A MOCK METAPLECTIC REPRESENTATION

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Abstract. We obtain necessary and sufficient conditions for the admissible vectors of a new unitary non irreducible representation U. The group G is an arbitrary semidirect product whose normal factor A is abelian and whose homogeneous factor H is a locally compact second countable group acting on a Riemannian manifold M. The key ingredient in the construction of U is a C^1 intertwining map between the actions of H on the dual group $\hat{A}$ and on M. The representation U generalizes the restriction of the metaplectic representation to triangular subgroups of $Sp(d, \mathbb{R})$, whence the name "mock metaplectic". For simplicity, we content ourselves with the case where $A = \mathbb{R}^n$ and $M = \mathbb{R}^d$. The main technical point is the decomposition of U as direct integral of its irreducible components. This theory is motivated by some recent developments in signal analysis, notably shearlets. Many related examples are discussed.

1. Introduction

Unitary representations of semidirect products have been thoroughly studied by many authors and are useful in a wide variety of applications. In particular, they play a central rôle in the harmonic analysis of the continuous wavelet transform, as discussed in [18]. From the point of view of applications, a unitary representation U of a locally compact group G (with Haar measure dg) is particularly useful if it yields a reproducing formula, that is, a weak reconstruction of the form

$$f = \int_G \langle f, U_g\eta \rangle \, U_g\eta \, dg, \tag{1}$$

valid for every f in the representation space $\mathcal{H}$, for some *admissible* vector $\eta \in \mathcal{H}$. In this case (G, U, η) is called a *reproducing* system. Alternatively, we simply say that G is a reproducing group. If U is irreducible, this is nothing else but the classical concept of square integrable representation [12], [13]. Typically, $\mathcal{H} = L^2(\mathbb{R}^d)$, and in this case an admissible vector η is sometimes called a *generating function* or *wavelet*. Apart from direct use, formula (1) is important also because it is the starting point for its discrete counterparts, an aspect that we shall not develop in the present paper. It is actually rather interesting to observe that most formulae of the above type that appear in applications, either in their continuous or discrete versions, turn out to be expressible by taking the restriction of the metaplectic representation to some parabolic subgroup G of the symplectic group $Sp(d, \mathbb{R})$. This is the main theme in the papers [9], [7], [8] and the present contribution is an outgrowth thereof.

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We will be concerned with groups G that are semidirect products, where the normal factor is an abelian group A and the homogeneous factor is a locally compact second countable group H. Our main object of study is a unitary representation U of G whose construction is based on the following ingredients: a Riemannian manifold M on which H acts by C^1 diffeomorphisms and a C^1 map $\Phi : M \to \hat{A}$ (the dual group of A) that intertwines the actions of H on M and on $\hat{A}$. The representation $g \mapsto U_g$ acts on $L^2(M)$ as pointwise multiplication by the character $\langle \Phi(\cdot), g\rangle$ if $g \in A$ and quasi regularly if $g \in H$, as clarified below in (10). For simplicity, we take $A = \mathbb{R}^n$ and $M = \mathbb{R}^d$ and we also suppose that the Jacobian of the action on $\mathbb{R}^d$ is constant. We call U the "mock" metaplectic representation because its definition is inspired by the case where $\mathbb{R}^n$ is a vector space of $d \times d$ symmetric matrices on which a closed subgroup H of $GL(d, \mathbb{R})$ acts by $\sigma \mapsto {}^t h^{-1} \sigma h^{-1}$. Under these circumstances, G can be identified with a triangular subgroup of $Sp(d, \mathbb{R})$ and U is the restriction to G of the metaplectic representation (see Example 1).

General admissibility criteria for type-I groups have been proved in [18]. Given the representation U on $\mathcal{H}$, his theory stems from knowledge of a direct integral decomposition $U = \int_{\widehat{G}} m_\sigma \sigma \, d\nu(\sigma)$ into irreducible components, and the corresponding decomposition $\mathcal{H} = \int_{\widehat{G}} m_\sigma \mathcal{H}_\sigma \, d\nu(\sigma)$. With these data at hand, Führ proves that if G is non-unimodular, then (1) holds true for some η if and only if ν has density with respect to $\mu_{\widehat{G}}$, the Plancherel measure of G; if G is unimodular, then one has to add the extra conditions that $m_\sigma \leq \dim \mathcal{H}_\sigma$ for ν-almost every σ and $\int_{\widehat{G}} m_\sigma \, d\nu(\sigma) < +\infty$. Observe that the measure ν is known to exist [12], but one has to find it, together with the measurable field $\{\mathcal{H}_\sigma\}$ and the multiplicity function $\sigma \mapsto m_\sigma$. The explicit knowledge of $\mu_{\widehat{G}}$ is also non trivial, in general, but is understood for semidirect products [23]. Without using the remarkable machinery of [18], we explicitly decompose U and thereby obtain, as a byproduct, computable admissibility criteria in terms of the intertwining map Φ.

Our finer results are Theorem 8 and Theorem 9, which deal with the cases where G is unimodular or non-unimodular, respectively. They both hold under the standard technical assumption that the H-orbits are locally closed in $\Phi(\mathbb{R}^d)$ and assuming also that almost all H-stabilizers in $\Phi(\mathbb{R}^d)$ are compact. The latter assumption may be removed and yields the weaker conclusion given in Theorem 6. Theorem 9 actually contains the following result: if G is non-unimodular U is reproducing if and only if the set of critical points of Φ has Lebesgue measure zero. This is of course very easy to check in the examples in which Φ is explicitly known. In the case where $n = d$ and where Φ is a homogeneous polynomial, circumstances that happen in many examples, then U is reproducing if and only if G is non unimodular and the stabilizers are almost all compact (see Theorem 10). This last result settles the problem that was the original motivation of this work.

Here is an outline of the other results contained in the paper.

- Theorem 1, which establishes an important necessary condition for a reproducing formula (1) to hold true: Φ must map sets of positive measure into sets of positive measure, hence the critical points $\mathcal{C}$ have zero Lebesgue measure and $n \leq d$. Thus we introduce an open H-invariant subset X of $\mathbb{R}^d$ with negligible

Lebesgue complement whose image is denoted by $Y = \Phi(X) \subseteq \mathbb{R}^n$ in such a way that Φ is a submersion of X onto Y. The fibers $\Phi^{-1}(y)$ are therefore Riemannian submanifolds of X and play a crucial rôle in what follows. All the results except Theorem 1 will be formulated for X and Y, namely, for the map $\Phi : X \to Y$, and hold true under the assumption that $\mathcal{C}$ has zero Lebesgue measure (see Assumption 1).

