U)$ for $G$ based on $X$ and the representations $V$ of the semidirect product $A \times \gamma G$, where $A$ is a $G$-space with respect to the dual action. Moreover, such correspondence preserves the irreducibility. By the third observation in Remark 4, any Haar measure $dx$ of $X$ is relatively invariant. With this choice, the square integrability of $(M, U)$ is equivalent to the square-integrability of the corresponding representation $V$ of $A \times \gamma G$.

Indeed, since $dx$ is relatively invariant, there is a positive character $\rho$ of $G$ such that $\lambda(g,x) = \rho(g)$ for all $g \in G$ (and $x \in X$). A simple calculation shows that a left invariant Haar measure of $A \times \gamma G$ is $\rho(g)dadg$, where $da$ is a Haar measure of $A$. Moreover, fixed $u,v \in \mathcal{H}$, for all $g \in G$, we denote by $v_g$ the bounded measure on $X$,

$$f \mapsto (u,M(\overline{f})U_g v).$$

Then, since, for all $a \in A, V_a = \int_X \langle x, a \rangle dP^M(x)$ and, for all $g \in G, V_g = U_g$, we have that

$$\langle u, V_{a,g} v \rangle = \int_X \langle x, a \rangle d\nu_g(x) = \mathcal{F}(\nu_g)(a) \quad a \in A, g \in G,$$
where $\mathcal{F}$ is the Fourier transform defined on the space of bounded measures on $X$.

Let $u,v$ be nonzero vectors in $\mathcal{H}$. By Fubini theorem, the continuous function $(a,g) \mapsto (u,V_{ag}v)$ is square-integrable with respect to $\rho(g)dadg$ if and only if

$$\int_G \rho(g) \int_A |(u,V_{ag}v)|^2 dalg = \int_G \rho(g) \int_A |\mathcal{F}(v_g)(a)|^2 dalg < \infty.$$ 

By a standard result of Fourier analysis (and Fubini theorem), this condition is equivalent to the fact that, for $dg$-almost all $g \in G$, the measure $v_g$ has a density $\eta_g$ with respect to $dx$ and

$$\int_G \rho(g) \int_X |\eta_g(x)|^2 dx dg < \infty.$$ 

The equivalence of the above conditions and the square-integrability of $(M, U)$ is given by Lemma 4.

The properties of square-integrable representations can be extended to square-integrable imprimitivity systems. First of all, we define a canonical imprimitivity system playing the role of the left regular representation for groups.

Let $(M^L, L)$ be the imprimitivity system acting in $L^2(G \times X, \mu)$ defined by

$$(M^L(f)(g,x)) = f(x) \psi(g,x) \quad f \in C_c(X),$$

$$(L_h \psi)(g,x) = \psi(h^{-1}g, h^{-1}[x]) \quad h \in G,$$

where $\psi \in L^2(G \times X, \mu)$ and the equalities hold for $\mu$-almost all $(g,x) \in G \times X$. It is clear that $(M^L, L)$ is an imprimitivity system.

The following theorem shows that a square-integrable imprimitivity system $(M, U)$ acting in $\mathcal{H}$ defines an isometry from $\mathcal{H}$ to $L^2(G \times X, \mu)$ intertwining $(M, U)$ with $(M^L, L)$.

**Theorem 1:** Let $(M, U)$ be an imprimitivity system acting in $\mathcal{H}$ and square-integrable with respect to $dx$. Given an admissible vector $v \in \mathcal{H}$, $v \neq 0$, then

1. For all $u \in \mathcal{H}$ the linear form $c_{u,v}$ is a function in $L^2(G \times X, \mu)$;
2. The map $W_v$ from $\mathcal{H}$ to $L^2(G \times X, \mu)$

$$W_vu = c_{u,v}$$

is a (nonzero) multiple of an isometry that intertwines $(M, U)$ with $(M^L, L)$;
3. For all $\phi \in C_c(G \times X) \subset L^2(G \times X, \mu)$, $W_v^* \phi = \cup (\phi) v$.

**Proof:** We follow the proof of Duflo and Moore for representations. By definition of square-integrability and of admissible vector, there is a nonzero vector $u_0 \in \mathcal{H}$ such that $c_{u_0,v} \in L^2(G \times X, \mu)$. We claim that $c_{u_0,v} \neq 0$. Suppose the opposite, by definition of $U$, it follows that, for $dg$-almost all $g \in G$ and all $f \in C_c(X)$, $\langle u, M(f) U_g v \rangle = 0$. By continuity, this relation holds for all $g \in G$ and, by irreducibility and since $v \neq 0$, it follows that $u_0 = 0$. This is a contradiction and, hence, $c_{u_0,v} \neq 0$.

Define $W_v$ as the operator in $L^2(G \times X, \mu)$ with domain

$$\{ u \in \mathcal{H} : c_{u,v} \in L^2(G \times X, \mu) \}$$

and $W_vu = c_{u,v}$. The domain of $W_v$ is not the null space since $c_{u_0,v}$ is square-integrable. We claim that $W_v$ is a closed operator. Indeed, let $(u_n)$ be a sequence in $\mathcal{H}$ converging to $u$ such that $(W_v u_n)$ tends to $\psi$. Then, for all $\phi \in C_c(G \times X)$,
\[
\langle u, U(f) v \rangle = \lim_{n \to \infty} \langle u_n, U(f) v \rangle = \lim_{n \to \infty} \int_{G \times X} (W_c u_n)(g,x) \phi(g,x) d\mu(g,x) = \int_{G \times X} \psi(g,x) \phi(g,x) d\mu(g,x).
\]

So that \( u \in \text{dom } W_v \) and \( W_v u = \psi \).

Moreover, we show that \( W_v \) intertwines \((M, U)\) and \((M^f, L)\). Indeed, let \( f \in C_0(X) \) and \( h \in G \), we have to prove that, given \( u \in \text{dom } W_v \), then \( M(f) U_h u \in \text{dom } W_v \) and \( W_v M(f) U_h u = M^f(f) L_h W_v u \). Let \( \phi \in C_{c}(G \times X) \),

\[
\langle M(f) U_h u, U(f) v \rangle = \int_{G} \langle u, U_{h^{-1}} M(f \phi(g,\cdot)) U_{h^{-1}} v \rangle dg
= \int_{G} \langle u, M(h^{-1}(f \phi(g,\cdot)) U_{h^{-1}} v \rangle dg
= \int_{G} \langle u, M(h^{-1}(f \phi(h g,\cdot)) U_{h^{-1}} v \rangle dg
= \int_{G} \langle u, M(h^{-1}(f \phi(h g,\cdot)) U_{h^{-1}} v \rangle dg
= \int_{G} \langle u, M(h^{-1} g, h^{-1}(h x)) \phi(h g, h x) c_{u,v}(g,x) d\mu(g,x)
= \int_{G} \langle u, M(h^{-1} g, h^{-1}(h x)) \phi(g,x) c_{u,v}(g,x) d\mu(g,x)
= \int_{G} \phi(g,x) (M^f(f) L_h c_{u,v})(g,x) d\mu(g,x).
\]

Since \((M^f(f) L_h c_{u,v}) \in L^2(G \times X, \mu)\), the claim follows.

Moreover, since the domain of \( W_v \) is invariant with respect to the action of \((M, U)\) and it is not the null space, \( \text{dom } W_v \) is dense in \( \mathcal{H} \). Hence, by Lemma 2, since \((M, U)\) is irreducible, \( \text{dom } W_v = \mathcal{H} \) and \( W_v \) is a nonzero multiple of an isometry.

The claim in item (3) easily follows from the definition of \( W_v \).

The above theorem shows that a square-integrable imprimitivity system is equivalent to the restriction of the canonical imprimitivity system \((M^f, L)\) to an irreducible invariant closed subspace of \( L^2(G \times X, \mu) \). The converse implication is showed by the following result.

**Theorem 2:** Let \( \mathcal{H} \) be a closed (non-null) subspace of \( L^2(G \times X, \mu) \) that is invariant and irreducible with respect to the action of \((M^f, L)\). Then, the restriction to \( \mathcal{H} \) of \((M^f, L)\) is square-integrable with respect to \( dx \).

