Analysis and Applications

Vector valued reproducing kernel Hilbert spaces of integrable functions and Mercer theorem

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We characterize the reproducing kernel Hilbert spaces whose elements are p-integrable functions in terms of the *boundedness* of the integral operator whose kernel is the reproducing kernel. Moreover, for $p = 2$ we show that the spectral decomposition of this integral operator gives a complete description of the reproducing kernel, extending Mercer theorem.

Keywords: Reproducing kernel Hilbert space; Mercer theorem; integral operator.

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1. Introduction

The aim of this paper is the characterization of the reproducing kernel Hilbert spaces (RKH spaces) whose elements are vector valued p-integrable functions. We show that, if $\mathcal{H}$ is such a space and Γ its reproducing kernel, the functions in $\mathcal{H}$ are p-integrable if and only if the integral operator of kernel Γ is bounded from $L^{\frac{p}{p-1}}$ to L^p. Moreover, for $p = 2$, we prove a generalized version of Mercer theorem, that is, the fact that the reproducing kernel can be expressed in terms of the spectral measure of the integral operator. Our results hold for RKH spaces of functions $f : X \to \mathcal{K}$ where X is a measurable set and $\mathcal{K}$ is a Hilbert space, following the general setting of vector valued RKH spaces outlined in [1,2,3].

The characterization of the regularity properties of the RKH spaces in terms of corresponding properties of the reproducing kernel is already discussed in the literature. In [3] there is a complete characterization of RKH spaces whose elements are continuous or smooth complex functions, see also [4], whereas in [1] there is a discussion of RKH spaces of holomorphic vector valued functions. However, a similar treatment for RKH spaces of p-integrable functions has not yet been exploited. The problem of square-integrability is discussed in the framework of harmonic analysis in connection with square-integrable representations (there is a large literature on the topic, see for example [5,6] and references therein); in a general setting there are some sufficient conditions in [4].

The motivation of the present work is twofold. In recent years there is a new interest for the theory of RKH spaces in different frameworks, like quantum mechanics [7], signal analysis [8,6], probability theory [9] and statistical learning theory [10,11]. In particular, for these applications there is often the need of RKH spaces whose elements are square-integrable (possibly vector valued) functions. However, most of references are mainly devoted to characterize *operations* between RKH spaces (like sum, restriction, tensor product), whereas few papers discuss the correspondence between regularity properties of RKH spaces and features of the associated kernels.

This paper is both a research article and a self-contained survey about RKH spaces whose elements are functions that take value in a separable Hilbert space $\mathcal{K}$ and are p-integrable according to a σ-finite measure. The article is organized as follows. At the beginning of each section we briefly introduce the main notations we need. In Section 2, following [3,4] we review the connection between

(1) RKH spaces of functions from a set X into a Hilbert space $\mathcal{K}$;
(2) kernels of positive type on $X \times X$ and taking value in the space of bounded operators on $\mathcal{K}$;
(3) maps on X taking values in the space of bounded operators from $\mathcal{K}$ into an arbitrary Hilbert space.

In Section 3 we study the problem of measurability under the assumption that both $\mathcal{K}$ and the RKH space are separable. Our proof is an easy consequence of the equivalence between weak and strong measurability for operator valued maps. In Section 4 we assume that X is a measurable set endowed with a σ-finite measure μ and we show that a RKH space $\mathcal{H}$ is a subspace of $L^p(X, \mu; \mathcal{K})$ if and only if the integral operator whose kernel is the reproducing kernel of $\mathcal{H}$ is bounded from $L^{\frac{p}{p-1}}(X, \mu; \mathcal{K})$ into $L^p(X, \mu; \mathcal{K})$. This is the main result of the paper. In Section 4.4 we give additional conditions on the reproducing kernel Γ ensuring that the inclusion of $\mathcal{H}$ into $L^p(X, \mu; \mathcal{K})$ is compact. In Section 5 we assume that X is a locally compact space and we prove that the RKH space $\mathcal{H}$ is a subspace of $\mathcal{C}(X; \mathcal{K})$ if and only if the reproducing kernel is locally bounded and separately continuous. As before we also discuss the compactness of the inclusion. For the scalar case the results we present are due to [3], however we give an elementary proof which holds also for the vector case.

Finally, in Section 6 we assume that X is a measurable space endowed with a σ-finite measure μ and $\mathcal{H}$ is a separable RKH space such that $\mathcal{H} \subset L^2(X, \mu; \mathcal{K})$. We characterize the space $\mathcal{H}$ and the reproducing kernel Γ in terms of the spectral decomposition of the corresponding integral operator. When X is a compact subset of $\mathbb{R}^n$ endowed with the Lebesgue measure, this kind of result is known as *Mercer theorem* [12]. Extensions of Mercer theorem can be found in [10,13,14] and references therein.

2. Reproducing kernel Hilbert spaces

In this section we give the definition of vector valued RKH spaces, we show the correspondence between such spaces and vector valued kernels of positive type and we analyse the relation between the vector and scalar case. The results we present in this section are well known for the scalar case, see [4,15,9] for updated references. For the vector case we refer to [3,2,1].

2.1. *Notations*

Given two sets X and Y, the vector space of functions from X into Y is denoted by Y^X endowed with the topology of point-wise convergence. If $\mathcal{H}$ is a Hilbert space[a], the corresponding norm and scalar product are denoted by $\|\cdot\|_{\mathcal{H}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, respectively. The scalar product is linear in the first argument. If $\mathcal{H}$, $\mathcal{K}$ are Hilbert spaces, $\mathcal{B}(\mathcal{H};\mathcal{K})$ is the Banach space of bounded operators from $\mathcal{H}$ to $\mathcal{K}$ (with $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H};\mathcal{H})$) and $\|\cdot\|_{\mathcal{H},\mathcal{K}}$ denotes the uniform norm in $\mathcal{B}(\mathcal{H};\mathcal{K})$. If $A \in \mathcal{B}(\mathcal{H};\mathcal{K})$, $\operatorname{Ker} A$ denotes the kernel, $\operatorname{Im} A$ the image and $A^* \in \mathcal{B}(\mathcal{K};\mathcal{H})$ the adjoint.
Finally we let $\mathcal{B}_0(\mathcal{H};\mathcal{K})$ be the Banach space of compact operators with the uniform norm and $\mathcal{B}_1(\mathcal{H};\mathcal{K})$ the Banach space of trace class operators with the trace norm.

2.2. *Definitions and main properties*

We recall the definitions of RKH space and of kernel of positive type for vector valued functions. Let X be a set and $\mathcal{K}$ a Hilbert space.

Definition 2.1. A $\mathcal{K}$-valued *reproducing kernel Hilbert space* on X is a Hilbert space $\mathcal{H}$ such that

(1) the elements of $\mathcal{H}$ are functions from X to $\mathcal{K}$;
(2) for all $x \in X$ there exists a positive constant C_x such that

$$\|f(x)\|_{\mathcal{K}} \leq C_x \|f\|_{\mathcal{H}} \qquad \forall f \in \mathcal{H}. \tag{2.1}$$

[a]We only consider the case of complex Hilbert spaces, however almost all the results hold in the real case.

Definition 2.2. A $\mathcal{K}$-kernel of positive type on $X \times X$ is a map $\Gamma : X \times X \longrightarrow \mathcal{B}(\mathcal{K})$ such that, for all $N \in \mathbb{N}$, $x_1, \ldots, x_N \in X$ and $c_1, \ldots, c_N \in \mathbb{C}$,

$$\sum_{i,j=1}^{N} c_i \overline{c_j} \langle \Gamma(x_j, x_i)v, v \rangle_{\mathcal{K}} \geq 0 \qquad \forall v \in \mathcal{K}.$$

As in the scalar case any RKH space $\mathcal{H}$ canonically defines a $\mathcal{K}$-kernel of positive type. Indeed, given $x \in X$, (2.1) ensures that the evaluation map at x

$$\mathrm{ev}_x : \mathcal{H} \longrightarrow \mathcal{K} \qquad \mathrm{ev}_x(f) = f(x)$$

is a bounded operator and the *reproducing kernel* associated to $\mathcal{H}$ is defined as the map

$$\Gamma : X \times X \longrightarrow \mathcal{B}(\mathcal{K}) \qquad \Gamma(x, y) = \mathrm{ev}_x \mathrm{ev}_y^*.$$

Since for all $v \in \mathcal{K}$

$$\left\langle \sum_{i,j=1}^{N} c_i \overline{c_j} \Gamma(x_j, x_i)v, v \right\rangle_{\mathcal{K}} = \left\langle \sum_{i=1}^{N} c_i \mathrm{ev}_{x_i}^* v, \sum_{j=1}^{N} c_j \mathrm{ev}_{x_j}^* v \right\rangle_{\mathcal{K}} \geq 0,$$

the map Γ is $\mathcal{K}$-kernel of positive type.

To study the regularity properties of the elements of $\mathcal{H}$ it is useful to introduce the map

$$\gamma : X \longrightarrow \mathcal{B}(\mathcal{K}; \mathcal{H}), \qquad \gamma(x) = \mathrm{ev}_x^*,$$

so that $\Gamma(x, y) = \gamma(x)^* \gamma(y)$.

The following properties are simple consequences of the definition.

(1) The kernel Γ *reproduces* the value of a function $f \in \mathcal{H}$ at a point $x \in X$. Indeed, for all $x \in X$ and $v \in \mathcal{K}$

$$\mathrm{ev}_x^* v = \Gamma(\cdot, x)v$$

so that

$$\langle f(x), v \rangle_{\mathcal{K}} = \langle f, \Gamma(\cdot, x)v \rangle_{\mathcal{H}}. \tag{2.2}$$

The inclusion of $\mathcal{H}$ into $\mathcal{K}^X$ can be written as the linear operator $\imath_\Gamma : \mathcal{H} \to \mathcal{K}^X$

$$(\imath_\Gamma f)(x) = \gamma(x)^* f \qquad f \in \mathcal{H}, \, x \in X \tag{2.3}$$

and (2.1) is equivalent to the fact that $\imath_\Gamma$ is continuous from $\mathcal{H}$ into $\mathcal{K}^X$. This point of view is developed in full generality in [3] where $\mathcal{K}^X$ is replaced by any locally convex topological vector space.

(2) The set $\{\mathrm{ev}_x^* v \mid x \in X, \ v \in \mathcal{K}\}$ is total in $\mathcal{H}$, that is,

$$\left(\cup_{x \in X} \mathrm{Im}\, \mathrm{ev}_x^*\right)^\perp = \{0\}. \tag{2.4}$$

Indeed, if $f \in \left(\cup_{x \in X} \mathrm{Im}\, \mathrm{ev}_x^*\right)^\perp$, then $f \in (\mathrm{Im}\, \mathrm{ev}_x^*)^\perp = \mathrm{Ker}\, \mathrm{ev}_x$ for all $x \in X$, so that $f(x) = 0$ for all $x \in X$, i.e. $f = 0$.

(3) Since $\|\mathrm{ev}_x\|_{\mathcal{H},\mathcal{K}} = \|\mathrm{ev}_x^*\|_{\mathcal{K},\mathcal{H}} = \|\Gamma(x,x)\|_{\mathcal{K},\mathcal{K}}^{\frac{1}{2}}$

$$\|f(x)\|_{\mathcal{K}} \leq \|\Gamma(x,x)\|_{\mathcal{K},\mathcal{K}}^{\frac{1}{2}} \|f\|_{\mathcal{H}} \qquad x \in X f \in \mathcal{H}.$$

Hence, if a sequence $(f_n)_{n\in\mathbb{N}}$ converges to f in $\mathcal{H}$, it converges uniformly on any subset $C \subset X$ such that $\sup_{x\in C} \|\Gamma(x,x)\|_{\mathcal{K},\mathcal{K}}$ is finite. In particular, $(f_n)_{n\in\mathbb{N}}$ converges point-wise to f on X.

The next proposition proves that any $\mathcal{K}$-kernel Γ of positive type on X defines a unique $\mathcal{K}$-valued RKH space whose reproducing kernel is Γ. For the scalar case, it has been obtained by many authors, see [16,17,18,19,20,21] and, for a complete list of references, [15,22,23,4]. For the vector case see [2,3].

