SIGNAL ANALYSES IN 2D, PART I.

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Abstract. We classify the connected Lie subgroups of $Sp(2,\mathbb{R})$ whose elements have the triangular form (1.2). The classification is up to conjugation within the symplectic group $Sp(2,\mathbb{R})$. Their study is motivated by the need of a unified approach to continuous 2D signal analyses, as those provided by wavelets and shearlets.

1. Introduction

The continuous wavelet transform [6, 10, 19, 20] and its many variants, such as, for example, the shearlet transform [4, 5, 13, 16], lie in the background of a growing body of techniques, that may be collectively referred to as *signal analysis*, whose common feature is perhaps the decomposition of functions, primarily in $L^2(\mathbb{R}^d)$, by means of superpositions of projections along selected "directions". Symmetry and finite dimensional geometry often play a prominent rôle in the way in which these directions are generated or selected, and hence, with this notion of signal analysis, topological transformation groups and their representations provide a natural setup. In particular, the restriction of the metaplectic representation of $Sp(d,\mathbb{R})$ to its Lie subgroups produces a wealth of useful reproducing formulae [2, 3], all based on linear geometric actions either in the time or in the frequency domain, and is thus one of the most natural environments both for a unified approach and for the search of new strategies. In fact, the deep connections of the metaplectic representation with harmonic analysis in phase space is thoroughly investigated [9, 12], and one of the keys to its understanding is the Wigner transform.

The central importance of the symplectic group has motivated both a general theory of "mock" metaplectic representations (and the abstract harmonic analysis thereof [7]), and a more applications-oriented approach, where the main focus is the actual study of these formulae in connection with the classical themes of signal analysis [11]. In this work, that consists of two parts, we introduce the class $\mathcal{E}$ of Lie subgroups of $Sp(d,\mathbb{R})$ that we believe is the "right" class for signal analysis and we illustrate its relevance in $2D$-analysis by exhibiting the full list of reproducing formulae that it yields, up to the appropriate notion of equivalence. In some sense, therefore, we obtain a complete picture, at least as far as continuous "geometric" transforms are concerned, of reasonable $2D$ signal analyses. In the first part (this paper) we classify the groups, modulo conjugation within $Sp(2,\mathbb{R})$. In part II we address the analytic issues: by appealing to the theory developed in [7] we are able to show exactly which groups are reproducing and which are not. The full description of the associated admissible vectors is also achieved. In part II, we

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also introduce a rather subtle notion of analytic equivalence, that we call *orbit* equivalence.

We say that a Lie subgroup G of $Sp(d,\mathbb{R})$ is a *reproducing group* if there exists $\eta \in L^2(\mathbb{R}^d)$, to be called an *admissible vector*, such that the *reproducing formula*

$$f = \int_G \langle f, \mu_g \eta \rangle \mu_g \eta \, dg, \tag{1.1}$$

holds (weakly) for every $f \in L^2(\mathbb{R}^d)$, where dg is a left Haar measure of G and μ is the metaplectic representation restricted to G. For simplicity, we actually restrict ourselves to connected subgroups. As pointed out in previous work [2, 3, 7], many known continuous formulae (notably those associated to wavelets, shearlets and some of their variants) arise in this way, or are at least equivalent to them via natural intertwining operators such as the Fourier transform, perhaps combined with geometric (affine) transformations of phase space. But much more is true. All the reproducing groups that we are aware of, share a structural feature: they are block triangular[1] semidirect products of a particular type. Written as $d \times d$ blocks, their elements have the form

$$g(\sigma, h) = \begin{bmatrix} h & 0 \\ \sigma h & {}^t h^{-1} \end{bmatrix} \tag{1.2}$$

where σ ranges in a non trivial vector space Σ of symmetric $d \times d$ matrices (the *vector* components) and h ranges, independently of σ, in a non trivial connected Lie subgroup H of $GL(d,\mathbb{R})$ (the *homogeneous* component), that acts on Σ via

$$h^\dagger[\sigma] = {}^t h^{-1} \sigma h^{-1}.$$

From the point of view of analysis, one should think of Σ as encoding translations and H as the group of geometric "deformations" such as, for example, shearings or possibly anisotropic dilations, or combinations of both. Thus, a group in the class $\mathcal{E}$ is, by definition, a connected semidirect product $G = \Sigma \rtimes H$. All these groups lie inside the standard maximal parabolic subgroup Q of $Sp(d,\mathbb{R})$ described in (2.2), but, in general, they are not parabolic, nor do they fill up the class of connected Lie subgroups of Q, as we show below. Hence, this is a non trivial class and we actually conjecture that if G is a connected reproducing subgroup of $Sp(d,\mathbb{R})$, then, modulo extensions by compact factors, G is conjugate within $Sp(d,\mathbb{R})$ to a closed subgroup of Q; for any such group G, in turn, there exists a naturally associated group in the class $\mathcal{E}$ that is reproducing if and only if G is such (see part two).

In the two papers, of which this is the first, we accomplish one of the main objectives of our research project, namely the classification, for $d = 2$, of all the reproducing groups in $\mathcal{E}$, together with the relevant analytic information. The classification we are after, of course, must be done modulo some reasonable and pertinent notion of equivalence. This is a rather delicate issue, as we now illustrate, and is one of the central points of our work. The most natural notion of equivalence is algebraic. In Lie theoretic terms, it is just conjugation modulo MA, where MAN is the Langlands decomposition of Q. The matrices in MA are the block diagonal elements in Q and conjugation by them preserves the class $\mathcal{E}$. As explained in Proposition 3.4, every $y \in MA$ sends any $G \in \mathcal{E}$ into $yGy^{-1} \in \mathcal{E}$ and actually maps vector components into vector components (i.e. Σ to Σ', because MA normalizes

[1]By conjugating with a suitable permutation one can either adopt the lower or upper triangular shape, as desired.

N) and homogeneous components into homogeneous components (i.e. H to H', because MA normalizes itself). No other symplectic matrix has this property on all of $\mathcal{E}$. Furthermore, this equivalence yields the equivalence of the restrictions of the metaplectic representation, groups in the same equivalence class are either all reproducing or none of them is, and the sets of admissible vectors in a reproducing class are in one-to-one correspondence via the unitary equivalences induced by $\mu(y)$.

Another natural equivalence is conjugation by *any* element in $Sp(d,\mathbb{R})$. It is very important because, although not adapted to $\mathcal{E}$, any conjugation induces equivalence of the restrictions of the metaplectic representation, and transfers the reproducing property, with admissible vectors that correspond to eachother via natural unitary equivalence. In Section 3.4 we analyze in full detail this general conjugation problem and we finally prove the classiciation below; for notation see Section 4.

Theorem 1.1. *The following is a complete list, up to $Sp(2,\mathbb{R})$-conjugation, of the groups in $\mathcal{E}_2$ with $1 \leq \dim\Sigma \leq 2$.*
Two dimensional groups:

(2.1) $\Sigma_1 \rtimes H_\alpha(\sigma_1)$, $\alpha \in [0,+\infty]$
(2.2) $\Sigma_2 \rtimes H_\alpha(\sigma_2)$, $\alpha \in [0,+\infty]$
(2.3) $\Sigma_3 \rtimes H_0(\sigma_3)$
(2.4) $\Sigma_3 \rtimes H_1(\sigma_3)$
(2.5) $\Sigma_3 \rtimes H_{\alpha,0}(\sigma_3)$, $\alpha \in [-1,0]$

Three dimensional groups:

(3.1) $\Sigma_1 \rtimes H^0(\sigma_1)$
(3.2) $\Sigma_2 \rtimes H^0(\sigma_2)$
(3.3) $\Sigma_3 \rtimes K_0(\sigma_3)$
(3.4) $\Sigma_3 \rtimes K_\infty(\sigma_3)$
(3.5) $\Sigma_3 \rtimes L_\gamma(\sigma_3)$, $\gamma \in \mathbb{R}$
(3.6) $\Sigma_1^\perp \rtimes H_\alpha(\sigma_1)$, $\alpha \in [0,+\infty]$
(3.7) $\Sigma_2^\perp \rtimes H_\alpha(\sigma_2)$, $\alpha \in [0,+\infty]$
(3.8) $\Sigma_3^\perp \rtimes {}^tH_0(\sigma_3)$
(3.9) $\Sigma_3^\perp \rtimes {}^tH_1(\sigma_3)$

Four dimensional groups:

(4.1) $\Sigma_3 \rtimes H^0(\sigma_3)$
(4.2) $\Sigma_1^\perp \rtimes H^0(\sigma_1)$
(4.3) $\Sigma_2^\perp \rtimes H^0(\sigma_2)$
(4.4) $\Sigma_3^\perp \rtimes {}^tL_\gamma(\sigma_3)$, $\gamma \in [-1,0]$

Five dimensional groups:

(5.1) $\Sigma_3^\perp \rtimes {}^tH^0(\sigma_3)$.

2. The parabolic group Q and its subgroups

We fix the size d where the L^2-signals live. The symplectic group is

$$Sp(d,\mathbb{R}) = \{g \in GL(2d,\mathbb{R}) : {}^tgJg = J\},$$

where $J = \begin{bmatrix} & I_d \\ -I_d & \end{bmatrix}$ is the standard symplectic form. Its Lie algebra is evidently

$$\mathfrak{sp}(d,\mathbb{R}) = \left\{g \in \mathfrak{gl}(2d,\mathbb{R}) : {}^tgJ + Jg = 0\right\},$$

and its elements are of the form

$$X = \begin{bmatrix} A & B \\ C & -{}^tA \end{bmatrix}, \tag{2.1}$$

where A is an arbitrary $d \times d$ matrix and $B, C \in \mathrm{Sym}(\mathrm{d}, \mathbb{R})$, the vector space of $d \times d$ real symmetric matrices. The standard maximal parabolic subgroup Q of the symplectic group that we are interested in is the closed Lie group

$$Q = \left\{ \begin{bmatrix} h & 0 \\ \sigma h & h^\sharp \end{bmatrix} : h \in GL(d, \mathbb{R}),\ \sigma \in \mathrm{Sym}(\mathrm{d}, \mathbb{R}) \right\}, \tag{2.2}$$

whose Lie algebra is:

$$\mathfrak{q} = \left\{ \begin{bmatrix} A & 0 \\ \sigma & -{}^tA \end{bmatrix} : A \in \mathfrak{gl}(d, \mathbb{R}),\ \sigma \in \mathrm{Sym}(\mathrm{d}, \mathbb{R}) \right\}. \tag{2.3}$$

The Langlands decomposition [15] $Q = MAN$ is easily checked to be

$$\begin{aligned} M &= \left\{ \begin{bmatrix} h & 0 \\ 0 & {}^th^{-1} \end{bmatrix} : \det h = \pm 1 \right\} \\ A &= \left\{ \begin{bmatrix} \lambda I_d & 0 \\ 0 & \lambda^{-1} I_d \end{bmatrix} : \lambda > 0 \right\} \\ N &= \left\{ \begin{bmatrix} I_d & 0 \\ \sigma & I_d \end{bmatrix} : \sigma \in \mathrm{Sym}(\mathrm{d}, \mathbb{R}) \right\}. \end{aligned}$$

We call $MA \simeq GL(d, \mathbb{R})$ the homogeneous component and $N \simeq \mathrm{Sym}(\mathrm{d}, \mathbb{R})$ the vector component. As is well-known, MA normalizes N, so that Q is the semidirect product of MA and the abelian normal factor N, namely

$$Q = \mathrm{Sym}(\mathrm{d}, \mathbb{R}) \rtimes \mathrm{GL}(\mathrm{d}, \mathbb{R}), \qquad \mathfrak{q} = \mathrm{Sym}(\mathrm{d}, \mathbb{R}) \rtimes \mathfrak{gl}(\mathrm{d}, \mathbb{R}). \tag{2.4}$$

To see this explicitly, notice that each element of Q is the product

$$g(\sigma, h) = \begin{bmatrix} I_d & 0 \\ \sigma & I_d \end{bmatrix} \begin{bmatrix} h & 0 \\ 0 & h^\sharp \end{bmatrix} = \begin{bmatrix} h & 0 \\ \sigma h & h^\sharp \end{bmatrix}, \tag{2.5}$$

where $\sigma \in \mathrm{Sym}(\mathrm{d}, \mathbb{R})$ and $h \in GL(d, \mathbb{R})$ and each such product is automatically symplectic. The above factorization is formally

$$g(\sigma, h) = g(\sigma, I_d) g(0, h). \tag{2.6}$$

Now, the product of two matrices in Q is

$$g(\sigma, h) g(\sigma', h') = \begin{bmatrix} hh' & 0 \\ (\sigma + h^\dagger[\sigma']) hh' & (hh')^\sharp \end{bmatrix} = g(\sigma + h^\dagger[\sigma'], hh'),$$

where

$$h^\dagger[\sigma] = {}^th^{-1} \sigma h^{-1}. \tag{2.7}$$

Thus, the group law is given by

$$g(\sigma, h) g(\sigma', h') = g(\sigma + h^\dagger[\sigma'], hh'), \tag{2.8}$$

the identity is $I_{2d} = g(0, I_d)$ and inverses are given by

$$g(\sigma, h)^{-1} = g(-{}^th \sigma h, h^{-1}) = g(-(h^{-1})^\dagger[\sigma], h^{-1}). \tag{2.9}$$

Notice that

$$^\dagger : GL(d, \mathbb{R}) \times \mathrm{Sym}(\mathrm{d}, \mathbb{R}) \to \mathrm{Sym}(\mathrm{d}, \mathbb{R}), \qquad ^\dagger(\mathrm{h}, \sigma) = \mathrm{h}^\dagger[\sigma]$$

is actually a group action and $\sigma \mapsto h^\dagger[\sigma]$ is a group automorphism of N. More generally, if we take a Lie subgroup H of $GL(d,\mathbb{R})$ and an additive Lie subgroup Σ of $\mathrm{Sym(d,\mathbb{R})}$ in such a way that H leaves Σ invariant under the action (2.7), then we obtain the semidirect product $\Sigma \rtimes H$, which is a Lie subgroup of Q. Clearly, in any such group, formulae (2.8) and (2.9) still hold true. If we fix Σ as above, then there is a largest group normalizing it, namely

$$H(\Sigma) = \big\{h \in GL(d,\mathbb{R}) : h^\dagger[\sigma] \in \Sigma,\ \text{for all } \sigma \in \Sigma\big\}. \tag{2.10}$$

Observe that $\Sigma \rtimes H$ is connected if and only if both Σ and H are connected, and, if Σ is connected, then it is a subspace of $\mathrm{Sym(d,\mathbb{R})}$.