**Proof:** Let \( Q \) be the orthogonal projection on \( \mathcal{H} \). Since \( \mathcal{H} \) is not the null space and \( C_{c}(G \times X) \) is dense in \( L^2(G \times X, d\mu) \), then there is \( \Psi_2 \in C_{c}(G \times X) \) such that \( Q \psi_2 \neq 0 \), where \( \psi_2 \in L^2(G \times X, \mu) \) is defined as

\[
\psi_2(g,x) := \lambda(g^{-1}, x)^{-\frac{1}{2}} \Psi_2(g,x), \quad (g,x) \in G \times X.
\]

We claim that \( Q \psi_2 \) is an admissible vector for the restriction to \( \mathcal{H} \) of \((M^f, L)\). Indeed, given \( \psi_1 \in \mathcal{H} \), we have to prove that the linear form \( c_{\psi_1, Q \psi_2} \) is in \( L^2(G \times X, \mu) \). Let \( g \in G \) and \( f \in C_{0}(X) \), then

\[
c_{\psi_1, Q \psi_2}(g) = \int_{G} \langle \psi_1, Q \psi_2 \rangle \phi(g,x) d\mu(g,x) = \int_{G} \psi_1(g,x) Q \Psi_2(g,x) d\mu(g,x) = \int_{G} \psi_1(g,x) \lambda(g^{-1}, x)^{-\frac{1}{2}} \Psi_2(g,x) d\mu(g,x).
\]
\[ \langle \psi_1, M^f(j)L_g \psi_2 \rangle = \langle \psi_1, Q M^f(j)L_g \psi_2 \rangle = \langle Q \psi_1, M^f(j)L_g \psi_2 \rangle = \langle \psi_1, M^f(j)L_g \psi_2 \rangle \]

\[ = \int_{G \times X} \lambda(h^{-1},x) \psi_1(h,x)f(x) \bar{\psi}_2(g^{-1}h, g^{-1}[x]) dh dx = \int_X f(x) \eta_g(x) dx, \]

where \( \eta_g \) is defined for \( dx \)-almost all \( x \in X \) as

\[ \eta_g(x) = \int_G \lambda(h^{-1},x) \psi_1(h,x) \bar{\psi}_2(g^{-1}h, g^{-1}[x]) dh \]

\[ = \int_G \lambda(h^{-1},g^{-1}h) \psi_1(g h, x) \bar{\psi}_2(h, g^{-1}[x]) dh \]

\[ = \lambda(g^{-1}, x) \int_G \omega(g,x,h) dh. \]

According to Lemma 4, the linear form \( c_{\psi_1, Q \psi_2} = c_{\psi_1, \psi_2} \) is a locally \( \mu \)-integrable function \( \Omega \) and for \( dg \)-almost all \( g \in G \),

\[ \Omega(g, x) = \lambda(g^{-1}, x)^{-1} \eta_g(x) = \int_G \omega(g,x,h) dh, \]

where, fixed \( g \in G \), the equalities hold for \( dx \)-almost all \( x \in X \). From Eq. (3), if follows that, for \( dg \)-almost all \( h \in G \),

\[ \lambda(g^{-1}, x) = \lambda(h^{-1}, h g^{-1}[x]) \lambda(h g^{-1}, x) \text{ } \mu \text{-a.e.} (g, x) \in G \times X. \] (7)

Fixed \( h \in G \) such that Eq. (7) holds, then

\[ \| \omega(\cdot, \cdot, h) \|^2_{L^2(G \times X, \mu)} = \int_{G \times X} |\omega(g,x,h)|^2 \lambda(g^{-1}, x) dg dx \]

\[ = \Delta_G(h)^{-1} \int_{G \times X} \{ |\psi_1(g,x) \bar{\psi}_2(h, h g^{-1}[x])|^2 \lambda^2(g^{-1}, x) \lambda(h g^{-1}, x) \} dg dx \]

\[ = \Delta_G(h)^{-1} \int_{G \times X} \{ |\psi_1(g,x) \bar{\psi}_2(h, h g^{-1}[x])|^2 \lambda(h^{-1}, h g^{-1}[x]) \} d \mu(g, x) \]

\[ = \Delta_G(h)^{-1} \int_{G \times X} |\psi_1(g,x) \Psi_2(h, h g^{-1}[x])|^2 d \mu(g, x), \]

where, by definition, \( \Psi_2 \in C_c(G \times X) \). Then, by Minkowski inequality for integrals,

\[ \| \Omega \|_{L^2(G \times X, \mu)} \leq \int_G \| \omega(\cdot, \cdot, h) \|_{L^2(G \times X, \mu)} dh \]

\[ = \int_G \Delta_G(h)^{-1/2} \| \psi_1(g,x) \Psi_2(h, h g^{-1}[x]) \|_{L^2(G \times X, \mu)} dh \]

\[ \leq \| \psi_1 \|_{L^2(G \times X, \mu)} \int_G \Delta_G(h)^{-1/2} \| \psi_2(h, \cdot) \|_{\text{sup}} dh. \]

Since \( \Psi_2 \in C_c(G \times X) \), then \( \Omega \in L^2(G \times X, \mu) \). So the thesis is proven. \( \Box \)
The following corollaries study how the square-integrability depends on the choice of the measure $dx$.

**Corollary 1:** Let $(M, U)$ be an imprimitivity system and $dx'$ be a quasi-invariant measure on $X$ having density with respect to $dx$. If $(M, U)$ is square-integrable with respect to $dx'$, then $(M, U)$ is square integrable with respect to $dx$, too. In particular the square-integrability of $(M, U)$ depends only on the equivalence class of $dx$.

**Proof:** Denote with a prime all the objects defined replacing $dx$ with $dx'$ and let $\alpha$ be the density of $dx'$ with respect to $dx$. Then, by Lemma 3, $\mu'$ has density $\beta(g,x) = \alpha(g^{-1}[x])$ with respect to $\mu$ and the map

$$\psi \rightarrow \beta^{1/2}\psi$$

is an isometry from $L^2(G \times X, \mu')$ into $L^2(G \times X, \mu)$.

Due to the particular form of the density, this isometry intertwines $(M^L, L')$ and $(M^L, L)$. Since $(M, U)$ is square-integrable with respect to $dx'$, by Theorem 1, it is equivalent to a subsystem of $(M^L, L')$ and, so, to a subsystem of $(M^L, L)$. Theorem 2 proves the claim. \qed

Nevertheless, by direct check, one can show that, in passing from the measure $dx'$ to $dx$, the set of admissible vectors can change.

Consider an imprimitivity system $(M, U)$ acting in $H$. The above result suggests that there could be a *minimal* quasi-invariant measure class such that $(M, U)$ is square-integrable with respect to a measure belonging to this class. To this aim, we recall that, as a consequence of the spectral multiplicity theorem applied to $M$, there is a positive measure $\nu^M$ on $X$ such that $\nu^M(E) = 0$ if and only if $P^M(E) = 0$; this measure is uniquely defined by $M$, up to an equivalence, and is quasi-invariant.

**Corollary 2:** With the previous notation, let $(M, U)$ be an imprimitivity system square-integrable with respect to $dx$, then $\nu^M$ has density with respect to $dx$.

Moreover, if $\alpha$ is such a density and $Y = \{x \in X : \alpha(x) > 0\}$, then, for all admissible vectors $v \in H$ and all $u \in H$,

$$(W_v u)(g,x) = \chi_Y(x)(W_v)(g,x) \quad \mu - d.e.$$  

Finally, the system $(M, U)$ is square-integrable with respect to $\nu^M$.

**Proof:** By means of Theorem 1, there is an admissible vector $v \in H$ such that the operator $W_v$ is an isometry intertwining $P^M$ with $P^L$. Let $E \in B(X)$ be $dx$-negligible, then, taking into account that $P^L(E)$ is the multiplicative operator by the characteristic function of the set $G \times E$, then $P^L(E) = 0$ and, hence, $0 = P^L(E) W_v = W_v P^M(E)$. Then, $P^M(E) = 0$ and, hence, $\nu^M(E) = 0$. This implies that $\nu^M$ has density with respect to $dx$.

Fix now an admissible vector $v \in H$. With the same arguments, $P^L(Y) W_v = W_v P^M(Y) = W_v$ since the complement of $Y$ is $\nu^M$-negligible and the second claim follows.