Proposition 2.1. *Given a $\mathcal{K}$-kernel of positive type $\Gamma : X \times X \to \mathcal{K}$, there is a unique $\mathcal{K}$-valued RKH space $\mathcal{H}$ on X with reproducing kernel Γ.*

Proof. We report the proof of [3], see also [15]. For all $x \in X$ and $v \in \mathcal{K}$, define the function $\Gamma_{x,v} = \Gamma(\cdot, x)v \in \mathcal{K}^X$ and

$$\mathcal{H}_0 = \mathrm{span}\ \{\Gamma_{x,v} \mid x \in X,\ v \in \mathcal{K}\} \subset \mathcal{K}^X.$$

If $f = \sum_i^n c_i \Gamma_{x_i,v_i}$ and $g = \sum_j^n d_j \Gamma_{y_j,w_j}$ are elements of $\mathcal{H}_0$, we have

$$\sum_j \overline{d_j} \langle f(y_j), w_j\rangle_{\mathcal{K}} = \sum_{ij} c_i \overline{d_j} \langle \Gamma(y_j, x_i) v_i, w_j\rangle_{\mathcal{K}} = \sum_i c_i \langle v_i, g(x_i)\rangle_{\mathcal{K}},$$

so the sesquilinear form on $\mathcal{H}_0 \times \mathcal{H}_0$

$$\langle f, g\rangle = \sum_{ij} c_i \overline{d_j} \langle \Gamma(y_j, x_i) v_i, w_j\rangle_{\mathcal{K}}$$

is well defined. The fact that Γ is a $\mathcal{K}$-kernel of positive type implies that $\langle f, f\rangle \geq 0$ for all $f \in \mathcal{H}_0$. The positivity ensures that the sesquilinear form is hermitian. Let now $x \in X$, the choice $g = \Gamma_{x,v}$ in the above definition gives

$$\langle f, \Gamma_{x,v}\rangle = \langle f(x), v\rangle_{\mathcal{K}} \qquad \forall x \in X$$

for all $f \in \mathcal{H}_0$.

We claim that the above sesquilinear form is a scalar product. If $f \in \mathcal{H}_0$, for all $v \in \mathcal{K}$ with $\|v\|_{\mathcal{K}} = 1$ by the Cauchy-Schwarz inequality we have

$$\begin{aligned} |\langle f(x), v\rangle_{\mathcal{K}}| &= |\,\langle f, \Gamma_{x,v}\rangle\,| \leq \langle f, f\rangle^{1/2} \langle \Gamma_{x,v}, \Gamma_{x,v}\rangle^{1/2} \\ &= \langle f, f\rangle^{1/2} \langle \Gamma(x,x)v, v\rangle_{\mathcal{K}}^{1/2} \leq \langle f, f\rangle^{1/2} \|\Gamma(x,x)\|_{\mathcal{K},\mathcal{K}}^{1/2}, \end{aligned}$$

implying

$$\|f(x)\|_{\mathcal{K}} \leq \langle f, f\rangle^{1/2} \|\Gamma(x,x)\|_{\mathcal{K},\mathcal{K}}^{1/2}.$$

Hence, if $\langle f, f\rangle = 0$, then $f = 0$ and, hence, $\langle\cdot,\cdot\rangle$ is a scalar product on $\mathcal{H}_0$.
Let $\mathcal{H}$ be the completion of $\mathcal{H}_0$ and define $\Gamma_x : \mathcal{K} \to \mathcal{H}$, $\Gamma_x v = \Gamma_{x,v}$, which is bounded by construction, and $A : \mathcal{H} \to \mathcal{K}^X$, $(Af)(x) = \Gamma_x^* f$. We claim that A is injective. Indeed, if $Af = 0$, then $f \in \ker \Gamma_x^* = \operatorname{Im} \Gamma_x{}^\perp$ for all $x \in X$ and, since the set $\cup_{x\in X} \operatorname{Im} \Gamma_x$ generates $\mathcal{H}_0$, $f = 0$. Due to the fact that A is injective, $\mathcal{H}$ can be canonically identified with a subspace of $\mathcal{K}^X$, so that $f(x) = \mathrm{ev}_x f = \Gamma_x^* f$ showing that $\mathcal{H}$ is a RKH space with reproducing kernel

$$\Gamma_{\mathcal{H}}(x, y)v = (\mathrm{ev}_y^* v)(x) = \Gamma(x, y)v.$$

The uniqueness of $\mathcal{H}$ is evident from the uniqueness of the completion. □

The above theorem holds also if $\mathcal{K}$ is a real vector space provided we add the assumption that Γ is symmetric, $\Gamma(x, y) = \Gamma(y, x)$. If $\mathcal{K}$ is a complex space, a kernel of positive type is always hermitian, $\Gamma(x, y)^* = \Gamma(y, x)$.

The following proposition shows another way to define a RKH space $\mathcal{H}$. This point of view is developed in [4].

Proposition 2.2. *Let $\widehat{\mathcal{H}}$ be an arbitrary Hilbert space and $A : \widehat{\mathcal{H}} \to \mathcal{K}^X$. The following facts are equivalent.*

(1) For any $x \in X$ there is a positive constant C_x satisfying

$$\|(Au)(x)\|_{\mathcal{K}} \le C_x \|u\|_{\widehat{\mathcal{H}}} \qquad u \in \widehat{\mathcal{H}}.$$

(2) There is a map $\gamma : X \to \mathcal{B}(\mathcal{K}; \widehat{\mathcal{H}})$ such that

$$(Au)(x) = \gamma(x)^* u \qquad u \in \widehat{\mathcal{H}},\ x \in X. \tag{2.5}$$

(3) The operator A is a partial isometry from $\widehat{\mathcal{H}}$ onto a RKH space $\mathcal{H} \subset \mathcal{K}^X$.

If one of the above conditions is satisfied, then

$$\ker A = (\cup_{x\in X} \operatorname{Im} \gamma(x))^\perp, \tag{2.6}$$

the reproducing kernel of $\mathcal{H}$ is

$$\Gamma(x, y) = \gamma(x)^* \gamma(y) \qquad x, y \in X$$

and the evaluation map at $x \in X$ is

$$\mathrm{ev}_x = (A\gamma(x))^* : \mathcal{H} \to \mathcal{K}. \tag{2.7}$$

Proof. Clearly 1. $\iff$ 2. and 3. $\Rightarrow$ 1.. We show 2. $\Rightarrow$ 3.. Indeed, (2.5) ensures that the kernel of A is $\mathcal{N} = \cap_{x\in X} \ker \gamma(x)^*$, which is closed. Moreover,

$$\mathcal{N} = \cap_{x\in X} \ker \gamma(x)^* = \cap_{x\in X} (\operatorname{Im} \gamma(x))^\perp = (\cup_{x\in X} \operatorname{Im} \gamma(x))^\perp ,$$

so (2.6) follows and the restriction of A to $\mathcal{N}^\perp$ is injective. Let $\mathcal{H} = \operatorname{Im} A$ as a vector space, and define on it the unique Hilbert space structure such that A becomes a partial isometry from $\widehat{\mathcal{H}}$ onto $\mathcal{H}$ and we denote this partial isometry again by A. We show that $\mathcal{H}$ is a RKH space. Since A^*A is the projection onto $\mathcal{N}^\perp$, given $f \in \mathcal{H}$ where $f = Au$ and $u \in \mathcal{N}^\perp$,

$$f(x) = (Au)(x) = \gamma(x)^* u = \gamma(x)^* A^* A u = (A\gamma(x))^* f \qquad x \in X,$$

so that the evaluation map $\mathrm{ev}_x = (A\gamma(x))^*$ is continuous and the reproducing kernel is given by

$$\Gamma(x, y) = \mathrm{ev}_x \mathrm{ev}_y^* = \gamma(x)^* A^* A \gamma(y) = \gamma(x)^* \gamma(y) \qquad x, y \in X,$$

since A^*A is the identity on $\operatorname{Im}\gamma(y)$. □

If the map γ is such that the set $\cup_{x\in X} \operatorname{Im}\gamma(x)$ is total in $\widehat{\mathcal{H}}$, then A is a unitary operator from $\widehat{\mathcal{H}}$ to the RKH space $\mathcal{H}$. It follows that, up to a unitary equivalence, there is a correspondence between $\mathcal{K}$-valued reproducing kernel Hilbert spaces $\mathcal{H}$, $\mathcal{K}$-kernels of positive type and operator valued maps $\gamma : X \to \mathcal{B}(\mathcal{K}; \mathcal{H})$ such that $\overline{\operatorname{span}}\, \{\gamma(x)v \mid x \in X,\, v \in \mathcal{K}\} = \mathcal{H}$. Hence the regularity properties of the elements of a RKH space can be characterized in terms of the corresponding properties of the inclusion $\imath_\Gamma$, the reproducing kernel Γ and the map γ. A first example is given by the following proposition, which discusses the problem of compactness of the inclusion.

Proposition 2.3. *With the above notation, the following facts are equivalent:*

(1) the inclusion $\imath_\Gamma$ is compact from $\mathcal{H}$ into $\mathcal{K}^X$;
(2) for all $x \in X$, $\Gamma(x, x) \in \mathcal{B}_0(\mathcal{K})$;
(3) for all $x, y \in X$, $\Gamma(x, y) \in \mathcal{B}_0(\mathcal{K})$;
(4) for all $x \in X$, $\gamma(x) \in \mathcal{B}_0(\mathcal{K}, \mathcal{H})$.

Proof. Since $\Gamma(x, y) = \gamma(x)^*\gamma(y)$ and, by polar decomposition, $\gamma(x) = U_x \Gamma(x, x)^{\frac{1}{2}}$ where U_x is a partial isometry, the equivalence between the last three conditions follows by the fact that the space of compact operators is an ideal and Schauder theorem [24].
We show that 1. $\iff$ 4.. The topology of $\mathcal{K}^X$ is the product topology and Tikhonov theorem implies that $\imath_\Gamma$ is compact if and only $f \mapsto f(x) = \gamma(x)^* f$ is a compact operator from $\mathcal{H}$ to $\mathcal{K}$. The claim follows again by Schauder theorem. □

We end the section recalling the correspondence between vector and scalar reproducing kernel Hilbert spaces [2,3].

Scalar RKH spaces correspond to the choice $\mathcal{K} = \mathbb{C}$ so that $\mathcal{B}(\mathbb{C}; \mathbb{C}) = \mathbb{C}$ and $\mathcal{B}(\mathbb{C}; \mathcal{H}) = \mathcal{H}$. Hence the reproducing kernel Γ takes value in $\mathbb{C}$ and is a function of positive type in the usual sense. Moreover $\gamma(x)$ is a vector $\gamma_x \in \mathcal{H}$ such that

$$\gamma_x = \Gamma(\cdot, x) \in \mathcal{H}$$

$$f(x) = \langle f, \gamma_x \rangle_{\mathcal{H}}$$
$$\Gamma(x,y) = \langle \gamma_y, \gamma_x \rangle_{\mathcal{H}}.$$

for all $x, y \in X$ and $f \in \mathcal{H}$.

The importance of the scalar case is due to the fact that the algebraic properties of any vector valued RKH space can be reduced to the corresponding properties of a scalar RKH space. Let $\mathcal{K}$ be a Hilbert space and $\mathcal{H}$ a $\mathcal{K}$-valued RKH space on X with reproducing kernel Γ.
We define the linear map $W : \mathcal{H} \longrightarrow \mathbb{C}^{X \times \mathcal{K}}$ as

$$(Wf)(x,v) = \langle f(x), v \rangle_{\mathcal{K}}$$

Proposition 2.4. *The map W is a unitary operator from $\mathcal{H}$ onto the scalar RKH space $\widetilde{\mathcal{H}}$ on $X \times \mathcal{K}$ whose reproducing kernel is*

$$\widetilde{\Gamma}(x,v;y,w) = \langle \Gamma(x,y)w, v \rangle_{\mathcal{K}} \qquad (x,v),\ (y,w) \in X \times \mathcal{K}.$$

Proof. By definition $(Wf)(x,v) = \langle f, \mathrm{ev}_x^* v \rangle_{\mathcal{H}}$ and (2.4) implies that W is injective, so the thesis follows applying Prop. 2.2 with $A = W$. □

The above construction is not as powerful as it can seem at first glance. If, for example, we are interested in the case in which the base space has some regularity property (e.g. local compactness) then it is not guaranteed that also $X \times \mathcal{K}$ shares this property. Usually in this case one resorts to the linearity of the second entry thus recovering the distinctive role played by $\mathcal{K}$. Moreover, no simplification arises in the proof of Prop. 2.1 and Prop. 2.2 considering the scalar case. Finally given a scalar RKH space $\widetilde{\mathcal{H}}$ on $X \times \mathcal{K}$, in general it does not exists a $\mathcal{K}$-valued RKH space $\mathcal{H}$ such that $W\mathcal{H} = \widetilde{\mathcal{H}}$, for a discussion see [2].

3. Measurability

In this section we assume that X is a measurable space and we characterize the conditions on the reproducing kernel ensuring that the elements of the corresponding RKH space are measurable functions. An assumption of separability will be essential.