We now characterize the Lie subgroups of Q. If G is such a group, we denote by $\pi : Q \to GL(d,\mathbb{R})$ the smooth group homomorphism $g(\sigma,h) \mapsto h$.

Proposition 2.1. *Take G be a Lie subgroup of Q and define*

$$H = \pi(G) \qquad \Sigma = \{\sigma \in \mathrm{Sym(d,\mathbb{R}) : g(\sigma, I_d) \in G}\}. \tag{2.11}$$

Then H is a Lie subgroup of $GL(d,\mathbb{R})$, Σ is a Lie subgroup of $\mathrm{Sym(d,\mathbb{R})}$ *which is invariant with respect to the action of H, $\sigma \mapsto h^\dagger[\sigma]$ and there exists a measurable map $\tau : H \to \mathrm{Sym(d,\mathbb{R})}$ that satisfies*

$$\begin{aligned} &\tau(I_d) = 0 \\ &\tau(h) + h^\dagger[\tau(h')] - \tau(hh') \in \Sigma \end{aligned} \tag{2.12}$$

for every $h, h' \in H$. The triple (Σ, H, τ) identifies the group, in the sense that

$$G = \{g(\sigma + \tau(h), h) : \sigma \in \Sigma,\ h \in H\}. \tag{2.13}$$

Conversely, if (Σ, H, τ) is any such triple, then G as in (2.13) *is a Lie subgroup of Q satisfying* (2.11).

Proof. The first part is essentially known, see [8], Proposition 1.11.8. We sketch the main steps. Take a Lie subgroup G of Q. A standard result on Lie groups, see e. g. Theorem 2.7.3 in [21], ensures that $H := \pi(G)$ is a Lie subgroup of $GL(d,\mathbb{R})$. Since $\ker(\pi)$ is closed in G, hence a Lie subgroup of Q, the set

$$\Sigma = \big\{\sigma \in \mathrm{Sym(d,\mathbb{R}) : g(\sigma, e) \in G}\big\} \simeq \ker(\pi)$$

is a Lie subgroup of $\mathrm{Sym(d,\mathbb{R})}$, and is contained in $H(\Sigma)$ (recall (2.10)) because $\ker(\pi)$ is normal in G. The quotient Lie group $H = G/\ker(\pi)$ admits a global measurable section $s : H \to G$ that maps I_d to $g(0, I_d)$ (see [18] or [22]). Since $G \subset Q$, we may write $s(h) = g(\tau(h), h)$. Therefore, if $g \in G$, then we may write $g = g(\sigma + \tau(h), h)$, where $h = \pi(g)$ and $\sigma \in \Sigma$. Since G is a group, the product (2.8) shows that

$$\sigma + h^\dagger[\sigma'] + \big(\tau(h) + h^\dagger[\tau(h')] - \tau(hh')\big) \in \Sigma \tag{2.14}$$

so that $\tau(h) + h^\dagger[\tau(h')] - \tau(hh') \in \Sigma$.

Conversely, fix a triple (Σ, H, τ) as in the statement. We prove that there exists a Lie subgroup G of Q such that (2.11) holds. Define G as in (2.13), a subgroup of $Sp(d,\mathbb{R})$ because

$$g(\sigma+\tau(h), h)g(\sigma'+\tau(h'), h') = g(\sigma+h^\dagger[\sigma']+\big(\tau(h) + h^\dagger[\tau(h')] - \tau(hh')\big)+\tau(hh'), hh')$$

and by the assumptions

$$\sigma + h^\dagger[\sigma'] + \big(\tau(h) + h^\dagger[\tau(h')] - \tau(hh')\big) \in \Sigma.$$

A similar argument applies to inverses. In order to prove that G is a Lie subgroup, we follow this strategy: first we show that G is a standard Borel group[2] with an invariant σ-finite measure. As a consequence of a theorem of Mackey's, we will be able to endow G with the Weil topology, so that G becomes a locally compact second countable group. Finally, applying a classical result on Lie groups we see that G admits a unique smooth structure converting it into a Lie subgroup of $Sp(d,\mathbb{R})$.

We claim that G is a Borel subset of $Sp(d,\mathbb{R})$. Since Σ and H are Lie groups, they are standard Borel spaces with respect to the corresponding Borel σ-algebras $\mathcal{B}(\Sigma)$ and $\mathcal{B}(H)$. Hence the product $\Sigma \times H$ is a standard Borel space with respect to $\mathcal{B}(\Sigma)\otimes\mathcal{B}(H)$ and the injection $\xi:\Sigma\times H \to Sp(d,\mathbb{R})$, $\xi(\sigma,h)=g(\sigma+\tau(h),h)$, is a Borel measurable map. Since ξ is a one-to-one map from a standard Borel space into another standard Borel space, its range G is a Borel subset of $Sp(d,\mathbb{R})$ and ξ is a Borel isomorphism from $\Sigma\times H$ onto G, the latter being endowed with the restriction of $\mathcal{B}(Sp(d,\mathbb{R}))$.

We choose (left) Haar measures $d\sigma$ and dh on Σ and H, respectively. For any fixed $h\in H$, the map $\sigma\mapsto h^\dagger[\sigma]$ is a group homomorphism of Σ onto itself, so that the image measure of $d\sigma$ under $h^\dagger[\cdot]$ is again a Haar measure. Hence there exists a unique $\alpha(h)>0$ such that for all positive Borel measurable functions φ on Σ

$$\int_\Sigma \varphi(h^\dagger[\sigma])d\sigma = \alpha(h)\int_\Sigma \varphi(\sigma)d\sigma.$$

Since $h\mapsto \int_\Sigma \varphi(h^\dagger[\sigma])d\sigma$ is Borel measurable, so is $h\mapsto\alpha(h)$. Furthermore, the uniqueness of $\alpha(h)$ implies that $h\mapsto\alpha(h)$ is a group homomorphism of H, that is, α is a continuous positive character of H. Write dg as the image measure of the measure $\alpha\cdot d\sigma\otimes dh$ under ξ. We claim that dg is a G-invariant σ-finite measure on G. Since both Σ and H are σ-compact and α is continuous, then $\alpha\cdot d\sigma\otimes dh$ is σ-finite as well as dg. Moreover, for any positive Borel measurable function φ on G and $g_0=g(\sigma_0+\tau(h_0),h_0)\in G$

$$\begin{aligned}
\int_G \varphi(g_0g)dg &= \int_{\Sigma\times H}\varphi(g(\sigma_0+h_0^\dagger[\sigma]+\tau(h_0)+h_0^\dagger[\tau(h)],h_0h))\alpha(h)d\sigma\,dh\\
&=\int_H\int_\Sigma \varphi(g(\sigma_0+\sigma'+\tau(h_0)+h_0^\dagger[\tau(h)],h_0h))\alpha(h_0h)d\sigma'\,dh\\
&=\int_H\int_\Sigma \varphi(g(\sigma''+\tau(h_0h),h_0h))\alpha(h_0h)d\sigma''\,dh\\
&=\int_\Sigma\int_H \varphi(g(\sigma''+\tau(h'),h'))\alpha(h)d\sigma''\,dh' = \int_G\varphi(g)dg,
\end{aligned}$$

where the equality in the second line is due to Fubini's theorem, the change of variable $h_0^\dagger[\sigma]=\sigma'$ and the fact that α is a character; the equality in the third line is a consequence of the fact that $d\sigma$ is the Haar measure on Σ; finally, the fourth line follows by Fubini's theorem, the change of variable $h'=h_0h$ and the H-invariance of dh.

Next we apply the theorem of Mackey's, see for example Theorem 8.41 of [22], that states that there exists exactly one topology on G which converts it into a locally compact second countable space whose Borel structure is the original one. From now on, we regard G as endowed with this topology.

[2]For notation and basic results on these issues, see [22], Chapter VIII.

Clearly, the inclusion i of G into $Sp(d,\mathbb{R})$ is a Borel measurable group homomorphism. Hence i is continuous, (see Lemma 8.28 of [22]). Finally, by Proposition 1 Ch. IV §. XIV in [1], there exists exactly one C^∞-structure on G which converts it into a Lie group and Proposition 1 Ch. IV §. XII in [1] implies that the inclusion is a C^∞-map. Hence, G is a Lie subgroup of $Sp(d,\mathbb{R})$. □

Remark 2.1. The correspondence between the triples (Σ, H, τ) and the Lie subgroups G of Q is not one-to-one. Indeed, two different maps τ and τ' define the same group G if and only if $\tau'(h) - \tau(h) \in \Sigma$ for all $h \in H$ and, if this happens, we say that τ and τ' are equivalent. From now on we thus parametrize the Lie subgroups of Q writing $G = (\Sigma, H, \tau)$, with the understanding that τ is only defined up to equivalence.

Remark 2.2. One could go about the proof of Proposition 2.1 in a different way, using the standard result according to which, under the foregoing assumptions, there exists a locally defined smooth section, hence one could assume τ to be smooth around the identity, and then use this to define a smooth atlas on G via translations.

Remark 2.3. The problem of characterizing the Lie subgroups of Q can be stated in a slightly different form, in the framework of Lie group extensions [14]. Since Q is the semi-direct product of $\mathrm{Sym(d,\mathbb{R})}$ and $GL(d,\mathbb{R})$, Q is a (Lie group) extension of $\mathrm{Sym(d,\mathbb{R})}$ by $GL(d,\mathbb{R})$. In the language of group extensions, $i_0 : \mathrm{Sym(d,\mathbb{R})} \to \mathrm{Q}$ (the canonical injection) and $\pi_0 : Q \to GL(d,\mathbb{R})$ (the canonical surjection) give rise to a short exact sequence, that is $i_0(\mathrm{Sym(d,\mathbb{R})}) = \ker \pi_0$.
Proposition 2.1 shows that any Lie subgroup G of Q is a Lie group extension of $\Sigma = \ker \pi$ (a Lie subgroup of $\mathrm{Sym(d,\mathbb{R})}$) by $H = \pi(G)$ (a Lie subgroup of $GL(d,\mathbb{R})$). Furthermore, the canonical inclusion j is a group homomorphism of G into Q compatible with i_0 and π_0, in the sense that the diagram

$$\begin{array}{ccccc} \Sigma & \xrightarrow{i} & G & \xrightarrow{\pi} & H \\ \downarrow & & \downarrow{\scriptstyle j} & & \downarrow \\ \mathrm{Sym(d,\mathbb{R})} & \xrightarrow{i_0} & Q & \xrightarrow{\pi_0} & GL(d,\mathbb{R}) \end{array}$$

commutes, where the vertical arrows are the natural inclusions. The factor sets corresponding to the extension G are: the map

$$(h,h') \mapsto \tau(h) + h^\dagger[\tau(h')] - \tau(hh')$$

from $H \times H$ into Σ and the map $h \mapsto h^\dagger[\cdot]$ from H into the group automorphisms of Σ. Conversely, for any pair (G,j) where G is a Lie group extension of a Lie subgroup of $\mathrm{Sym(d,\mathbb{R})}$ by a Lie subgroup of $GL(d,\mathbb{R})$ and where $j : G \to Q$ is a group homomorphism compatible with both i_0 and π_0, $j(G)$ turns out to be a Lie subgroup of Q.

For any fixed Σ and H, the maps τ satisfying (2.12) characterize all the extensions G of Σ by H for which there is a group homomorphism compatibile with i_0 and π_0.

Remark 2.4. Several special instances of (2.12) are of interest. The easiest is when τ is (equivalent to) zero, a case that plays a prominent rôle in our paper. When this happens, G becomes the semi-direct product $\Sigma \rtimes H$, because (2.14) reduces to $\sigma + h^\dagger[\sigma']$. The family of subgroups of Q for which $\tau = 0$ and both factors are connected and not trivial , will be denoted by $\mathcal{E}$. We shall formalize this below (see Definition 3.2).

