Wavelet transforms and discrete frames associated to semidirect products

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We consider a semidirect product $G = \mathbb{R}^n \times H$ and its unitary representations $U$ of the form $\text{Ind}_{G_0}^G(p_0 m)$ where $\text{Ind}$ is the unitary induction, $p_0$ is in the dual group of $\mathbb{R}^n$, $G_0$ is the stability group of $p_0$, and $m$ is a unitary representation of $G_0 \cap H$. We give sufficient conditions such that $U$ defines a wavelet transform and a discrete frame.

I. INTRODUCTION

In the framework of wavelet analysis it is well known that the groups that are semidirect products with normal Abelian factor play an essential role in the construction of both continuous wavelet transforms and of discrete frames.

The classical example, Ref. 1, is the group $ax + b$ and its quasi-regular representation $U$ acting on $L^2(\mathbb{R})$ as

$$(U_{ab}f)(x) = a^{-1/2} f\left(\frac{x-b}{a}\right), \quad x \in \mathbb{R}, \quad f \in L^2(\mathbb{R}), \quad a \in \mathbb{R}^+, \quad b \in \mathbb{R}.$$ 

We refer for this to the treatise of Ref. 2 and to the review paper Ref. 3.

In recent years there has been an interest in extending these kind of results to other groups: on one side, for example, Refs. 4–6, obtaining results valid for a fairly general class of groups; on the other side, discussing specific cases. For example, in Ref. 7, the group $\mathbb{R}^n \times (\mathbb{R}^+ \times \text{SO}(n))$ has been considered in order to define wavelet transforms on $L^2(\mathbb{R}^n)$.

In this paper we consider the case of groups $G$ that are semidirect product of $\mathbb{R}^n$ and a topological group $H$ and their representations $U$ of the form $U = \text{Ind}_{G_0}^G(p_0 m)$ where $\text{Ind}$ is the unitary induction à la Mackey, $p_0 \in \mathbb{P}^n$, the dual of $\mathbb{R}^n$, $G_0$ is the stability group of $p_0$ with respect to the dual action of $G$ on $\mathbb{P}^n$, and $m$ is a unitary representation of $G_0 \cap H$. For such groups we consider two kinds of problems.

First we describe the conditions under which the representations $\text{Ind}_{G_0}^G(p_0 m)$ are square-integrable and characterize for them the formal degree operator. Moreover we discuss a concrete example that extends the results of Ref. 7 to vector-valued functions.

Second we consider the problem of the existence of a countable set $(g_i)_{i \in I}$ of $G$ and a vector $\psi \in \mathcal{H}$ such that $(U_{g_i} \psi)_{i \in I}$ is a frame in $\mathcal{H}$, denoting by $\mathcal{H}$ the Hilbert space where $U$ acts. In particular we show that, if $H$ is the direct product of $\mathbb{R}^+$ (the multiplicative group of positive numbers) and a semisimple connected Lie group, and $m$ is a cyclic finite-dimensional representation

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II. PRELIMINARIES AND NOTATIONS

Let $G$ be a locally compact second countable (lcsc) topological group. We denote by $\mu_G$ and $\Delta_G$ respectively a left Haar measure on $G$ and its modular function.

We review the basic definition and properties of square integrable representations and of discrete frames. In this paper we will use the word representation to mean a continuous unitary representation of $G$ acting in a complex separable Hilbert space $\mathcal{H}$.

A representation $U$ of $G$ in $\mathcal{H}$ is square integrable if it is irreducible and there exist two nonzero vectors $\phi, \psi \in \mathcal{H}$ such that the map $c_{\phi,\psi}$ from $G$ to $\mathbb{C}$,

$$c_{\phi,\psi}(g) = \langle U_g\psi, \phi \rangle \neq 0,$$

is in $L^2(G, \mu_G)$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product (linear in the second variable). We recall this well-known result of Duflo and Moore (Theorem 3 of Ref. 8).