- Theorem 2, based on the classical coarea formula, shows how the Lebesgue measure of X disintegrates into a family of measures $\{\nu_y\}$ concentrated on the fibers $\Phi^{-1}(y)$, whose covariance with respect to the H-action is explicitly calculated in (22).
- Theorem 3, where a first reduction criterion for admissible vectors is given. One looks at the H-orbits in Y and takes their preimages under Φ in X. Upon selecting an origin y in each H-orbit in Y, one gets the fiber $\Phi^{-1}(y)$. The theorem states that it is necessary and sufficient to test that, for almost every H-orbit in Y, the L^2-norm with respect to ν_y of any $u \in L^2(X, \nu_y)$ can be reproduced by the (weighted) H-integral of the square modulus $|\langle u, \eta_y^h \rangle_{\nu_y}|^2$ of the components of u along the H-translates of the restriction to $\Phi^{-1}(y)$ of the admissible vector η. This is formula (25).
- Theorem 5, which exhibits a direct integral decomposition of U in terms of induced representations of isotropy subgroups of H, and is independent of any admissibility issue. This is achieved as follows.
  - First of all, we assume that the H-orbits are locally closed in Y. This is a standard assumption, without which most results in the current literature on these themes cannot be applied. In Section 3.4.1 we make some technical comments on this in relation to the recent results in [19].
  - Secondly, we derive a disintegration of the Lebesgue measure on Y *à la Mackey*, that is, $dy = \int_Z \tau_z \, d\lambda(z)$. Here λ is a pseudo-image measure on the locally compact second countable space Z which is a nice parametrization of the orbits (better than Y/H) and τ_z is concentrated on the orbit corresponding to $z \in Z$. This preliminary disintegration is carried out in Theorem 4, where the covariance of $\{\tau_z\}$ with respect to the H-action is also calculated in (28).
  - In Proposition 3 we use the measures $\{\tau_z\}$ in order to "glue" together the measures ν_y for all y in the same orbit, thereby producing new measures $\mu_z = \int_Y \nu_y \, d\tau_z(y)$ on X which, in turn, allow to disintegrate the Lebesgue measure on X as $dx = \int_Z \mu_z \, d\lambda(z)$. As before, the covariance of $\{\mu_z\}$ with respect to the H-action is calculated. The reason for introducing these measures are formulae (34) and (35): the representation space of U, namely $L^2(X)$, is formally the double direct integral
$$L^2(X) = \int_Z \Big(\int_Y L^2(X, \nu_y) \, d\tau_z(y) \Big) \, d\lambda(z),$$
where the inner integral is $L^2(X, \mu_z)$.
  - Next we show in Lemma 5 that $L^2(X, \mu_z)$ is unitarily equivalent to the representation space $\mathcal{H}_z$ of the representation W_z which is unitarily induced

to G by the quasi regular representation of the stabilizer $H_{o(z)}$ (naturally extended to the semidirect product $\mathbb{R}^n \rtimes H_{o(z)}$). Here it is important to select an origin $o(z)$ of the orbit in Y whose label is z.

The conclusion of Theorem 5 is that U is equivalent to $\int_Z W_z \, d\lambda(z)$, with an explicit intertwining isometry. The main technical ingredient of this part is the theory of disintegration of measures, as developed by Bourbaki and it is reviewed in Appendix A.1 under the simplifying assumption that the spaces are second countable.

- Theorem 7 assumes that the stabilizers of the H action on Y are almost all compact and it is based on the theory of von Neumann algebras. It takes care of a nontrivial measurability issue involved in the decomposition of the map $z \mapsto W_z$ as direct sum of its irreducible components.

Finally, in Section 4 we illustrate several examples.

2. Notation and assumptions

In this section we fix the notation and describe the setup. We start by recalling the notions of *reproducing group* and *admissible vector*. For a thorough discussion the reader is referred to [18].

Let G be a locally compact group with (left) Haar measure dg and U be a strongly continuous unitary representation of G acting on the complex separable Hilbert space $\mathcal{H}$. A vector $\eta \in \mathcal{H}$ is called admissible if

$$\|f\|^2 = \int_G |\langle f, U_g \eta \rangle|^2 \, dg \qquad \text{for all } f \in \mathcal{H}.$$

If such a vector exists, we say that G is a reproducing group and that U is a *reproducing representation*. Clearly, if U is reproducing, then it is a cyclic representation, but in general it is not irreducible. When U is irreducible, the representation is reproducing if and only if it is square integrable [13].

2.1. **The semidirect product.**

Let H be a locally compact second countable group acting on $\mathbb{R}^n$ by means of the continuous representation

$$y \mapsto h[y], \qquad h \in H. \tag{2}$$

Let G be the semidirect product $G = \mathbb{R}^n \rtimes H$ with group law

$$(a_1, h_1)(a_2, h_2) = (a_1 + h_1^\dagger[a_2], h_1 h_2) \qquad a_1, a_2 \in \mathbb{R}^n, h_1, h_2 \in H,$$

where $h^\dagger[\cdot]$ is the action given by the contragredient representation of H on $\mathbb{R}^n$ defined via the usual inner product by

$$\langle h^\dagger[a], y \rangle = \langle a, h^{-1}[y] \rangle, \qquad a, y \in \mathbb{R}^n. \tag{3}$$

Since $h[\cdot]$ is linear, the semidirect product is well defined and G is a locally compact second countable group. Conversely, any locally compact second countable group G that is the semidirect product of a closed subgroup and a normal subgroup, which is a real vector space of dimension n, is of the above form.