Remark 2.5. The next simpler case is perhaps when $\tau(h) = \tau_0 - h^\dagger[\tau_0]$ for some $\tau_0 \in \mathrm{Sym}(\mathrm{d}, \mathbb{R})$. The class of maps τ of this kind will be denoted by $\mathcal{T}$. This happens if and only if we conjugate a group $\Sigma \rtimes H$ by means of $g(\tau_0, I_d)$:

$$g(\tau_0, I_d)g(\sigma, h)g(\tau_0, I_d)^{-1} = g(\sigma + \tau_0 - h^\dagger[\tau_0], h).$$

We shall often identify the functions $\tau \in \mathcal{T}$ with the symmetric matrices that uniquely determine them. For example, we can take

$$\Sigma = \Big\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} : x \in \mathbb{R} \Big\}, \qquad H = \Big\{ \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} : y \in \mathbb{R} \Big\},$$

thereby obtaining $\Sigma \rtimes H$, consisting of the symplectic matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ y & 1 & 0 & 0 \\ x & 0 & 1 & -y \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we conjugate $\Sigma \rtimes H$ with $g(\tau_0, I_d)$, where τ_0 is the symmetric matrix

$$\tau_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we obtain

$$G = \Big\{ \begin{bmatrix} 1 & 0 & 0 & 0 \\ y & 1 & 0 & 0 \\ x + y^2 & y & 1 & -y \\ 2y & 0 & 0 & 1 \end{bmatrix} : x, y \in \mathbb{R} \Big\},$$

a subgroup of Q of the form (Σ, H, τ) for which τ is not equivalent to zero.

Remark 2.6. A slightly more general class of groups $G = (\Sigma, H, \tau)$, that includes the previous one, corresponds to maps τ that satisfy

$$\tau(h) + h^\dagger[\tau(h')] - \tau(hh') = 0.$$

Then $H_\tau = \{g(\tau(h), h) \in G : h \in H\}$ is a Lie subgroup of Q. Using the same arguments as those in the proof of Theorem 2.1, one sees that τ is a C^∞ map from H into $\mathrm{Sym}(\mathrm{d}, \mathbb{R})$. Furthermore, G is the semi-direct product of Σ and H_τ, and is isomorphic (as Lie group) to $\Sigma \rtimes H$ via the mapping $\Sigma \rtimes H \to G$ given by $(\sigma, h) \mapsto (\sigma + \tau(h), h)$. For example, take

$$G = \Big\{ \begin{bmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ se^t & -te^{-t} & e^{-t} & 0 \\ -te^t & 0 & 0 & e^t \end{bmatrix} : t, s \in \mathbb{R} \Big\}.$$

Here

$$h = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}, \qquad \sigma = \begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tau(h) = \begin{bmatrix} 0 & -t \\ -t & 0 \end{bmatrix}.$$

It is easily checked that τ is not of the form $\tau(h) = \tau_0 - h^\dagger[\tau_0]$ for any symmetric τ_0, but $\tau(h) + h^\dagger[\tau(h')] - \tau(hh') = 0$.

Remark 2.7. If $G = (\Sigma, H, \tau)$ is connected, then so is H, but Σ may well be disconnected. The statement concerning H is clear, since π is continuous. Consider

$$G = \{ \begin{bmatrix} R_\theta & 0 \\ \theta R_\theta & R_\theta \end{bmatrix} : \theta \in \mathbb{R} \} \subset Q,$$

where R_θ is the clockwise rotation matrix as in (3.3) below. Clearly G is connected, but (2.11) tells us that $\Sigma = 2\pi\mathbb{Z}$, and is therefore not connected.

Finally, take $\tau : H \to \mathrm{Sym}(2,\mathbb{R})$ such that $G = (\Sigma, H, \tau)$. We show that it is not possible to choose τ is such a way that $\tau(h) + h^\dagger[\tau(h')] - \tau(hh') = 0$ for all $h, h' \in H$. Assuming the converse, then $h \mapsto g(\tau(h), h)$ is an injective measurable (hence smooth) group homomorphism of the compact group H into G. However, G is isomorphic to $\mathbb{R}$, so that it does not have compact subgroups other than $\{0\}$.

3. Classification of $\mathcal{E}_2$

In part II of this paper we shall prove the following result, which illustrates our interest in the case when $\tau = 0$ (see Definition 3.2 below).

Proposition 3.1. *If $G = (\Sigma, H, \tau)$ is a subgroup of Q, then also $G_0 = (\Sigma, H, 0)$ is such. Furthermore, G is reproducing if and only if G_0 is reproducing and they both have the same set of admissible vectors.*

By Proposition 3.1, the study of reproducing formulae for subgroups of Q reduces to subgroups for which $\tau = 0$. Two critical and somehow opposite situations may still occur, namely when either $\Sigma = 0$ or when $H = 0$. In the latter case, the group $(\Sigma, \{0\}, 0)$ cannot possibly be reproducing because it is abelian, hence unimodular (see [7]). In the former case, the semi-direct product structure of $(\{0\}, H, 0)$ disappears and our results are not applicable. In fact, this case falls in the wider scope of generalized wavelet theory (see e.g. [17]). Other possible complications involve connectedness issues. Thus, for simplicity we restrict ourselves to connected groups.

Definition 3.2. We denote by $\mathcal{E}$ the collection of all connected subgroups of Q associated to triples of the form $(\Sigma, H, 0)$ with $\dim \Sigma > 0$ and $\dim H > 0$. Thus a group in $\mathcal{E}$ is a semidirect product $\Sigma \rtimes H$ where both Σ and H are connected and, in particular, Σ is a vector space. When necessary, we write $\mathcal{E}_d$ to specify the size.

The most natural equivalence relation in $\mathcal{E}$ is induced by conjugation modulo MA, because it sends $\mathcal{E}$ to itself. We thus start by deriving a full description of the set of equivalence classes modulo MA. Different equivalence classes, however, might still yield the same signal analysis, and in some instances this is indeed the case. This is due to two possible phenomena. The first is again algebraic, and is conjugation modulo some other $w \in Sp(d,\mathbb{R})$ (as we shall see, typically a Weyl group element). In general such conjugations do not preserve $\mathcal{E}$ but groups in different classes modulo MA can be equivalent modulo w. The second is analytic and subtler, and will be discussed in part II of this work.

3.1. **Classification modulo MA of $\mathcal{E}_2$.**

A first useful general reduction of the problem comes from analysis: it is shown in [7] that if $\Sigma \rtimes H \in \mathcal{E}$ is reproducing, then $n := \dim \Sigma \leq d$. We therefore assume $1 \leq n \leq d$. The second reduction is given by a suitable notion of "duality" that is induced by the orthogonality within $\mathrm{Sym}(\mathrm{d},\mathbb{R})$ relative to the usual inner product $\langle \sigma, \tau \rangle = \mathrm{tr}(\sigma\tau)$. For any subset Σ of $\mathrm{Sym}(d)$ we write

$$\Sigma^\perp = \{\tau \in \mathrm{Sym}(d) : \langle \sigma, \tau \rangle = 0 \text{ for all } \sigma \in \Sigma\}.$$

As seen below in Proposition 3.3, the natural companion notion for the homogeneous factor H is transposition. Hence for any subgroup H of $GL(d,\mathbb{R})$, we write

$$^{t}H = \{^{t}h : h \in H\}.$$

Proposition 3.3. *The following are equivalent:*

(i) $\Sigma \rtimes H \in \mathcal{E}$;
(ii) $\Sigma^{\perp} \rtimes {}^{t}H \in \mathcal{E}$.

Proof. If $\sigma \in \Sigma$, $\tau \in \Sigma^{\perp}$ and $h \in H$, then

$$\langle (^{t}h)^{\dagger}[\tau], \sigma\rangle = \mathrm{tr}(h^{-1}\tau\,^{t}h^{-1}\sigma) = \mathrm{tr}(\tau\,^{t}h^{-1}\sigma h^{-1}) = \langle \tau, h^{\dagger}[\sigma]\rangle.$$

Therefore $h^{\dagger}[\Sigma] = \Sigma$ if and only if $(^{t}h)^{\dagger}[\Sigma^{\perp}] = \Sigma^{\perp}$. □

The next proposition shows that conjugation via $g(0,h) \in MA$ maps $\mathcal{E}$ into itself and, more precisely, that it preserves both the homogeneous and the normal factors. It also records that it preserves the subclass of reproducing groups. We use the notation i_g for conjugation by g within any group $\mathcal{G}$, that is

$$i_g x = gxg^{-1}, \qquad g, x \in \mathcal{G}.$$

Proposition 3.4. *Take $\Sigma \rtimes H \in \mathcal{E}$ and $h \in GL(d,\mathbb{R})$. Then $i_{g(0,h)}(\Sigma \rtimes H) \in \mathcal{E}$. More precisely, if $\Sigma' \rtimes H' \in \mathcal{E}$, then the following are equivalent:*

(i) $i_{g(0,h)}(\Sigma \rtimes H) = \Sigma' \rtimes H'$
(ii) $h^{\dagger}[\Sigma] = \Sigma'$ *and* $i_h(H) = H'$.
(iii) $i_{g(0,h^{\sharp})}(\Sigma^{\perp} \rtimes {}^{t}H) = \Sigma'^{\perp} \rtimes {}^{t}H'$
(iv) $(h^{\sharp})^{\dagger}[\Sigma^{\perp}] = (\Sigma')^{\perp}$ *and* $i_{h^{\sharp}}(^{t}H) = {}^{t}(H')$.

In this case, conjugation by $g(0,h)$ establishes one to one correspondences between:

- *$\mathcal{E}$-subgroups of $\Sigma \rtimes H$ and $\mathcal{E}$-subgroups of $\Sigma' \rtimes H'$;*
- *reproducing $\mathcal{E}$-subgroups of $\Sigma \rtimes H$ and reproducing $\mathcal{E}$-subgroups of $\Sigma' \rtimes H'$.*

Proof. The equivalence of (i), (ii), (iii) and (iv) is a matter of writing down the various operations. Clearly, if $\Sigma_0 \rtimes H_0$ is a subgroup of $\Sigma \rtimes H$, then $i_{g(0,h)}$ maps it into the subgroup $(h^{\dagger}[\Sigma_0]) \rtimes (i_h(H_0))$ of $\Sigma' \rtimes H'$, and conversely. As for the reproducing property, we know that any conjugate image by some $g \in Sp(d,\mathbb{R})$ of a reproducing subgroup of $Sp(d,\mathbb{R})$ is reproducing (see part II). □

From now on $d = 2$. Our strategy for achieving the classification is the following:

- We start from the case $n = 1$ and therefore write $\Sigma = \mathrm{span}\{\sigma\}$. By Proposition 3.4, we assume that σ is in Sylvester canonical form (there are only three meaningful possibilities) and compute in each case $H(\Sigma)$ and its Lie algebra $\mathfrak{h}(\Sigma)$.
- We classify all the Lie subalgebras of $\mathfrak{h}(\Sigma)$ up to conjugation by $H(\Sigma)$ and compute the corresponding connected Lie subgroups, thereby obtaining all the subgroups in $\mathcal{E}_2$ with $n = 1$.
- We use Proposition 3.3 and describe all the subgroups in $\mathcal{E}_2$ with $n = 2$ as those that are dual to some G as before, with $n = 1$. Indeed, $\dim \mathrm{Sym}(2,\mathbb{R}) = 3$ and hence $\dim(\mathrm{span}\{\sigma\}^{\perp}) = 2$. This completes the picture because $n \leq d = 2$.

3.2. **Reduction to canonical form.** Whenever $\sigma \in \mathrm{Sym}(\mathrm{d},\mathbb{R})$, we write

$$H(\sigma) = \{h \in GL(d,\mathbb{R}) : h^\dagger[\sigma] = \lambda\sigma \text{ for some } \lambda \in \mathbb{R}^*\}$$

instead of $H(\mathrm{span}\{\sigma\})$, and also

$$F(\sigma) = \{h \in GL(d,\mathbb{R}) : h^\dagger[\sigma] = \pm\sigma\}.$$

Both $H(\sigma)$ and $F(\sigma)$ are subgroups of $GL(d,\mathbb{R})$. We make a first observation.