Theorem 1: (Duflo and Moore) Let $U$ be a square-integrable representation of $G$ acting in $\mathcal{H}$. There exists a unique positive self-adjoint operator $K_U$ in $\mathcal{H}$ such that

1. if $\phi, \psi, \phi' \in \mathcal{H}$, $\phi \neq 0$, the map $c_{\phi,\psi}$ is in $L^2(G, \mu_G)$ if and only if $\psi \in \text{Dom} \ K_U^{-1/2}$; and
2. if $\phi, \phi' \in \mathcal{H}$ and if $\phi, \psi' \in \text{Dom} \ K_U^{-1/2}$, then

$$\langle c_{\phi,\phi}, c_{\phi',\phi'} \rangle_{L^2(G, \mu_G)} = \langle \phi, \phi' \rangle_{\mathcal{H}} \langle K_U^{-1/2} \psi', K_U^{-1/2} \psi \rangle_{\mathcal{H}}.$$

The operator $K_U$ is called the formal degree of the representation and the vectors $\psi \in \text{Dom} \ K_U^{-1/2}$ are called admissible.

Let $\mathcal{H}$ be a (complex separable) Hilbert space and $\{\psi_i : i \in I\}$ a countable family of nonzero vectors in $\mathcal{H}$. We will say that $\{\psi_i : i \in I\}$ is a frame in $\mathcal{H}$ if there exist two positive numbers $\alpha$ and $\beta$ such that the following condition holds:

$$\alpha \| \phi \|^2 \leq \sum_{i \in I} |\langle \psi_i, \phi \rangle|^2 \leq \beta \| \phi \|^2 \quad \forall \phi \in \mathcal{H}.$$

The numbers $\alpha$ and $\beta$ are called frame bounds of the frame $\{\psi_i\}_{i \in I}$.

If there exists a representation of a group $G$ acting in $\mathcal{H}$, then it is of particular interest to determine frames of the following form:

$$\psi_i = U_{g_i} \psi,$$

where $\{g_i\}_{i \in I}$ is a countable family of elements of $G$ and $\psi$ is a fixed element in $\mathcal{H}$.

From now on let $G$ be the semidirect product $\mathbb{R}^n \ltimes H$, where $H$ is a lcsc topological group acting continuously on $\mathbb{R}^n$ (G is in a natural way a lcsc topological group). Explicitly, let $\Omega$ be the representation of $H$ that defines the action of $H$ on $\mathbb{R}^n$. Denoting $g = (x, h)$ the elements of $G$, with $x \in \mathbb{R}^n$ and $h \in H$, we have

$$(x, h)(x', h') = (x + \Omega(h)x', hh'),$$

and the inner action of $G$ on $\mathbb{R}^n$ is

$$g \cdot x = gxg^{-1} = \Omega(h)x, \quad x \in \mathbb{R}^n, \quad g = (y, h) \in G.$$

We denote by $d^nx$ the Lebesgue measure on $\mathbb{R}^n$. Let $\mathbb{P}^n$ be the dual group of $\mathbb{R}^n$, which we identify with $\mathbb{R}^n$ in a natural way, and $d^n\rho$ be the corresponding Lebesgue measure. By duality, $G$ acts continuously on $\mathbb{P}^n$, explicitly.
We denote by \( \text{Ind}^G_{G_0} \) properties of the commutative Fourier transform.

normal type I subgroup. In Ref. 11 we give an elementary proof of this theorem, based only on the sequence of the characterization of the Plancherel measure for a locally compact group with a closed normal type I subgroup. Let us assume that \( \text{Ind}^G_{G_0} \) has positive Lebesgue measure in \( \mathbb{R}^n \) and \( m \) is a square-integrable representation. This result was proven by Kleppner and Lipsman (Corollary 11.1 of Ref. 10) as a consequence of the characterization of the Plancherel measure for a locally compact group with a closed normal type I subgroup. In Ref. 11 we give an elementary proof of this theorem, based only on the properties of the commutative Fourier transform.

Let us assume that \( \text{Ind}^G_{G_0}(p_0 m) \) is square integrable. Since the square integrability of a representation depends only on its equivalence class, we can freely choose the explicit form of \( \text{Ind}^G_{G_0}(p_0 m) \). Due to Theorem 2 the measure of \( \mathcal{C} \) has to be positive, hence the restriction of the Lebesgue measure \( d^n p \) to \( \mathcal{C} \) is a nonzero \( \sigma \)-finite measure defined on \( \mathcal{C} \). We denote by \( \text{Ind}^G_{G_0}(p_0 m) \) the representation unitarily induced by \( p_0 m \) from \( G_0 \) to \( G \).

### III. SQUARE INTEGRABILITY

The following theorem solves the problem of the square integrability of the induced representations of \( G \).