Finally, we prove the square-integrability of $(M, U)$ with respect to $\nu^M$. Let $v \in H$ be an admissible nonzero vector and $u \in H$ nonzero. By means of Corollary 1, we can always suppose that $\alpha$ is the characteristic function of $Y$ so that, taking into account the previous result

$$(W_v u)(g,x) = \alpha(x)(W_v u)(g,x) \quad \mu - a.e.$$  

We claim that for $\mu$-almost all $(g, x) \in G \times X$, $\alpha(g^{-1}[x]) = \alpha(x)$. Indeed, taking into account that $\mu' = \lambda'(g^{-1}, x) dg \nu^M(x), \quad \mu = \lambda(g^{-1}, x) dg dx$ and $\mu' = \alpha(g^{-1}[x]) \mu$, it follows that, for $\mu$-almost all $(g, x) \in G \times X$,

$$\lambda'(g^{-1}, x) \alpha(x) = \lambda(g^{-1}, x) \alpha(g^{-1}[x]).$$

The claim follows since $\lambda$ and $\lambda'$ are strictly positive and $\alpha$ takes values 0 and 1. Fix now $\phi \in C_c(G \times X)$, then
\[
\int_{G \times X} (W_g u)(g,x) \phi(g,x) d\mu'(g,x) = \int_{G \times X} (W_g u)(g,x) \phi(g,x) \alpha(g^{-1}[x]) d\mu(g,x)
\]
\[
= \int_{G \times X} \alpha(x)(W_g u)(g,x) \phi(g,x) d\mu(g,x)
\]
\[
= \int_{G \times X} (W_g u)(g,x) \phi(g,x) d\mu(g,x) = \langle u, U(\tilde{\phi})v \rangle.
\]

The thesis is now clear. \qed

From the above two corollaries it follows that \((M, U)\) is square-integrable with respect to \(dx\) if and only if \(\nu^M \ll dx\) and \((M, U)\) is square-integrable with respect to \(\nu^M\). Since the equivalence class of \(\nu^M\) is uniquely defined by \(M\), the square-integrability of \((M, U)\) with respect to \(\nu^M\) is a property intrinsic to the imprimitivity system. However, in some cases, there is on \(X\) a natural quasi-invariant measure, which is independent of the imprimitivity system, and it is useful to study the square-integrability with respect to such a measure.

**Remark 6:** With the above notations, let \((M, U)\) be square-integrable with respect to \(dx\). Taking into account that \((M, U)\) is irreducible, one can obtain an explicit form of the operator \(W_g\).

Indeed, by irreducibility there is a Hilbert space \(\mathcal{K}\) such that, up to a unitary equivalence,

\[\mathcal{H} = L^2 \times (\nu^M, \mathcal{K})\]

where \(f \in C_c (X)\) and \(u \in \mathcal{H}\). Moreover, denoted by \(\lambda_{\nu^M}\) the cocycle of \(\nu^M\), there is a measurable function \(\Pi\) from \(G \times X\) to the unitary group of \(\mathcal{K}\) such that, for \(dg\)-almost all \(g \in G\),

\[(U_g u)(x) = \lambda_{\nu^M}(g^{-1}, x)^{1/2} \Pi(g, g^{-1}[x]) u(g^{-1}[x]),\]

for \(u \in \mathcal{H}\) and for \(dx\)-almost all \(x \in X\), see, for example, Theorem 6.10 of Ref. 16. Then, given \(u, v \in \mathcal{H}\) with \(v\) admissible, it easily follows from Lemma 4 that

\[(W_g u)(g,x) = \frac{\alpha(x) \lambda_{\nu^M}(g^{-1}, X)^{1/2}}{\lambda(g^{-1}, x)} (u(x), \Pi(g, g^{-1}[x]) v(g^{-1}[x]))_K \ \mu\text{-a.e.}\]

The next goal is to prove the existence of orthogonality relations. To this aim, given an imprimitivity system \((M, U)\) acting in \(\mathcal{H}\) and square-integrable with respect to \(dx\), we let

\[E_{(M, U)} := \text{span}\{W_g u : u, v \in \mathcal{H}, \text{ such that } v \text{ is admissible}\}\]

The following properties are immediate:

**Lemma 5:** Let \((M, U)\) be an imprimitivity system square-integrable with respect to \(dx\). With the above notations, \(E_{(M, U)}\) is a closed subspace of \(L^2 \times (G \times X, \mu)\) invariant with respect to the action of \((M', L)\).

Moreover, let \((M', U')\) be another imprimitivity system square-integrable with respect to \(dx\). If \((M, U)\) and \((M', U')\) are equivalent, then \(E_{(M, U)} = E_{(M', U')}\), whereas, if they are not equivalent, \(E_{(M, U)} \perp E_{(M', U')}\).

**Proof:** The first statement is consequence of Theorem 1. Assume now that \((M, U)\) and \((M', U')\) are equivalent and let \(J\) be the unitary operator intertwining them. If \(\mathcal{H}\) and \(\mathcal{H}'\) are the Hilbert spaces where the two imprimitivity systems act, then, given \(u', v' \in \mathcal{H}', v'\) admissible, for all \(\phi \in C_c (G \times X)\),
Then \( v = J^{-1}u' \) is admissible and \( W_v'W_v = W_u \), where \( u = J^{-1}u' \).

If \((M, U)\) and \((M', U')\) are not equivalent, given \( u, v \in \mathcal{H} \) with \( v \) admissible and \( u', v' \in \mathcal{H}' \) with \( v' \) admissible, then

\[
\langle W_v'W_v, u \rangle = \langle W_v'W_v, u' \rangle = 0,
\]

since, from Theorem 1, \( W_v'W_v \) intertwines \((M', U')\) with \((M, U)\), so it is the null operator and the claim follows.

The second step is to decompose each \( E_{(M, U)} \) into its irreducible components. We are able to obtain this result only with a weak assumption on the measure \( dx \).

**Theorem 3:** Assume that, for all \( g \in G \), \( \lambda(g, \cdot) \) is locally \( dx \)-essentially bounded on \( X \). Let \((M, U)\) be an imprimitivity system acting in \( \mathcal{H} \) and square-integrable with respect to \( dx \), then there exists a unique positive self-adjoint operator \( C \), called the *normalizing operator* of \((M, U)\), such that

1. the vector \( v \in \mathcal{H} \) is admissible for \((M, U)\) if and only if \( v \in \text{dom} C \);
2. for all \( u_1, u_2 \in \mathcal{H}, \ v_1, v_2 \in \text{dom} C \),

\[
\langle W_{(M, U)}u_1, W_{(M, U)}u_2 \rangle_{L^2(G \times X, \mu)} = \langle u_1, u_2 \rangle_{\mathcal{H}} \langle Cv_2, Cv_1 \rangle_{\mathcal{H}}.
\]

To prove the above theorem we need some preliminary results. We define \( T \) to be the operator acting on \( L^2(G \times X, \mu) \) as

\[
(T\psi)(g, x) = \sqrt{\frac{1}{\Delta_G(g)\lambda(g^{-1}, x)}} \psi(g^{-1}, g^{-1}[x]),
\]

where \( \psi \in L^2(G \times X, \mu) \) and for \( \mu \)-almost all \((g, x) \in G \times X \).

**Lemma 6:** The operator \( T \) is well-defined, unitary and \( T^2 = I \).

**Proof:** Let \( \psi \in L^2(G \times X, \mu) \). Then, \( T\psi \) is clearly \( \mu \)-measurable and

\[
\int_{G \times X} |(T\psi)(g, x)|^2 d\mu(g, x) = \int_{G \times X} \frac{|\psi(g^{-1}, g^{-1}[x])|^2}{\Delta_G(g)\lambda(g^{-1}, x)} \lambda(g^{-1}, x)dgdx
\]

\[
= \int_{G \times X} |\psi(g, x)|^2 \lambda(g^{-1}, x)dgdx
\]

\[
= \int_{G \times X} |\psi(g, x)|^2d\mu(g, x).
\]

Moreover, for \( \mu \)-almost all \((g, x) \in G \times X \),

\[
(T^2\psi)(g, x) = \sqrt{\frac{1}{\Delta_G(g)\lambda(g^{-1}, x)}}(T\psi)(g^{-1}, g^{-1}[x])
\]

\[
= \sqrt{\frac{1}{\Delta_G(g)\lambda(g^{-1}, x)}} \sqrt{\frac{1}{\Delta_G(g^{-1})\lambda(g, g^{-1}[x])}} \psi(g, x) = \psi(g, x),
\]

where we used Eq. (3).