Proposition 3.5. *Let $d = 2$. The map $\varphi : \mathbb{R}_+ \times F(\sigma) \to H(\sigma)$ defined by $\varphi(e^t, h) = he^{-t/2}$ is a group isomorphism.*

Proof. First of all, if $(e^t, h) \in \mathbb{R}_+ \times F(\sigma)$, then

$$(he^{-t/2})^\dagger[\sigma] = e^t h^\dagger[\sigma] = \pm e^t\sigma$$

and hence $he^{-t/2} \in H(\sigma)$. Clearly, φ is a group homomorphism. If $he^{-t/2} = I_d$, then $e^{t/2}I_d = h \in F(\sigma)$ and it follows that $e^t\sigma = \pm\sigma$. Therefore $t = 0$ and $h = I_d$. Hence φ is injective. Finally, take $h \in H(\sigma)$. Then $h^\dagger[\sigma] = \lambda\sigma$ for some $\lambda \in \mathbb{R}^*$. Upon writing $\lambda = \mathrm{sign}(\lambda)|\lambda| =: \varepsilon e^s$, with $\varepsilon = \pm 1$, we get

$$(e^{s/2}h)^\dagger[\sigma] = e^{-s}h^\dagger[\sigma] = \varepsilon\sigma,$$

so that $e^{s/2}h \in F(\sigma)$. But then $h = \varphi(e^s, e^{s/2}h)$, whence surjectivity. □

By Sylvester's law of inertia, there exists $g \in GL(d,\mathbb{R})$ such that $g^\dagger[I_{pqr}] = \sigma$, where $p+q+r = d$ and I_{pqr} is the canonical metric with signature (p,q,r), namely

$$I_{pqr} = \begin{bmatrix} I_p & 0 & 0 \\ 0 & -I_q & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We decompose $F(I_{pqr}) = O(p,q,r) \cup O^*(p,q,r)$, where

$$O(p,q,r) = \{g \in GL(d,\mathbb{R}) : {}^tgI_{pqr}g = I_{pqr}\}$$
$$O^*(p,q,r) = \{g \in GL(d,\mathbb{R}) : {}^tgI_{pqr}g = -I_{pqr}\}$$

and observe that $O^*(p,q,r)$ is empty whenever $p \neq q$ because ${}^tgI_{pqr}g$ has signature (p,q,r), whereas $-I_{pqr}$ has signature (q,p,r). The former is a group, the latter is not, and the product of two elements of $O^*(p,q,r)$ is in $O(p,q,r)$.

Corollary 3.6. $H(I_{pqr}) = \{e^sh : s \in \mathbb{R}, h \in O(p,q,r) \cup O^*(p,q,r)\}$.

Proof. Follows from Proposition 3.5 and the definitions of $O(p,q,r)$ and $O^*(p,q,r)$. □

By Proposition 3.4, $\sigma = I_{pqr}$, and since $\mathrm{span}\{\sigma\} = \mathrm{span}\{-\sigma\}$, we may assume $p \geq q$. In the case $d = 2$, there are exactly three interesting possibilities for (p,q,r), namely $(2,0,0)$, $(1,1,0)$ and $(1,0,1)$, because the case $(0,0,2)$ yields $\sigma = 0$. Correspondingly, we put

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.1}$$

3.3. **Classification.** As outlined earlier, we carry out the classification starting from the canonical forms (3.1). Hereafter, I denotes the identity matrix and J the standard symplectic form. Furthermore, we put

$$\sigma_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3.2}$$

As for rotations and boosts we put

$$R_\theta = \exp \theta J = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \qquad A_t = \exp t\sigma_5 = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}. \tag{3.3}$$

Evidently, $SO(2) = \{R_\theta : \theta \in [0, 2\pi)\}$ and $SO^0(1,1) = \{A_t : t \in \mathbb{R}\}$. Occasionally, we write Λ in place of σ_2, when we want to regard it as a "rotation with negative determinant", rather than a canonical representative in the space of symmetric matrices, as in (3.1). In fact, the full orthogonal group $O(2) = O(2,0,0)$ decomposes

$$O(2) = SO(2) \cup \Lambda \cdot SO(2).$$

Also, we denote by T the group of lower triangular matrices in $GL(2,\mathbb{R})$. Finally, we often write ε for a number in $\{\pm 1\}$.

3.3.1. *Signature* $(2,0,0)$. Corollary 3.6 gives

$$H(\sigma_1) = \mathbb{R}_+ \times O(2).$$

The Lie algebra of $H(\sigma_1)$ is

$$\mathfrak{h}(\sigma_1) = \mathfrak{so}(2) \oplus \mathbb{R} = \{\alpha J + \beta I : \alpha, \beta \in \mathbb{R}\}.$$

Both $H(\sigma_1)$ and $\mathfrak{h}(\sigma_1)$ are abelian direct sums. The nontrivial Lie subalgebras of $\mathfrak{h}(\sigma_1)$ are its one-dimensional subspaces. We put

$$\mathfrak{h}_\infty(\sigma_1) = \operatorname{span}\{J\}$$

and, for $\alpha \in \mathbb{R}$,

$$\mathfrak{h}_\alpha(\sigma_1) = \operatorname{span}\{I + \alpha J\}.$$

Proposition 3.7. *Take $\alpha_1, \alpha_2 \in \mathbb{R} \cup \{\infty\}$. Then $\mathfrak{h}_{\alpha_1}(\sigma_1)$ is conjugate to $\mathfrak{h}_{\alpha_2}(\sigma_1)$ by an element of $H(\sigma_1)$ if and only if $\alpha_1 = \pm\alpha_2$.*

Proof. Take $g \in H(\sigma_1)$. Since scalars commute with everything, we can assume that $g \in O(2) = SO(2) \cup \Lambda \cdot SO(2)$. Observe that $R_\theta J R_{-\theta} = J$ and $\Lambda J \Lambda = -J$, so that $\operatorname{span}\{J\}$ is fixed under conjugation by g. It follows that $\mathfrak{h}_\infty(\sigma_1)$ is not conjugate to any other algebra in the class. Finally,

$$R_\theta(\alpha J + I)R_{-\theta} = (\alpha J + I), \qquad \Lambda\, R_\theta\,(\alpha J + I)\, R_{-\theta}\,\Lambda = \Lambda\,(\alpha J + I)\,\Lambda = -\alpha J + I$$

imply the result. □

Next, we identify the connected Lie subgroups corresponding to the various Lie algebras and then apply duality, in the sense of Proposition 3.3. To this end we set

$$\begin{aligned} H^0(\sigma_1) &= SO(2) \times \mathbb{R}_+ \\ H_\infty(\sigma_1) &= SO(2) \\ H_\alpha(\sigma_1) &= \{e^t R_{\alpha t} : t \in \mathbb{R}\}, \qquad \alpha \in [0, +\infty). \end{aligned}$$

These are the connected Lie subgroups of Q whose Lie algebras are $\mathfrak{h}(\sigma_1)$, $\mathfrak{h}_\infty(\sigma_1)$ and $\mathfrak{h}_\alpha(\sigma_1)$, respectively. Notice that

$$\Sigma_1^\perp = \Big\{ \begin{bmatrix} u & v \\ v & -u \end{bmatrix} : u, v \in \mathbb{R} \Big\}.$$

Proposition 3.8. *The following is a complete list, up to MA-conjugation, of the groups in $\mathcal{E}_2$ whose normal factor is equal or orthogonal to $\Sigma_1 = \mathrm{span}\{\sigma_1\}$:*

(1.i) $\Sigma_1 \rtimes H^0(\sigma_1)$

(1.ii) $\Sigma_1 \rtimes H_\alpha(\sigma_1)$, *with* $\alpha \in [0, +\infty]$

(1.iii) $\Sigma_1^\perp \rtimes H^0(\sigma_1)$

(1.iv) $\Sigma_1^\perp \rtimes H_\alpha(\sigma_1)$, *with* $\alpha \in [0, +\infty]$.

Proof. Items (1.i) and (1.ii) are clear, and arise by taking first the full two-dimensional algebra $\mathfrak{h}(\sigma_1)$ and then its one-dimensional subalgebras. Now, the groups $H^0(\sigma_1)$ and $H_\infty(\sigma_1)$ are closed under transposition, whereas ${}^tH_\alpha(\sigma_1) = H_{-\alpha}(\sigma_1)$. However, $\Lambda^\dagger[\Sigma_1^\perp] = \Lambda\Sigma_1^\perp\Lambda = \Sigma_1^\perp$ and $\Lambda H_{-\alpha}(\sigma_1)\Lambda^{-1} = H_\alpha(\sigma_1)$. Hence, applying Proposition 3.3 we obtain the groups in (1.iii) and (1.iv). □

3.3.2. *Signature* $(1,1,0)$. Here the relevant group is $O(1,1) = O(1,1,0)$ together with

$$O^*(1,1) = \{h \in GL(2,\mathbb{R}) : {}^thI_{1,-1}h = -I_{1,-1}\}.$$

By Corollary 3.6, we obtain

$$H(\sigma_2) = \mathbb{R}_+ \times \big(O(1,1) \cup O^*(1,1)\big).$$

Its Lie algebra $\mathfrak{h}(\sigma_1)$ can be written as

$$\mathfrak{h}(\sigma_2) = \mathfrak{so}(1,1) \oplus \mathbb{R} = \{\alpha\sigma_5 + \beta I : \alpha, \beta \in \mathbb{R}\}.$$

The non trivial subalgebras are the vector subspaces of $\mathfrak{h}(\sigma_2)$ of dimension 1. Put

$$\mathfrak{h}_\infty(\sigma_2) = \mathrm{span}\{\sigma_5\}$$

and, for $\alpha \in \mathbb{R}$,

$$\mathfrak{h}_\alpha(\sigma_2) = \mathrm{span}\{I + \alpha\sigma_5\}.$$

Proposition 3.9. *Take $\alpha_1, \alpha_2 \in \mathbb{R} \cup \{\infty\}$. Then $\mathfrak{h}_{\alpha_1}(\sigma_2)$ is conjugate to $\mathfrak{h}_{\alpha_2}(\sigma_2)$ by an element of $H(\sigma_2)$ if and only if $\alpha_1 = \pm\alpha_2$.*

Proof. Take $g \in H(\sigma_2)$. Since scalars commute with everything, we can assume that $g \in O(1,1) \cup O^*(1,1)$. The following relations are straightforward:

$$O(1,1) = \{\pm A_t, \pm\Lambda A_t : t \in \mathbb{R}\}, \quad O(1,1)^* = \sigma_5 \cdot O(1,1).$$

Since $A_t\,\sigma_5\,A_t^{-1} = \sigma_5$ and $\Lambda\,\sigma_5\,\Lambda = -\sigma_5$, the algebra $\mathfrak{h}_\infty(\sigma_2)$ is not conjugate to any other one in the class. Finally, we have

$$\begin{aligned} A_t(I + \alpha\sigma_5)A_t^{-1} &= I + \alpha\sigma_5 \\ \Lambda A_t(I + \alpha\sigma_5)A_t^{-1}\Lambda &= \Lambda(I + \alpha\sigma_5)\Lambda = I - \alpha\sigma_5 \\ \sigma_5(I + \alpha\sigma_5)\sigma_5 &= I + \alpha\sigma_5, \end{aligned}$$

whence the result. □

Finally, it follows from $\exp t(I+\alpha\sigma_5)=e^tA_{\alpha t}$ that the connected subgroups of Q whose Lie algebras are $\mathfrak{h}(\sigma_2)$, $\mathfrak{h}_\infty(\sigma_2)$ and $\mathfrak{h}_\alpha(\sigma_2)$, respectively, are

$$\begin{aligned} H^0(\sigma_2)&=SO^0(1,1)\times\mathbb{R}_+\\ H_\infty(\sigma_2)&=SO^0(1,1)\\ H_\alpha(\sigma_2)&=\{e^tA_{\alpha t}:t\in\mathbb{R}\},\qquad \alpha\in[0,\infty). \end{aligned}$$

Notice that

$$\Sigma_2^\perp=\Big\{\begin{bmatrix}u & v\\ v & u\end{bmatrix}:u,v\in\mathbb{R}\Big\}.$$

Proposition 3.10. *The following is a complete list, up to MA-conjugation, of the groups in $\mathcal{E}_2$ whose normal factor is equal or orthogonal to $\Sigma_2=\operatorname{span}\{\sigma_2\}$:*

(2.i) $\Sigma_2\rtimes H^0(\sigma_2)$
(2.ii) $\Sigma_2\rtimes H_\alpha(\sigma_2)$, *with* $\alpha\in[0,+\infty]$
(2.iii) $\Sigma_2^\perp\rtimes H^0(\sigma_2)$
(2.iv) $\Sigma_2^\perp\rtimes H_\alpha(\sigma_2)$, *with* $\alpha\in[0,+\infty]$.