**Theorem 2**: The representation \( \text{Ind}^G_{G_0}(p_0 m) \) is square integrable if and only if \( \mathcal{C} \) has positive Lebesgue measure in \( \mathbb{R}^n \) and \( m \) is a square-integrable representation.

This result was proven by Kleppner and Lipsman (Corollary 11.1 of Ref. 10) as a consequence of the characterization of the Plancherel measure for a locally compact group with a closed normal type I subgroup. In Ref. 11 we give an elementary proof of this theorem, based only on the properties of the commutative Fourier transform.

Let \( \mathcal{C} \) be the formal degree of \( \omega_0 \). Since the square integrability of a representation depends only on its equivalence class, we can freely choose the explicit form of \( \text{Ind}^G_{G_0}(p_0 m) \). Due to Theorem 2 the measure of \( \mathcal{C} \) has to be positive, hence the restriction of the Lebesgue measure \( d^n p \) to \( \mathcal{C} \) is a nonzero \( \sigma \)-finite measure defined on \( \mathcal{C} \). In fact it is \( H \)-quasi-invariant, due to the following elementary lemma (the proof is given, for example, in Ref. 11). Let \( \rho(h) = \Delta_H(h) / \Delta_G(h) \).

**Lemma 1**: For any \( h \in H \) and \( E \in \mathcal{B}(\mathbb{R}^n) \) we have

\[
(d^n p)(h[E]) = \rho(h^{-1})(d^n p)(E).
\]

With the above notations, a suitable choice is the representation \( U \) acting in \( L^2(\mathcal{C}, \mathcal{H}) \) as

\[
(U_{x,h} \phi)(p) = \rho(h)^{-1/2} e^{-ip \cdot \phi(q(p)^{-1} h q(h^{-1}[p]))} \phi(h^{-1}[p]),
\]

(1)

where \( p \in \mathcal{C}, x \in \mathbb{R}^n, h \in H, \phi \in L^2(\mathcal{C}, \mathcal{H}), \) and \( q \) is a measurable section for the action of \( H \) on \( \mathcal{C} \), that is, a measurable map \( q: \mathcal{C} \rightarrow H \) such that

\[
q(p_0) = e \quad \text{where} \quad e \quad \text{is the identity},
\]

\[
q(p)[p_0] = p, \quad p \in \mathcal{C}
\]

(we recall that the orbit \( \mathcal{C} \) is a Borel subset of \( \mathbb{R}^n \)).

With this choice the condition of admissibility is given by the following corollary (for the proof see Corollary 2 of Ref. 11). Let \( K_n \) be the formal degree of \( m \).

**Corollary 1**: With the above assumptions, \( \psi \in L^2(\mathcal{C}, \mathcal{H}) \) is an admissible vector for \( U \) if and only if the following two conditions hold:
\[ \psi(p) \in \text{Dom } K_m^{-1/2} \text{ for almost all } p \in C, \]
\[ \int_{C} \Delta_G(G(p))^{-1} K_m^{-1/2} \psi(p) \|_{\mathcal{H}}^2 \, d^m p < \infty. \]

If \( H_0 \) is compact, all its irreducible representations are square integrable, and one can easily show that \( \text{Dom } K_m^{-1/2} = \overline{H} \), \( K_m \) is a multiple of the identity, and \( \Delta_G(q(p)) \) is independent of the choice of the section \( q \), precisely
\[ \Delta_G(q(p)) = \Delta_G(g_p) \quad \forall g_p \in G \quad \text{such that } g_p[p_0] = p. \]

Going back to the general case, if \( U \) is square integrable, we can define a corresponding wavelet transform. Let \( \psi \in L^2(\mathcal{C}, \overline{H}) \) be an admissible vector normalized in such a way that
\[ \int_{\mathcal{C}} \Delta_G(G(p))^{-1} K_m^{-1/2} \psi(p) \|_{\mathcal{H}}^2 \, d^m p = 1. \]

Let \( \mathcal{H}_\psi \) be the map from \( L^2(\mathcal{C}, \overline{H}) \) to \( L^2(G, \mu_G) \) defined by
\[ (\mathcal{H}_\psi \phi)(g) = (U_g \psi, \phi) L^2(\mathcal{C}, \overline{H}), \quad g \in G, \phi \in L^2(\mathcal{C}, \overline{H}). \]