The (left) Haar measures of G and H are written dg and dh, and, similarly, da is the Lebesgue measure on $\mathbb{R}^n$. The modular functions of G and H are denoted by Δ_G and Δ_H, respectively. The following relations are easily established

$$dg = \frac{1}{\alpha(h)} da\, dh \tag{4}$$

$$\Delta_G(a,h) = \frac{\Delta_H(h)}{\alpha(h)} \tag{5}$$

where $\alpha : H \to (0,+\infty)$ is the character of H defined by

$$\alpha(h) = |\det(a \mapsto h^\dagger[a])| = |\det(y \mapsto h^{-1}[y])|. \tag{6}$$

The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is defined by

$$(\mathcal{F}f)(y) = \int_{\mathbb{R}^n} e^{-2\pi i \langle y,a\rangle} f(a)\, da, \qquad f \in L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n).$$

In general, if G is any locally compact second countable group, $L^2(G)$ will denote the Hilbert space of square integrable functions with respect to left Haar measure. Finally, if X is a locally compact second countable topological space, the Borel σ-algebra on X is denoted $\mathcal{B}(X)$ and $C_c(X)$ denotes the space of complex continuous functions on X with compact support. By *measure* we mean a σ-additive function μ on $\mathcal{B}(X)$ with values in $[0,+\infty]$ which is finite on compact sets. The hypothesis on X implies that any such measure is automatically inner and outer regular [26]. A function $f : X \to X'$ between two such spaces will be called Borel measurable if $f^{-1}(B) \in \mathcal{B}(X)$ for every $B \in \mathcal{B}(X')$ and μ-measurable if $f^{-1}(B) \in \mathcal{B}_\mu(X)$, where $\mathcal{B}_\mu(X)$ denotes the completion of $\mathcal{B}(X)$ with respect to μ. When dealing with open subsets of Euclidean spaces endowed with the Lebesgue measure, however, we say measurable to mean Lebesgue measurable. Finally, if $E \in \mathcal{B}(\mathbb{R}^d)$ we write $|E|_d$ for it Lebesgue measure or simply $|E|$ if no confusion arises.

2.2. **The mock metaplectic representation.** Suppose we are given:

(H1) a continuous action of H on $\mathbb{R}^d$ by smooth maps denoted $x \mapsto h.x$, whose Jacobian is constant and equal to $\beta(h)$; for $h \in H$ and $E \in \mathcal{B}(\mathbb{R}^d)$ we thus have

$$|h.E| = \beta(h)|E|, \tag{7}$$

that is, for every $\varphi \in C_c(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \varphi(h^{-1}.x)\, dx = \beta(h) \int_{\mathbb{R}^d} \varphi(x)\, dx. \tag{8}$$

(H2) a C^1-map $\Phi : \mathbb{R}^d \to \mathbb{R}^n$ intertwining the two actions of H on $\mathbb{R}^d$ and $\mathbb{R}^n$:

$$\Phi(h.x) = h[\Phi(x)] \qquad x \in \mathbb{R}^d,\ h \in H. \tag{9}$$

For $g = (a,h) \in G$ we define $U_g : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$(U_g f)(x) = \beta(h)^{-\frac{1}{2}} e^{-2\pi i \langle \Phi(x),a\rangle} f(h^{-1}.x) \tag{10}$$

for almost every $x \in \mathbb{R}^d$. We show below that this is indeed a representation, that we call the *mock metaplectic* representation. For a motivation for the choice of this name, see Example 3 below.

Remark 1. The representation (2) of H on $\mathbb{R}^n$ plays no direct rôle in the definition of U; its purpose is to construct the semidirect product G.

Remark 2. Occasionally, we shall write $f^h(x)$ for $f(h^{-1}.x)$.

Remark 3. At this stage there are no limitations on the relative sizes of n and d, but we shall see later (Theorem 1) that in the situations that are of interest to us $n \le d$.

The next proposition records that (10) is a good definition.

Proposition 1. *The map $g \mapsto U_g$ is a strongly continuous unitary representation of G acting on $L^2(\mathbb{R}^d)$.*

Proof. Clearly, U_g is a unitary operator and U is a representation of $\mathbb{R}^n$ and H separately. In order to prove that it is a representation of G, it is enough to show that $U_h U_a U_{h^{-1}} = U_{h^\dagger[a]}$ for $a \in \mathbb{R}^n$ and $h \in H$. For $f \in L^2(\mathbb{R}^d)$, and almost every $x \in \mathbb{R}^d$

$$\begin{aligned}\left(U_h U_a U_{h^{-1}} f\right)(x) &= \beta(h)^{-\frac{1}{2}} e^{-2\pi i \langle \Phi(h^{-1}.x), a\rangle} \left(U_{h^{-1}} f\right)(h^{-1}.x) \\ &= e^{-2\pi i \langle \Phi(h^{-1}.x), a\rangle} f(x) = e^{-2\pi i \langle h^{-1}[\Phi(x)], a\rangle} f(x) \\ &= e^{-2\pi i \langle \Phi(x), h^\dagger[a]\rangle} f(x) = (U_{h^\dagger[a]} f)(x)\end{aligned}$$

To show strong continuity, it is enough to prove that $g \mapsto \langle U_g f_1, f_2\rangle$ is continuous at the identity whenever f_1, f_2 are continuous functions with compact support, and this is an easy consequence of the dominated convergence theorem. $\square$

2.3. **Examples.** There are many interesting examples of the setup we are considering. We will focus on some situations in which most relevant features occur.

Example 1. Let H be a closed subgroup of $GL(d, \mathbb{R})$ and assume $n = d$. Since the group H acts naturally on $\mathbb{R}^d$, define

$$h.x = h[x] = {}^t h^{-1} x \qquad x \in \mathbb{R}^d,\ h \in H.$$

Choosing $\Phi(x) = x$, the representation U is equivalent to the quasi regular representation of G via the Fourier transform. Necessary and sufficient conditions for U to be reproducing are given in [18]. It is worth observing that if $n = 1$, then $H = \mathbb{R}_+$ and hence G is the "$ax + b$" group, whereby the dilations are parametrized by H. In this case U is

$$U_{(b,a)} f(x) = \sqrt{a} e^{-2\pi i b x} f(ax)$$

which, after conjugation with the Fourier transform, is the usual wavelet representation. It may be generalized to higher dimension [27].