Proof. Argue as in the proof of Proposition 3.8, but notice that this time $H^0(\sigma_2)$, $H_\infty(\sigma_2)$ and $H_\alpha(\sigma_2)$ are all closed under transposition. □

3.3.3. *Signature* $(1,0,1)$. The group $O(1,0,1)$ is easily computed to be

$$O(1,0,1)=\Big\{\begin{bmatrix}\pm1 & 0\\ b & a\end{bmatrix}:a,b\in\mathbb{R},a\neq0\Big\},$$

and $O^*(1,0,1)=\emptyset$. The Lie algebra of $O(1,0,1)$ is

$$\mathfrak{so}(1,0,1)=\Big\{\begin{bmatrix}0 & 0\\ b & a\end{bmatrix}:a,b\in\mathbb{R}\Big\}.$$

Clearly, the identity component $O^0(1,0,1)$ is isomorphic to the "$ax+b$" group. By Corollary 3.6, the symmetrizers are

$$\begin{aligned} H(\sigma_3)&=\Big\{\ell_{a,b,c}=\begin{bmatrix}c & 0\\ b & a\end{bmatrix}:a,b,c\in\mathbb{R},\ ac\neq0\Big\}=T,\\ \mathfrak{h}(\sigma_3)&=\Big\{\begin{bmatrix}c & 0\\ b & a\end{bmatrix}:a,b,c\in\mathbb{R}\Big\} \end{aligned}$$

that is, the group of all nonsingular lower triangular matrices and its Lie algebra. We choose $\{I,\sigma_4,B\}$ as a basis of $\mathfrak{h}(\sigma_3)$, where

$$\sigma_4=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}. \tag{3.4}$$

First, we analyze the one-dimensional subalgebras in $\mathfrak{h}(\sigma_3)$ up to conjugation by $H(\sigma_3)$. To this end, parametrizing as in real projective space $\mathbb{RP}^2$, we put

$$\begin{aligned} \mathfrak{h}_\infty(\sigma_3)&=\operatorname{span}\{I\}\\ \mathfrak{h}_\gamma(\sigma_3)&=\operatorname{span}\{\gamma I+B\}, && \gamma\in\mathbb{R}\\ \mathfrak{h}_{\gamma,\beta}(\sigma_3)&=\operatorname{span}\{\gamma I+\beta B+\sigma_4\}, && \gamma,\beta\in\mathbb{R}. \end{aligned}$$

Proposition 3.11. *Among the one dimensional Lie algebras listed above, the only conjugacies by elements in $H(\sigma_3)=T$ are the following:*

(a) *$\mathfrak{h}_\gamma(\sigma_3)$ is conjugate to $\mathfrak{h}_1(\sigma_3)$, for every real number $\gamma\neq0$,*
(b) *$\mathfrak{h}_{\gamma,\beta}(\sigma_3)$ is conjugate to $\mathfrak{h}_{\gamma,\beta'}(\sigma_3)$, for every $\gamma,\beta,\beta'\in\mathbb{R}$.*

Proof. A direct computation gives

$$\ell_{a,b,c}\, B\, \ell_{a,b,c}^{-1} = \frac{a}{c}\, B, \tag{3.5a}$$

$$\ell_{a,b,c}\, \sigma_4\, \ell_{a,b,c}^{-1} = -\frac{b}{c}\, B + \sigma_4. \tag{3.5b}$$

From (3.12) we infer that $\mathfrak{h}_0(\sigma_3)$ cannot be conjugate to either $\mathfrak{h}_\infty(\sigma_3)$ or to any of the algebras $\mathfrak{h}_\gamma(\sigma_3)$, for any $\gamma \neq 0$. Also, (3.12) yields

$$\ell_{\gamma,0,1}\, (\gamma I + B)\, \ell_{\gamma,0,1}^{-1} = \gamma(I + B)$$

and statement (a) follows. Again, (3.12) yields

$$\ell_{a,b,c}\, (\gamma I + B)\, \ell_{a,b,c}^{-1} = \gamma I + \frac{a}{c}\, B,$$

which shows that none of the algebras $\mathfrak{h}_\gamma(\sigma_3)$ can possibly be conjugate to any of the algebras $\mathfrak{h}_{\gamma,\beta}(\sigma_3)$. Finally, from (3.12) and (3.13) we have

$$\ell_{a,b,c}\, (\gamma I + \beta B + \sigma_4)\, \ell_{a,b,c}^{-1} = \gamma I + \frac{\beta a - b}{c}\, B + \sigma_4,$$

whence (b). □

By the above proposition, the relevant one-dimensional subalgebras of $\mathfrak{h}(\sigma_3)$ are $\mathfrak{h}_0(\sigma_3)$, $\mathfrak{h}_1(\sigma_3)$, $\mathfrak{h}_\infty(\sigma_3)$ and the family $\{\mathfrak{h}_{\gamma,0}(\sigma_3) : \gamma \in \mathbb{R}\}$. The corresponding one-dimensional connected Lie subgroups of $H(\sigma_3)$ are

$$\begin{aligned}
H_0(\sigma_3) &= \Big\{ \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} : t \in \mathbb{R} \Big\} \\
H_1(\sigma_3) &= \Big\{ e^t \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} : t \in \mathbb{R} \Big\} \\
H_\infty(\sigma_3) &= \Big\{ e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : t \in \mathbb{R} \Big\} \\
H_{\gamma,0}(\sigma_3) &= \Big\{ \begin{bmatrix} e^{\gamma t} & 0 \\ 0 & e^{(\gamma+1)t} \end{bmatrix} : t \in \mathbb{R} \Big\}, \qquad \gamma \in \mathbb{R}.
\end{aligned}$$

Next we put

$$\begin{aligned}
\mathfrak{k}_0(\sigma_3) &= \operatorname{span}\{I, \sigma_4\} \\
\mathfrak{k}_\infty(\sigma_3) &= \operatorname{span}\{I, B\} \\
\mathfrak{l}_\gamma(\sigma_3) &= \operatorname{span}\{B, \gamma I + \sigma_4\}, \qquad \gamma \in \mathbb{R}.
\end{aligned}$$

Proposition 3.12. *Up to conjugation by elements in $H(\sigma_3)$, there are no two dimensional Lie subalgebras of $\mathfrak{h}(\sigma_3)$ other than those listed above, which are mutually not conjugate.*

Proof. We begin by observing that the only non trivial bracket among the elements in $\{I, \sigma_4, B\}$ is of course $[\sigma_4, B] = B$. Assume that $\mathfrak{h}$ is a two dimensional subalgebra of $\mathfrak{h}(\sigma_3)$ and suppose that $\mathfrak{h} = \operatorname{span}\{X_1, X_2\}$, with

$$\begin{aligned}
X_1 &= \alpha_1 \sigma_4 + \beta_1 B + \gamma_1 I \\
X_2 &= \alpha_2 \sigma_4 + \beta_2 B + \gamma_2 I.
\end{aligned}$$

Evidently, requiring that $\mathfrak{h}$ is a Lie algebra is equivalent to asking that

$$[X_1, X_2] = (\alpha_1 \beta_2 - \alpha_2 \beta_1) B \tag{3.6}$$

belongs to $\mathfrak{h}$. If $B \in \mathfrak{h}$, then this is obvious. In this case we may suppose that $X_1 = B$ and consequently $X_2 = \alpha\sigma_4 + \gamma I$. If $\alpha = 0$ we get $\mathfrak{k}_\infty(\sigma_3)$, otherwise we set $\alpha = 1$ and we get $\mathfrak{l}_\gamma(\sigma_3)$. If $B \notin \mathfrak{h}$, then (3.6) yields $(\alpha_1\beta_2 - \alpha_2\beta_1) = 0$. This means that the vectors $\alpha_1\sigma_4 + \beta_1 B$ and $\alpha_2\sigma_4 + \beta_2 B$ are linearly dependent; hence there exists a linear combination $\lambda X_1 + \mu X_2$ that is equal to I, which we choose as basis vector for $\mathfrak{h}$ in place, say, of X_2. By subtracting off $\gamma_1 I$ from X_1, we may thus suppose that the other basis vector is $X_1 = \alpha_1\sigma_4 + \beta_1 B$. If $\alpha_1 = 0$, then we get again $\mathfrak{k}_\infty(\sigma_3)$. Hence we put $\alpha_1 = 1$ and obtain that

$$\mathfrak{h} = \operatorname{span}\{I, \beta B + \sigma_4\}$$

for some $\beta \in \mathbb{R}$. But this is conjugate to $\mathfrak{k}_0(\sigma_3)$, because by (3.12) and (3.13) we have

$$\ell_{a,b,c}\,(\beta B + \sigma_4)\,\ell_{a,b,c}^{-1} = \frac{\beta a - b}{c}\,B + \sigma_4,$$

which can be made equal to σ_4 because $a \neq 0$.

It remains to be shown that there are no conjugate pairs in the list. This follows by inspection, taking into account that the only possibilities are given by (3.12) and (3.13). $\square$

By the above proposition, the relevant two-dimensional subalgebras of $\mathfrak{h}(\sigma_3)$ are $\mathfrak{k}_0(\sigma_3)$, $\mathfrak{k}_\infty(\sigma_3)$, and the family $\{\mathfrak{l}_\gamma(\sigma_3) : \gamma \in \mathbb{R}\}$. The corresponding two-dimensional connected Lie subgroups of $H(\sigma_3)$ are

$$\begin{aligned}
K_0(\sigma_3) &= \Big\{ \begin{bmatrix} e^t & 0 \\ 0 & e^s \end{bmatrix} : s,t \in \mathbb{R} \Big\} \\
K_\infty(\sigma_3) &= \Big\{ \begin{bmatrix} e^t & 0 \\ s & e^t \end{bmatrix} : s,t \in \mathbb{R} \Big\} \\
L_\gamma(\sigma_3) &= \Big\{ \begin{bmatrix} e^{\gamma t} & 0 \\ s & e^{(\gamma+1)t} \end{bmatrix} : s,t \in \mathbb{R} \Big\}, \qquad \gamma \in \mathbb{R}.
\end{aligned}$$

Finally, in this case

$$\Sigma_3^\perp = \Big\{ \begin{bmatrix} 0 & v/\sqrt{2} \\ v/\sqrt{2} & u \end{bmatrix} : u,v \in \mathbb{R} \Big\}.$$

Proposition 3.13. *The following is a complete list, up to MA-conjugation, of the groups in $\mathcal{E}_2$ whose normal factor is equal or orthogonal to $\Sigma_3 = \operatorname{span}\{\sigma_3\}$:*

(3.i) $\Sigma_3 \rtimes H^0(\sigma_3)$
(3.ii) $\Sigma_3 \rtimes H_0(\sigma_3)$
(3.iii) $\Sigma_3 \rtimes H_1(\sigma_3)$
(3.iv) $\Sigma_3 \rtimes H_\infty(\sigma_3)$
(3.v) $\Sigma_3 \rtimes H_{\gamma,0}(\sigma_3),\ \gamma \in \mathbb{R}$
(3.vi) $\Sigma_3 \rtimes K_0(\sigma_3)$
(3.vii) $\Sigma_3 \rtimes K_\infty(\sigma_3)$
(3.viii) $\Sigma_3 \rtimes L_\gamma(\sigma_3),\ \gamma \in \mathbb{R}$
(3.ix) $\Sigma_3^\perp \rtimes {}^tH^0(\sigma_3)$
(3.x) $\Sigma_3^\perp \rtimes {}^tH_0(\sigma_3)$
(3.xi) $\Sigma_3^\perp \rtimes {}^tH_1(\sigma_3)$
(3.xii) $\Sigma_3^\perp \rtimes {}^tH_\infty(\sigma_3)$
(3.xiii) $\Sigma_3^\perp \rtimes H_{\gamma,0}(\sigma_3),\ \gamma \in \mathbb{R}$
(3.xiv) $\Sigma_3^\perp \rtimes K_0(\sigma_3)$
(3.xv) $\Sigma_3^\perp \rtimes {}^tK_\infty(\sigma_3)$
(3.xvi) $\Sigma_3^\perp \rtimes {}^tL_\gamma(\sigma_3),\ \gamma \in \mathbb{R}$.

3.4. **Classification modulo $Sp(d,\mathbb{R})$ of $\mathcal{E}_2$.** The question we want to answer is: when are two groups in $\mathcal{E}_2$ conjugate by $g \in Sp(d,\mathbb{R})$? We now state the main technical lemma, which is a consequence of the Bruhat decomposition. Remember that σ_4 is as in (3.4) and T is the group of lower triangular matrices in $GL(2,\mathbb{R})$. We use the following notation

$$w_0 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Lemma 3.14. *Suppose that $\Sigma_1 \rtimes H_1, \Sigma_2 \rtimes H_2 \in \mathcal{E}_2$ are not conjugate modulo MA. If $g \in Sp(d,\mathbb{R})$, is such that $g(\Sigma_1 \rtimes H_1)g^{-1} = \Sigma_2 \rtimes H_2$, then g is of the form*

$$g = g(\sigma', h')^{-1} w_0 g(a_0\sigma_4, h), \tag{3.7}$$

for some $\sigma' \in \mathrm{Sym(d,\mathbb{R})}$, $h, h' \in GL(2,\mathbb{R})$, some $a_0 \in \mathbb{R}$. This can only happen if $h^\dagger[\Sigma_1] \subseteq \sigma_4^\perp$ and $hH_1h^{-1} \subseteq T$. Furthermore $a_0 \neq 0$ only if

$$hH_1h^{-1} \subseteq \{\left[\begin{smallmatrix} \alpha & 0 \\ \beta & 1 \end{smallmatrix}\right] : \alpha > 0, \beta \in \mathbb{R}\}.$$

The proof of Lemma 3.14 is based on the Bruhat decomposition of $Sp(2,\mathbb{R})$, that expresses $Sp(2,\mathbb{R})$ as the disjoint union

$$Sp(2,\mathbb{R}) = \bigcup_{w \in W} PwP$$

of the double cosets PwP of the minimal parabolic group P, parametrized by the elements in the Weyl group W. More precisely,

$$P = \left\{ \begin{bmatrix} \ell & 0 \\ \sigma\ell & \ell^\sharp \end{bmatrix} : \ell \in T,\ \sigma \in \mathrm{Sym(d,\mathbb{R})} \right\},$$

and, with slight abuse of notation, a representative[3] of the Weyl group element $w \in W$ may be taken in in $Sp(2,\mathbb{R})$ as a matrix of the form

$$w = \begin{bmatrix} S_+ & -S_- \\ S_- & S_+ \end{bmatrix} \begin{bmatrix} \pi & 0 \\ 0 & \pi \end{bmatrix}$$

where π is either I_2 or σ_5, and where $S_- = I_2 - S_+$, with S_+ one of

$$s_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad s_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

As is well-known, W has 8 elements. Evidently, w_0 corresponds to $S_+ = s_0$ and $\pi = I_2$. Notice that, W is a semidirect product and in particular

$$\begin{bmatrix} \sigma_5 & 0 \\ 0 & \sigma_5 \end{bmatrix} \begin{bmatrix} s_0 & -s_1 \\ s_1 & s_0 \end{bmatrix} \begin{bmatrix} \sigma_5 & 0 \\ 0 & \sigma_5 \end{bmatrix} = \begin{bmatrix} s_1 & -s_0 \\ s_0 & s_1 \end{bmatrix}. \tag{3.8}$$

Also, notice that $\begin{bmatrix} \sigma_5 & 0 \\ 0 & \sigma_5 \end{bmatrix} \in MA$.