3.1. *Notations*

Let $\mathcal{K}$ be a Hilbert space. A function $f : X \longrightarrow \mathcal{K}$ is *measurable* if it is measurable as a function from X to $\mathcal{K}$, $\mathcal{K}$ being endowed with its Borel σ-algebra; f is *weakly measurable* if each function $x \mapsto \langle f(x), v \rangle_{\mathcal{K}}$, $v \in \mathcal{K}$, is measurable. If $\mathcal{K}$ is separable, the two definitions are equivalent. Let $\mathcal{H}$ be another Hilbert space, a function $\gamma : X \longrightarrow \mathcal{B}(\mathcal{K};\mathcal{H})$, is *strongly* (resp. *weakly*) *measurable* if the map $x \mapsto \gamma(x)u$ is measurable (resp. weakly measurable) for all $u \in \mathcal{K}$. The function γ is *measurable* if it is measurable as a map taking values in the Banach space $\mathcal{B}(\mathcal{K};\mathcal{H})$ with its uniform norm. If both $\mathcal{H}$ and $\mathcal{K}$ are separable, weak and strong measurability of γ are equivalent and ensure that $x \mapsto \gamma(x)^*$ is strongly measurable, the function $x \mapsto \|\gamma(x)\|_{\mathcal{K},\mathcal{H}}$ is measurable and the map $X \ni x \mapsto \gamma(x)\phi(x) \in \mathcal{H}$ is measurable for any measurable function $\phi : X \longrightarrow \mathcal{K}$ [25].

3.2. *Main results*

Let X be a measurable space and $\mathcal{K}$ a *separable* Hilbert space. Let $\mathcal{H}$ be a $\mathcal{K}$-valued RKH space with reproducing kernel Γ.

The following result is an elementary consequence of the properties of measurable functions.

Proposition 3.1. *Assume that the RKH space $\mathcal{H}$ is* separable*. The following conditions are equivalent:*

(1) the elements of $\mathcal{H}$ are [weakly] measurable functions $f : X \to \mathcal{K}$;
(2) the map $\Gamma : X \times X \longrightarrow \mathcal{B}(\mathcal{K})$ is strongly [weakly] measurable;
(3) for all $x \in X$, the map $X \ni y \mapsto \Gamma(y,x) \in \mathcal{B}(\mathcal{K})$ is strongly [weakly] measurable;
(4) the map $\gamma : X \longrightarrow \mathcal{B}(\mathcal{K};\mathcal{H})$ is strongly [weakly] measurable.

Proof. Clearly, 2. ⇒ 3. and we show the other implications.

1. ⇒ 4. Given $f \in \mathcal{H}$ the map

$$x \mapsto \gamma(x)^* f = f(x)$$

is measurable by assumption. This means that γ^* and, hence, γ are strongly measurable.

4. ⇒ 2. By assumption the map $x \mapsto \gamma(x)v$ is measurable and $x \mapsto \gamma(x)^*$ is strongly measurable, so

$$(x,y) \mapsto \Gamma(x,y)v = \gamma(x)^*\gamma(y)v$$

is measurable, that is, Γ is strongly measurable.

3. ⇒ 1. By assumption, for all $x \in X$ and $v \in \mathcal{K}$ the functions $\mathrm{ev}_x^* v = \Gamma(\cdot,x)v \in \mathcal{H}$ are measurable. Let now $f \in \mathcal{H}$. By (2.4) there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in span $\{\mathrm{ev}_x^* v \mid x \in X,\, v \in \mathcal{K}\}$ converging to f in $\mathcal{H}$. The functions f_n are measurable and by (2.1) the sequence converges point-wise to f, so f is measurable. □

The following example (see [25]) shows that the separability of $\mathcal{H}$ is essential in the above proposition.

Example 3.1. Let $X = \mathbb{R}$ with its Borel σ-algebra $\Sigma(\mathbb{R})$. Fix a subset $A \subset \mathbb{R}$ such that $A \notin \Sigma(\mathbb{R})$. Let

$$\mathcal{H} = \left\{ f : \mathbb{R} \longrightarrow \mathbb{C} \mid f(x) = 0 \,\forall x \notin A,\ \sum_{x\in X} |f(x)|^2 < \infty \right\}$$

where $\sum_{x \in X}$ denotes the summability. The space $\mathcal{H}$ is a Hilbert space with respect to the scalar product

$$\langle f, g \rangle_{\mathcal{H}} = \sum_{x \in X} f(x)\overline{g(x)}$$

and $\mathcal{H}$ is not separable. It is a scalar RKH space on X with reproducing kernel

$$\Gamma(x,y) = \begin{cases} 1 & \text{if } x = y \in A \\ 0 & \text{otherwise} \end{cases}$$

Given $f \in \mathcal{H}$, the condition $\sum_{x \in X} |f(x)|^2 < +\infty$ implies that $f(x) = 0$ for all but denumerable number of $x \in X$, so f is measurable. However, since A is not measurable, Γ is not measurable, so that in the statement of the above proposition 1. does not imply 2. .

If Γ takes values in the space of compact operators, Prop. 3.1 can be improved, as shown in the next result.

Proposition 3.2. *Assume that $\mathcal{H}$ is separable. If $\Gamma(x,x) \in \mathcal{B}_0(\mathcal{K})$ for all $x \in X$, then the following facts are equivalent:*

(1) the elements of $\mathcal{H}$ are measurable functions;
(2) the map $\gamma : X \longrightarrow \mathcal{B}(\mathcal{K};\mathcal{H})$ is measurable;
(3) the map $\Gamma : X \times X \longrightarrow \mathcal{B}(\mathcal{K})$ is measurable.

Proof. Prop. 2.3 ensures that $\gamma(x) \in \mathcal{B}_0(\mathcal{K};\mathcal{H})$ and $\Gamma(x,y) \in \mathcal{B}_0(\mathcal{K})$ for all $x, y \in X$. Moreover Prop. 3.1 implies that 1. is equivalent to the fact that Γ or γ are strongly measurable. It follows that 3. $\Rightarrow$ 1. .

1) $\Rightarrow$ 2) Since $\mathcal{B}_0(\mathcal{K};\mathcal{H})^* = \mathcal{B}_1(\mathcal{H};\mathcal{K})$ is separable, we only need to prove that the map $x \mapsto \mathrm{tr}_{\mathcal{K}}(T\gamma(x))$ is measurable for every $T \in \mathcal{B}_1(\mathcal{H};\mathcal{K})$. In a basis $(e_n)_{n \in \mathbb{N}}$ of $\mathcal{K}$ we have

$$\mathrm{tr}_{\mathcal{K}}(T\gamma(x)) = \sum_n \langle T\gamma(x)e_n, e_n \rangle_{\mathcal{K}} = \sum_n \langle \gamma(x)e_n, T^* e_n \rangle_{\mathcal{H}}.$$

Since γ is strongly measurable, each term in the sum is a measurable function of x, hence $x \mapsto \mathrm{tr}_{\mathcal{K}}(T\gamma(x))$ is measurable, as claimed.

2) $\Rightarrow$ 3) Since the map $\mathcal{B}(\mathcal{K};\mathcal{H}) \times \mathcal{B}(\mathcal{K};\mathcal{H}) \ni (A,B) \mapsto A^*B \in \mathcal{B}(\mathcal{K})$ is continuous in the uniform norm topology, the map Γ is measurable by measurability of γ. □

4. Integrability

In this section we assume that X is a measurable space endowed with a σ-finite positive measure μ and we characterize the RKH spaces whose elements are p-integrable functions with respect to the measure μ for any $1 \leq p \leq \infty$. We always assume that the Hilbert space $\mathcal{K}$ is separable.

4.1. *Notations*

Given $1 \leq p < \infty$, $L^p(X, \mu; \mathcal{K})$ denotes the Banach space of (equivalence classes of) measurable functions $f : X \to \mathcal{K}$ such $\|f\|_{\mathcal{K}}^p$ is μ-integrable, whereas $L^\infty(X, \mu; \mathcal{K})$ is the Banach space of measurable functions $f : X \to \mathcal{K}$ that are μ-essentially bounded. The corresponding norm in $L^p(X, \mu; \mathcal{K})$ is denoted by $\|\cdot\|_p$. If $\mathcal{K} = \mathbb{C}$, we let $L^p(X, \mu) := L^p(X, \mu; \mathbb{C})$.
We let $q = \frac{p}{p-1}$ with the convention $\frac{p}{p-1} = \infty$ if $p = 1$, and $\frac{p}{p-1} = 1$ if $p = \infty$. We regard the spaces $L^p(X, \mu; \mathcal{K})$ and $L^q(X, \mu; \mathcal{K})$ in duality with respect to the pairing

$$\langle \phi, \psi \rangle_p = \int \langle \phi(x), \psi(x) \rangle_{\mathcal{K}} \, \mathrm{d}\mu(x) \qquad \phi \in L^p(X, \mu; \mathcal{K}), \; \psi \in L^q(X, \mu; \mathcal{K}).$$

Notice that the pairing is linear in the first argument and antilinear in the second. If $\mathcal{H}$ is a Hilbert space and $A : \mathcal{H} \longrightarrow L^p(X, \mu; \mathcal{K})$ is a bounded linear operator, we let $A^* : L^q(X, \mu; \mathcal{K}) \longrightarrow \mathcal{H}$ be the adjoint of A with respect to the pairing above. The operator A^* always exists and is bounded.
Finally we denote by $\int f(x) \mathrm{d}\mu(x)$ and by $\mathrm{w}{-}\int f(x) \mathrm{d}\mu(x)$ respectively the Bochner integral and the Pettis (weak) integral of a vector valued function f with respect to the measure μ.

4.2. *Bounded kernels*

We now extend to the vector valued case the definition of bounded kernel from the theory of integral operators [26]. The following definition holds for arbitrary kernels (not necessarily of positive type).

Definition 4.1. Let $\Gamma : X \times X \longrightarrow \mathcal{B}(\mathcal{K})$ be a strongly measurable function. Given $1 \leq p \leq \infty$, the kernel Γ is called *p-bounded* if

(1) for μ-almost all $x \in X$

$$\int \|\Gamma(x, y)^* v\|_{\mathcal{K}}^q \, \mathrm{d}\mu(y) < +\infty \qquad \forall v \in \mathcal{K};$$

(2) for all $\phi \in L^p(X, \mu; \mathcal{K})$, the map

$$X \ni x \longmapsto \mathrm{w}{-}\int \Gamma(x, y) \phi(y) \, \mathrm{d}\mu(y)$$

is in $L^q(X, \mu; \mathcal{K})$.

Condition 1. implies that, given $\phi \in L^p(X, \mu; \mathcal{K})$ and $v \in \mathcal{K}$, the function $y \mapsto \langle \Gamma(x, y) \phi(y), v \rangle_{\mathcal{K}}$ is integrable for μ-almost all $x \in X$, so Condition 2. makes sense. Indeed, since Γ is strongly measurable, $\langle \Gamma(x, \cdot) \phi(\cdot), v \rangle_{\mathcal{K}}$ is measurable. Moreover,

$$\begin{aligned} \int |\langle \Gamma(x, y) \phi(y), v \rangle_{\mathcal{K}}| \, \mathrm{d}\mu(y) &\leq \int \|\phi(y)\|_{\mathcal{K}} \, \|\Gamma(x, y)^* v\|_{\mathcal{K}} \, \mathrm{d}\mu(y) \\ &\leq \|\phi\|_p \, \|\Gamma(x, \cdot)^* v\|_q \, . \end{aligned} \tag{4.1}$$

Hence, the weak integral $\mathrm{w}\!-\!\int \Gamma(x,y)\phi(y)\,\mathrm{d}\mu(y)$ exists and is an element of $\mathcal{K}$ for μ-almost all $x \in X$.

In the above definition, *boundedness* refers to the fact that the operator L_Γ is bounded from $L^p(X,\mu;\mathcal{K})$ to $L^q(X,\mu;\mathcal{K})$, as shown in the next proposition. However one can show that the condition of p-bounded kernel is not strictly necessary to have a bounded integral operator (for a discussion see [26], where for $p = 2$ our definition coincides with the notion of Carleman bounded kernel).

Proposition 4.1. *Let $\Gamma : X \times X \longrightarrow \mathcal{B}(\mathcal{K})$ be a p-bounded kernel ($1 \le p \le \infty$). The operator $L_\Gamma : L^p(X,\mu;\mathcal{K}) \longrightarrow L^q(X,\mu;\mathcal{K})$*

$$(L_\Gamma\phi)(x) = \mathrm{w}\!-\!\int \Gamma(x,y)\phi(y)\,\mathrm{d}\mu(y) \qquad \text{for } \mu\text{−a.a. } x \in X \tag{4.2}$$

is bounded.