We let \((M^R, R)\) be the imprimitivity system acting in \( L^2(G \times X, \mu) \) defined by...
\[ R_g = TL_g T \quad g \in G, \]
\[ M^R(f) = TM^L(f) T \quad f \in C_0(X). \]

Explicitly, if \( g \in G \) and \( f \in C_0(X) \),
\[ (R_g \psi)(h,x) = \sqrt{\Delta_G(g)} \lambda(g^{-1}h^{-1}[x]) \psi(hg,x), \]
\[ (M^R(f) \psi)(h,x) = f(h^{-1}[x]) \psi(h,x), \]
for all \( \psi \in L^2(G \times X, \mu) \) and for \( \mu \)-almost all \((h,x) \in G \times X\). The above imprimitivity system plays the role of the right regular representation.

We are now ready to prove the theorem.

**Proof of Theorem 3:** We follow the proof of Ref. 1. Let \( B \subseteq \mathcal{H} \) be the subspace of admissible vectors and, for all \( u \in \mathcal{H}, u \neq 0 \), define \( B_u \) as the operator from \( B \) to \( L^2(G \times X, \mu) \) given by
\[ B_u v = c_{u,v}. \]

By Theorem 1, the domain of \( B_u \) is the set of \( v \in \mathcal{H} \) such that \( c_{u,v} \) is square-integrable and it is not the null space by definition of square integrability.

Since \( \lambda(g, \cdot) \) is defined up to a \( dx \)-negligible set, due to the hypothesis on \( \lambda \), we can always assume \( \lambda(g, \cdot) \) to be locally bounded. Hence, for all \( g \in G \) and \( f \in C_c(X) \), let \( S_g(f) \) be the function on \( X \) given by
\[ (S_g(f))(x) = \sqrt{\Delta_G(g^{-1})} \lambda(g^{-1}x)f(x). \]

It is clear that \( S_g(f) \) is a measurable bounded function and
\[ M^R(S_g(f)) = \int_X S_g(f)(x) dP^R(x) \]
is a well defined bounded operator on \( L^2(G \times X, \mu) \). We claim that, given \( f \in C_c(X) \) and \( g \in G \),
\[ M^R(S_g(f)) R_g B_u \subset B_u M^R(f) U_g. \tag{8} \]

Indeed, let \( v \in \text{dom} B_u \). By Lemma 4, there is a \( dg \)-negligible set \( A \), such that for all \( h \in G, h \notin A \), the measure
\[ f_1 \mapsto \langle u, M(\tilde{f}_1) U_g v \rangle \]
has a density \( \lambda(h^{-1}, \cdot) c_{u,v}(h, \cdot) \) with respect to \( dx \).

Let now \( f \in C_c(X) \) and \( g \in G \), then for all \( h \notin Ag^{-1} \),
\[ \langle u, M(\tilde{f}_1) U_g M(f) U_g v \rangle = \langle u, M(\tilde{f}_1 h f) U_h g v \rangle \]
\[ = \int_X f_1(x) \bar{f}(h^{-1}[x]) c_{u,v}(hg,x) \lambda(g^{-1}h^{-1},x) dx \]
\[ = \int_X \{ f_1(x) \bar{f}(h^{-1}[x]) c_{u,v}(hg,x) \lambda(g^{-1}h^{-1},x) \} dx = \int_X f_1(x) \]
\[ \times (M^R(S_g(f)) R_g c_{u,v})(h,x) \lambda(h^{-1},x) dx \]
\[ = \int_X f_1(x) (M^R(S_g(f)) R_g c_{u,v})(h,x) \lambda(h^{-1},x) dx. \]
Using Lemma 4 with the fact that $Ag^{-1}$ is $dg$-negligible, it follows that $M(f)U_g v \in \text{dom} \ B_u$ and $B_u M(f) U_g v = M^R (s(f)) R_u B_u v$, so the claim is proven.

Hence, since $(M, U)$ is irreducible, $B$ is dense in $H$ and, as in the proof of Theorem 1, we can prove that $B_u$ is a closed operator and $B_u \neq 0$. 

Fix $u_0 \in H$ such that $\|u_0\| = 1$ and define $C = \sqrt{B_u B_u^*}$, so that $\text{dom} C = B$ and, for all $v_1, v_2 \in B$,

$$\langle B_u v_2, B_u v_1 \rangle = \langle Cv_2, Cv_1 \rangle.$$

Let now $u_1, u_2 \in H$ and $v_1, v_2 \in B$. Then, by Theorem 1, we have that

$$\langle c_{u_1, v_1}, c_{u_2, v_2} \rangle = \langle W_{v_1} u_1, W_{v_2} u_2 \rangle = \langle W_{v_2} W_{v_1} u_1, u_2 \rangle = \epsilon_{v_2, v_1} \langle u_1, u_2 \rangle,$$

where $\epsilon_{v_2, v_1} \in C$ since $W_{v_2} W_{v_1}$ is in the commuting ring of $(M, U)$ so that it is a multiple of a scalar.

With the choice $u_1 = u_2 = u_0$ in the above equation, one obtains that

$$\epsilon_{v_2, v_1} = \langle c_{u_0, v_1}, c_{u_0, v_2} \rangle = \langle B_u v_2, B_u v_1 \rangle = \langle Cv_2, Cv_1 \rangle.$$

The unicity of $C$ is evident and this ends the proof.

The condition on the cocycle $\lambda$ given in the previous proposition clearly holds if $dx$ is relatively invariant or, more generally, if the cocycle $\lambda$ is continuous on $G \times X$ (in this case $dx$ is said to be strongly quasi-invariant). This happens, for instance, when $X$ is transitive, see, for example Ref. 14. Moreover, we use the above condition only in order to prove that the domain of $C$, which coincides with the set of admissible vectors, is a dense subspace of $H$.

Comparing the orthogonality relations for representations with the ones for imprimitivity systems, we see that $C^{-1}$ plays the role of the formal degree operator. The main difference is that the formal degree operator is semi-invariant with respect to the action of the representation, see Theorem 3 of Ref. 1, whereas our Theorem 3 does not give any information about the covariance properties of the normalizing operator $C$ with respect to the action of the imprimitivity system.

However, when $dx$ is relatively invariant (so that $\lambda(g, x) = \lambda(g)$) we have the following result.

We denote by $\chi$ the character of $G$ given by

$$\chi(g) = \sqrt{\Delta_G(g)} \lambda(g).$$

**Corollary 3:** Assume the measure $dx$ is relatively invariant. Let $(M, U)$ be an imprimitivity system acting in $H$ and square-integrable with respect to $dx$ and let $C$ be the corresponding normalizing operator, then

$$M(f) C \subseteq CM(f) \quad f \in C_0(X),$$

$$U_g C U_g^{-1} = \chi(g) C \quad g \in G.$$

Moreover, there is a unique isometry $\sigma$ from the Hilbert space $L^2(H)$ of Hilbert-Schmidt operators in $H$ into $L^2(G \times X, \mu)$ such that for all $u \in H$ and $v \in \text{dom} \ C^{-1}$,

$$\sigma(v^* \otimes u) = W_{C^{-1}} u.$$

Finally,

1. the range of $\sigma$ is $E_{(M, U)}$;
2. for all $f_1, f_2 \in C_0(X), g_1, g_2 \in G$ and $A \in L^2(H)$,

$$\sigma(M(f_1) U_{g_1} A U_{g_2}^{-1} M(f_2)) = M^R(f_1) M^R(f_2) L_{g_1} R_{g_2} \sigma(A);$$

\[ (9) \]
Indeed, let \( f \) proves that
\[
\lim_{X} g \in \text{orthogonality relations and of the covariance relation of } \sim M, U
\]
representations, see Lemma 4 of Ref. 1.

Remark 7: With respect to Eq. (9), notice that, since \( \lambda \) does not depend on \( x \), the imprimitivity system \((M^L,L)\) commutes with \((M^R,R)\). Hence, an imprimitivity system for \( G \times G \) based on \( X \times X \) is canonically defined on \( L^2(G \times X, \mu) \).