Proof of Lemma 3.14. First of all, put $G_1 = \Sigma_1 \rtimes H_1$, $G_2 = \Sigma_2 \rtimes H_2$ and, according to the Bruhat decomposition, write $g = p_2^{-1} w p_1$ with $p_1, p_2 \in P$ and $w \in W$. Therefore

$$p_2 G_2 p_2^{-1} = w(p_1 G_1 p_1^{-1}) w^{-1}. \tag{3.9}$$

[3]Formally, $W = N(D)/D$ where D is the maximal torus in $Sp(2,\mathbb{R})$ consisting of its positive diagonal matrices, and $N(D)$ is its normalizer. We are indicating a set of representatives in $N(D)$.

Clearly, $F_j := p_j G_j p_j^{-1}$ is a subgroup of Q, for $j = 1, 2$. Also, we can assume that the permutation factor π in w is the identity, because it belongs to $MA \subset Q$. By the same token, by (3.8), we can suppose that $S_+ \neq s_1$. Our assumption is thus

$$F_2 = wF_1w^{-1}.$$

The proof now proceeds by inspecting the three remaining cases for w.

Suppose $w = -J$, that is $S_+ = 0$. Upon writing $F_1 = (\Sigma, H, \tau)$ and taking any element with $h = I_2 \in H$, a straightforward computation gives

$$-J \begin{bmatrix} I_2 & 0 \\ \sigma & I_2 \end{bmatrix} J = \begin{bmatrix} I_2 & -\sigma \\ 0 & I_2 \end{bmatrix},$$

in contradiction with $F_2 \subseteq Q$ unless $\sigma = 0$. In this case, though, $G_1 \notin \mathcal{E}_2$. Hence we may exclude $w = -J$.

Next, suppose $w = I_4$, that is $S_+ = I_2$. Going back to (3.9), we have then $G_2 = pG_1p^{-1}$ for some $p \in P$. But this yields $p \in MA$, against the hypothesis.

Finally, suppose $w = w_0$, namely $S_+ = s_0$. The conjugation $gG_1g^{-1} = G_2$ can be formally written as in (3.9), with the understanding that under the assumption $w = w_0$ we might have to absorb into p_1 a permutation term π coming from the Weyl group. We factor

$$p_1 = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ c & b & 1 & 0 \\ b & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ 0 & 0 & 1 & 0 \\ 0 & a & 0 & 1 \end{bmatrix} \begin{bmatrix} h & \\ & h^\sharp \end{bmatrix} = g(b\sigma_5 + c\sigma_3, I_2)g(a\sigma_4, h).$$

As already observed, we cannot assume that $h \in T$. Now, it is easy to check that

$$w_0 g(b\sigma_5 + c\sigma_3, I_2) w_0^{-1} = \begin{bmatrix} 1 & 0 & & \\ -b & 1 & & \\ c & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{bmatrix} \in Q. \tag{3.10}$$

Therefore, we have

$$\begin{aligned} p_2G_2p_2^{-1} &= w(p_1G_1p_1^{-1})w^{-1} \\ &= [w_0g(b\sigma_5 + c\sigma_3, I_2)w_0^{-1}]w_0g(a\sigma_4, h)G_1g(a\sigma_4, h)^{-1}w_0^{-1}[w_0g(b\sigma_5 + c\sigma_3, I_2)w_0^{-1}]^{-1} \end{aligned}$$

and by (3.10) we can absorb the term in square brackets into $p_2 \in Q$. This proves (3.7), because $p_2 = g(\sigma', h')$ and $p_1 = g(a_0\sigma_4, h)$ for some $a_0 \in \mathbb{R}$.

So far we thus have that (3.7) holds with $p_2 \in Q$, $w = w_0$ and $p_1 = g(a_0\sigma_4, h)$. Looking at the right hand side of this version of (3.7), we observe that

$$p_1G_1p_1^{-1} = (\Sigma', H', \tau_1) = G'$$

where $h^\dagger[\Sigma_1] = \Sigma'$, $H' = hH_1h^{-1}$ and, by Remark 2.5,

$$\tau_1(h') = a_0\left(\sigma_4 - h'^\dagger[\sigma_4]\right), \qquad h' \in H'.$$

We start by writing the elements in G' as

$$g_{\tau_1}(\sigma', h') = \begin{bmatrix} I_2 & \\ \sigma' & I_2 \end{bmatrix} \begin{bmatrix} I_2 & \\ \tau_1(h') & I_2 \end{bmatrix} \begin{bmatrix} h' & \\ & h'^\sharp \end{bmatrix}$$

and then we study the effect of conjugation by w_0. We thus parametrize

$$\sigma' = \begin{bmatrix} c & b \\ b & a \end{bmatrix}, \qquad h' = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix},$$

where we must interpret $a(\sigma'), b(\sigma'), c(\sigma')$ and similarly $\alpha(h'), \beta(h'), \gamma(h'), \delta(h')$. Computing, we see that

$$w_0 g_{\tau_1}(\sigma', I_2) w_0^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -b & 1 & 0 & -a \\ c & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is in Q if and only if $a = a(\sigma') \equiv 0$ as a function on Σ'. This is formulated by $\Sigma' = h^\dagger[\Sigma_1] \subseteq \sigma_4^\perp$. Next

$$w_0 g_{\tau_1}(0, h') w_0^{-1} = \begin{bmatrix} * & x \\ * & * \end{bmatrix}$$

with

$$x = s_0 h' s_1 - s_1 \tau_1(h') h' s_1 - s_1 h'^\sharp s_0.$$

Now, only the first summand has a nonzero entry in the upper-left corner, and it is equal to γ. Therefore $\gamma = \gamma(h') \equiv 0$ s a function on H'. This is $hH_1h^{-1} \subseteq T$. Similarly, only the second summand has a nonzero entry in the lower-left corner. Since $\tau_1(h') = a_0(\sigma_4 - h'^\dagger[\sigma_4])$, we have

$$\tau_1(h')h' = a_0(\sigma_4 h' - h'^\sharp \sigma_4) = a_0 \begin{bmatrix} 0 & \beta/\alpha\delta \\ \beta & \delta - 1/\delta \end{bmatrix}, \tag{3.11}$$

whose lower-right corner is $a_0(\delta - 1/\delta)$. Therefore, a_0 can be different from zero only if the continuous function δ is $\delta(h') = \pm 1$. However, we are working with connected groups, hence $\delta = 1$ and so $H' = hH_1h^{-1} \subseteq \{\left[\begin{smallmatrix} \alpha & 0 \\ \beta & 1 \end{smallmatrix}\right] : \alpha > 0, \beta \in \mathbb{R}\}$. □

Remark 3.1. The last statement of Lemma 3.14 has a consequence for the classification problem. Fix $G = \Sigma \rtimes H \in \mathcal{E}$, say for example one among the canonical groups determined in the previous section. If G is conjugate to some other group via an element *not* in MA, then there must exist $h \in GL(2, \mathbb{R})$ such that $h^\dagger[\Sigma] \subseteq \sigma_4^\perp$ and $hHh^{-1} \subseteq T$. The first of these conditions forces the determinant of all elements in Σ to be less than or equal to zero. Therefore, remembering (3.1), only the following cases can be considered:

(a) if $n = 1$, then either $\Sigma = \Sigma_2$ or $\Sigma = \Sigma_3$;
(b) if $n = 2$, then $\Sigma = \Sigma_3^\perp$.

Remark 3.2. Lemma 3.14 may be formulated in a different way. Given $G \in \mathcal{E}$, if there exists $g \notin MA$ such that $gGg^{-1} \in \mathcal{E}$, then there must exist $h_0 \in GL(2, \mathbb{R})$ such that

$$G = g(0, h_0)^{-1}(\Sigma, H, \tau) g(0, h_0) \tag{3.12}$$

with $\Sigma \subseteq \sigma_4^\perp$, $H \subseteq T$ and $\tau(h) = a_0(\sigma_4 - h^\dagger[\sigma_4])$, hence $\tau \in \mathcal{T}$ (see Remark 2.5). In this case there exists $\tau' \in \mathcal{T}$ such that

$$w_0(\Sigma, H, \tau) w_0^{-1} = (\Sigma', H', \tau'). \tag{3.13}$$

The next lemma shows that if (3.12) and (3.13) hold for some h_0, then they also hold for th_0, for all $t \in T$. This will be used to put (Σ, H, τ) in canonical form.

Lemma 3.15. *Suppose that $\Sigma \subseteq \sigma_4^\perp$, $H \subseteq T$ and $\tau(h) = a_0(\sigma_4 - h^\dagger[\sigma_4])$ are such that (3.13) holds with $\tau' \in \mathcal{T}$, for some symmetric τ_0. Then for all $t \in T$ there exists $\tau'' \in \mathcal{T}$ such that*

$$w_0 g(0, t)(\Sigma, H, \tau) g(0, t)^{-1} w_0^{-1} = (\Sigma'', H'', \tau'').$$

Proof. We parametrize the lower triangular matrices in T by

$$t = \begin{bmatrix} \alpha & \\ \alpha\beta & \delta \end{bmatrix}. \tag{3.14}$$

Then

$$w_0 g(0,t) w_0^{-1} = g(\begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix}, \begin{bmatrix} \alpha & 0 \\ 0 & \delta^{-1} \end{bmatrix}) := g'$$

and so $g'(\Sigma', H', \tau')g'^{-1} = (\Sigma'', H'', \tau'')$, where

$$\Sigma'' = \begin{bmatrix} \alpha & 0 \\ 0 & \delta^{-1} \end{bmatrix}^{\dagger} [\Sigma']$$

$$H'' = \begin{bmatrix} \alpha & 0 \\ 0 & \delta^{-1} \end{bmatrix} H' \begin{bmatrix} \alpha & 0 \\ 0 & \delta^{-1} \end{bmatrix}^{-1}$$

$$\tau'' = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} + \begin{bmatrix} \alpha & 0 \\ 0 & \delta^{-1} \end{bmatrix}^{\dagger} [\tau'].$$

In the last line we have identified $\tau', \tau'' \in \mathcal{T}$ with the corresponding symmetric matrices. □

We apply the above lemmata to our classification problem as follows. Take any group G in canonical form. If there exists $g \notin MA$ that conjugates G to another group in the class $\mathcal{E}$, then, by Lemma 3.14, there exists $h \in MA$ that maps the vector part Σ inside $\sigma_4^{\perp}$. By Lemma 3.15 we know that any other $ht \in MA$ can be used for this purpose, with $t \in T$, and hence we can reduce the analysis to three possible cases: $\sigma_4^{\perp}$ itself if Σ is bidimensional, and two cases if $n = 1$, as the following proposition clarifies.

Proposition 3.16. *Suppose that Σ is a one dimensional vector subspace of $\sigma_4^{\perp}$. Then there exists $t \in T$ such that $t^{\dagger}[\Sigma]$ is generated by:*

(i) *σ_5 if the signature is $(1,1,0)$, and $H(\sigma_5) \cap T$ are the diagonal matrices in $GL(2,\mathbb{R})$;*
(ii) *σ_3 if the signature is $(1,0,1)$, and $H(\sigma_3) = T$.*

Proof. Denote by

$$\sigma_0 = \begin{bmatrix} c & b \\ b & 0 \end{bmatrix}$$

the generator of Σ and parametrize as in (3.14) the elements in T. Then

$$t^{\dagger}[\sigma_0] = \begin{bmatrix} (c\delta - 2b\alpha\beta)/\alpha^2\delta & b/\alpha\delta \\ b/\alpha\delta & 0 \end{bmatrix}.$$

If $b = 0$ we get case (ii), otherwise we put $b = 1$ and we get (i). □

Next we perform the conjugation by w_0, as in (3.13), under the necessary conditions on Σ and H that must be satisfied, but without assuming that the conjugation produces a group in $\mathcal{E}$, so that for the resulting triple (Σ', H', τ') we don't know that $\tau' \in \mathcal{T}$. We start by computing Σ' and H'.