Example 2. The Schrödinger representation of the Heisenberg group $\mathbb{H}^1$ may be included in this setup, by regarding $\mathbb{H}^1$ as a closed subgroup of $GL(3, \mathbb{R})$:

$$\mathbb{H}^1 = \left\{ \begin{bmatrix} 1 & q & t \\ 0 & 1 & p \\ 0 & 0 & 1 \end{bmatrix} : q, p, t \in \mathbb{R} \right\}.$$

It is easy to see that $\mathbb{H}^1$ is isomorphic to the semidirect product $A \rtimes H$, where $A = \left\{\left[\begin{smallmatrix} p \\ t \end{smallmatrix}\right] : p, t \in \mathbb{R}\right\}$ and $H = \left\{\left[\begin{smallmatrix} 1 & 0 \\ q & 1 \end{smallmatrix}\right] : q \in \mathbb{R}\right\}$. Indeed, the group H has the natural representation on $\mathbb{R}^2$:

$$q \mapsto {}^t\begin{bmatrix} 1 & 0 \\ q & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -q \\ 0 & 1 \end{bmatrix}$$

and acts on $\mathbb{R}$ via the translations $q.x = x + q$. The smooth map $\Phi : \mathbb{R} \to \mathbb{R}^2$ defined by $\Phi(x) = \left[\begin{smallmatrix} -x \\ 1 \end{smallmatrix}\right]$ satisfies the intertwining property (9). The mock metaplectic representation takes the form

$$U_{(q,p,t)} f(x) = e^{-2\pi i \langle \Phi(x), \left[\begin{smallmatrix} p \\ t \end{smallmatrix}\right] \rangle} f(q^{-1}.\, x) = e^{-2\pi i (t - px)} f(x - q)$$

and it thus coincides with the Schrödinger representation, which is irreducible but notoriously not square integrable (i.e. not reproducing). Notice that $n > d$.

Example 3. This class of examples is where our investigation started. It will be transparent that the mock metaplectic representation is a generalization of the metaplectic representation as restricted to this class of subgroups of $Sp(d, \mathbb{R})$ which includes all the parabolic subgroups. Let $G = \Sigma \rtimes H \subset Sp(d, \mathbb{R})$ be a subgroup of the form

$$G = \left\{ \begin{bmatrix} h & 0 \\ \sigma h & {}^t h^{-1} \end{bmatrix} : h \in H,\ \sigma \in \Sigma \right\}, \tag{11}$$

where H is a closed subgroup of $GL(d, \mathbb{R})$ and Σ is an n-dimensional subspace of $\mathrm{Sym}(d, \mathbb{R})$, the space of symmetric $d \times d$ matrices. We call any such group a triangular subgroup.

Inner conjugation within G yields the H-action on Σ

$$h^\dagger[\sigma] := {}^t h^{-1} \sigma h^{-1} \qquad \sigma \in \Sigma, h \in H, \tag{12}$$

under which Σ must be invariant. As the notation suggests, (12) can be seen as a contragredient action. Indeed, we endow $\mathrm{Sym}(d, \mathbb{R})$ with the natural inner product $\langle \sigma_1, \sigma_2 \rangle = \mathrm{tr}(\sigma_1 \sigma_2)$, whose restriction to Σ will be denoted $\langle \cdot, \cdot \rangle_\Sigma$. If $\sigma \mapsto h[\sigma]$ is the representation whose contragredient version is (12), then for $\sigma, \tau \in \Sigma$ we have

$$\langle \tau, h[\sigma] \rangle_\Sigma = \langle {}^t h \tau h, \sigma \rangle_\Sigma = \mathrm{tr}(\tau h \sigma {}^t h) = \langle \tau, P_\Sigma(h \sigma {}^t h) \rangle_\Sigma,$$

where P_Σ is the orthogonal projection from $\mathrm{Sym}(d, \mathbb{R})$ onto Σ. Thus

$$h[\sigma] = P_\Sigma(h \sigma\, {}^t h) \qquad \sigma \in \Sigma, h \in H, \tag{13}$$

and if ${}^t H = H$ there is no need of the projection.

The group H acts naturally on $\mathbb{R}^d$, that is, $h.x = hx$. Given $x \in \mathbb{R}^d$, let $\Phi(x) \in \Sigma$ be defined by

$$\mathrm{tr}(\Phi(x)\sigma) = -\frac{1}{2} \langle \sigma x, x \rangle \qquad x \in \mathbb{R}^d. \tag{14}$$

Identifying $\mathbb{R}^n \simeq \widehat{\Sigma} \simeq \Sigma$, we can interpret $\Phi(x)$ either as the linear functional on Σ whose action on σ is $-\frac{1}{2} \langle \sigma x, x \rangle$ or as the symmetric matrix associated to it via the usual inner product on symmetric matrices. Condition (9) is satisfied, since, upon observing that $\sigma = P_\Sigma(\sigma)$ and that P_Σ is self-adjoint,

$$\operatorname{tr}(\Phi(h.x)\sigma) = -\frac{1}{2}\left\langle {}^t h\sigma h x, x\right\rangle = \operatorname{tr}(\Phi(x)^t h\sigma h) = \operatorname{tr}(h\Phi(x)^t h\sigma) = \operatorname{tr}(h[\Phi(x)]\sigma).$$

The representation (10) is

$$U_{(\sigma,h)}f(x) = |\det h|^{-1/2}\, e^{\pi i\langle\sigma x,x\rangle} f(h^{-1}x) \tag{15}$$

and hence it coincides with the restriction of the metaplectic representation to the group G. Various properties of U are analyzed in [7, 9].

An important explicit example in this class is connected to the theory of shearlets initiated in [20]. Here the group G parametrizes the two-dimensional phase-space operations of translation, dilation and shear and is thus sometimes denoted $TDS(2)$. We shall do so and call it the shearlet group.

Precisely, $G = \mathbb{R}^2 \rtimes H$ in the following way. Fix a parameter $\gamma > 0$ (usually $\gamma = 1/2$). The abelian normal subgroup $\Sigma \simeq \mathbb{R}^2$ consists of the 2×2 symmetric matrices $\left[\begin{smallmatrix} a_1 & a_2/2 \\ a_2/2 & 0\end{smallmatrix}\right]$. The homogeneous group H contains all the 2×2 matrices of the form $S_\ell A_t$ where $\ell \in \mathbb{R}$, $t \in \mathbb{R}_+$ and

$$S_\ell = \begin{bmatrix} 1 & 0 \\ -\ell & 1\end{bmatrix}, \qquad A_t = \begin{bmatrix} t^{-\frac{1}{2}} & 0 \\ 0 & t^{\frac{1}{2}-\gamma}\end{bmatrix}$$

with Haar measure $dh = t^{\gamma-2}d\ell dt$ and modular function $\Delta_H(\ell,t) = t^{\gamma-1}$. For any $h = (\ell,t)$ the linear action on the abelian normal factor $\mathbb{R}^2$ is

$$h^\dagger[\cdot] = \begin{bmatrix} 1 & \ell \\ 0 & 1\end{bmatrix}\begin{bmatrix} t & 0 \\ 0 & t^\gamma\end{bmatrix}$$

and the group law of G is

$$(a,\ell,t)(a',\ell',t') = (a + \left[\begin{smallmatrix} t & t^\gamma\ell \\ 0 & t^\gamma\end{smallmatrix}\right] a', \ell + t^{1-\gamma}\ell', tt').$$

It is easy to see that formula (14) implies that $\Phi(x_1,x_2) = -\frac{1}{2}(x_1^2, x_1x_2)$. The mock metaplectic representation U restricted to Σ is equivalent to translations and restricted to $\{A_t\}$ it amounts to dilations, as shown in [7], where necessary and sufficient conditions for admissible vectors are given in the case $\gamma = 1$. Admissibility conditions are also given in [10] for $\gamma = 1/2$. Observe that $d = n$.