As a consequence of Theorem 1, one can easily prove that \( \mathcal{H}_\psi \) is an isometry and the projection on its range is a convolution operator, precisely
\[ F \in \text{Ran } \mathcal{H}_\psi \Leftrightarrow F \ast \mathcal{H}_\psi \psi = F, \]

where \( \ast \) denotes the convolution in \( L^2(G, \mu_G) \). Moreover, the inverse map of \( \mathcal{H}_\psi \) from \( \text{Ran}(\mathcal{H}_\psi) \) onto \( L^2(\mathcal{C}, \overline{H}) \) is given by
\[ F \mapsto \int_{G} F(g)(\mathcal{H}_\psi \psi)(g) \, d\mu_G(g), \]

where the integral is in the weak sense.

To present this kind of result, we consider a specific example. Let \( G = \mathbb{R}^n \rtimes \{R^+ \times \text{SO}(n)\} \), where \( \text{SO}(n) \) is the \( n \)-dimensional special orthogonal group. The direct product \( H = \mathbb{R}^+ \times \text{SO}(n) \) acts on \( \mathbb{R}^n \) as
\[ (aR)[x] = aRx, \]

where \( Rx \) is the usual matrix action and \( a \) acts as a scalar. The group \( H \) is unimodular and the Haar measure is \( a^{-1} d\alpha dR \), where \( d\alpha \) is the normalized Haar measure on \( \text{SO}(n) \). The modular function of \( G \) is \( \Delta_G(x aR) = a^{-n} \). The dual action is
\[ (aR)[p] = a^{-1} Rp. \]

With respect to this action, \( \mathbb{P}^n \) is the union of two orbits \( \{0\} \) and \( \mathcal{C} = \mathbb{P}^n \setminus \{0\} \). The first orbit has zero Lebesgue measure, so that the corresponding representations are not square integrable. We consider the orbit \( \mathcal{C} \) and we fix \( p_0 = (0, \ldots, 0, 1) \in \mathcal{C} \). The stability group is \( G_0 = \mathbb{R}^n \rtimes \text{SO}(n-1) \) where \( \text{SO}(n-1) \) is regarded as a subgroup of \( \text{SO}(n) \) in the following way:
\[ \text{SO}(n-1) \ni R \mapsto \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \in \text{SO}(n). \]

Since \( \text{SO}(n-1) \) is compact, every representation of \( G \) induced on the orbit \( \mathcal{C} \) is square integrable. The usual way to construct wavelets in \( L^2(\mathbb{R}^n) \) is to choose \( m \) as the trivial representation. We suggest an alternative way in order to obtain vector-valued wavelets.

Let \( m \) be a representation of \( \text{SO}(n-1) \). As a consequence of the Frobenious reciprocity theorem and of the fact that \( \text{SO}(n) \) is compact, there exists an (in general nonunique) represen-
tation $D$ of $\text{SO}(n)$ such that $m$ is a subrepresentation of $D$, when $D$ is restricted to $\text{SO}(n-1)$. This allows us to use a covariant realization of $\text{Ind}^G_{H_0}(p_0m)$, instead of the formula (1). It is defined in the following way. Let $D$ act in $W$ and $m$ act in the subspace $\mathcal{H}$ of $W$. For all $p \in \mathcal{C}$ let $\mathcal{H}_p = D(R_p)\mathcal{H}$ where $R_p \in \text{SO}(n)$ is such that $R_p p_0 = p|p|$. Define the Hilbert space $\mathcal{H}$

$$\mathcal{H} = \{ \phi \in L^2(\mathbb{R}^n, W) : \phi(p) \in \mathcal{H}_p \text{ for almost all } p \}$$

and the representation $V$ acting in $\mathcal{H}$ as

$$(V_{x,k}\phi)(p) = a^{-n/2} e^{ip|x|} D(R) \phi(aR^{-1}p).$$

It is possible to show that $V$ is equivalent to $\text{Ind}^G_{H_0}(p_0m)$. Moreover, a vector $\psi \in \mathcal{H}$ is admissible if and only if

$$\int |p|^{-n} \| \psi(p) \|_W^2 d^n p < \infty.$$ 

The Fourier–Plancherel transform intertwines $V$ with a geometric representation $\hat{V}$ acting on a closed subspace $\hat{\mathcal{H}}$ of $L^2(\mathbb{R}^n, W)$ as