Proof of Corollary 3: We use the same notation of the proof of Theorem 3. Fixed \( u_0 \in \mathcal{H}, \|u_0\| = 1 \), one has that \( B_u B_u = C^2 \). Taking into account Eq. (8) and the fact that \( S_\epsilon(f) = x(g^{-1})f \), one has, for all \( f \in C_0(X) \) and \( g \in G \),
\[
\chi(g^{-1})^2 M(f) U_g C^2 \subseteq C^2 M(f) U_g .
\]

In the above relation, let \( g = e \) and \( f \in C_0(X) \), then, as a consequence of the spectral theorem, for all \( f \in C_0(X) \),
\[
 M(f) C \subseteq CM(f) .
\]

Clearly, the above equation holds for all \( f \in C_0(X) \). Fix now \( g \in G \), let \( (f_n) \) be a bounded sequence in \( C_0(X) \) converging to 1 pointwise then, for all \( u, v \in \text{dom} \, C^2 \), taking into account that \( \lim_n M(f_n) u = u \),
\[
\langle C^2 u, U_g v \rangle = \lim_{n} \langle C^2 u, M(f_n) U_g v \rangle = \chi(g^{-1})^2 \lim_{n} \langle u, M(f_n) U_g C^2 v \rangle = \chi(g^{-1})^2 \langle u, U_g C^2 v \rangle .
\]

Hence, \( U_g \) leaves \( \text{dom} \, C^2 \) invariant and \( C^2 U_g = \chi(g^{-1})^2 U_g C^2 \). Finally, the spectral theorem proves that \( C = \chi(g^{-1}) U_g C U_g^{-1} \).

Since \( \|W_{C^{-1}u}\|_{L^2(G \times X, \mu)} = \|u\| \|v\| \), then \( \sigma \) is a well-defined isometry on \( L^2(\mathcal{H}) \). The fact that it is onto \( E_{(M, U)} \) follows by the definition of \( E_{(M, U)} \). The other properties are consequences of the orthogonality relations and of the covariance relation of \( C \), as in the case of square-integrable representations, see Lemma 4 of Ref. 1.

As in the case of square-integrable representations, a reproducing formula follows from the orthogonality relations.

Corollary 4: Assume that, for all \( g \in G, \lambda(g, \cdot) \) is locally, \( dx \)-essentially bounded on \( X \) and let \( (M, U) \) be an imprimitivity system square-integrable with respect to \( dx \). Let \( C \) be the normalizing operator. Let \( v, v_1, v_2 \in \text{dom} \, C \) and \( u \in \mathcal{H} \), then
\[
\langle Cv_2, Cv_1 \rangle W_v u = W_v u \ast W_v v_2 .
\]

Proof: First of all, we claim that, fixed \( u, v \in \text{dom} \, C \) and \( g \in G \), for \( \mu \)-almost all \( (h, x) \in G \times X \),
\[
W_v u (g^{-1}[(h, x)]) \lambda(h^{-1}x) = W_v (h^{-1}[(g, x)]) \lambda(g^{-1}, x) .
\]

Indeed, let \( f \in C_0(X) \). Since \( v \in \text{dom} \, C \), by Lemma 4, there is a \( dg \)-negligible set \( \Omega \), depending on \( u, v \), but not on \( f \), such that, for all \( g^{-1} h \notin \Omega \),
\[
\langle u, M(l_{g^{-1}f}) U_{g^{-1}u} v \rangle = \int_X \overline{f(g[x])}(W_v u)(g^{-1}h, x) \lambda(h^{-1}g, x) dx
\]
\[= \int_X \overline{f(x)}(W_v u)(g^{-1}[(h, x)]) \lambda(h^{-1}g, g^{-1}[x]) \lambda(g^{-1}, x) dx \]
\[= \int_X f(x)(W_v u)(g^{-1}[(h, x)]) \lambda(h^{-1}, x) dx \].
due to Eq. (3). Interchanging the role of $g \mapsto h$ and $u \mapsto v$, since $u \in \text{dom } C$, for all $h^{-1} g \in \Omega'$, where $\Omega'$ is $dg$-negligible, then

$$\langle v, M(l_{h^{-1}f})U_{h^{-1}g}u \rangle = \int_X f(x)(W_u v)(h^{-1}[(g,x)]) \lambda(g^{-1},x)dx.$$ 

On the other hand, for all $g, h \in G$,

$$\langle v, M(l_{h^{-1}f})U_{h^{-1}g}u \rangle = \langle U_{h^*v}, M(f) U_{g}u \rangle = \langle M(f) U_{g}u, U_{h^*v} \rangle = \langle u, M(l_{g^{-1}f})U_{g^{-1}h^*v} \rangle.$$ 

So, fixed $g \in G$, for almost all $h \in g \Omega \cup g \Omega^{-1}$, one has that

$$\int_X f(x)(W_u u)(g^{-1}[(h,x)]) \lambda(h^{-1},x)dx = \int_X f(x)(W_u v)(h^{-1}[(g,x)]) \lambda(g^{-1},x)dx.$$ 

This relation holds for all $f \in C_0(X)$, hence, for all $h \notin g \Omega \cup g \Omega^{-1}$ and for all $x \notin Y_h$, where $Y_h$ is a $dx$-negligible set, one has that

$$(W_u u)(g^{-1}[(h,x)]) \lambda(h^{-1},x) = (W_u v)(h^{-1}[(g,x)]) \lambda(g^{-1},x).$$

Since the two sides are $\mu$-measurable functions and $g \Omega \cup g \Omega^{-1}$ is $dg$-negligible, the claim follows.

Let now $u \in H$, $v, v_1, v_2 \in \text{dom } C$. Fixed $g \in G$, for all $f \in C_0(X)$, one has that, by the orthogonality relations,

$$\langle u, M(f) U_g v \rangle (Cv_2, Cv_1) = \langle W_{v^*} u, W_{v^*} M(f) U_g v \rangle$$

$$= \langle W_{v^*} u, M(f)(f) L_g W_{v^*} v \rangle$$

$$= \int_{G \times X} \{(W_{v^*} u)(h,x)f(x)(W_{v^*} v)(g^{-1}[(h,x)]) \lambda(h^{-1},x)\}dhdx$$

$$= \int_{G \times X} \{f(x)\lambda(g^{-1},x)(W_{v^*} u)(h,x)(W_{v^*} v)(h^{-1}[(g,x)])\}dhdx$$

$$= \int_X f(x)\lambda(g^{-1},x) \int_G (W_{v^*} u)(h,x)(W_{v^*} v)(h^{-1}[(g,x)])dh dx$$

$$= \int_X f(x)\lambda(g^{-1},x)(W_{v^*} u) \star (W_{v^*} v)(g,x)dx,$$

where we used the claim stated at the beginning of the proof. Moreover, Fubini theorem assures that $W_{v^*} u \star W_{v^*} v$ is well defined. By Lemma 4 and the above relation, it follows that, for almost all $g \in G$,

$$\langle Cv_2, Cv_1 \rangle (W_g u)(g, \cdot) = (W_{v^*} u) \star (W_{v^*} v)(g, \cdot).$$

The thesis is now clear. 

\[\square\]

**IV. TRANSITIVE IMPRIMITIVITY SYSTEMS**

In this section we study the square-integrability of imprimitivity systems in the case that $X$ is transitive. With this assumption, on $X$ there is only one quasi-invariant measure class, so that the square-integrability does not depend on the measure and we can always choose (and we do) the measure $dx$ to be strongly quasi-invariant and the corresponding cocycle $\lambda$ to be a continuous
function on $G \times X$, see, for example Ref. 14. The importance of transitive imprimitivity systems is twofold. First of all, their square-integrability can be characterized in terms of square-integrability of a representation of the stability subgroup; this latter representation is uniquely defined by the imprimitivity system by means of the inducing functor of Mackey. On the other hand, if $X$ is not transitive, but its orbits with respect to the action of $G$ are locally closed, any irreducible imprimitivity system based on $X$ is completely defined in terms of an imprimitivity system based on an orbit of $X$ and, hence, the study of square-integrability can be done by the use of the results about the transitive case.

Hence, we assume now that $X$ is a transitive space and we fix $x_0 \in X$, so that $X = G[x_0]$. We let $H$ be the stability subgroup of $G$ at $x_0$. If $(M, U)$ is an imprimitivity system, by the Mackey theorem there is, up to unitary equivalence, a unique representation $m$ of $H$ such that $(M, U)$ is equivalent to the imprimitivity system $(M^m, U^m)$ induced by $m$. We denote such representation $m$ by $\text{Res}_{H}^{G}(U)$.