Example 4. This is a case where $n < d$. Let $H = \mathbb{R}_+ \times \mathbb{T}$. Here $\mathbb{T}$ is the one-dimensional torus, parametrized by $\theta \in [0,2\pi)$, with Haar measure $d\theta/2\pi$, and $\mathbb{R}_+$ is the multiplicative group with Haar measure $t^{-1}dt$ where dt is the restriction to $\mathbb{R}_+$ of the Lebesgue measure on the real line. Hence H has Haar measure $dt\,d\theta/2\pi t$ and modular function $\Delta_H(h) = 1$. The representation of H on $\mathbb{R}$ is

$$h[y] = t^2 y \qquad y \in \mathbb{R},$$

where $h = (t,\theta)$. Hence in particular $\alpha(h) = t^{-2}$. The group law in $G = \mathbb{R}\rtimes H$ is

$$(a_1,t_1,\theta_1)(a_2,t_2,\theta_2) = (a_1 + t_1^{-2}a_2, t_1t_2, \theta_1+\theta_2).$$

The resulting Haar measure is $tdt\,d\theta/2\pi$ and the modular function is easily seen to be $\Delta_G(a,t,\theta)=t^2$. The action of $h=(t,\theta)\in H$ on $\mathbb{R}^2$ is given by

$$h.(x_1,x_2)=t(\cos\theta\,x_1-\sin\theta\,x_2,\sin\theta\,x_1+\cos\theta\,x_2)\qquad (x_1,x_2)\in\mathbb{R}^2$$

so that $\beta(h)=t^2$. Finally, $\Phi:\mathbb{R}^2\to\mathbb{R}$ is given by $\Phi(x_1,x_2)=x_1^2+x_2^2$. The mock metaplectic representation $U$of G on $L^2(\mathbb{R}^2)$ is

$$\begin{aligned}U_{(a,t,\theta)}f(x_1,x_2)&=t^{-1}e^{-2\pi i(x_1^2+x_2^2)a}\times\\&\qquad\times f\left(t^{-1}(\cos\theta\,x_1+\sin\theta\,x_2),t^{-1}(-\sin\theta\,x_1+\cos\theta\,x_2)\right).\end{aligned}$$

Example 5. The point of this example, where again $n<d$, will become clearer later, when H-stabilizers enter into the picture: this is a case where they are not compact. Let $H=\mathbb{R}^*\times\mathbb{R}$ where $\mathbb{R}^*$ is the (non-connected) multiplicative group of non-zero real numbers and $\mathbb{R}$ is the additive group with Haar measures $|t|^{-1}dt$ and db respectively. The Haar measure of H is $|t|^{-1}dtdb$ and $\Delta_H=1$. An element $h=(t,b)\in H$ acts on $\mathbb{R}$ and $\mathbb{R}^2$ by means of

$$\begin{aligned}h[y]&=ty && y\in\mathbb{R}\\ h.(x_1,x_2)&=(x_1+b,tx_2) && (x_1,x_2)\in\mathbb{R}^2\end{aligned}$$

so that $\alpha(h)=|t|^{-1}$ and $\beta(h)=|t|$. Finally $\Phi:\mathbb{R}^2\to\mathbb{R}$ is defined by $\Phi(x_1,x_2)=x_2$, which clearly satisfies (9).

3. Main results

3.1. **Dimensional constraints.** Our first result, Theorem 1, states that if G is reproducing, then $n\le d$. The interpretation of this statement in the case of wavelets is that the dimension of the space of translations cannot exceed that of the "ground" space. In order to prove the theorem we need a technical lemma, in the proof of which we use a standard result in harmonic analysis on locally compact abelian groups (see Theorem (31.33) in [22]). This is the fact that if a bounded measure ν on the locally compact abelian group $\mathcal{G}$ has Fourier transform that coincides almost everywhere (on the character group $\widehat{\mathcal{G}}$) with the Fourier transform of an $L^p(\mathcal{G})$-function F, with $1\le p\le 2$, then $F\in L^1(\mathcal{G})$, ν is absolutely continuous with respect to Haar measure and its Radon-Nikodym derivative is F. We apply this to a bounded measure on $\mathbb{R}^n$.

Lemma 1. *For any $f,\eta\in L^2(\mathbb{R}^d)$ the following facts are equivalent:*

(i) $\int_G|\langle f,U_g\eta\rangle|^2\,dg<+\infty$;

(ii) *for almost every $h\in H$ the bounded measure on $\mathbb{R}^n$*

$$\Omega_h(E)=\int_{\Phi^{-1}(E)}f(x)\overline{\eta(h^{-1}.x)}\,dx,\qquad E\in\mathcal{B}(\mathbb{R}^n),\tag{16}$$

has a density $\omega_h\in L^2(\mathbb{R}^n)$ for which

$$\int_H\left(\int_{\mathbb{R}^n}|\omega_h(y)|^2\,dy\right)\frac{dh}{\alpha(h)\beta(h)}<+\infty.\tag{17}$$

Under the above circumstances

$$\int_G |\langle f, U_g\eta\rangle|^2\,dg = \int_H \left(\int_{\mathbb{R}^n} |\omega_h(y)|^2\,dy\right) \frac{dh}{\alpha(h)\beta(h)}. \tag{18}$$

Proof. Observe that Ω_h is the image measure, induced by Φ, of the bounded measure with density $f\overline{\eta^h} \in L^1(\mathbb{R}^d)$ with respect to dx (see e.g. Sec. 39 in [21]). Since Ω_h is bounded, the basic integration formula for image measures, (see Theorem C, p.161 in [21]) and (10) imply that

$$\begin{aligned}\langle f, U_{(a,h)}\eta\rangle &= \beta^{-\frac{1}{2}}(h)\int_{\mathbb{R}^d} e^{2\pi i\langle\Phi(x),a\rangle} f(x)\overline{\eta^h}(x)\,dx \\ &= \beta^{-\frac{1}{2}}(h)\int_{\mathbb{R}^n} e^{2\pi i\langle y,a\rangle}\,d\Omega_h(y).\end{aligned}$$