$$(\hat{V}_{x,k}f)(y) = a^{-n/2} D(R) f \left( \frac{R^{-1}(y-x)}{a} \right).$$

To give an explicit description of the subspace $\hat{\mathcal{H}}$ let us choose, for example, $n = 3$ and $D$ the representation of $\text{SO}(3), D(R) = R$, acting in $W = \mathbb{C}^3$ in a natural way. Let $\mathcal{H} = \{(0,0,e) : e \in \mathbb{C}\}$, hence the restriction $m$ of $D$ to $\text{SO}(2)$ leaves invariant $\mathcal{H}$ ($m$ is the trivial representation). An easy calculation shows that

$$\mathcal{H} = \{ f \in L^2(\mathbb{R}^3, \mathbb{C}^3) : (\nabla \wedge f) = 0 \},$$

where $\nabla \wedge$ is the curl operator (in the sense of distributions). Moreover, we observe that the orthogonal complement of $\mathcal{H}$,

$$(\mathcal{H})^\perp = \{ f \in L^2(\mathbb{R}^3, \mathbb{C}^3) : \nabla \cdot f = 0 \},$$

where $\nabla \cdot$ is the divergence operator (in the sense of distributions), is the direct sum of two closed subspaces where $\hat{V}$ acts irreducibly. These subspaces correspond to the choice of the subspaces of $W$,

$$\mathcal{H}_1 = \{(e,ic,0) : c \in \mathbb{C}\}, \quad \mathcal{H}_2 = \{(e,-ic,0) : c \in \mathbb{C}\},$$

and of the representations of $\text{SO}(2)$,

$$m_1(\theta) = e^{i\theta}, \quad m_2(\theta) = e^{-i\theta},$$

where $\theta \in [0,2\pi]$ is the usual parameter of $\text{SO}(2)$.

**IV. CONSTRUCTION OF DISCRETE FRAMES**

In this section we consider the problem of the existence of discrete frames associated to the induced representations of $G = \mathbb{R}^n \times H$, when $H = \mathbb{R}^+ \times S$ and $S$ is a semisimple connected Lie group.
The action of $H$ on $\mathbb{R}^n$ is

$$h[x]= (as)[x] := a(s[x]),$$

where $x \mapsto s[x]$ is a smooth action of $S$ on $\mathbb{R}^n$. Let $\Gamma^m$ denote the dual group of $\mathbb{R}^n$ and fix $p_0 \in \Gamma^m$. Assume that the orbit $\mathcal{O}=G[p_0]$ for the dual action of $G$ on $\Gamma^m$ is open and that the subgroup $H_0 = H \cap G_0$ is compact (notice that $\mathbb{R}^+ \cap H_0 = \emptyset$). In these hypotheses $\mathcal{O}$ is homeomorphic to $G/G_0$ and to $H/H_0$.

We will prove the existence of frames associated to the representation $\text{Ind}^G_{G_0}(p_0 m)$ where $m$ is any unitary finite dimensional cyclic representation of $H_0$.

We need to establish some preliminary topological results.

**Lemma 2:** Let $\Gamma$ be a discrete subgroup of $H$. Then, for any compact subset $C$ in $\mathcal{O}$, there exists a finite partition $\Gamma_1, ..., \Gamma_d$ of $\Gamma$ such that

$$h', h'' \in \Gamma_j \Rightarrow h'[C] \cap h''[C] = \emptyset.$$

**Proof:** Let $\pi : H \rightarrow \mathcal{O}$ be the canonical projection. First of all, we prove that $\pi^{-1}(C)$ is compact. By Theorem 5.11 of Ref. 12, there exists a section $q : \mathcal{O} \rightarrow H$ such that the set $q(C)$ is relatively compact in $H$. Since $H_0$ is compact also, $\pi^{-1}(C) = q(C)H_0$ is relatively compact. Since $\pi^{-1}(C)$ is closed, it is compact.

By lemma 3.3 of Ref. 5, there exists a finite partition $\Gamma_1, ..., \Gamma_d$ of $\Gamma$ such that

$$h', h'' \in \Gamma_j \Rightarrow h' \pi^{-1}(C) \cap h'' \pi^{-1}(C) = \emptyset.$$

Hence

$$\pi^{-1}(h'[C] \cap h''[C]) = \emptyset,$$

and $h'[C] \cap h''[C] = \emptyset$, since $\pi$ is surjective.