**Remark 8:** A realization of $(M^m, U^m)$ is the following one. Let $c$ be a regular section, from $X$ to $G$ such that $c(x_0) = e$, and $K$ the Hilbert space where $m$ acts. Then $(M^m, U^m)$ is given by

$$\mathcal{H} = L^2(X, dx, K),$$

$$(M^m(f)u)(x) = f(x)u(x),$$

$$(U^m_g(u)(x) = \lambda^{1/2}(g^{-1}.x)m(c(x)^{-1}g(c^{-1}[x]))u(g^{-1}[x]),$$

where $u \in \mathcal{H}$, $f \in C_{0}(X)$, $g \in G$ and the equalities hold for $dx$-almost all $x \in X$.

The next theorem characterizes the square-integrability of transitive imprimitivity systems.

**Theorem 4:** Assume that there is $x_0 \in X$ such that $X = G[x_0]$. Let $dx$ be strongly-quasi-invariant and let $(M, U)$ be an irreducible imprimitivity system. The following conditions are equivalent:

1. $(M, U)$ is a square-integrable imprimitivity system with respect to $dx$;
2. $\text{Res}_{H}^{G}(U)$ is a square-integrable representation of the stability subgroup $H$ of $G$ at $x_0$.

Moreover, if $m$ is a square-integrable representation of $H$ and $(M^m, U^m)$ is the imprimitivity system given in Remark 8, then the corresponding normalizing operator $C$ is given by

$$\text{dom} \ C = \left\{ u \in \mathcal{H} : u(x) \in \text{dom} \ K_{m}^{-1/2}, \int_{X} \gamma(x)\|K_{m}^{-1/2}v(x)\|^2dx < \infty \right\},$$

$$(Cv)(x) = \gamma(x)^{1/2}K_{m}^{-1/2}v(x),$$

where $\gamma(x) = 1/\Delta_{G}(c(x))\lambda(c(x), x_0)$ and $K_{m}$ is the formal degree of $m$ for a suitable choice of the Haar measure of $H$.

**Proof:** Since the square integrability depends only on the equivalence class of $(M, U)$, we can always assume that $(M, U)$ is of the form $(M^m, U^m)$, where $m = \text{Res}_{H}^{G}(U)$ is an irreducible representation of $H$. Moreover, we recall that the cocycle $\lambda$ satisfies

$$\lambda(h, x_0) = \frac{\Delta_{H}(h)}{\Delta_{G}(h)} \quad h \in H,$$

where $\Delta_{H}$ is the modular function of $H$, see, for example Ref. 14.

First of all, we prove the equivalence between the two conditions in the statement of the theorem.

Let $f \in C_{0}(X)$, $g \in G$ and $u, v \in \mathcal{H}$, then
\[ \langle u, M(f) U_g v \rangle = \int_X f(x) \lambda^{1/2}(g^{-1} x) \langle u(x), m(c(x)^{-1} g c(g^{-1}[x])) v(g^{-1}[x]) \rangle dx, \]

so that, for all \( g \in G \), the measure \( f \mapsto \langle u, M(f) U_g v \rangle \) has density
\[ \lambda^{1/2}(g^{-1} x) \langle u(x), m(c(x)^{-1} g c(g^{-1}[x])) v(g^{-1}[x]) \rangle. \]

Moreover, let \( I_{u,v} := \int_X \int_G \langle u(x), m(c(x)^{-1} g c(g^{-1}[x])) v(g^{-1}[x]) \rangle^2 dg dx \), where \( I \in [0, \infty]. \)

By Lemma 4, the system \((M, U)\) is square integrable, i.e., condition (1) of the theorem holds, if and only if there exist \( u, v \in \mathcal{H} \), such that \( 0 < I_{u,v} < \infty \).

Assume the existence of such vectors \( u \) and \( v \). Since
\[ \langle g(x) \mapsto \langle u(x), m(c(x)^{-1} g c(g^{-1}[x])) v(g^{-1}[x]) \rangle^2 \]
is \( dg dx \)-measurable and positive, due to the Fubini theorem,
\[ I_{u,v} = \int_X \int_G \int_X \int_G \langle u(x), m(c(x)^{-1} g c(g^{-1}[x])) v(g^{-1}[x]) \rangle^2 dg dx dV \]
\[ = \int_X \int_G \int_X \int_G \langle u(x), m(g^{-1}[x_0]) v(g^{-1}[x_0]) \rangle^2 dg dx \]
\[ = \int_X \int_G \langle u(x), m(g^{-1} c(g(x_0)) v(g(x_0)) \rangle^2 \Delta_G(g^{-1}) dg dx. \]

The Mackey–Bruhat formula implies that, if we identify \( G \) with \( X \times H \) by means of \( g = c(y) h \), where \( y \in X \), \( h \in H \), then
\[ dg = \frac{\Delta_G(h)}{\Delta_H(h) \lambda(c(y), x_0)} dx dh, \]
where \( dh \) is a suitable Haar measure on \( H \). Hence,
\[ I_{u,v} = \int_X \int_X \int_X \int_H \langle u(x), m(h^{-1} v(y)) \rangle^2 \frac{1}{\Delta_G(c(y)) \lambda(c(y), x_0) \Delta_H(h)} dh dy dx \]
\[ = \int_X \int_X \int_X \int_H \langle u(x), m(h) v(y) \rangle^2 dh dy dx, \]
where \( \gamma(x) = 1/\Delta_G(c(y)) \lambda(c(y), x_0) \) and the map \( \gamma \) is bounded on the compact sets, due to the regularity of \( c \).

Obviously, \( 0 < I_{u,v} < \infty \) if and only if
(a) for \( dx \)-almost all \( x \in X \) and, fixed \( x \in X \), for \( dx \)-almost all \( y \in X \),
\[ 0 < \int_H \|u(x), m(h) v(y)\|^2 dh < \infty. \]

This condition is equivalent to the fact that \( m \) is square-integrable and, due to the orthogonality relations for representations, we have that
\[ v(y) \in \text{dom} K^{-1/2}_m \int_H \|u(x), m(h) v(y)\|^2 dh = \|u(x)\|^2 \|K^{-1/2}_m v(y)\|^2, \]
where \( K_m \) is the formal degree of \( m \);
(b) for $dx$-almost all $x \in X$,

$$0 < \|u(x)\|^2 \int_X \gamma(y) \|K_m^{-1/2}v(y)\|^2 dy < \infty.$$ 

This is equivalent to

$$0 < \int_X \gamma(y) \|K_m^{-1/2}v(y)\|^2 dy < \infty,$$

and $u \neq 0$;

(c) 

$$0 < \int_X \|u(x)\|^2 dx < \infty.$$ 

Hence, the fact that $0 < I_{u,v} < \infty$ implies that $m$ is square-integrable.

Conversely, assume that $m$ is square-integrable. Choose any $u \in \mathcal{H}$, $u \neq 0$, and define

$$v(x) = \chi_K v_0,$$

where $K$ is a compact non-negligible subset of $X$ and $v_0 \in \text{dom } K_m^{-1/2}$, $v_0 \neq 0$. Then, conditions (a), (b), and (c) hold, so that $0 < I_{u,v} < \infty$. This prove the equivalence between the two condition.

Finally, we consider the case when $X$ is not transitive, but, nevertheless, its orbits are locally closed. With this assumption, every orbit $Y$ is a lcsc transitive $G$-space and we say that $M$ lives on $Y$ if $P^M(Y) = I$. From a theorem of Glimm, see, for example, Proposition 6.6 of Ref. 14, given an irreducible imprimitivity system $(M, U)$, there exist an orbit $Y = G[x_0]$ and an irreducible representation $m$ of $H$, the stability subgroup at $x_0$, such that $M$ lives on $Y$ and $U$ is equivalent to the induced representation $\text{Ind}^G_H(m)$ by $m$ from $H$ to $G$. Moreover, we define the measure $dx^Y$ as the restriction of $dx$ to $Y$ and $(M^Y, U^Y)$ as the imprimitivity system for $G$ based on $Y$ given by

$$M^Y(f) = \int_Y f(x) dP^M(x),$$

and $U^Y_g = U_g$, 

where $f \in C_0(Y)$ and $g \in G$. Notice that $(M^Y, U^Y)$ acts in the same Hilbert space of $(M, U)$ and, since $P^M(Y) = I$, $(M^Y, U^Y)$ is irreducible. The following corollary characterizes the square-integrability of $(M, U)$ in terms of the square-integrability of $(M^Y, U^Y)$, which is an imprimitivity system based on a transitive space.