Assume that $\int_G |\langle f, U_g\eta\rangle|^2\,dg < \infty$. Since $dg = \frac{da\,dh}{\alpha(h)}$, Fubini's theorem implies that, for almost every $h \in H$,

$$\int_{\mathbb{R}^n} |\langle f, U_{(a,h)}\eta\rangle|^2\,da = \beta(h)^{-1}\int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} e^{2\pi i\langle y,a\rangle}\,d\Omega_h(y)|^2\,da < +\infty.$$

This says that the inverse Fourier transform of Ω_h is in $L^2(\mathbb{R}^n)$, and the aforementioned Theorem (31.33) in [22] ensures that the latter condition is equivalent to saying that Ω_h has an $L^2(\mathbb{R}^n)$-density ω_h with respect to dy. Furthermore, by Plancherel's theorem

$$\int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} e^{2\pi i\langle y,a\rangle}\,d\Omega_h(y)|^2\,da = \int_{\mathbb{R}^n} |\omega_h(y)|^2\,dy.$$

Applying again Fubini's theorem, (18) follows and hence (17) holds. Therefore (i) implies (ii). The converse statement is shown by applying the same argument backwards. $\square$

We are now in a position to state our first result.

Theorem 1. *If U is a reproducing representation, then the image under Φ of any Borel subset of $\mathbb{R}^d$ with positive measure has positive measure. Hence*

(i) *$n \le d$;*
(ii) *the set $\mathcal{C}$ of critical points*[1] *of Φ has measure zero.*

Proof. By contradiction, suppose that there exists a Borel subset A of $\mathbb{R}^d$ with positive measure such that $\Phi(A)$ is negligible. Since $|A|_d > 0$ and the Lebesgue measure is regular, there exists a compact subset $K \subset A$ with $|K|_d > 0$. Clearly, $\Phi(K)$ is also compact, but $|\Phi(K)|_n = 0$. Take an admissible vector η for U. The reproducing formula for $f = \chi_K$ and (18) imply that

$$0 < |K|_d = \int_H \left(\int_{\mathbb{R}^n} |\omega_h(y)|^2\,dy\right) \frac{dh}{\alpha(h)\beta(h)},$$

[1] A point $x \in \mathbb{R}^d$ is critical for $\Phi : \mathbb{R}^d \to \mathbb{R}^n$ if the rank of the differential map Φ_{*x} is less than n.

so that, on a subset of H of positive Haar measure we have $\omega_h \neq 0$. Take then $h \in H$ such that $\Omega_h = \omega_h dy \neq 0$. Now, if E is a Borel subset of $\mathbb{R}^n$, the definition of Ω_h gives

$$\Omega_h(E) = \Omega_h(E \cap \Phi(K)) = \int_{E\cap\Phi(K)} \omega_h(y)dy = 0$$

because $|\Phi(K)|_n = 0$. Hence $\Omega_h = 0$, a contradiction.

To show (i), assume that $n > d$ and apply the above result to $A = \mathbb{R}^d$. Since Φ is of class C^1 we have $|\Phi(A)|_n = 0$, so that U cannot be reproducing.

To show (ii), denote by $\mathcal{C}$ the set of critical points of Φ. Sard's theorem [32] implies that $\Phi(\mathcal{C})$ has measure zero. But then, by (i), also $\mathcal{C}$ has measure zero. □

3.2. **Measures concentrated on the preimages under Φ.** Given any $x \in \mathbb{R}^d$, let $J(\Phi)(x) = \sqrt{\det(\Phi_{*x} \cdot {}^t\Phi_{*x})}$ be the Jacobian of Φ at x and denote by $\mathcal{R}$ the set of regular points of Φ, namely

$$\mathcal{R} = \left\{x \in \mathbb{R}^d : J(\Phi)(x) > 0\right\} = \mathbb{R}^d \setminus \mathcal{C}.$$

Lemma 2. *The set $\mathcal{R}$ satisfies the following properties:*

(i) *it is open;*
(ii) *it is H-invariant and has H-invariant image under Φ;*
(iii) *the restriction of Φ to it is an open mapping;*
(iv) *for every y in its image, $\Phi^{-1}(y)$ is a Riemannian submanifold of $\mathbb{R}^d$;*
(v) *a subset $E \subset \Phi(\mathcal{R})$ is negligible if and only if $\Phi^{-1}(E)$ is negligible.*

Proof. (i) Since Φ has continuous derivatives, $\mathcal{R}$ is an open set. (ii) The H-invariance follows from

$$\Phi_{*h.x}(h_*.v) = h[\Phi_{*x}v], \qquad x, v \in \mathbb{R}^d, \tag{19}$$

where h_* denotes the differential of the action $x \mapsto h.x$ and is therefore linear. Indeed, (19) and the fact that $u \mapsto h[u]$ is a linear isomorphism, show that $v \in \ker \Phi_{*x}$ if and only $h_*.v \in \ker \Phi_{*h.x}$, so that $\dim\ker\Phi_{*x} = \dim\ker\Phi_{*h.x}$. Since $x \in \mathcal{R}$ if and only if $\dim\ker\Phi_{*x} = d - n$, the claim follows. To prove (19), fix $x \in \mathbb{R}^d$, a tangent vector $v \in T_x(\mathbb{R}^d) \simeq \mathbb{R}^d$ and a smooth curve $v(t)$ passing through x at time zero with tangent vector v. Evidently, $h.v(t)$ is smooth and has tangent $h_*.v$ at time zero. By (9) and again by the linearity of $u \mapsto h[u]$

$$\begin{aligned}\Phi_{*h.x}(h_*.v) &= \frac{d}{dt}\Phi(h.v(t))\Big|_{t=0} = \frac{d}{dt}h[\Phi(v(t))]\Big|_{t=0}\\ &= h[\frac{d}{dt}\Phi(v(t))\Big|_{t=0}] = h[\Phi_{*x}v],\end{aligned}$$

as desired.

Finally, (iii) and (iv) are standard consequences of the fact that, by definition of $J(\Phi)$, the differential Φ_{*x} is surjective whenever $x \in \mathcal{R}$.