Proof. The definition of p-bounded kernel ensures that L_Γ is everywhere defined, so by the closed graph theorem it suffices to show that L_Γ is a closed operator. So, suppose that $\phi_n \to \phi$ in L^p and $L_\Gamma\phi_n \to \psi$ in L^q. Eq. (4.1) implies that

$$\left|\left\langle \left[\mathrm{w}\!-\!\int \Gamma(x,y)\left(\phi_n(y) - \phi(y)\right)\,\mathrm{d}\mu(y)\right], v\right\rangle_{\mathcal{K}}\right| \le \|\phi_n - \phi\|_p \,\|\Gamma(x,\cdot)^* v\|_q \quad \forall v \in \mathcal{K}$$

for μ-almost all $x \in X$, so that the weak limit of $(L_\Gamma\phi_n)(x)$ is $(L_\Gamma\phi)(x)$ μ-almost everywhere. By the uniqueness of the limit, $\psi = L_\Gamma\phi$ so that the graph of L_Γ is closed, as claimed. □

The following corollary gives a sufficient condition to have a p-bounded kernel.

Corollary 4.1. *Let $\Gamma : X \times X \to \mathcal{B}(\mathcal{K})$ be a strongly measurable function such that*

$$\int \|\Gamma(x,y)\|^q_{\mathcal{K},\mathcal{K}}\ \mathrm{d}(\mu\otimes\mu)(x,y) < +\infty, \tag{4.3}$$

then Γ is a p-bounded kernel and

$$(L_\Gamma\phi)(x) = \int \Gamma(x,y)\phi(y)\,\mathrm{d}\mu(y).$$

Proof. Notice that, since $\mathcal{K}$ is separable, the map $x \mapsto \|\Gamma(x,y)\|_{\mathcal{K},\mathcal{K}}$ is measurable, so (4.3) makes sense. Assume for example $p > 1$. Since $\|\Gamma(x,y)^* v\|_{\mathcal{K}} \le \|\Gamma(x,y)\|_{\mathcal{K},\mathcal{K}} \|v\|_{\mathcal{K}}$, Fubini theorem ensures that, for μ-almost all $x \in X$, the function $y \mapsto \|\Gamma(x,y)^* v\|_{\mathcal{K}}$ is in L^q for all $v \in \mathcal{K}$, so that Condition 1 of Definition 4.1 follows. We now show that, if $\phi \in L^p(X,\mu;\mathcal{K})$, the map

$$X \ni y \mapsto \Gamma(x,y)\phi(y) \in \mathcal{K}$$

is integrable and $L_\Gamma \phi \in L^q(X,\mu;\mathcal{K})$. Indeed,

$$\begin{aligned}
&\int \left\| \int \Gamma(x,y)\phi(y)\,\mathrm{d}\mu(y) \right\|_{\mathcal{K}}^q \mathrm{d}\mu(x) \\
&\quad \le \int \left(\int \|\Gamma(x,y)\|_{\mathcal{K},\mathcal{K}} \|\phi(y)\|_{\mathcal{K}} \,\mathrm{d}\mu(y) \right)^q \mathrm{d}\mu(x) \\
&\quad \le \int \left(\int \|\Gamma(x,y)\|_{\mathcal{K},\mathcal{K}}^q \,\mathrm{d}\mu(y) \right) \left(\int \|\phi(y)\|_{\mathcal{K}}^p \,\mathrm{d}\mu(y) \right)^{\frac{1}{p-1}} \mathrm{d}\mu(x) \\
&\quad = \|\phi\|_p^q \int \|\Gamma(x,y)\|_{\mathcal{K},\mathcal{K}}^q \,\mathrm{d}(\mu\otimes\mu)(x,y) < \infty.
\end{aligned}$$

The case $p = 1$ is treated in a similar manner. □

4.3. *Main results*

Let X be a measurable space endowed with a σ-finite measure μ and $\mathcal{K}$ a separable Hilbert space. Let $\mathcal{H}$ be a $\mathcal{K}$-valued RKH space with reproducing kernel Γ.

Proposition 4.2. *Assume that $\mathcal{H}$ is a separable RKH space of measurable functions. Given $1 \le p \le \infty$, the following conditions are equivalent.*

(1) the elements of $\mathcal{H}$ belongs to $L^p(X,\mu;\mathcal{K})$;
(2) the reproducing kernel Γ of $\mathcal{H}$ is q-bounded with $q = \frac{p}{p-1}$.

If one of the above conditions holds, then

(i) the inclusion $\imath_\Gamma : \mathcal{H} \to L^p(X,\mu;\mathcal{K})$ is a bounded linear map;
(ii) its adjoint $\imath_\Gamma^* : L^q(X,\mu;\mathcal{K}) \to \mathcal{H}$ is given by

$$\imath_\Gamma^* \phi = \mathrm{w}\!-\!\int \gamma(x)\phi(x)\,\mathrm{d}\mu(x). \tag{4.4}$$

(iii) $\imath_\Gamma \imath_\Gamma^* = L_\Gamma$, where L_Γ is the integral operator of kernel Γ given by (4.2).

Proof.

1. ⇒ 2. We prove that the inclusion $\imath_\Gamma : \mathcal{H} \to L^p(X,\mu;\mathcal{K})$ is bounded. If $f_n \to f$ in $\mathcal{H}$ is such that $\imath_\Gamma f_n \to \phi$ in L^p, then $(\imath_\Gamma f_n)(x) \to f(x)$ for all $x \in X$, and so $\phi = \imath_\Gamma f$ by the uniqueness of the limit. The closed graph theorem ensures that $\imath_\Gamma$ is continuous.

We show (4.4). Given $\phi \in L^q(X,\mu;\mathcal{K})$, for all $f \in \mathcal{H}$

$$\left\langle \imath_\Gamma^*\phi, f \right\rangle_{\mathcal{H}} = \left\langle \phi, \imath_\Gamma f \right\rangle_p = \int \left\langle \phi(x), f(x) \right\rangle_{\mathcal{K}} \mathrm{d}\mu(x) = \int \left\langle \gamma(x)\phi(x), f \right\rangle_{\mathcal{H}} \mathrm{d}\mu(x).$$

It follows that the map $x \mapsto \gamma(x)\phi(x)$ is weakly integrable and $\imath_\Gamma^*\phi = \mathrm{w}\!-\!\int \gamma(x)\phi(x)\,\mathrm{d}\mu(x)$.

We now show that Γ is a q-bounded kernel. For all $x \in X$ and $v \in \mathcal{K}$, the function $\Gamma(x, \cdot)^* v = \mathrm{ev}_x^* v$ belongs to $\mathcal{H}$ and, by assumption, is p-integrable, so that condition 1 of Definition 4.1 is satisfied. Moreover, if $\phi \in L^q(X, \mu; \mathcal{K})$

$$\mathrm{w-}\!\int \Gamma(x, y)\phi(y)\ \mathrm{d}\mu(y) = \mathrm{w-}\!\int \gamma(x)^* \gamma(y)\phi(y)\ \mathrm{d}\mu(y) = \mathrm{ev}_x \imath_\Gamma^* \phi = (\imath_\Gamma \imath_\Gamma^* \phi)(x)$$

for μ-almost all x. Since $\imath_\Gamma \imath_\Gamma^* \phi \in L^p(X, \mu; \mathcal{K})$, condition 2 of Definition 4.1 holds and, in particular, $\imath_\Gamma \imath_\Gamma^* = L_\Gamma$.

2. ⇒ 1. Since μ is σ-finite, there is an increasing sequence $(X_n)_{n\in\mathbb{N}}$ of measurable subsets of X such that $\mu(X_n) < +\infty$ and $\cup_{n\in\mathbb{N}} X_n = X$. Given $n \in \mathbb{N}$, let

$$C_n = \{x \in X_n \mid \|\gamma(x)\|_{\mathcal{K},\mathcal{H}} \le n\}.$$

The subsets C_n are measurable, $C_n \subset C_{n+1}$, $\cup_{n\in\mathbb{N}} C_n = X$, and $\mu(C_n) < \infty$.

Let $f \in \mathcal{H}$. Define

$$f_n(x) = \chi_{C_n}(x) f(x),$$

χ_{C_n} being the characteristic function of the set C_n. Then

$$\|f_n(x)\|_{\mathcal{K}} \le \chi_{C_n}(x) \|\gamma(x)^* f\|_{\mathcal{K}} \le n\chi_{C_n}(x) \|f\|_{\mathcal{H}},$$

so $f_n \in L^p(X, \mu; \mathcal{K})$. If $\phi \in L^q(X, \mu; \mathcal{K})$, we have

$$\begin{aligned} \langle f_n, \phi\rangle_p &= \int \langle \chi_{C_n}(x) f(x), \phi(x)\rangle_{\mathcal{H}}\, \mathrm{d}\mu(x) \\ &= \left\langle f, \int \chi_{C_n}(x)\gamma(x)\phi(x)\, \mathrm{d}\mu(x)\right\rangle_{\mathcal{H}}. \end{aligned}$$

The norm of the second term in the scalar product has the following upper bound

$$\begin{aligned} &\left\| \int \chi_{C_n}(x)\gamma(x)\phi(x)\, \mathrm{d}\mu(x)\right\|_{\mathcal{H}}^2 \\ &\quad= \int \left(\int \langle \chi_{C_n}(y)\gamma(y)\phi(y), \chi_{C_n}(x)\gamma(x)\phi(x)\rangle_{\mathcal{H}}\ \mathrm{d}\mu(y)\right) \mathrm{d}\mu(x) \\ &\quad= \int \left\langle \mathrm{w-}\!\int \chi_{C_n}(y)\Gamma(x, y)\phi(y)\, \mathrm{d}\mu(y), \chi_{C_n}(x)\phi(x)\right\rangle_{\mathcal{K}} \mathrm{d}\mu(x) \\ &\quad= \langle L_\Gamma(\chi_{C_n}\phi), (\chi_{C_n}\phi)\rangle_p \le \|L_\Gamma\|_{q,p} \|\phi\|_q^2, \end{aligned}$$

since by assumption and Prop. 4.1 L_Γ is an everywhere defined bounded operator. We thus have

$$|\langle f_n, \phi\rangle_p| \le \|f\|_{\mathcal{H}} \|L_\Gamma\|_{q,p}^{1/2} \|\phi\|_q. \tag{4.5}$$

For $1 \le p < \infty$, we take the supremum over $\phi \in L^q = (L^p)^*$ with $\|\phi\|_q \le 1$ and we get

$$\|f_n\|_p \le \|L_\Gamma\|_{q,p}^{1/2} \|f\|_{\mathcal{H}}.$$

By monotone convergence theorem, this implies $\|f\|_{\mathcal{K}} \in L^p(X,\mu)$, so that $f \in L^p(X,\mu;\mathcal{K})$. For $p=\infty$, (4.5) implies that $L^1 \ni \phi \mapsto \langle f_n, \phi\rangle_p \in \mathbb{C}$ is continuous so $f_n \in L^\infty$ and

$$\|f_n\|_\infty \leq \|L_\Gamma\|_{q,p}^{1/2} \|f\|_{\mathcal{H}}.$$

This implies $f \in L^\infty$. □

The fact that the elements of $L^p(X,\mu;\mathcal{K})$ are equivalence classes implies that, in general, the inclusion operator $\imath_\Gamma : \mathcal{H} \longrightarrow L^p(X,\mu;\mathcal{K})$ is not injective. The following result characterizes $\operatorname{Ker} \imath_\Gamma$ under the assumption that $\gamma(x)$ is compact.

Proposition 4.3. *Let $\mathcal{H}$ be a separable $\mathcal{K}$-valued RKH space with a q-bounded reproducing kernel Γ. Assume that $\Gamma(x,x) \in \mathcal{B}_0(\mathcal{K})$ for all $x \in X$ and define*

$$\mathcal{S} = \{x \in X \mid \mu(B_{x,\epsilon}) > 0 \ \forall \epsilon > 0\},$$

where $B_{x,\epsilon} = \{y \in X \mid \|\Gamma(y,y)+\Gamma(x,x)-\Gamma(x,y)-\Gamma(y,x)\|_{\mathcal{K},\mathcal{K}} < \epsilon^2\}$. Let $\imath_\Gamma : \mathcal{H} \to L^p(X,\mu;\mathcal{K})$ be the inclusion, then

$$\operatorname{Ker} \imath_\Gamma = \{f \in \mathcal{H} \mid f(x) = 0 \quad \forall x \in \mathcal{S}\}. \tag{4.6}$$

Proof. First of all, notice that the definition of Γ gives that

$$B_{x,\epsilon} = \{y \in X \mid \|\gamma(y)-\gamma(x)\|_{\mathcal{K},\mathcal{H}} < \epsilon\},$$

which is measurable since γ is measurable by Propositions 2.3 and 3.2. Since $\mathcal{H}$ and $\mathcal{K}$ are separable, the space $\mathcal{B}_0(\mathcal{K};\mathcal{H})$ is separable. Observing that $\gamma(x) \in \mathcal{B}_0(\mathcal{K};\mathcal{H})$ for all $x \in X$, it follows there is a denumerable family $\{B_{x_n,\epsilon_n} \mid n \in I\}$ such that, if $x \in X$ and $\epsilon > 0$,

$$B_{x,\epsilon} = \cup_{n\in J} B_{x_n,\epsilon_n}$$

where $J \subset I$. Hence $X \setminus \mathcal{S} = \{x \in X \mid \exists \epsilon > 0 \ \mu(B_{x,\epsilon}) = 0\}$ has null measure being the denumerable union of null sets.