**Corollary 5:** Assume that the orbits of $X$ are locally closed and let $(M, U)$ be an irreducible imprimitivity system acting in $\mathcal{H}$. Denote by $Y$ the orbit where $M$ lives and by $m$ the representation of $H$, the stability subgroup at $x_0 \in Y$, such that $U$ is equivalent to $\text{Ind}^G_H(m)$.

With this notations, the following conditions are equivalent:

(1) $(M, U)$ is square-integrable with respect to $dx$;

(2) $Y$ is not negligible with respect to $dx$ and $m$ is a square-integrable representation of $H$;

(3) $Y$ is not negligible and $(M^Y, U^Y)$ is square-integrable with respect to $dx^Y$.

If any of the above equivalent conditions is satisfied and $dx^Y$ is strongly quasi-invariant, then $v \in \mathcal{H}$ is an admissible vector for $(M, U)$ if and only if $v \in \text{dom } C$, where $C$ is the normalizing operator of $(M^Y, U^Y)$. Moreover, if $u_1, u_2 \in \mathcal{H}$ and $v_1, v_2 \in \text{dom } C$,

$$\langle W_{v_1}u_1, W_{v_2}u_2 \rangle = \langle u_1, u_2 \rangle \langle Cv_2, Cv_1 \rangle.$$
Proof: We prove the equivalence between the first and third condition. If \((M, U)\) is square-integrable, with the notation of Corollary 2, the measure \(\nu^M\) has density with respect to \(dx\) and, since \(P^M(Y) = I\), \(\nu^M(Y) > 0\), then \(Y\) is not \(dx\)-negligible. Hence, from now on, we assume that this condition is satisfied.

Let \(u, v \in \mathcal{H}\) and, given \(g \in G\), let \(\omega_g\) be the bounded measure on \(X\),

\[
f \mapsto \langle u, M(f)U_g v \rangle
\]

and \(\omega^Y_g\) the measure on \(Y\)

\[
f \mapsto \langle u, M^Y(f)U^Y_g v \rangle.
\]

By definition of \((M^Y, U^Y)\), \(\omega^Y_g\) is the restriction to \(Y\) of the measure \(\omega_g\). On the other hand, taking into account that \(\omega_g\) has density with respect to \(\nu^M\), the complement of \(Y\) is \(\omega_g\)-negligible, so that \(\omega_g\) is the natural extension of \(\omega^Y_g\).

By Lemma 4, the linear form \(c_{u,v}\) on \(C_c(G \times X)\) is a locally \(\mu\)-integrable function if and only if, for \(dg\)-almost all \(g \in G\), \(\omega_g\) has density \(\eta_g\) with respect to \(dx\). This last condition, taking into account that, for \(dx\)-almost all \(x \in Y\), \(\eta_g(x) = 0\), is equivalent to the fact that \(\omega^Y_g\) has density \((\eta_g)|_Y\) with respect to \(dx^Y\). Moreover, if one of the above equivalent conditions is satisfied, then, for \(\mu\)-almost all \((g, x) \in G \times X, x \in Y\),

\[
c_{u,v}(g, x) = 0,
\]

and, for \(\mu\)-almost all \((g, x) \in G \times Y\),

\[
c_{u,v}(g, x) = c^Y_{u,v}(g, x),
\]

where the apex \(Y\) refers to the quantities defined in terms of \((M^Y, U^Y)\).

Taking into account the above relations and Lemma 4, if follows that \(c_{u,v} \in L^2(G \times X, \mu)\) if and only if \(c^Y_{u,v} \in L^2(G \times Y, \mu^Y)\). Hence, the vector \(v\) is admissible for \((M, U)\) if and only if it is so for \((M^Y, U^Y)\) and, in this case, for \(\mu\)-almost all \((g, x) \in G \times Y\),

\[
(W_{u})(g, x) = (W^Y_{u})(g, x)
\]

and, for \(\mu\)-almost all \((g, x) \in G \times X, x \in Y\),

\[
(W_{u})(g, x) = 0.
\]

The equivalence of the first and third condition is clear as well as, by means of Theorem 3 and Theorem 4, the last statement of the theorem.

The equivalence of the second and third condition is a restatement of the second part of Theorem 4, taking into account that \(m = \text{Res}^G_{\text{id}}(U^Y)\).

\[\square\]

V. IMPRIMITIVITY SYSTEMS BASED ON AN ABELIAN GROUP

In the case of square-integrable representations of a locally compact group \(G\), the Hilbert space carrying the representation is canonically embedded in \(L^2(G, dg)\) as a subspace of continuous functions. This is no longer true in the case of square-integrable imprimitivity systems. However, when \(X\) is an Abelian group, we can regularize the image of the operator \(W_u\) by means of Fourier transform.

In the following, we assume the \(G\)-space \(X\) to be an abelian lcsc group, the measure \(dx\) to be a Haar measure of \(X\) (considering \(X\) as an Abelian group), and \(dx\) to be relatively invariant with respect to the action of \(G\). We denote by \(\hat{X}\) the dual group of \(X\), by \(d\xi\) a Haar measure of \(\hat{X}\) and by \(\mathcal{F}\) the Fourier-Plancherel operator from \(L^2(X, dx)\) onto \(L^2(\hat{X}, d\xi)\) (we normalize \(d\xi\) in such a
Hence, we define \( J \) as the unitary operator from \( L^2(G \times X, \mu) \) onto \( L^2(G \times \hat{X}, \hat{\mu}) \) given by
\[
(J \psi)(g, \hat{x}) = (\mathcal{F} \psi(g, \cdot))(\hat{x}) \quad g \in G, \quad \hat{x} \in \hat{X}, \quad \psi \in L^2(G \times X, \mu).
\]

Finally, let \((M, U)\) be an imprimitivity system square-integrable with respect to \( dx \). For all \( \hat{x} \in \hat{X} \), let \( V_\hat{x} = \int_X \langle x, \hat{x} \rangle dP^M(x) \), where the integral is in the weak operator topology. Moreover, we denote by \( C \) the corresponding normalizing operator. Hence, given \( v \in \text{dom} \ C \), we define \( \hat{W}_v = JW_v \).

**Theorem 5:** Assume that \( X \) is an Abelian lcsc group and its Haar measure \( dx \) is relatively invariant. Let \((M, U)\) be an imprimitivity system acting in \( \mathcal{H} \). If \((M, U)\) is square-integrable with respect to \( dx \), then

1. given \( v \in \text{dom} \ C \), for all \( u \in \mathcal{H} \),
   \[
   (\hat{W}_v u)(g, \hat{x}) = \lambda(g)(u, V_\hat{x} U_g v) \quad \hat{\mu}\text{-a.e.},
   \]
   in particular, \( \hat{W}_v u \) has a continuous representative in \( L^2(G \times \hat{X}, \hat{\mu}) \);
2. for all \( u_1, u_2 \in \mathcal{H} \) and \( v_1, v_2 \in \text{dom} \ C \),
   \[
   \langle \hat{W}_{v_1} u_1, \hat{W}_{v_2} u_2 \rangle_{L^2(G \times \hat{X}, \hat{\mu})} = \langle u_1, u_2 \rangle \langle Cv_2, Cv_1 \rangle.
   \]

**Proof:** We prove the first item. For \( dg\)-almost all \( g \in G \),
\[
\langle u, V_\hat{x} U_g v \rangle = \int_X \langle x, \hat{x} \rangle \langle u, dP^M(x) U_g v \rangle
\]
\[
= \int_X \langle x, \hat{x} \rangle \lambda(g^{-1})(W_v u)(g, x) dx = \lambda(g^{-1})(\mathcal{F}(W_v u)(g, \cdot))(\hat{x})
\]
\[
= \lambda(g^{-1})(\hat{W}_v u)(g, \hat{x}),
\]
where we used the fact that \( (W_v u)(g, \cdot) \in L^1(X, dx) \cap L^2(X, dx) \). The claim now easily follows.

The second item is consequence of the orthogonality relations and the fact that \( J \) is unitary. \( \square \)

**Remark 9:** In the previous theorem, we do not assume that the action of \( G \) preserves the group law of \( X \). Compare with item 3 of Remark 4. In Ref. 19, there is a partial overlap with the results contained in the above theorem.

**VI. EXAMPLES**

In this section we give some examples of square integrable imprimitivity systems. The first one clarifies the contacts between our construction and the Gabor analysis on Abelian groups, whereas the other two examples are simple toy-models. Some more examples, in a slightly different framework, can be found in Ref. 19, where we discuss the relation between square-integrability of imprimitivity systems and square-integrability of representations on quotient spaces.