**Lemma 3:** There exist a discrete subgroup $\Gamma$ of $H$ and a compact $C$ in $\mathcal{O}$ such that

$$\mathcal{O} = \bigcup \{h[C] : h \in \Gamma\}.$$

**Proof:** Since $H$ is the direct product of $\mathbb{R}^+$ and a semisimple connected Lie group, as a consequence of Theorem C of Ref. 13 $H$ has a discrete subgroup $\Gamma$ such that $H/\Gamma$ is compact. Denote by $\pi'$ the canonical projection of $H$ onto $H/\Gamma$. There exists a compact $K' \subset H$ such that $\pi'(K') = H/\Gamma$, hence

$$H = \bigcup \{hK : h \in \Gamma\},$$

where $K = (K')^{-1}$ is compact in $H$. Define $C = \pi(K)$. Then $C$ is compact in $\mathcal{O}$ and

$$\bigcup \{h[C] : h \in \Gamma\} = \bigcup \{\pi(hK) : h \in \Gamma\} = \mathcal{O}.$$

**Lemma 4:** For any nonvoid open set $X \subset \mathcal{O}$ there exists a finite set $\{h_1, ..., h_L\}$ in $H$, such that

$$\bigcup \{h_i[X] : h_i \in \Gamma, \ 1 \leq k \leq L\} = \mathcal{O},$$

where $\Gamma$ is the discrete subgroup of $H$ given by the previous lemma.

**Proof:** The family of sets $\{h[X] : h \in H\}$ is an open covering of the compact subset $C$ of $\mathcal{O}$ singled out by the previous lemma. Hence there exists a finite subcovering $\{h_i[X] : 1 \leq k \leq L\}$ and we have the thesis.

Having established these results, let $m$ be a unitary cyclic finite dimensional representation of the compact group $H_0$ acting in the Hilbert space $\mathcal{H}$ of dimension $M$. Let $v$ be a cyclic vector for $m$. Then there exist $M$ elements $s_1, ..., s_M$ of $H_0$ such that $\{m(s_i)v : i = 1, ..., s_M\}$ is an algebraic basis of $\mathcal{H}$. Fix these elements. Let $c : \mathcal{O} \rightarrow H$ be a section for the action of $H$ on $\mathcal{O}$. Since $H$ is a Lie group acting smoothly on $\mathcal{O}$ we can assume that $c$ is locally continuous around $p_0$. For all $h \in H$ and $p \in \mathcal{O}$ define $m(h, p) := m(c(p)^{-1}hc(h^{-1}[p])).$ Then we have the following.
Lemma 5: There exist a closed ball \( B \) with center \( p_0 \), contained in \( \mathcal{C} \), and two positive numbers \( \alpha, \beta \) such that

\[
\alpha \|w\|_{\mathcal{H}}^2 = \sum_{i=1}^{M} |\langle m(s_i, p) v, w \rangle_{\mathcal{H}}|^2 \leq \beta \|v\|_{\mathcal{H}}^2
\]

for all \( w \in \mathcal{H} \) and \( p \in B \).

Proof: Let \( \{e_i\}_{i=1}^{M} \) be an orthonormal basis of \( \mathcal{H} \) and, for all \( p \in \mathcal{C} \), \( A(p) \) be the operator

\[
A(p)w = \sum_{i=1}^{M} \langle m(s_i, p) v, w \rangle_{\mathcal{H}} e_i, \ w \in \mathcal{H}.
\]

Since the vectors \( m(s_i) = m(s_i, p_0) v \) form a basis of \( \mathcal{H} \), \( A(p_0) \) is invertible and

\[
\|A(p_0)^{-1}\| \leq \sum_{i=1}^{M} |\langle m(s_i, p_0) v, w \rangle_{\mathcal{H}}|^2 \leq \|A(p_0)^{-1}\| \|v\|_{\mathcal{H}}^2.
\]

By the properties of \( c \) there exists a neighborhood of \( p_0 \) where the function \( p \mapsto A(p) \) is continuous. The thesis easily follows. \( \square \)

We consider the explicit realization of \( \text{Ind}_{G_0}^{G}(p_0 m) \) acting in \( L^2(\mathcal{C}, \mathcal{H}) \) as

\[
(U_{shf}(p)) = \rho(h) e^{i\pi \cdot p m(h,p)} f(h^{-1}[p]).
\]