Lemma 3.17. *Take $\Sigma \subseteq \sigma_4^{\perp}$, $H \subseteq T$ and $\tau(h) = a_0(\sigma_4 - h^{\dagger}[\sigma_4])$ such that $w_0(\Sigma, H, \tau)w_0^{-1} \subset Q$ and parametrize*

$$\sigma = \begin{bmatrix} c(\sigma) & b(\sigma) \\ b(\sigma) & 0 \end{bmatrix} \in \Sigma, \qquad h = \begin{bmatrix} \alpha(h) & 0 \\ \beta(h)\alpha(h) & \delta(h) \end{bmatrix} \in H.$$

Then $(\Sigma', H', \tau') := w_0(\Sigma, H, \tau)w_0^{-1}$ *is as follows:* Σ' *consists of all the matrices*

$$\sigma' = \begin{bmatrix} c(\sigma)+b(\sigma)\beta(h) & \beta(h) \\ \beta(h) & 0 \end{bmatrix} \tag{3.15}$$

as $g_\tau(\sigma, h)$ *varies in the subset of* (Σ, H, τ) *whose elements have the form*

$$\sigma = \begin{bmatrix} c(\sigma) & -a_0\beta(h) \\ -a_0\beta(h) & 0 \end{bmatrix} \qquad h = \begin{bmatrix} 1 & 0 \\ -\beta(h) & 1 \end{bmatrix}, \tag{3.16}$$

the group H' *consists of all the matrices*

$$h' = \begin{bmatrix} \alpha(h) & 0 \\ -(a_0\beta(h)+b(\sigma))\alpha(h) & \delta(h)^{-1} \end{bmatrix} \tag{3.17}$$

as $g_\tau(\sigma, h)$ *varies freely in* (Σ, H, τ)*, and* τ' *is not necessarily*[4] *in* $\mathcal{T}$*.*

Proof. With our notation, but omitting the various dependencies, we have

$$g_\tau(\sigma, h) = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ \beta\alpha & \delta & 0 & 0 \\ (c+b\beta)\alpha & b\delta + a_0\beta\delta^{-1} & \alpha^{-1} & -\beta\delta^{-1} \\ (b+a_0\beta)\alpha & a_0(\delta-\delta^{-1}) & 0 & \delta^{-1} \end{bmatrix},$$

and hence

$$w_0 g_\tau(\sigma, h) w_0^{-1} = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ -(b+a_0\beta)\alpha & \delta^{-1} & 0 & a_0(\delta-\delta^{-1}) \\ (c+b\beta)\alpha & \beta\delta^{-1} & \alpha^{-1} & b\delta + a_0\beta\delta^{-1} \\ \beta\alpha & 0 & 0 & \delta \end{bmatrix}, \tag{3.18}$$

Rember that the hypothesis $w_0(\Sigma, H, \tau)w_0^{-1} \subset Q$, hence of the form (Σ', H', τ'), is equivalent to requiring that $a_0(\delta - \delta^{-1}) = 0$ (see the proof of Lemma 3.14). The upper-left 2×2 block is as in (3.17), and by setting it to be equal to I_2, the lower-left 2×2 block is (3.15), and this happens if and only if $g_\tau(\sigma, h)$ is as in (3.16). □

Remark 3.3. Observe that case (i) of Proposition 3.16 is ruled out from our classification problem by the above lemma. Indeed, in that case, $c(\sigma) = 0$ and h is diagonal, so the group elements satisfying (3.16) have $\beta(h) = 0$, whence $\sigma' = 0$.

Remark 3.4. In both the remaining two cases ($\Sigma = \Sigma_3$ and $\Sigma = \sigma_4^\perp$), we can always take $\beta(h) = 0$ and $c(\sigma) = 1$ in (3.16). Therefore we always obtain that $\sigma_3 \in \Sigma'$. As a result, the only canonical groups that are possibly conjugate to other groups in the class $\mathcal{E}$ are those listed in Proposition 3.13, because $\sigma_5\Sigma_3^\perp\sigma_5 = \sigma_4^\perp$.

Finally, we complete the picture drawn in Lemma 3.17.

Lemma 3.18. *Hypotheses and notation as in Lemma 3.17, with either* $\Sigma = \Sigma_3$ *or* $\Sigma = \sigma_4^\perp$*. For every pair of real numbers* a', b' *define the function*

$$\Psi(\sigma, h) = \beta(h) - b'\left(1 - \delta(h)\alpha(h)^{-1}\right) + a'\left(a_0\beta(h) + b(\sigma)\right)\delta(h)^2. \tag{3.19}$$

Then $\tau' \in \mathcal{T}$ *if and only if there exist* a', b' *such that for all* $h \in H$ *and all* $\sigma \in \Sigma$

$$a'(1-\delta^2(h)) = 0, \qquad \begin{bmatrix} 1 & 0 \\ \Psi(\sigma, h) & 1 \end{bmatrix} \in H, \qquad a_0\Psi(\sigma, h)\sigma_5 \in \Sigma. \tag{3.20}$$

In this case, the symmetric matrix associated to τ' *is* $\begin{bmatrix} 0 & b' \\ b' & a' \end{bmatrix}$.

[4] See the following Lemma 3.18 for further information on τ'.

Proof. Look at (3.18). The lower-left 2×2 block factors as

$$\begin{bmatrix} c(\sigma) + 2b(\sigma)\beta(h) + a_0\beta(h)^2 & \beta(h) \\ \beta(h) & 0 \end{bmatrix} \begin{bmatrix} \alpha(h) & 0 \\ -(b(\sigma) + a_0\beta(h))\alpha(h) & \delta(h)^{-1} \end{bmatrix},$$

where the second is evidently $h'(\sigma, h)$. Now, $\tau' \in \mathcal{T}$ if and only if the first factor, that we denote by $\omega(\sigma, h)$, satisfies

$$\omega(\sigma, h) - \tau'(h'(\sigma, h)) = \omega(\sigma, h) - \left(\tau' - h'(\sigma, h)^{\dagger}[\tau']\right) \in \Sigma' \tag{3.21}$$

for some symmetric (constant) matrix τ'. For any such

$$\tau' = \begin{bmatrix} c' & b' \\ b' & a' \end{bmatrix}$$

a direct computation gives that $h'(\sigma, h)^{\dagger}[\tau']$ is equal to

$$\begin{bmatrix} c'\alpha^{-2} + 2b'\alpha^{-1}\delta\left(a_0\beta + b\right) + a'\delta^2\left(a_0\beta + b\right)^2 & \delta\left[b'\alpha^{-1} + a'\delta\left(a_0\beta + b\right)\right] \\ \delta\left[b'\alpha^{-1} + a'\delta\left(a_0\beta + b\right)\right] & a'\delta^2 \end{bmatrix}$$

Now, the lower-right entry of $\omega(\sigma, h) - \left(\tau' - h'(\sigma, h)^{\dagger}[\tau']\right)$ is $a'(\delta^2 - 1)$, and must vanish. This is the first of (3.20). The upper-right entry of $\omega(\sigma, h) - \left(\tau' - h'(\sigma, h)^{\dagger}[\tau']\right)$ is precisely $\Psi(\sigma, h)$. As we have already observed, $\sigma_3 \in \Sigma'$. Therefore (3.21) holds true if and only if $\Psi(\sigma, h)\sigma_5 \in \Sigma'$. By Lemma 3.17, this occurs if and only if the remaining two conditions in (3.20) are satisfied. □

Remark 3.5. Notice that if there exists $h \in H$ such that $\delta(h) \neq 1$ then both a_0 and $a' = 0$.

Remark 3.6. Lemma 3.18 expresses necessary and sufficient conditions for the conjugation via w_0 to send a group (Σ, H, τ) with $\tau \in \mathcal{T}$, in a group of the same kind. The image group, however, is determined in Lemma 3.17

Remark 3.7. Notice that if we choose $a' = b' = 0$, and $a_0 = 0$, then (3.20) is satisfied if and only if for every $h \in H$

$$\begin{bmatrix} 1 & 0 \\ \beta(h) & 1 \end{bmatrix} \in H.$$

In this case, $\tau = \tau' = 0$ and conjugation by w_0 sends the group $\Sigma \rtimes H \in \mathcal{E}$ into another group in $\mathcal{E}$, namely $\sigma_4^{\perp} \rtimes H'$.

We are in a position to apply these results to our classification problem. We take a group $G \in \mathcal{E}$ in canonical form and we want to know if it is conjugate to another such, or not. By Lemma 3.14, we must find $g \in MA$ (and Lemma 3.15 tells us that any such choice is legitimate) such that $gGg^{-1} = \Sigma \rtimes H$, where either $\Sigma = \Sigma_3$ or $\Sigma = \sigma_4^{\perp}$, by Remark 3.3. Hence it is enough to consider the groups in the list of Proposition 3.13. At this point we look at H and check whether there are entries $\delta(h) \neq 1$, in which case the condition $a_0(1 - \delta(h)^2) = 0$ forces $a_0 = 0$ and the first of (3.20) forces $a' = 0$. If not, we must allow for $a_0 \neq 0$ and $a' \neq 0$. Next we verify if the various conditions in (3.20) are satisfied for some a', b'. In this case, we know that $w_0(\Sigma, H, \tau)w_0^{-1} = (\Sigma', H', \tau')$ with both $\tau, \tau' \in \mathcal{T}$, which is equivalent to saying that $\Sigma \rtimes H$ is conjugate to a $\Sigma' \rtimes H'$ yet to be determined. Finally, using Lemma 3.17 we find all the elements of the form (3.16) and thus compute Σ' and H' by means of (3.15) and (3.17), respectively. The last step is to identify the MA-canonical form of $\Sigma' \rtimes H'$.

We now apply the above procedure to the groups in the list of Proposition 3.13. With slight abuse of notation, we write σ_5 in place of $g(0,\sigma_5)$ and we write $G_1 \sim G_2$ to mean that they are conjugate. Recall that $\sigma_5\Sigma_3^{\perp}\sigma_5^{-1} = \sigma_4^{\perp}$.

(3.i) and (3.xiv). $\Sigma_3 \rtimes T^0 \sim \Sigma_3^{\perp} \rtimes K_0(\sigma_3)$.

(i) There is $h \in H$ such that $\delta(h) \neq 1$, hence $a_0 = a' = 0$;
(ii) with the choice $\tau' = 0$, $\Psi(\sigma, [\begin{smallmatrix} \alpha & 0 \\ \beta\alpha & \delta \end{smallmatrix}]) = \beta$ with $\beta \in \mathbb{R}$ and $[\begin{smallmatrix} 1 & 0 \\ \beta & 1 \end{smallmatrix}] \in H$, hence (3.20) are satisfied;
(iii) $\Sigma' = \sigma_4^{\perp}$, $H' = K_0(\sigma_3)$ and $\sigma_5(\sigma_4^{\perp} \rtimes K_0(\sigma_3))\sigma_5^{-1} = \Sigma_3^{\perp} \rtimes K_0(\sigma_3)$.

(3.ii). $\Sigma_3 \rtimes H_0(\sigma_3)$ has only (not trivial) MA-conjugations.

(i) If $a_0 = 0$, then $H' = \{I_2\}$ since $\alpha(h) = \beta(h) = 1$ and $b(\sigma) = 0$; hence and $\Sigma' \rtimes \{I_2\} = \Sigma'$ can not be conjugate to an element of the class $\mathcal{E}$ with $q \in Q$;
(ii) if $a_0 \neq 0$, then $\Sigma' = \Sigma_3$ and $H' = H_0(\sigma_3)$.

(3.iii). $\Sigma_3 \rtimes H_1(\sigma_3)$ has only MA-conjugations.

(i) There is $h \in H$ such that $\delta(h) \neq 1$, hence $a_0 = a' = 0$.
(ii) $\Psi(\sigma, [\begin{smallmatrix} e^t & 0 \\ te^t & e^t \end{smallmatrix}]) = t$ with $t \in \mathbb{R}$, but $[\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}] \notin H$ if $t \neq 0$, hence (3.20) are not satisfied.

(3.iv) and (3.v) with $\gamma = -\frac{1}{2}$. $\Sigma_3 \rtimes H_\infty(\sigma_3) \sim \Sigma_3 \rtimes H_{-\frac{1}{2},0}(\sigma_3)$.

(i) There is $h \in H$ such that $\delta(h) \neq 1$, hence $a_0 = a' = 0$;
(ii) with the choice $\tau' = 0$, $\Psi(\sigma, [\begin{smallmatrix} e^t & 0 \\ 0 & e^t \end{smallmatrix}]) = 0$, hence (3.20) are trivially satisfied;
(iii) $\Sigma' = \Sigma_3$ and $H' = \{[\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}] : t \in \mathbb{R}\} = H_{-\frac{1}{2},0}(\sigma_3)$.

(3.v) with $\gamma \neq -\frac{1}{2}$: $\Sigma_3 \rtimes H_{\gamma,0}(\sigma_3) \sim \Sigma_3 \rtimes H_{-\frac{\gamma}{2\gamma+1},0}(\sigma_3)$.