In order to prove (v), put $X = \mathcal{R}$ and $Y = \Phi(\mathcal{R})$. Since Φ is a submersion from X onto Y and since X is locally compact second countable space, there exists a countable family of diffeomorphisms $\Psi_i : U_i \times V_i \to W_i$ such that $\{W_i\}$ is an open covering of X, $\{V_i\}$ is an open covering of Y, $\{U_i\}$ is a family of open sets of $\mathbb{R}^{d-n}$ and

$$\Phi(\Psi_i(z, y)) = y \qquad (z, y) \in U_i \times V_i. \tag{20}$$

Assume that E is a Borel subset of Y, then $|\Phi^{-1}(E)|_d = 0$ if and only if $|\Phi^{-1}(E) \cap W_i|_d = 0$ for all i. Since Ψ_i is a diffeomorphism, by the chain rule this is equivalent to $|\Psi_i^{-1}(\Phi^{-1}(E) \cap W_i)|_d = 0$, that is, by (20), $U_i \times (E \cap V_i)$ is a negligible set of $\mathbb{R}^{d-n} \times \mathbb{R}^n$. Since U_i is an open non-void set, the last condition is equivalent to $|E \cap V_i|_n = 0$, that is, to $|E|_n = 0$. □

Assumption 1. Motivated by Theorem 1, in the following we assume that $\mathcal{C}$ has Lebesgue measure zero. In particular, we assume that $n \leq d$. Furthermore, we fix an open H-invariant subset X of $\mathcal{R}$ whose complement also has measure zero and we denote by Y its image under Φ, namely $Y = \Phi(X)$. Clearly, X satisfies the properties (i)–(v) described in Lemma 2 and so its complement is negligible.

The next results are based on several kinds of disintegration formulae and their covariance properties with respect to the H-action. In Section A.1 we review the general theory of disintegration of measures and introduce the pertinent notation. As for the induced H-action on measures, and the resulting covariance properties, we recall that, if ν is a measure on X and $h \in H$, ν^h is the measure given by $\nu^h(E) = \nu(h.E)$ whenever $E \in \mathcal{B}(X)$. Equivalently,

$$\int_X \varphi(x)\, d\nu^h(x) = \int_X \varphi(h^{-1}.x)\, d\nu(x) \tag{21}$$

for every $\varphi \in C_c(X)$. The first disintegration we discuss arises from the Coarea Formula for submersions.

Theorem 2. *There exists a unique family $\{\nu_y\}$ of measures on X, labeled by the points of Y, with the following properties:*

(i) *ν_y is concentrated on $\Phi^{-1}(y)$ for all $y \in Y$;*
(ii) *$dx = \int_Y \nu_y dy$;*
(iii) *for any $\varphi \in C_c(X)$ the map $y \mapsto \int_X \varphi(x)\, d\nu_y(x) \in \mathbb{C}$ is continuous.*

Furthermore,

$$\nu^h_{h[y]} = \alpha(h)\beta(h)\, \nu_y \tag{22}$$

for all $h \in H$ and all $y \in Y$.

Proof. The proof is based on the classical Coarea Formula. In Section A.3 we give a short proof adapted to the situation at hand and we introduce the notation used in this proof. The reader is thus referred to Theorem 12 below.

For every $y \in Y$, define ν_y by (75). Property (i) is then obvious and (ii) is the content of Theorem 12.

To prove (iii), fix $\varphi \in C_c(X)$ and $y_0 \in Y$. If $y_0 \notin \Phi(\operatorname{supp} \varphi)$, there is an open neighborhood V of y_0 such that $V \cap \Phi(\operatorname{supp} \varphi) = \emptyset$. Thus $\int_X \varphi(x)\, d\nu_y(x) = 0$ for all $y \in V$ because ν_y is concentrated on $\Phi^{-1}(y)$. If $y_0 \in \Phi(\operatorname{supp} \varphi)$, taking a finite covering if necessary, we can always assume that there exists a diffeomorphism $\Psi : U \times V \mapsto W$

such that (77) holds, where U is an open subset of $\mathbb{R}^{d-n}$, V is an open neighborhood of y_0 and W is an open subset of X containing $\operatorname{supp}\varphi$. The definition of ν_y gives

$$\int_X \varphi(x)\,d\nu_y(x) = \int_U \varphi(\Psi(z,y))(J\Psi)(z,y)\,dz$$

and the map $y \mapsto \int_U \varphi(\Psi(z,y))(J\Psi)(z,y)\,dz$ is continuous on V by the dominated convergence theorem.

In order to show (22), fix $h \in H$. Since the action of H on X is continuous, $\{\nu^h_{h[y]}\}_{y\in Y}$ is a family of measures on X and each of them is concentrated on $\Phi^{-1}(y)$, as shown by

$$\nu^h_{h[y]}(X \setminus \Phi^{-1}(y)) = \nu_{h[y]}(X \setminus \Phi^{-1}(h[y])) = 0,$$

where the last equality is due to (i). Furthermore the family $\{\nu^h_{h[y]}\}_{y\in Y}$ is scalarly integrable with respect to dy because for all $\varphi \in C_c(X)$

$$\begin{aligned}
\int_Y \Bigl(\int_X \varphi(x)\,d\nu^h_{h[y]}(x)\Bigr)dy &= \int_Y \Bigl(\int_X \varphi(h^{-1}.x)\,d\nu_{h[y]}(x)\Bigr)dy \\
(\ y \mapsto h^{-1}[y]\ ) &= \alpha(h) \int_Y \Bigl(\int_X \varphi(h^{-1}.x)\,d\nu_y(x)\Bigr)dy \\
&= \alpha(h) \int_X \varphi(h^{-1}.x)\,dx \\
(\ x \mapsto h.x\ ) &= \alpha(h)\beta(h) \int_X \varphi(x)\,dx,
\end{aligned}$$

where the third line follows from (ii). Hence

$$dx = \int_X \alpha(h^{-1})\beta(h^{-1})\nu^h_{h[y]}\,dy$$

and (iv) of Theorem 11 implies that for almost all $y \in Y$ (22) holds true. Item (iii) tells us that for any fixed $\varphi \in C_c(X)$, the mappings $y \mapsto \int_X \varphi(x)\,d\nu_y(x)$ and $y \mapsto \int_X \varphi(x)\,d\nu^h_{h[y]}(x)$ are continuous and hence the almost everywhere equality is really an equality. $\square$

In view of the previous result, we may apply the theory developed in Section A.2. In particular we obtain (73) in the case in which ω and ρ are the Lebesgue measures:

$$L^2(X) = \int_Y L^2(X,\nu_y)dy, \qquad f = \int_Y f_y dy. \tag{23}$$

Here the equalities must be interpreted in $M(X)$ and the second integral is a scalar integral relative to the duality of $M(X)$ and $C_c(X)$. For a discussion of the details see the Appendix, where it is also explained that in particular

$$\|f\|^2 = \int_Y \|f_y\|^2_{\nu_y}\,dy. \tag{24}$$

One of the reasons for introducing the measures $\{\nu_y\}$ is because, via the coarea formula, they provide a very useful description of the density ω_h discussed in Lemma 1.