We notice that, since \( S \) is semisimple and connected, \( \rho \) has the following expression

\[
\rho(as) = a^n, \ a \in \mathbb{R}^+, \ s \in S.
\]

Fix the closed ball \( B \) with center \( p_0 \) as in Lemma 5 and the discrete subgroup \( \Gamma \) of \( H \) as in Lemma 3. Let \( Y \) be an open subset of \( \mathcal{C} \) such that \( p_0 \in Y \) and \( s \in Y \subset B \) for all \( i = 1, \ldots, M \).

The set \( X := \cap \{s_i[Y]: 1 \leq i \leq M\} \) is not empty (since it contains \( p_0 \)) and open, hence, by Lemma 4, there exists a finite family \( \{h_1, \ldots, h_L\} \subset H \) such that

\[
\bigcup \{hh_k[X]: h \in \Gamma, \ 1 \leq k \leq L\} = \mathcal{C}.
\]

Since the set \( \bigcup \{h_k[B]: 1 \leq k \leq L\} \) is compact, due to Lemma 2, there exists a finite partition \( \Gamma_1, \ldots, \Gamma_n \) of \( \Gamma \) such that\n
\[
h', h'' \in \Gamma_j \Rightarrow h' \left( \bigcup_{k=1}^{L} h_k[B] \right) \cap h'' \left( \bigcup_{k=1}^{L} h_k[B] \right) = \emptyset.
\]

Let \( \nu \) be an integer such that \( B \subset R \) where

\[
R := \{x \in \mathbb{R}^n: -\pi \nu \leq x_i \leq \pi \nu, \ 1 \leq i \leq n\}
\]

and, for all \( l \in \mathbb{Z} \), \( x_j = 2\pi l \in \mathbb{R}^n \).

We observe that the countable family of functions

\[
\{e_i|l \in \mathbb{Z}^n\}
\]

defined as

\[
e_i(p) = (2\pi \nu)^{-n/2} e^{i\nu \cdot p}, \ p \in R,
\]

is a Hilbert basis of \( L^2(R) \).

Now we are in a position to describe a covariant frame in \( L^2(\mathcal{C}, \mathcal{H}) \). Let \( \psi \in L^2(\mathcal{C}, \mathcal{H}) \) be the vector

\[
\psi(p) := \chi_Y(p)v, \ p \in \mathcal{C},
\]
and consider the countable set of vectors

\[ \psi_{h,k,i,l} := U_h U_{h_k} U_{s_i} U_{s_l} \psi \]

where \( h \in \Gamma, \ 1 \leq k \leq L, \ 1 \leq i \leq M \) and \( l \in \mathbb{Z}^n \) (\( \Gamma \) is countable, since it is discrete). With these notations we have the following result.

**Theorem 3:** The set \( \{ \psi_{h,k,i,l} \} \) is a frame in \( L^2(\mathcal{C}, \mathcal{H}) \).

Proof: Given \( f \in L^2(\mathcal{C}, \mathcal{H}) \), we have

\[
\langle \psi_{h,k,i,l}, f \rangle_{L^2(\mathcal{C}, \mathcal{H})} = \langle U_{s_i} \psi, U_{(hh_k)}^{-1} f \rangle_{L^2(\mathcal{C}, \mathcal{H})} = \int_{\mathcal{C}} e^{-ix_1 \cdot p} \langle \psi(p), (U_{(hh_k)}^{-1} f)(p) \rangle_{\mathcal{H}} \, d^np
\]

\[
= (2\pi\nu)^{\frac{n}{2}} \int_{\mathcal{R}} \overline{e(p)} \langle \psi(p), (U_{(hh_k)}^{-1} f)(p) \rangle_{\mathcal{H}} \times (p)_{\mathcal{H}} \, d^np.
\]