Let now $f \in \mathcal{H}$ such that $f(x) = 0$ for all $x \in \mathcal{S}$, then $f = 0$ in $L^p(X,\mu)$. Conversely, suppose there exists $x \in \mathcal{S}$ such that $f(x) \neq 0$, that is, $\gamma(x)^* f \neq 0$. For ϵ sufficiently small, we have that $\gamma(y)^* f \neq 0$ for all $y \in B_{x,\epsilon}$. In particular, $f(y) = \gamma(y)^* f \neq 0$ for all $y \in B_{x,\epsilon}$, which has nonzero measure by definition of $\mathcal{S}$. It follows that $f \neq 0$ in $L^p(X,\mu)$. □

For $p = 2$ we can compute $\imath_\Gamma^* \imath_\Gamma$, which is known as frame operator in the context of frame theory (see, for example, [27]).

Corollary 4.2. *Let $\mathcal{H}$ be a separable $\mathcal{K}$-valued RKH space whose elements are square integrable functions. Then*

$$\imath_\Gamma^* \imath_\Gamma = \text{w-}\int \gamma(x)\gamma(x)^* \, d\mu(x). \tag{4.7}$$

In particular, the following conditions are equivalent:

(i) $\imath_\Gamma$ *is a Hilbert-Schmidt operator;*
(ii) $\Gamma(x,x)$ *is a trace class operator for almost all* $x \in X$ *and*

$$\int \mathrm{tr}_{\mathcal{K}}\, \Gamma(x,x)\ \mathrm{d}\mu(x) < +\infty;$$

(iii) L_Γ *is a trace class operator.*

If one of the above conditions holds, the integral in (4.4) *converges in norm and the integral in* (4.7) *converges in trace norm.*

Proof. Eq. (4.7) follows from (4.4) and (2.3). We now prove that $(i) \iff (ii)$. The separability of $\mathcal{K}$ and the strong measurability of Γ ensure that $X \ni x \mapsto \mathrm{tr}_{\mathcal{K}}\, \Gamma(x,x) \in [0,+\infty]$ is measurable. Let $(f_n)_{n\in\mathbb{N}}$ be a Hilbert basis of $\mathcal{H}$. Since $\imath_\Gamma^* \imath_\Gamma$ is a positive operator and $x \mapsto |\langle \gamma(x)\gamma(x)^* f_n, f_n\rangle_{\mathcal{H}}|^2$ are positive functions the monotone convergence theorem gives

$$\begin{aligned}
\mathrm{tr}_{\mathcal{H}}\, \imath_\Gamma^* \imath_\Gamma &= \sum_n \int \langle \gamma(x)\gamma(x)^* f_n, f_n\rangle_{\mathcal{H}}\, \mathrm{d}\mu(x) \\
&= \int \mathrm{tr}_{\mathcal{H}}\, \gamma(x)\gamma(x)^*\, \mathrm{d}\mu(x) \\
&= \int \mathrm{tr}_{\mathcal{K}}\, \gamma(x)^*\gamma(x)\, \mathrm{d}\mu(x) \\
&= \int \mathrm{tr}_{\mathcal{K}}\, \Gamma(x,x)\, \mathrm{d}\mu(x).
\end{aligned}$$

The equivalence of (i) and (ii) follows. The equivalence between (i) and (iii) is trivial since $L_\Gamma = \imath_\Gamma \imath_\Gamma^*$.

Now we prove the statements about (4.4) and (4.7). Since $\gamma : X \longrightarrow \mathcal{B}(\mathcal{K};\mathcal{H})$ is strongly measurable by Proposition 3.1, for $\phi \in L^2(X,\mu;\mathcal{K})$ the map $x \mapsto \gamma(x)\phi(x)$ is measurable. Moreover,

$$\begin{aligned}
\|\gamma(x)\phi(x)\|_{\mathcal{H}}^2 = \langle \Gamma(x,x)\phi(x), \phi(x)\rangle_{\mathcal{K}} &\le \|\Gamma(x,x)\|_{\mathcal{K},\mathcal{K}}\, \|\phi(x)\|_{\mathcal{K}}^2 \\
&\le \mathrm{tr}_{\mathcal{K}}\, \Gamma(x,x)\, \|\phi(x)\|_{\mathcal{K}}^2 .
\end{aligned}$$

Condition (ii) ensures that $x \mapsto \gamma(x)\phi(x)$ is in $L^1(X,\mu;\mathcal{K})$.

We come to (4.7). The strong measurability of γ ensures that $x \mapsto \gamma(x)\gamma(x)^*$ is measurable as a map from X into $\mathcal{B}_1(\mathcal{H})$. Indeed, since $\mathcal{B}_1(\mathcal{H})$ is separable, it is enough to show that for all $B \in \mathcal{B}(\mathcal{H}) = \mathcal{B}_1(\mathcal{H})^*$ the map $x \mapsto \mathrm{tr}_{\mathcal{H}}\,(B\gamma(x)\gamma(x)^*)$ is measurable. Indeed,

$$\begin{aligned}
\mathrm{tr}_{\mathcal{H}}\,(B\gamma(x)\gamma(x)^*) &= \sum_n \langle B\gamma(x)\gamma(x)^* f_n, f_n\rangle_{\mathcal{H}} \\
&= \sum_n \langle \gamma(x)^* f_n, \gamma(x)^* B^* f_n\rangle_{\mathcal{K}}.
\end{aligned}$$

and the maps $x \mapsto \gamma(x)^* f_n$ and $\gamma(x)^* B^* f_n$ are measurable. Since $\gamma(x)\gamma(x)^*$ is a positive operator, its norm in $\mathcal{B}_1(\mathcal{H})$ is $\mathrm{tr}_{\mathcal{H}}\,(\gamma(x)\gamma(x)^*) = \mathrm{tr}_{\mathcal{K}}\,\Gamma(x,x)$. Convergence of the integral (4.7) in $\mathcal{B}_1(\mathcal{H})$ then follows immediately from condition (ii). □

4.4. *Compactness*

We now discuss the problem of the compactness of the inclusion of the RKH space $\mathcal{H}$ into $L^p(X,\mu;\mathcal{K})$. If $\mathcal{K}=\mathbb{C}$, the next proposition is an easy consequence of a well known fact in the framework of integral operators (see, for example, [26] for a complete discussion about the compactness of integral operators in $L^2(X,\mu)$).

Proposition 4.4. *Suppose that $\mathcal{H}$ is a separable RKH space such that $\Gamma(x,x)\in\mathcal{B}_0(\mathcal{K})$ for all $x\in X$ and $x\mapsto\Gamma(x,x)$ is measurable. Let $1\le p<\infty$, if*

$$\int_X \|\Gamma(x,x)\|_{\mathcal{K},\mathcal{K}}^{p/2}\ \mathrm{d}\mu(x)<+\infty,$$

then $\mathcal{H}\subset L^p(X,\mu;\mathcal{K})$ and the inclusion $\imath_\Gamma:\mathcal{H}\to L^p(X,\mu;\mathcal{K})$ is compact.

Proof. Prop. 3.1 ensures that the elements of $\mathcal{H}$ are measurable functions. Moreover, the map $x\mapsto\|\gamma(x)\|_{\mathcal{K},\mathcal{H}}=\|\Gamma(x,x)\|_{\mathcal{K},\mathcal{K}}^{1/2}$ is in $L^p(X,\mu)$. For $f\in\mathcal{H}$, we have $\|f(x)\|_{\mathcal{K}}\le\|\gamma(x)\|_{\mathcal{K},\mathcal{H}}\|f\|_{\mathcal{H}}$, thus showing that $f\in L^p(X,\mu;\mathcal{K})$. If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{H}$ which converges weakly to 0, then $f_n(x)=\gamma(x)^*f_n\to0$ in $\mathcal{K}$ for all $x\in X$, since $\gamma(x)$ is compact by Prop. 2.3. Since $\|f_n(x)\|_{\mathcal{K}}\le\|\gamma(x)\|_{\mathcal{K},\mathcal{H}}\|f_n\|_{\mathcal{H}}$ and $\sup_n\|f_n\|_{\mathcal{H}}<\infty$, it follows by dominated convergence theorem that $f_n\to0$ in $L^p(X,\mu;\mathcal{K})$. This shows that $\imath_\Gamma$ maps weakly convergent sequences into norm convergent sequences, so $\imath_\Gamma$ is compact. □

Suppose μ is a finite measure. If $\mathcal{H}$ is a separable RKH space of scalar valued measurable functions, and $\mathcal{H}\subset L^1(X,\mu)$, the inclusion $\imath_\Gamma:\mathcal{H}\longrightarrow L^1(X,\mu)$ is compact as an easy consequence of a result due to [28]. In the general case, with no restriction on μ and the dimension of $\mathcal{K}$, we have the following fact.

Proposition 4.5. *Suppose that $\mathcal{H}$ is a separable RKH space such that $\mathcal{H}\subset L^1(X,\mu;\mathcal{K})$. If $\Gamma(x,x)\in\mathcal{B}_0(\mathcal{K})$ for all $x\in X$, then the inclusion $\imath_\Gamma:\mathcal{H}\to L^1(X,\mu;\mathcal{K})$ is compact.*

Proof. We divide the proof in three steps.

(1) Suppose that there exists a disjoint sequence of measurable subsets $(E_j)_{j\in\mathbb{N}}$, with $\mu(E_j)<\infty$, and operators $\gamma_j\in\mathcal{B}_0(\mathcal{K};\mathcal{H})$ such that

$$\gamma(x)=\sum_{j\in\mathbb{N}}\chi_{E_j}(x)\gamma_j\quad\forall x\in X.\tag{4.8}$$

The condition that $\mathcal{H}\subset L^1(X,\mu;\mathcal{K})$ implies that for all $f\in\mathcal{H}$

$$\sum_j\mu(E_j)\left\|\gamma_j^*f\right\|_{\mathcal{K}}=\int\|\gamma(x)^*f\|_{\mathcal{K}}\ \mathrm{d}\mu(x)=\int\|f(x)\|_{\mathcal{K}}\ \mathrm{d}\mu(x)=\|f\|_1,$$

i.e. the sequence $\left(\mu(E_j)\gamma_j^*f\right)_{j\in\mathbb{N}}$ is in $\ell^1(\mathcal{K})$. The linear operator $T:\mathcal{H}\to\ell^1(\mathcal{K})$

$$(Tf)_j=\mu(E_j)\gamma_j^*f$$

is bounded since

$$\|Tf\|_{\ell^1(\mathcal{K})} = \|f\|_1 \leq \|\imath_\Gamma\|_{\mathcal{H},1} \|f\|_{\mathcal{H}} \tag{4.9}$$

Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{H}$ converging weakly to 0. For all $j \in \mathbb{N}$, by compactness of γ_j, $(Tf_n)(j) \to 0$ in $\mathcal{K}$. Moreover, by continuity of T, $Tf_n \to 0$ weakly. Thus, by Lemma Appendix A.1 in the appendix, $\|Tf_n\|_{\ell^1(\mathcal{K})} \to 0$. This fact and (4.9) show that $\imath_\Gamma$ maps weakly convergent sequences in $\mathcal{H}$ into norm convergent sequences in L^1, hence $\imath_\Gamma$ is compact.

(2) Now, without making any assumption on the map γ, we claim that there exist maps $\gamma_1, \gamma_2 : X \longrightarrow \mathcal{B}(\mathcal{K};\mathcal{H})$ such that: (*i*) γ_1 is as in (4.8); (*ii*) $\gamma_2(x) \in \mathcal{B}_0(\mathcal{K};\mathcal{H})$ for all x, and the map $x \mapsto \|\gamma_2(x)\|_{\mathcal{K},\mathcal{H}}$ is in L^1; (*iii*) $\gamma = \gamma_1 + \gamma_2$.