**A. Short time Fourier transform.**

Consider a lcsc Abelian group \( G \) acting on itself by translation, so that \( X = G \) and the action is transitive and free. Let \( dg \) be a Haar measure of \( G \). With the choice \( dx = dg \), the measure \( \mu \) on \( G \times G \) of Lemma 3 is the product measure \( dgdx \).
Let now \((M, U)\) be the \textit{Weyl–Heisenberg imprimitivity system} for \(G\) based on \(G\) explicitly given by

\[
\mathcal{H} = L^2(G, dg),
\]

\[
M(f)u = fu,
\]

\[
U_au = u^a,
\]

where \(f \in C_0(G), \ a \in G\) and \(u \in L^2(G, dg)\). In Gabor analysis, \(U\) is the time-shift operator and \(P^M\) is the projection valued measure associated with the frequency-shift operator.\(^9\)

Taking into account that the action is transitive and free, one has that \((M, U)\) is irreducible and, as a consequence of Theorem 4, it is square-integrable with respect to \(dg\). Moreover, the normalizing operator \(C\) is the identity, so that any vector is admissible. Explicitly, fixed \(v \in \mathcal{H}\), \(W_v\) is an operator from \(L^2(G, dg)\) into \(L^2(G \times \hat{G}, dg dg)\) given by

\[
(W_v u)(a, x) = u(x)\overline{v(x-a)} \, dg \, d\hat{g} \text{-a.e.}
\]

Let \(\hat{G}\) be the dual group of \(G\) and \(d\hat{g}\) the Haar measure of \(\hat{G}\) such that the Fourier–Plancherel transform is unitary. Using Theorem 5, \(\hat{W}_v\) is an operator from \(L^2(G, dg)\) into \(L^2(G \times \hat{G}, dg dg)\) given by

\[
(\hat{W}_v u)(a, \omega) = \int_X u(x)\overline{v(x-a)\omega(x)} \, dx \quad \forall (a, \omega) \in G \times \hat{G}
\]

and \(\hat{W}_u\) is nothing but the short time Fourier transform, see Ref. 9 and, in the case \(G = \mathbb{R}\), for example, Chap. 2.7 of Ref. 20, where it is called \textit{windowed Fourier transform}.

Notice that, since \((M, U)\) is the unique, up to an equivalence, cyclic imprimitivity system for \(G\) based on \(G\), then \(E_{(M, U)}\) is equal to \(L^2(G \times \hat{G}, dg dg)\) and any orthonormal basis \((e_n)\) of \(L^2(G, dg)\) gives rise to a decomposition into irreducible subspaces of \(L^2(G \times \hat{G}, dg dg)\) as

\[
L^2(G \times \hat{G}, dg dg) = \bigoplus_{n} E_{\hat{e}_n} \mathcal{H}.
\]

### B. Vector valued Euclidean transforms

Given \(n \in \mathbb{N}\), let \(G\) be the Euclidean group \(\mathbb{R}^n \times 'SO(n)\) acting on \(X = \mathbb{R}^n\) in a natural way, so that the action on \(X\) is transitive. The Haar measure of \(G\) is \(dg = d^nxdR\), where \(d^n x\) is the Lebesgue measure on \(\mathbb{R}^n\) and \(dR\) is the normalized Haar measure of \(SO(n)\). The Lebesgue measure \(d^n x\) is invariant with respect to the action of \(G\) and, with the choice \(dx = d^n x\), the measure \(\mu\) on \(G \times X\) is \(d^nxdRd^n x\).

Given an irreducible representation \(\tau\) of \(SO(n)\) acting in a finite dimensional vector space \(K\), let \((M, U)\) be the imprimitivity system for \(G\) based on \(X\) given by

\[
\mathcal{H} = L^2(\mathbb{R}^n, d^n x, K),
\]

\[
M(f)u = fu,
\]

\[
(U_{(a, R)}u)(x) = \tau(R)u(R^{-1}(x-a)) \, d^n x \text{-a.e.,}
\]

where \(f \in C_0(\mathbb{R}^n), \ (a, R) \in G\) and \(u \in L^2(\mathbb{R}^n, d^n x, K)\). Since \(\tau\) is irreducible \((M, U)\) is irreducible. Applying Theorem 4, since \(SO(n)\) is compact, \((M, U)\) is square integrable with respect to \(d^n x\) and the normalizing operator is the identity. Explicitly, fixed \(v \in \mathcal{H},\)
\[(W_a u)(a, R, x) = \langle u(x), \tau(R)v(R^{-1}(x-a)) \rangle_E,\]

\[
(W_{e_x} u)(a, R, p) = \int \langle u(x), \tau(R)v(R^{-1}(x-a)) \rangle_E e^{-ixpd^n x},
\]

where \((a, R) \in \mathbb{R}^n \times SO(n), x, p \in \mathbb{R}^n\) and \(u \in \mathcal{H}\).

Notice that, since \(SO(n)\) admits many inequivalent irreducible representations, \(E_{(M, U)}\) is a proper closed subspace of \(L^2(G \times X, \mu)\).

### C. Spin transforms

Consider the compact group \(SU(2)\) with the normalized Haar measure \(dh\). The group \(SU(2)\) acts on the three-dimensional sphere \(S^2 \subset \mathbb{R}^3\) by means of the covering homomorphism \(\delta\) from \(SU(2)\) onto \(SO(3)\). The action on \(S^2\) is transitive and an invariant measure on it is the area element \(d\Omega\). With this choice, \(\mu = d\Omega\).

Given \(j\) such that \(2j\) is in \(\mathbb{N}\), let \(D^j\) be the unique, up to an equivalence, irreducible representation of \(SU(2)\) acting on \(L^2(S^2, \Omega)\). We define \((M^j, U^j)\) to be the imprimitivity system for \(SU(2)\) based on \(S^2\) and acting in \(L^2(S^2, d\Omega, C^2j+1)\) as

\[(M^j(f))u(x) = f(x)u(x) \quad f \in C_0(S^2)\]

\[(U^j_0 u)(x) = D^j(h)u(\delta(h)^{-1}x) \quad h \in SU(2),\]

for \(u \in L^2(S^2, d\Omega, C^2j+1)\) and for \(d\Omega\)-almost all \(x \in S^2\).

The system \((M^j, U^j)\) is not irreducible and, by the Mackey theorem, its irreducible components are

\[L^2(S^2, d\Omega, C^2j+1) = \bigoplus_{m=-j}^j L^2(S^2, d\Omega)_m,\]

where the index \(m\) refers to the fact that the subspace \(L^2(S^2, d\Omega)_m\) is unitarily equivalent to \(L^2(S^2, d\Omega)\) carrying the representation of \(SU(2)\) induced by the character of the torus \(z \rightarrow z^m\).

Clearly, the restriction to \(L^2(S^2, d\Omega)_m\) of \((M^j, U^j)\) is square-integrable with respect to \(d\Omega\) and the corresponding normalizing operator \(C^m\) is proportional to the identity. Fix an admissible vector \(v_m \in L^2(S^2, d\Omega)_m\) such that \(\|C^m v_m\| = 1\), then the corresponding isometry \(W^m_{v_m}\) from \(L^2(S^2, d\Omega)_m\) in \(L^2(SU(2) \times S^2, d\Omega)\) is given by

\[(W^m_{v_m} u)(h, x) = \langle u(x), D^j(h)v_m(\delta(h)^{-1}x) \rangle_{C^2j+1},\]

for \(u \in L^2(S^2, d\Omega)_m\) and for \(d\Omega\)-almost all \((h, x) \in SU(2) \times S^2\).

Moreover, we can define an isometry from \(L^2(S^2, d\Omega, C^2j+1)\) into \(L^2(SU(2) \times S^2, d\Omega)\) as \(W^j = \bigoplus_{m=-j}^j W^m_{v_m} P_m\), where \(P_m\) is the projection onto \(L^2(S^2, d\Omega)_m\). Clearly we have that, for all \(u \in L^2(S^2, d\Omega, C^2j+1)\),

\[(W^j u)(h, x) = \sum_{m=-j}^j \langle u(x), D^j(h)v_m(\delta(h)^{-1}x) \rangle_{C^2j+1},\]

for \(d\Omega\)-almost all \((h, x) \in SU(2) \times S^2\).