Since \( \psi \) is bounded, the function

\[ \mathcal{C} \ni p \mapsto \langle \psi(p), (U_{(hh_k)}^{-1} f)(p) \rangle_{\mathcal{H}} \]

is in \( L^2(\mathcal{H}) \), hence

\[
\sum_{hkl} |\langle \psi_{h,k,i,l}, f \rangle_{L^2(\mathcal{C}, \mathcal{H})}|^2 = (2\pi\nu)^n \sum_{hkl} \int_{\mathcal{C}} |\langle \psi(p), (U_{(hh_k)}^{-1} f)(p) \rangle_{\mathcal{H}}|^2 \, d^np
\]

\[
= (2\pi\nu)^n \sum_{hkl} \int_{\mathcal{C}} |\langle m(s_i, p) \psi(s_i^{-1}[p]), (U_{(hh_k)}^{-1} f)(p) \rangle_{\mathcal{H}}|^2 \, d^np
\]

(changing variable \( p \mapsto s_i^{-1}[p] \))

\[
= (2\pi\nu)^n \sum_{hkl} \int_{s_i[\mathcal{Y}]} |\langle m(s_i, p) \psi(s_i^{-1}[p]), (U_{(hh_k)}^{-1} f)(p) \rangle_{\mathcal{H}}|^2 \, d^np.
\]

Recalling that \( X = \cap \{ s_i[\mathcal{Y}] : 1 \leq i \leq M \} \), for any positive function \( \phi \) we have

\[
\int_X \phi \, d^np \leq \int_{s_i[\mathcal{Y}]} \phi \, d^np \leq \int_B \phi \, d^np
\]

for all \( 1 \leq i \leq M \). Hence, since the sum on the index \( i \) is finite,

\[
(2\pi\nu)^n \sum_{hkl} \int_{s_i[\mathcal{Y}]} |\langle m(s_i, p) \psi(s_i^{-1}[p]), (U_{(hh_k)}^{-1} f)(p) \rangle_{\mathcal{H}}|^2 \, d^np
\]

\[
\geq (2\pi\nu)^n \sum_{hkl} \int_X \sum_i |\langle m(s_i, p) \psi(s_i^{-1}[p]), (U_{(hh_k)}^{-1} f)(p) \rangle_{\mathcal{H}}|^2 \, d^np
\]

\[
\geq \alpha (2\pi\nu)^n \sum_{hkl} \int_X \| (U_{(hh_k)}^{-1} f)(p) \|_{\mathcal{H}}^2 \, d^np
\]

\[
= \alpha (2\pi\nu)^n \sum_{hkl} \int_X \rho(hh_k)^{-1} \| f(hh_k[p]) \|_{\mathcal{H}}^2 \, d^np
\]

(changing variable \( p \mapsto (hh_k)^{-1}[p] \))
\[ = \alpha (2\pi \nu) \sum_{hk} \int_{hh_k[X]} |f(p)|^2 \, d^n p \alpha (2\pi \nu) \|f\|_{L^2(\mathcal{C}, \mathcal{E})}^2, \]

taking into account that
\[ \cup \{hh_k[X] : h \in \Gamma', \ 1 \leq k \leq L\} = \mathcal{C}. \]

On the other hand
\[ (2\pi \nu) ^{n} \sum_{hk} \int_{s_{[Y]}} \|f(s_i, p)v, (U_{(hh_k)^{-1}}f)(p)\|_{\mathcal{E}}^2 \, d^n p \]
\[ \leq (2\pi \nu) ^{n} \sum_{hk} \int_{B} \sum_{i} \|f(s_i, p)v, (U_{(hh_k)^{-1}}f)(p)\|_{\mathcal{E}}^2 \, d^n p \]
(reasoning in the same way as before)
\[ \leq \beta (2\pi \nu) ^{n} \sum_{hk} \int_{hh_k[B]} \|f(p)\|_{\mathcal{E}}^2 \, d^n p \]
(by property (2))
\[ = \beta (2\pi \nu) ^{n} \sum_{j=1}^{d} \sum_{k=1}^{L} \int_{hh_k[B]} \|f(p)\|_{\mathcal{E}}^2 \, d^n p \]
\[ = d\beta (2\pi \nu) ^{n} \int_{\mathcal{C}} \|f(p)\|_{\mathcal{E}}^2 \, d^n p = \beta (2\pi \nu) ^{n} \|f\|_{L^2(\mathcal{C}, \mathcal{E})}^2. \]

Summarizing we have
\[ \alpha (2\pi \nu) ^{n} \|f\|_{L^2(\mathcal{C}, \mathcal{E})}^2 \leq \beta (2\pi \nu) ^{n} \|f\|_{L^2(\mathcal{C}, \mathcal{E})}^2. \]