To this aim, let $(X_n)_{n\in\mathbb{N}}$ be an increasing sequence of measurable subsets of X such that $\mu(X_n) < \infty$ and $X = \bigcup_n X_n$. For all $n \in \mathbb{N}$ define by induction

$$A_0 = \emptyset, \qquad A_n = \left\{ x \in X_n \mid x \notin A_{n-1} \text{ and } \|\gamma(x)\|_{\mathcal{K},\mathcal{H}} \leq n \right\}.$$

Each A_n is measurable, $\mu(A_n) < \infty$ for all n, $A_n \cap A_m = \emptyset$ if $n \neq m$, and $\bigcup_n A_n = X$. By Prop. 2.3, $\gamma(x) \in \mathcal{B}_0(\mathcal{K};\mathcal{H})$ for all x, and the map $\gamma : X \longrightarrow \mathcal{B}_0(\mathcal{K};\mathcal{H})$ is measurable by Prop. 3.2. The function $\chi_{A_n}\gamma$ is thus integrable as a map taking values in $\mathcal{B}_0(\mathcal{K};\mathcal{H})$, so there is a step function $\eta_n : X \to \mathcal{B}_0(\mathcal{K};\mathcal{H})$ supported in A_n such that

$$\int_{A_n} \|\gamma(x) - \eta_n(x)\|_{\mathcal{K},\mathcal{H}} \, \mathrm{d}\mu(x) \leq \frac{1}{2^n}.$$

The map

$$\gamma_1 = \sum_{n\in\mathbb{N}} \chi_{A_n}\eta_n$$

(which is well defined since the sets in the sequence $(A_n)_{n\in\mathbb{N}}$ are disjoint) is as in (4.8). Let $\gamma_2 = \gamma - \gamma_1$, then $\gamma_2(x) \in \mathcal{B}_0(\mathcal{K};\mathcal{H})$ for all x, and

$$\int \|\gamma_2(x)\|_{\mathcal{K},\mathcal{H}} \, \mathrm{d}\mu(x) = \sum_n \int_{A_n} \|\gamma(x) - \eta_n(x)\|_{\mathcal{K},\mathcal{H}} \, \mathrm{d}\mu(x) \leq \sum_n \frac{1}{2^n} = 2,$$

so that the claim follows.

(3) Let $\gamma = \gamma_1 + \gamma_2$ be as in 2.. For $i = 1, 2$, define $\Gamma_i(x, y) = \gamma_i(x)^*\gamma_i(y)$, and let $\mathcal{H}_i$ be the RKH spaces with reproducing kernel Γ_i. By Prop. 2.2, we have two partial isometries

$$A_i : \mathcal{H} \longrightarrow \mathcal{H}_i, \qquad (A_i f)(x) = \gamma_i(x)^* f.$$

If $f \in \mathcal{H}$, then

$$\begin{aligned} \int \|(A_1 f)(x)\|_{\mathcal{K}} \, \mathrm{d}\mu(x) &= \int \left\|(\gamma(x) - \gamma_2(x))^* f\right\|_{\mathcal{K}} \mathrm{d}\mu(x) \\ &\leq \int \|f(x)\|_{\mathcal{K}} \, \mathrm{d}\mu(x) + \|f\|_{\mathcal{H}} \int \|\gamma_2(x)\|_{\mathcal{K},\mathcal{H}} \, \mathrm{d}\mu(x) < \infty, \end{aligned}$$

which shows that $\mathcal{H}_1 \subset L^1(X, \mu; \mathcal{K})$. Using the expression (2.7) for the evaluation map in $\mathcal{H}_1$, we see that $\mathcal{H}_1$ is as in step 1), hence the inclusion $\imath_{\Gamma_1} : \mathcal{H}_1 \longrightarrow L^1$ is compact. On the other hand, by Prop. 4.4 $\mathcal{H}_2 \subset L^1(X, \mu; \mathcal{K})$ and the inclusion $\imath_{\Gamma_2} : \mathcal{H}_2 \longrightarrow L^1$ is compact. In conclusion, $\imath_\Gamma = \imath_{\Gamma_1} A_1 + \imath_{\Gamma_2} A_2$ is compact. □

It is easy to check that the requirement $\Gamma(x, x) \in \mathcal{B}_0(\mathcal{K})$ for all x is essential in the above proposition, as illustrated by the following simple example.

Example 4.1. Suppose $\mathcal{K}$ is infinite dimensional and choose X to be a single point $\{x\}$. The space of functions $\mathcal{K}^X$, naturally identified with $\mathcal{K}$, is a RKH space of $\mathcal{K}$ valued functions with reproducing kernel $\Gamma(x, x) = I$. Letting μ be a non-null measure on X, $L^p(X, \mu; \mathcal{K})$ is identified as a Banach space with $\mathcal{K}^X$ endowed with this structure of RKH space. But the identity map $\mathcal{K} \simeq \mathcal{K}^X \to L^p(X, \mu; \mathcal{K}) \simeq \mathcal{K}$ is not compact.

5. Continuity

In this section we assume that X is a topological space and we characterize the RKH spaces whose elements are continuous functions. For the scalar case see [3].

5.1. *Notations*

Let X be a locally compact topological space and $\mathcal{K}$ a Hilbert space (in this section we do not assume that $\mathcal{K}$ is separable). We denote by $\mathcal{C}\,(X; \mathcal{K})$ the vector space of continuous functions $f : X \to \mathcal{K}$. The space $\mathcal{C}\,(X; \mathcal{K})$ is endowed with the topology of compact convergence, so that a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{C}\,(X; \mathcal{K})$ converges to a function f if

$$\lim_{n \to +\infty} \sup_{x \in C} \|f_n(x) - f(x)\|_{\mathcal{K}} = 0$$

for every compact set C in X.
If $\mathcal{H}$ is another Hilbert space, a map $\gamma : X \to \mathcal{B}(\mathcal{K}; \mathcal{H})$ is *strongly continuous* if the function $x \mapsto \gamma(x)v$ is continuous from X to $\mathcal{H}$ for all $v \in \mathcal{K}$.

5.2. *Main result*

Let X be a locally compact topological space and $\mathcal{K}$ a Hilbert space. Let $\mathcal{H}$ be a $\mathcal{K}$-valued RKH space with reproducing kernel Γ.

Proposition 5.1. *The following facts are equivalent:*

(1) the elements of $\mathcal{H}$ are continuous functions;
(2) the kernel Γ is locally bounded and, for all $x \in X$, the map $\Gamma(\cdot, x)$ is strongly continuous.

If one of the above conditions holds, the inclusion operator $\imath_\Gamma : \mathcal{H} \to \mathcal{C}(X;\mathcal{K})$ *is continuous.*

Proof.

1) ⇒ 2) Given $x \in X$ and $v \in \mathcal{K}$, by definition

$$\Gamma(\cdot,x)v = \mathrm{ev}_x^* v \in \mathcal{H} \subset \mathcal{C}(X;\mathcal{K}).$$

so that $\Gamma(\cdot,x)$ is strongly continuous. We show that Γ is locally bounded. Given $x_0 \in X$, let C be a compact neighbourhood of x_0 (C exists since X is locally compact). For any $f \in \mathcal{H}$, the continuity of f ensures that

$$\sup_{x\in C} \|\mathrm{ev}_x(f)\|_{\mathcal{K}} = \sup_{x\in C} \|f(x)\|_{\mathcal{K}} \leq M_f.$$

The principle of uniform boundedness implies

$$\sup_{x\in C} \|\mathrm{ev}_x\|_{\mathcal{H},\mathcal{K}} \leq M.$$

The claim follows observing that

$$\sup_{x,y\in C} \|\Gamma(x,y)\|_{\mathcal{K},\mathcal{K}} = \sup_{x,y\in C} \left(\left\|\mathrm{ev}_x \mathrm{ev}_y^*\right\|_{\mathcal{K},\mathcal{K}} \right) \leq \sup_{x,y\in C} \left(\|\mathrm{ev}_x\|_{\mathcal{H},\mathcal{K}} \|\mathrm{ev}_y\|_{\mathcal{H},\mathcal{K}} \right) \leq M^2.$$

2) ⇒ 1) Let

$$\mathcal{H}_0 = \mathrm{span}\,\{\Gamma(\cdot,x)\,v \mid x \in X,\, v \in \mathcal{K}\}.$$

The elements of $\mathcal{H}_0$ are continuous by hypothesis and (2.4) ensures that $\mathcal{H}_0$ is total.

Given $f \in \mathcal{H}$ and $x_0 \in X$, we prove that f is continuous in x_0. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}_0$ converging to f. Since Γ is locally bounded the convergence is uniform on a neighbourhood of x_0, so f is continuous at x_0.

In particular the inclusion operator is continuous, since a sequence of functions $(f_n)_{n\in\mathbb{N}}$ converging to f in $\mathcal{H}$ converges uniformly to f on each compact subset of X. □

The following corollary gives a simple condition ensuring that $\mathcal{H}$ is separable

Corollary 5.1. *Let* $\mathcal{H}$ *is a* $\mathcal{K}$*-valued RKH space of continuous functions. Assume that* X *and* $\mathcal{K}$ *are separable, then* $\mathcal{H}$ *is separable.*

Proof. The separability of X ensures that there is a denumerable dense subset $X_0 \subseteq X$ and, since $\mathcal{K}$ is separable,

$$\mathcal{S} = \bigcup_{x\in X_0} \mathrm{Im}\,\gamma(x) \subset \mathcal{H}$$

is separable, too. We show that $\mathcal{S}$ is total, so that $\mathcal{H}$ is separable. Indeed, let $f \in \mathcal{S}^\perp$, then $f \in \ker \gamma(x)^*$ for all $x \in X_0$, that is, $f(x) = \mathrm{ev}_x f = 0$. Since f is continuous and X_0 is dense, $f = 0$. □

We now come to the problem of characterizing the compactness of the inclusion operator.

Proposition 5.2. *Let $\mathcal{H}$ be a $\mathcal{K}$-valued RKH space with reproducing kernel Γ. The following facts are equivalent:*

(1) the inclusion $\imath_\Gamma : \mathcal{H} \to \mathcal{C}(X; \mathcal{K})$ is compact;
(2) Γ is continuous with respect to the uniform norm topology and $\Gamma(x, x)$ is a compact operator for all $x \in X$.

Proof. We denote by B the unit ball in $\mathcal{H}$. Condition 1 is equivalent to show that $\imath_\Gamma(B)$ is precompact in $\mathcal{C}(X; \mathcal{K})$. Due to the local compactness of X this is equivalent (Ascoli-Arzelá theorem) to

a) $\{f(x) = \gamma^*(x)f \mid f \in B\}$ is precompact in $\mathcal{K}$ for every $x \in X$;
b) $\imath_\Gamma(B)$ is equicontinuous.

Condition a) is equivalent to the fact that $\gamma(x)^*$ is compact, so is $\Gamma(x, x)$ for all $x \in X$. Moreover, since

$$\sup_{f \in B} \|(\imath_\Gamma f)(x) - (\imath_\Gamma f)(y)\|_{\mathcal{K}} = \|\gamma(x)^* - \gamma^*(y)\|_{\mathcal{H},\mathcal{K}}$$
$$= \left(\|\Gamma(x, x) + \Gamma(y, y) - \Gamma(x, y) - \Gamma(y, x)\|_{\mathcal{K},\mathcal{K}} \right)^{\frac{1}{2}}$$

condition b) is equivalent to the continuity of Γ with respect to operator norm topology. □

Notice that the correspondence given by Prop. 2.4 is not useful since $X \times \mathcal{K}$ is not locally compact and Ascoli-Arzelá theorem is no longer true (as an equivalence).

5.3. *Integrability and continuity*

In many examples $\mathcal{K}$ is a separable Hilbert space and X is a locally compact second countable Hausdorff space endowed with a positive Radon measure μ. Hence X is separable and μ is a σ-finite measure.

If Γ is a $\mathcal{K}$-kernel of positive type such that

(1) Γ is $\frac{p}{p-1}$-bounded with respect to μ for some $1 \leq p \leq \infty$,
(2) Γ is locally bounded and strongly continuous in the first entry,

the results of Section 4 and 5 ensure that the elements of the corresponding RKH space $\mathcal{H}$ are continuous p-integrable functions $f : X \to \mathcal{K}$ and these conditions are also necessary.
By Corollary 5.1 $\mathcal{H}$ is a separable Hilbert space and the inclusion $\imath_\Gamma$ can be regarded as a bounded operator from $\mathcal{H}$ either to $L^p(X, \mu; \mathcal{K})$ or to $\mathcal{C}(X, \mathcal{K})$. In the second case $\imath_\Gamma$ is injective, whereas in the first one

$$\operatorname{Ker} \imath_\Gamma = \{f(x) = 0 \mid x \in \operatorname{supp} \mu\}, \tag{5.1}$$

where $\operatorname{supp}\mu$ is the support of the measure μ.

Finally, assume that X is a compact set, Γ is bounded, $\Gamma(\cdot, x)$ is strongly continuous and $\Gamma(x, x)$ is a compact operator for all $x \in X$. Since μ is finite, the map $x \mapsto \Gamma(x, x)$ is p-integrable, so Γ is q-bounded for all $1 \leq p < \infty$ and the inclusion $\imath_\Gamma$ is always compact as a map in $L^p(X, \mu; \mathcal{K})$. However, in order $\imath_\Gamma$ be compact as a map in $\mathcal{C}(X; \mathcal{K}) \subset L^p(X, \mu; \mathcal{K})$, it is necessary (and sufficient) that Γ is continuous from $X \times X$ into $\mathcal{B}_0(\mathcal{K})$ with the uniform norm topology .