(i) There is $h \in H$ such that $\delta(h) \neq 1$, hence $a_0 = a' = 0$;
(ii) with the choice $\tau' = 0$, $\Psi(\sigma, [\begin{smallmatrix} e^{\gamma t} & 0 \\ 0 & e^{(\gamma+1)t} \end{smallmatrix}]) = 0$, hence (3.20) are trivially satisfied;
(iii) $\Sigma' = \Sigma_3$ and $H' = \{[\begin{smallmatrix} e^{\gamma t} & 0 \\ 0 & e^{-(\gamma+1)t} \end{smallmatrix}] : t \in \mathbb{R}\} = H_{-\frac{\gamma}{2\gamma+1},0}(\sigma_3)$.

(3.vi). $\Sigma_3 \rtimes K_0(\sigma_3)$ has only MA (not trivial) conjugations.

(i) There is $h \in H$ such that $\delta(h) \neq 1$, hence $a_0 = a' = 0$;
(ii) with the choice $\tau' = 0$, $\Psi(\sigma, [\begin{smallmatrix} e^t & 0 \\ 0 & e^s \end{smallmatrix}]) = 0$, hence (3.20) are trivially satisfied;
(iii) $\Sigma' = \Sigma_3$ and $H' = K_0(\sigma_3)$, so that $w_0(\Sigma_3 \rtimes K_0(\sigma_3))w_0^{-1} = \Sigma_3 \rtimes K_0(\sigma_3)$.

(3.vii) and (3.xiii) with $\gamma = -\frac{1}{2}$. $\Sigma_3 \rtimes K_\infty(\sigma_3) \sim \Sigma_3^{\perp} \rtimes H_{-\frac{1}{2},0}(\sigma_3)$.

(i) There is $h \in H$ such that $\delta(h) \neq 1$, hence $a_0 = a' = 0$;
(ii) with the choice $\tau' = 0$, $\Psi(\sigma, [\begin{smallmatrix} e^t & 0 \\ se^t & e^t \end{smallmatrix}]) = s$ with $s \in \mathbb{R}$ and $[\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}] \in H$, hence (3.20) are satisfied;
(iii) $\Sigma' = \sigma_4^{\perp}$, $H' = \{[\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}] : t \in \mathbb{R}\}$ and $\sigma_5(\sigma_4^{\perp} \rtimes H')\sigma_5^{-1} = \Sigma_3^{\perp} \rtimes H_{-\frac{1}{2},0}(\sigma_3)$.

(3.viii) with $\gamma \neq -\frac{1}{2}$ and (3.xiii) with $\gamma \neq -\frac{1}{2}$, (3.viii) with $\gamma = -\frac{1}{2}$ and (3.xii). If $\gamma \neq -\frac{1}{2}$, then $\Sigma_3 \rtimes L_\gamma(\sigma_3) \sim \Sigma_3^{\perp} \rtimes H_{-\frac{\gamma+1}{2\gamma+1},0}(\sigma_3)$; if $\gamma = -\frac{1}{2}$, then $\Sigma_3 \rtimes L_{-1/2}(\sigma_3) \sim \Sigma_3^{\perp} \rtimes H_\infty(\sigma_3)$.

(i) If $\gamma \neq -1$, there is $h \in H$ such that $\delta(h) \neq 1$, hence $a_0 = a' = 0$; if $\gamma = 1$, we choose $a_0 = 0$ (see iv);
(ii) with the choice $\tau' = 0$, $\Psi(\sigma, [\begin{smallmatrix} e^{\gamma t} & 0 \\ se^{\gamma t} & e^{(\gamma+1)t} \end{smallmatrix}]) = s$ with $s \in \mathbb{R}$ and $[\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}] \in H$, hence (3.20) are satisfied;

(iii) $\Sigma' = \sigma_4^\perp$, $H' = \{\left[\begin{smallmatrix} e^{\gamma t} & 0 \\ 0 & e^{-(\gamma+1)t} \end{smallmatrix}\right] : t \in \mathbb{R}\}$ and $\sigma_5(\sigma_4^\perp \rtimes H')\sigma_5^{-1} = \Sigma_3^\perp \rtimes H_{-\frac{\gamma+1}{2\gamma+1},0}(\sigma_3)$ if $\gamma \neq -1/2$ and $\sigma_5(\sigma_4^\perp \rtimes H')\sigma_5^{-1} = \Sigma_3^\perp \rtimes H_\infty(\sigma_3)$ if $\gamma = -1/2$;

(iv) if $\gamma = -1$ and $a_0 \neq 0$, choose $b' = 0$ and $a' = -a_0^{-1}$ so that $\Psi(\sigma, h) = 0$, but $\Sigma' = \Sigma_3$ and $H' = L_{-1}$ so that $w_0(\Sigma_3 \rtimes L_{-1}(\sigma_3))w_0^{-1} = \Sigma_3 \rtimes L_{-1}(\sigma_3)$.

(3.ix). $\Sigma_3^\perp \rtimes {}^tH^0(\sigma_3)$ has only (not trivial) MA-conjugations.

(i) $\sigma_5^t H^0(\sigma_3)\sigma_5^{-1} = T^0$;
(ii) $\Sigma' = \sigma_4^\perp$ and $H' = T^0$.

(3.x). $\Sigma_3^\perp \rtimes {}^tH_0(\sigma_3)$ has only (not trivial) MA-conjugations.

(i) $\sigma_5^t H_0(\sigma_3)\sigma_5^{-1} = H_0(\sigma_3)$;
(ii) $\Sigma' = \sigma_4^\perp$ and $H' = H_0(\sigma_3)$.

(3.xi). $\Sigma_3^\perp \rtimes {}^tH_1(\sigma_3)$ has only MA-conjugations.

(i) $\sigma_5^t H_1(\sigma_3)\sigma_5^{-1} = H_1(\sigma_3)$;
(ii) There is $h \in H$ such that $\delta(h) \neq 1$, hence $a_0 = a' = 0$;
(iii) $\Psi(\sigma, \left[\begin{smallmatrix} e^t & 0 \\ te^t & e^t \end{smallmatrix}\right]) = t$ with $t \in \mathbb{R}$, but $\left[\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right] \notin H$ if $t \neq 0$, hence (3.20) are not satisfied.

(3.xii) and (3.viii) with $\gamma = -\frac{1}{2}$. See above.
(3.xiii) with $\gamma \neq -\frac{1}{2}$ and (3.viii) with $\gamma \neq -\frac{1}{2}$. See above.
(3.xiv) and (3.i). See above.
(3.xv) and (3.xvi) with $\gamma = -\frac{1}{2}$. $\Sigma_3^\perp \rtimes {}^tK_\infty(\sigma_3) \sim \Sigma_3^\perp \rtimes {}^tL_{-\frac{1}{2}}(\sigma_3)$

(i) $\sigma_5^t K_\infty(\sigma_3)\sigma_5^{-1} = K_\infty(\sigma_3)$;
(ii) There is $h \in H$ such that $\delta(h) \neq 1$, hence $a_0 = a' = 0$;
(iii) with the choice $\tau' = 0$, $\Psi(\sigma, \left[\begin{smallmatrix} e^t & 0 \\ se^t & e^t \end{smallmatrix}\right]) = s$ with $s \in \mathbb{R}$ and $\left[\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right] \in H$, hence (3.20) are satisfied;
(iv) $\Sigma' = \sigma_4^\perp$, $H' = \{\left[\begin{smallmatrix} e^t & 0 \\ se^t & e^{-t} \end{smallmatrix}\right] : t \in \mathbb{R}\}$ and $\sigma_5(\sigma_4^\perp \rtimes H')\sigma_5^{-1} = \Sigma_3^\perp \rtimes {}^tL_{-\frac{1}{2}}(\sigma_3)$.

(3.xvi) with $\gamma \neq -\frac{1}{2}$. $\Sigma_3^\perp \rtimes {}^tL_\gamma(\sigma_3) \sim \Sigma_3^\perp \rtimes {}^tL_{-\frac{\gamma}{2\gamma+1}}(\sigma_3)$

(i) $\sigma_5^t L_\gamma(\sigma_3)\sigma_5^{-1} = \{\left[\begin{smallmatrix} e^{(\gamma+1)t} & 0 \\ se^{(\gamma+1)t} & e^{\gamma t} \end{smallmatrix}\right] : s, t \in \mathbb{R}\} = L_{-(\gamma+1)}(\sigma_3)$;
(ii) with the choice $\tau' = 0$, $\Psi(\sigma, \left[\begin{smallmatrix} e^{(\gamma+1)t} & 0 \\ se^{(\gamma+1)t} & e^{\gamma t} \end{smallmatrix}\right]) = s$ with $s \in \mathbb{R}$ and $\left[\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right] \in H$, hence (3.20) are satisfied;
(iii) $\Sigma' = \sigma_4^\perp$, $H' = \{\left[\begin{smallmatrix} e^{(\gamma+1)t} & 0 \\ se^{(\gamma+1)t} & e^{-\gamma t} \end{smallmatrix}\right] : t, s \in \mathbb{R}\}$ and, since $\gamma \neq -1/2$,

$$\sigma_5 H' \sigma_5^{-1} = \{\left[\begin{smallmatrix} e^{-\gamma t} & se^{(\gamma+1)t} \\ 0 & e^{(\gamma+1)t} \end{smallmatrix}\right] : t, s \in \mathbb{R}\} = {}^tL_{-\frac{\gamma}{2\gamma+1}}(\sigma_3).$$

This proves Theorem 1.1.

4. Notation and symbols

We shall use the following non standard notation. The letters g, h are to be regarded as invertible matrices and σ as a symmetric matrix.

$$g^\sharp = {}^tg^{-1}$$
$$i_g(h) = ghg^{-1}$$
$$g^\dagger[\sigma] = {}^tg^{-1}\sigma g^{-1}.$$

If G is Lie group, the connected component of the identity will be denoted G^0. We consider the symmetric matrices

$$\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \sigma_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For $i = 1, 2, 3$ we write $\Sigma_i = \mathrm{span}\{\sigma_i\}$. Their orthogonal complements are

$$\begin{aligned} \Sigma_1^\perp &= \Big\{ \begin{bmatrix} u & v \\ v & -u \end{bmatrix} : u, v \in \mathbb{R} \Big\} = \mathrm{span}\{\sigma_2, \sigma_5\}, \\ \Sigma_2^\perp &= \Big\{ \begin{bmatrix} u & v \\ v & u \end{bmatrix} : u, v \in \mathbb{R} \Big\} = \mathrm{span}\{\sigma_1, \sigma_5\}, \\ \Sigma_3^\perp &= \Big\{ \begin{bmatrix} 0 & v \\ v & u \end{bmatrix} : u, v \in \mathbb{R} \Big\} = \mathrm{span}\{\sigma_4, \sigma_5\}. \end{aligned}$$

For $t \in \mathbb{R}$ we set

$$R_t = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}, \quad A_t = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}.$$

The notation relative to the Lie subgroups of $GL(2, \mathbb{R})$ is as follows:

$$\begin{aligned} &SO(2) = \{R_t : t \in \mathbb{R}\} \\ &SO^0(1,1) = \{A_t : t \in \mathbb{R}\} \\ &T = \big\{ \left[\begin{smallmatrix} c & 0 \\ b & a \end{smallmatrix}\right] : a, b, c \in \mathbb{R},\ ac \neq 0 \big\} \\ &H^0(\sigma_1) = SO(2) \times \mathbb{R}_+ \\ &H_\infty(\sigma_1) = SO(2) \\ &H_\alpha(\sigma_1) = \{e^t R_{\alpha t} : t \in \mathbb{R}\}, \qquad \alpha \in \mathbb{R} \\ &H^0(\sigma_2) = SO^0(1,1) \times \mathbb{R}_+ \\ &H_\infty(\sigma_2) = SO^0(1,1) \\ &H_\alpha(\sigma_2) = \{e^t A_{\alpha t} : t \in \mathbb{R}\}, \qquad \alpha \in \mathbb{R} \\ &H^0(\sigma_3) = T^0 \\ &H_0(\sigma_3) = \big\{ \left[\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right] : t \in \mathbb{R} \big\} \\ &H_1(\sigma_3) = \big\{ e^t \left[\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right] : t \in \mathbb{R} \big\} \\ &H_\infty(\sigma_3) = \big\{ e^t \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] : t \in \mathbb{R} \big\} \\ &H_{\gamma,0}(\sigma_3) = \Big\{ \begin{bmatrix} e^{\gamma t} & 0 \\ 0 & e^{(\gamma+1)t} \end{bmatrix} : t \in \mathbb{R} \Big\}, \qquad \gamma \in \mathbb{R} \\ &K_0(\sigma_3) = \big\{ \left[\begin{smallmatrix} e^t & 0 \\ 0 & e^s \end{smallmatrix}\right] : s, t \in \mathbb{R} \big\} \\ &K_\infty(\sigma_3) = \Big\{ \begin{bmatrix} e^t & 0 \\ s & e^t \end{bmatrix} : s, t \in \mathbb{R} \Big\} \\ &L_\gamma(\sigma_3) = \Big\{ \begin{bmatrix} e^{\gamma t} & 0 \\ s & e^{(\gamma+1)t} \end{bmatrix} : s, t \in \mathbb{R} \Big\}, \qquad \gamma \in \mathbb{R}. \end{aligned}$$

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