6. Mercer theorem

In this section we characterize the RKH spaces of $\mathcal{K}$-valued functions that are subspaces of $L^2(X, \mu; \mathcal{K})$ in terms of the spectral decomposition of the integral operator L_Γ.

6.1. *Notations*

If $\mathcal{K}$ is a Hilbert space and $v_1, v_2 \in \mathcal{K}$, we let $v_1 \otimes v_2$ be the rank one operator in $\mathcal{K}$ defined by

$$(v_1 \otimes v_2)(w) = \langle w, v_2 \rangle_{\mathcal{K}} v_1 \qquad w \in \mathcal{K}.$$

If Σ is a σ-algebra and $\Sigma \ni E \mapsto P(E) \in \mathcal{B}(\mathcal{K})$ is a projection valued measure, for all $v, w \in \mathcal{K}$ we denote by $\langle \mathrm{d}P(\lambda)v, w \rangle_{\mathcal{K}}$ the bounded complex measure defined by $E \mapsto \langle P(E)v, w \rangle_{\mathcal{K}}$. If $(v_i)_{i \in I}$ is a summable family in $\mathcal{K}$, we denote by $\sum_{i \in I} v_i$ the sum with respect to the notion of summability.

6.2. *Main result*

The following proposition extends Mercer theorem to a noncompact set, compare with [13].
Let X be a measurable space endowed with a σ finite measure μ and $\mathcal{K}$ a separable Hilbert space. Let $\mathcal{H}$ be a $\mathcal{K}$-valued RKH space on X with reproducing kernel Γ. We assume that $\mathcal{H}$ is separable, Γ is 2-bounded with respect to μ and the inclusion $\imath_\Gamma : \mathcal{H} \to L^2(X, \mu; \mathcal{K})$ is injective.

We let

$$L_\Gamma = \int_{\sigma_\Gamma} \lambda \, \mathrm{d}P(\lambda)$$

be the spectral decomposition of the integral operator $L_\Gamma = \imath_\Gamma \imath_\Gamma^*$, where σ_Γ is the spectrum of L_Γ and $E \mapsto P(E)$ is the spectral measure (since L_Γ is a positive bounded operator, σ_Γ is a compact subset of $[0, +\infty)$).

Proposition 6.1. *With the above assumptions the following facts hold*

$$\imath_\Gamma(\mathcal{H}) = \{\phi \in L^2(X, \mu; \mathcal{K}) \mid \int_{\sigma_\Gamma} \frac{1}{\lambda} \langle \mathrm{d}P(\lambda)\phi, \phi \rangle_2 < +\infty\} \tag{6.1}$$

$$\langle f, g \rangle_{\mathcal{H}} = \int_{\sigma_\Gamma} \frac{1}{\lambda} \langle \mathrm{d}P(\lambda)\imath_\Gamma f, \imath_\Gamma g \rangle_2 \qquad \forall f, g \in \mathcal{H}. \tag{6.2}$$

Proof. The polar decomposition of the adjoint $\imath_\Gamma^*$ gives

$$\imath_\Gamma^* = W(\imath_\Gamma \imath_\Gamma^*)^{\frac{1}{2}} = W L_\Gamma^{\frac{1}{2}},$$

where W is a partial isometry from $L^2(X, \mu; K)$ to $\mathcal{H}$ with

$$W^*W = P(\sigma_\Gamma \setminus \{0\}) \qquad \text{and} \qquad WW^* = I_{\mathcal{H}} \tag{6.3}$$

where the last equality holds since $\imath_\Gamma$ is injective. It follows

$$\imath_\Gamma = L_\Gamma^{\frac{1}{2}} W^*, \tag{6.4}$$

so that $\imath_\Gamma(\mathcal{H})$ is the range of $L_\Gamma^{\frac{1}{2}}$ and the spectral theorem implies (6.1).

To show (6.2) let $f, g \in \mathcal{H}$. Recalling (6.3),

$$\langle f, g\rangle_{\mathcal{H}} = \langle W^*f, W^*g\rangle_2 = \int_{\sigma_\Gamma} \langle \mathrm{d}P(\lambda) W^* f, W^* g\rangle_2 = \int_{\sigma_\Gamma} \frac{1}{\lambda} \langle \mathrm{d}P(\lambda) \imath_\Gamma f, \imath_\Gamma g\rangle_2$$

where the last integral makes sense since, by (6.4),

$$\langle \mathrm{d}P(\lambda)\imath_\Gamma f, \imath_\Gamma g\rangle_2 = \left\langle \mathrm{d}P(\lambda) L_\Gamma^{\frac{1}{2}} W^* f, L_\Gamma^{\frac{1}{2}} W^* g \right\rangle_2 = \lambda \langle \mathrm{d}P(\lambda) W^* f, W^* g\rangle_2 . \qquad \square$$

If X is a locally compact second countable Hausdorff space endowed with a positive Radon measure μ such that $\operatorname{supp} \mu = X$, to ensure both that $\mathcal{H}$ is separable and that $\imath_\Gamma$ is injective as a map into $L^2(X, \mu; \mathcal{K})$, it is sufficient that Γ is 2-bounded, locally bounded and strongly continuous in the first entry. In this setting, (6.1) allows us to identify the elements of $\mathcal{H}$ with the only continuous functions on X whose equivalence class belongs to the range of $L_\Gamma^{\frac{1}{2}}$. With this identification, (6.2) implies that $L_\Gamma^{\frac{1}{2}}$ is a unitary operator from $\operatorname{Ker} L_\Gamma^\perp$ onto $\mathcal{H}$, compare with [10].
If $\imath_\Gamma$ is not injective, $L_\Gamma^{\frac{1}{2}}$ is a unitary operator from $\operatorname{Ker} L_\Gamma^\perp$ onto $\operatorname{Ker} \imath_\Gamma^\perp$ (see (4.6) and (5.1) for a characterization).

As a consequence of the above result we have the following version of Mercer theorem. Let ν be a positive σ-finite measure defined on the Borel σ-algebra $\Sigma(\sigma_\Gamma)$ such that $\nu(E) = 0$ if and only if $P(E) = 0$ (it exists and is unique, up to an equivalence, by Hellinger-Hahn theorem).

Theorem 6.1. *With the assumptions of Th. 6.1, for all $x, y \in X$ and $v, w \in \mathcal{K}$ there is a complex measurable function $\rho_{x,y;v,w}$ defined on σ_Γ such that*

$$\langle \Gamma(x, y)v, w\rangle_{\mathcal{K}} = \int_{\sigma_\Gamma} \lambda \; \rho_{x,y;v,w}(\lambda) \mathrm{d}\nu(\lambda). \tag{6.5}$$

Given $E \in \Sigma(\sigma_\Gamma)$ with $0 \notin \overline{E}$, any basis of $\operatorname{Im} P(E)$ is of the form $(\imath_\Gamma \phi_n)_{n\in I}$, the family $(\langle v, \phi_n(y)\rangle_{\mathcal{K}} \langle \phi_n(x), w\rangle_{\mathcal{K}})_{n\in I}$ is summable, the function $\chi_E \rho_{x,y;v,w}$ is ν-integrable and

$$\int_E \rho_{x,y;v,w}(\lambda) \mathrm{d}\nu(\lambda) = \sum_{n\in I} \langle v, \phi_n(y)\rangle_{\mathcal{K}} \langle \phi_n(x), w\rangle_{\mathcal{K}}. \tag{6.6}$$

If $x = y$ and $v = w$, given a basis for $\overline{\operatorname{Im} L_\Gamma}$ of the form $(\imath_\Gamma \phi_n)_{n\in I}$, the following two conditions are equivalent

(1) the function $\rho_{x,x;v,v}$ *is* ν*-integrable;*
(2) $\imath_\Gamma(\gamma(x)v) \in \operatorname{Im} L_\Gamma$.

If one of the above conditions holds, the family $\left(|\langle \phi_n(x), v\rangle_{\mathcal{K}}|^2\right)_{n\in I}$ *is summable and*

$$\int_{\sigma_\Gamma} \rho_{x,x;v,v}(\lambda)\mathrm{d}\nu(\lambda) = \left\| L_\Gamma^{-1}\imath_\Gamma \gamma(x)v\right\|_2^2 = \sum_{n\in I} |\langle \phi_n(x), v\rangle_{\mathcal{K}}|^2. \tag{6.7}$$

Proof. Let W be the partial isometry defined in the previous proof so that $\imath_\Gamma = L_\Gamma^{\frac{1}{2}} W^*$. Given $x, y \in X$ and $v, w \in \mathcal{K}$, the definition of ν implies that the bounded complex measure $\langle \mathrm{d}P(\lambda)W^*\gamma(y)v, W^*\gamma(x)w\rangle_2$ has density $\pi_{x,y;v,w} \in L^1(\sigma_\Gamma, \nu)$ with respect to ν. Let

$$\rho_{x,y;v,w}(\lambda) = \begin{cases} \frac{1}{\lambda}\pi_{x,y;v,w}(\lambda) & \lambda \neq 0 \\ 0 & \lambda = 0 \end{cases},$$

then $\rho_{x,y;v,w}$ is measurable and $\lambda \mapsto \lambda\rho_{x,y;v,w}(\lambda)$ is ν-integrable, so that

$$\begin{aligned} \int_{\sigma_\Gamma} \lambda\rho_{x,y;v,w}(\lambda)\mathrm{d}\nu(\lambda) &= \int_{\sigma_\Gamma\setminus\{0\}} \langle \mathrm{d}P(\lambda)W^*\gamma(y)v, W^*\gamma(x)w\rangle_2 \\ &= \langle P(\sigma_\Gamma \setminus \{0\})W^*\gamma(y)v, W^*\gamma(x)w\rangle_2 \\ &= \langle \gamma(y)v, \gamma(x)w\rangle_{\mathcal{H}} = \langle \Gamma(x,y)v, w\rangle_{\mathcal{K}} \end{aligned}$$

since (6.3).

Let now $E \in \Sigma(\sigma_\Gamma)$ be such that $0 \notin \overline{E}$. This last fact and the spectral theorem imply that $\operatorname{Im} P(E) \subset \operatorname{Im} L_\Gamma^{\frac{1}{2}} = \imath_\Gamma(\mathcal{H})$. Hence, any basis of $\operatorname{Im} P(E)$ is of the form $(\imath_\Gamma\phi_n)_{n\in I}$. Since $0 \notin \overline{E}$, $\chi_E\, \rho_{x,y;v,w}$ is ν-integrable and

$$\begin{aligned} \int_E \rho_{x,y;v,w}(\lambda)\mathrm{d}\nu(\lambda) &= \int_{\sigma_\Gamma} \frac{\chi_E(\lambda)}{\lambda} \langle \mathrm{d}P(\lambda)W^*\gamma(y)v, W^*\gamma(x)w\rangle_2 \\ &= \int_{\sigma_\Gamma} \frac{1}{\lambda} \langle \mathrm{d}P(\lambda)P(E)W^*\gamma(y)v, P(E)W^*\gamma(x)w\rangle_2 \\ &= \left\langle L_\Gamma^{-\frac{1}{2}} P(E)W^*\gamma(y)v, L_\Gamma^{-\frac{1}{2}} P(E)W^*\gamma(x)w \right\rangle_2, \end{aligned} \tag{6.8}$$

where $P(E)W^*\gamma(x)w$ and $P(E)W^*\gamma(y)v$ are in $\operatorname{Im} L_\Gamma^{\frac{1}{2}}$.

Let now J a finite subset of I. Since $\imath_\Gamma\phi_n = L_\Gamma^{\frac{1}{2}} W^*\phi_n$ and $WW^* = I_{\mathcal{H}}$

$$\begin{aligned} &\sum_{n\in J} \left\langle L_\Gamma^{-\frac{1}{2}} P(E)W^*\gamma(y)v, \imath_\Gamma\phi_n \right\rangle_2 \left\langle \imath_\Gamma\phi_n, L_\Gamma^{-\frac{1}{2}} P(E)W^*\gamma(x)w \right\rangle_2 \\ &\qquad = \sum_{n\in J} \left\langle W^*\gamma(y)v, L_\Gamma^{-\frac{1}{2}}\imath_\Gamma\phi_n \right\rangle_2 \left\langle L_\Gamma^{-\frac{1}{2}}\imath_\Gamma\phi_n, W^*\gamma(x)w \right\rangle_2 \\ &\qquad = \sum_{n\in J} \langle W^*\gamma(y)v, W^*\phi_n\rangle_2 \langle W^*\phi_n, W^*\gamma(x)w\rangle_2 \\ &\qquad = \sum_{n\in J} \langle \gamma(y)v, \phi_n\rangle_{\mathcal{H}} \langle \phi_n, \gamma(x)w\rangle_{\mathcal{H}} = \sum_{n\in J} \langle v, \phi_n(y)\rangle_{\mathcal{K}} \langle \phi_n(x), w\rangle_{\mathcal{K}}, \end{aligned}$$

where we used that $\gamma(x)^* = \mathrm{ev}_x$. Since the family $(\imath_\Gamma \phi_n)_{n\in I}$ is basis for $\mathrm{Im}\, P(E)$

$$\sum_{n\in I} \left\langle L_\Gamma^{-\frac{1}{2}} P(E) W^* \gamma(y) v, \imath_\Gamma \phi_n \right\rangle_2 \left\langle \imath_\Gamma \phi_n, L_\Gamma^{-\frac{1}{2}} P(E) W^* \gamma(x) w \right\rangle_2$$

is summable with sum $\left\langle L_\Gamma^{-\frac{1}{2}} P(E) W^* \gamma(y) v, L_\Gamma^{-\frac{1}{2}} P(E) W^* \gamma(x) w \right\rangle_2$, and (6.6) follows by means of (6.8).

Finally, if $x \in X$ and $v \in \mathcal{K}$, the measure $\langle \mathrm{d}P(\lambda) W^* \gamma(x) v, W^* \gamma(x) v \rangle_2$ is positive, so $\rho_{x,x;v,v}$ is positive ν-almost everywhere. The spectral theorem implies that $\rho_{x,x;v,v}$ is ν-integrable if and only if $W^* \gamma(x) v \in \mathrm{Im}\, L_\Gamma^{\frac{1}{2}}$. By means of (6.4), this condition is equivalent to $\imath_\Gamma \gamma(x) v \in \mathrm{Im}\, L_\Gamma$ and, if it is satisfied,

$$\int_{\sigma_\Gamma} \rho_{x,x;v,v}(\lambda) \mathrm{d}\nu(\lambda) = \left\| L_\Gamma^{-\frac{1}{2}} W^* \gamma(x) v \right\|_2^2. \tag{6.9}$$

Let $(\phi_n)_{n\in I}$ be a family in $\mathcal{H}$ such that $(\imath_\Gamma \phi_n)_{n\in I}$ is a basis of $\overline{\mathrm{Im}\, L_\Gamma}$ (such a basis exists since the closure of $\mathcal{H}$ in $L^2(X,\mu;\mathcal{K})$ is $\overline{\mathrm{Im}\, L_\Gamma}$). Reasoning as above

$$\sum_{n\in I} |\langle \phi_n(x), v \rangle_{\mathcal{K}}|^2 = \sum_{n\in I} \left| \left\langle L_\Gamma^{-\frac{1}{2}} \imath_\Gamma \phi_n, W^* \gamma(x) v \right\rangle_2 \right|^2.$$

The sum in the right side is finite since $W^* \gamma(x) v \in \mathrm{Im}\, L_\Gamma^{\frac{1}{2}}$ and its sum is $\left\| L_\Gamma^{-\frac{1}{2}} W^* \gamma(x) v \right\|_2^2$. Eq. (6.9) implies (6.7). □

Eq. (6.5) can be seen as an application of the result of Section 10 of [16] applied to the projection measure $E \mapsto W P(E) W^*$.

Assume now that the integral operator L_Γ has a pure point spectrum. Eq. (6.1) implies that there is a family $(\phi_n)_{n\in I} \in \mathcal{H}$ such that $(\imath_\Gamma \phi_n)_{n\in I}$ is a basis of $\mathrm{Ker}\, L_\Gamma^\perp$ and

$$L_\Gamma = \sum_{n\in I} \lambda_n \; \imath_\Gamma \phi_n \otimes_2 \imath_\Gamma \phi_n,$$

where $\lambda_n > 0$ and the sum converges in the strong operator topology. In this setting (6.2) becomes

$$\imath_\Gamma(\mathcal{H}) = \{\phi \in L^2(X,\mu;\mathcal{K}) \mid \sum_n \frac{1}{\lambda_n} |\langle \phi, \phi_n \rangle_2|^2 < +\infty\}.$$

Moreover, by (6.2) the family $(\sqrt{\lambda_n}\phi_n)_{n\in I}$ is a basis of $\mathcal{H}$, so

$$\Gamma(x,y) = \sum_{n\in I} \lambda_n \mathrm{ev}_x(\phi_n \otimes_{\mathcal{H}} \phi_n) \mathrm{ev}_y^* = \sum_{n\in I} \lambda_n \phi_n(x) \otimes_{\mathcal{K}} \phi_n(y) = \sum_{n\in I} \Gamma_n(x,y), \tag{6.10}$$

where $\Gamma_n(x,y) = \lambda_n \phi_n(x) \otimes_{\mathcal{K}} \phi_n(y)$ is a $\mathcal{K}$-kernel of positive type, the sum converges in the strong operator topology and absolutely in the weak operator topology. Finally, if $f, g \in \mathcal{H}$, (6.2) gives

$$\langle f, g \rangle_{\mathcal{H}} = \sum_{n\in I} \frac{1}{\lambda_n} \langle \imath_\Gamma f, \imath_\Gamma \phi_n \rangle_2 \langle \imath_\Gamma \phi_n, \imath_\Gamma g \rangle_2 = \sum_{n\in I} \frac{1}{\lambda_n^2} \int \langle \Gamma_n(y,x) f(x), g(y) \rangle_{\mathcal{K}} \, \mathrm{d}\mu(x) \mathrm{d}\mu(y)$$

since, by definition of Γ_n, $\langle \Gamma_n(y,x)f(x), g(y)\rangle_{\mathcal{K}} = \lambda_n \langle f(x), \phi_n(x)\rangle_{\mathcal{K}} \langle \phi_n(y), g(y)\rangle_{\mathcal{K}}$. Hence Mercer theorem can be seen as the decomposition of the RKH $\mathcal{H}$ in the direct sum of RKH spaces $\mathcal{H}_n$ with reproducing kernel Γ_n and this decomposition is defined by the spectral structure of L_Γ, see Prop. 19 of [3].

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Appendix A. Vector Valued Shur lemma

The following lemma is needed for the proof of Prop. 4.5 and it is well known for $\mathcal{K} = \mathbb{C}$ (Schur lemma). We denote by $\ell^1(\mathcal{K})$ the Banach space $L^1(\mathbb{N}, \nu; \mathcal{K})$, ν being the counting measure of $\mathbb{N}$. Similarly, we write $\ell^\infty(\mathcal{K})$ for the Banach dual $L^\infty(\mathbb{N}, \nu; \mathcal{K})$ of $\ell^1(\mathcal{K})$.

Lemma Appendix A.1. *Suppose $(f_n)_{n\in\mathbb{N}}$ is a sequence of elements in $\ell^1(\mathcal{K})$ such that*

(i) for all $j \in \mathbb{N}$, $f_n(j) \to 0$ in $\mathcal{K}$;
(ii) $f_n \to 0$ weakly in $\ell^1(\mathcal{K})$.

Then $f_n \to 0$ in $\ell^1(\mathcal{K})$.

Proof. We report a rearrangement of the proof given in [24, p. 135] for $\mathcal{K} = \mathbb{C}$.

Let $\operatorname{ball}\ell^\infty(\mathcal{K})$ be the unit ball of $\ell^\infty(\mathcal{K})$ endowed with the weak-$*$ topology. Since $\ell^1(\mathcal{K})$ is separable, $\operatorname{ball}\ell^\infty(\mathcal{K})$ is metrizable. Fix a sequence $(v_h)_{h\in\mathbb{N}}$ which is dense in the unit ball of $\mathcal{K}$. If $\phi, \psi \in \operatorname{ball}\ell^\infty(\mathcal{K})$, define

$$d(\phi, \psi) = \sum_{j=0}^{\infty} 2^{-j} \sum_{h=0}^{\infty} 2^{-h} \left|\langle v_h, \phi(j) - \psi(j)\rangle_{\mathcal{K}}\right|.$$

Then, $d(\phi, \psi) < \infty$, and d is a metric in $\operatorname{ball}\ell^\infty(\mathcal{K})$. We claim that d defines the weak-$*$ topology of $\operatorname{ball}\ell^\infty(\mathcal{K})$. Indeed, given $(\phi_n)_{n\in\mathbb{N}}$ and ψ in $\operatorname{ball}\ell^\infty(\mathcal{K})$, $d(\phi_n, \psi) \to 0$ if and only if

$$\lim_{n\to\infty} \left|\langle v_h, \phi_n(j) - \psi(j)\rangle_{\mathcal{K}}\right| = 0 \quad \forall j, h \in \mathbb{N}$$

and this is in turn equivalent to

$$\text{w}-\lim_{n\to\infty} \phi_n(j) = \psi(j) \quad \forall j \in \mathbb{N}.$$

It is then easy to check that if $\phi_n \to \psi$ in the weak-$*$ topology, then $d(\phi_n, \psi) \to 0$. Conversely, suppose $d(\phi_n, \psi) \to 0$, and let $f \in \ell^1(\mathcal{K})$, $\epsilon > 0$. Fix $j_\epsilon > 0$ such that $\sum_{j \geq j_\epsilon} \|f(j)\|_{\mathcal{K}} < \epsilon/4$. Let n_ϵ be such that for all $n \geq n_\epsilon$

$$|\langle f(j), \phi_n(j) - \psi(j) \rangle_{\mathcal{K}}| < \epsilon/2j_\epsilon \quad \forall j \leq j_\epsilon - 1.$$

For $n \geq n_\epsilon$

$$|\langle f, \phi_n - \psi \rangle_{\ell^1}| \leq \sum_j |\langle f(j), \phi_n(j) - \psi(j) \rangle_{\mathcal{K}}| < j_\epsilon \frac{\epsilon}{2j_\epsilon} + 2\frac{\epsilon}{4} = \epsilon.$$

The claim is thus proved.

Suppose now $(f_n)_{n \in \mathbb{N}}$ is as in the statement of the lemma, and let $\epsilon > 0$. For all $m \in \mathbb{N}$, set

$$F_m = \{\phi \in \operatorname{ball} \ell^\infty(\mathcal{K}) \mid |\langle f_n, \phi \rangle_{\ell^1}| \leq \epsilon/3 \ \forall n \geq m\}.$$

F_m is a closed subset in $\operatorname{ball} \ell^\infty(\mathcal{K})$, and $\cup_{m \in \mathbb{N}} F_m = \operatorname{ball} \ell^\infty(\mathcal{K})$. Since $\operatorname{ball} \ell^\infty(\mathcal{K})$ is metrizable and compact, hence complete, by Baire category theorem there are $m_0 \in \mathbb{N}$, $\delta > 0$ and $\phi \in F_{m_0}$ such that $\{\psi \in \operatorname{ball} \ell^\infty(\mathcal{K}) \mid d(\psi, \phi) < \delta\} \subset F_{m_0}$. Fix $N \in \mathbb{N}$ such that $\sum_{j \geq N} 2^{-j} < \delta/4$. For all $n \geq m_0$, define $\psi_n \in \operatorname{ball} \ell^\infty(\mathcal{K})$ as follows

$$\psi_n(j) = \begin{cases} \phi(j) & \text{if } j \leq N-1 \\ f_n(j)/\|f_n(j)\|_{\mathcal{K}} & \text{if } j \geq N \end{cases}$$

(with $0/0 = 0$). We have $d(\psi_n, \phi) < \delta$, and so $\psi_n \in F_{m_0}$. It follows that for $n \geq m_0$

$$\left| \sum_{j=0}^{N-1} \langle f_n(j), \phi(j) \rangle_{\mathcal{K}} + \sum_{j=N}^{\infty} \|f_n(j)\|_{\mathcal{K}} \right| = |\langle f_n, \psi_n \rangle_{\ell^1}| < \epsilon/3.$$

Since $\lim_{n \to \infty} \|f_n(j)\|_{\mathcal{K}} = 0$ for all j by hypothesis, there exists $m_1 \geq m_0$ such that

$$\sum_{j=0}^{N-1} \|f_n(j)\|_{\mathcal{K}} < \epsilon/3 \quad \forall n \geq m_1.$$

If $n \geq m_1$, we thus have

$$\begin{aligned} \|f\|_{\ell^1} \leq \sum_{j=0}^{N-1} \|f_n(j)\|_{\mathcal{K}} &+ \left| \sum_{j=N}^{\infty} \|f_n(j)\|_{\mathcal{K}} + \sum_{j=0}^{N-1} \langle f_n(j), \phi(j) \rangle_{\mathcal{K}} \right| \\ &+ \left| \sum_{j=0}^{N-1} \langle f_n(j), \phi(j) \rangle_{\mathcal{K}} \right| < \epsilon, \end{aligned}$$

and the lemma is proved. □