Corollary 1. *Given $f, \eta \in L^2(X)$, the function $y \mapsto \langle f_y, \eta_y^h \rangle_{\nu_y}$ coincides almost everywhere with the density ω_h of the measure Ω_h defined by* (16).

Proof. Item (iii) of Theorem 11, together with Theorem 2, applied to $f\bar{\eta} \in L^1(X)$ and any $\xi \in C_c(Y)$ gives

$$\int_X \xi\left(\Phi(x)\right) f(x)\bar{\eta}(h^{-1}.x)\, dx = \int_Y \xi(y) \int_X f(x)\bar{\eta}(h^{-1}.x)\, d\nu_y(x)\, dy.$$

The left hand side is nothing else but the integral $\int_Y \xi(y)\, d\Omega_h(y)$ because Ω_h is the image measure, induced by Φ, of $f\overline{\eta^h}\, dx$. The corollary follows. □

3.3. **Reduction to fibers.** Much of our analysis stems from decomposing the representation space $L^2(\mathbb{R}^d)$ in terms of the measures $\{\nu_y\}$, and from a rather detailed understanding of the H-action on Y. We thus introduce the usual notation for group actions: if $y \in Y$, then H_y is the stabilizer of y, $H[y] = \{h[y] : h \in H\}$ is the corresponding orbit and Y/H the orbit space. At this stage we need a hypothesis ensuring that the Y/H is not a pathological measurable space. It is worth mentioning that this hypothesis is satisfied in all the significant examples that we are aware of. Below we further comment on this.

Assumption 2. We assume that for every $y \in Y$ the H-orbit $H[y]$ is locally closed in Y, i.e., that it is open in its closure or, equivalently, that $H[y]$ is the intersection of an open and a closed set.

The above assumption is not enough to guarantee that the orbit space Y/H is a Hausdorff space, hence locally compact, with respect to the quotient topology. However, it is possible to bypass this topological obstruction by choosing a different parametrization of the H-orbits of Y. Indeed, a result of Effros (Theorem 2.9 in [14]) shows that Assumption 2 is equivalent to the fact that the orbit space Y/H is a standard Borel space. Hence there is a locally compact second countable space Z and a Borel measurable (hence Lebesgue measurable) map $\pi : Y \to Z$ such that $\pi(y) = \pi(y')$ if and only if y and y' belong to the same orbit. To see this, observe that, by definition of standard Borel space, Y/H with the quotient σ-algebra is Borel isomorphic to a Borel subset of a Polish space Z. By Kuratowski's theorem [24], we may assume that $Z = [0, 1]$. Define $\pi(y) = i(\dot{y})$, where $\dot{y}$ is the equivalence class of y in Y/H and i is the Borel isomorphism of Y/H into $[0, 1]$.

In the following we fix the space Z whose points will label the orbits of Y and we choose on Z a pseudo-image measure[2] λ of the Lebesgue measure under the map π. We note that λ is concentrated on $\pi(Y)$ and a subset E is λ-negligible if and only if $|\pi^{-1}(E)|_n = 0$, which is equivalent to $|(\pi \circ \Phi)^{-1}(E)|_d = 0$ (item (v) in Lemma 2).

Theorem 3. *The following facts are equivalent:*

(i) *the vector $\eta \in L^2(\mathbb{R}^d)$ is admissible for U;*

[2] It is a measure on Z whose sets of measure zero are exactly the sets whose preimage with respect to π have measure zero in Y. It always exists since Y is σ-compact: it is enough to take first a finite measure on Y equivalent to the Lebesgue measure (just choose a positive L^1 density), and then to consider the image measure on Z induced by π (see e.g. Chap. VI, Sect. 3.2 in [3]).

(ii) *for λ-almost every $z \in Z$, there exists a point $y \in \pi^{-1}(z)$ such that*

$$\|u\|_{\nu_y}^2 = \int_H |\langle u, \eta_y^h \rangle_{\nu_y}|^2 \frac{dh}{\alpha(h)\beta(h)}, \qquad u \in L^2(X, \nu_y). \tag{25}$$

If (25) *holds true for y, then it holds true for every point in $H[y]$.*

Proof. Given $\eta \in L^2(X, dx)$, write $\eta = \int_Y \eta_y \, dy$ where $\eta_y \in L^2(X, \nu_y)$. Fix $y \in Y$ and put

$$\mathcal{D}_y = \{u \in L^2(X, \nu_y) : \int_H |\langle u, \eta_y^h \rangle_{\nu_y}|^2 \frac{dh}{\alpha(h)\beta(h)} < +\infty\}.$$

The map $\mathcal{W}_y : \mathcal{D}_y \to L^2(H, \alpha(h^{-1})\beta(h^{-1})dh)$, defined by $(\mathcal{W}_y u)(h) = \langle u, \eta_y^h \rangle$ for almost all $h \in H$, is a closed linear operator (the proof is standard [13]). Hence it is enough to prove (25) for a dense countable subset of $L^2(X, \nu_y)$. Hence we fix a countable family of functions $\{\varphi_\ell\}$ in $C_c(X)$ with the following property: given an arbitrary $\varphi \in C_c(X)$, there exists a subsequence $(\varphi_{\ell_k})_{k \in \mathbb{N}}$ such that

$$|\varphi_{\ell_k}| \leq |\varphi_0|, \qquad \lim_{k \to \infty} \sup_{x \in X} |\varphi_{\ell_k}(x) - \varphi(x)| = 0. \tag{26}$$

The existence of such a family is clarified in Footnote 5 in the Appendix. Clearly, for any $y \in Y$, the family $\{\varphi_\ell\}$ is dense in $L^2(X, \nu_y)$.
Assume that U is reproducing and take an admissible $\eta \in L^2(X)$. For any ℓ we thus have

$$\int_G |\langle \varphi_\ell, U_g \eta \rangle|^2 \, dg = \int_X |\varphi_\ell(x)|^2 \, dx = \int_Y \big(\int_X |\varphi_\ell(x)|^2 \, d\nu_y(x) \big) \, dy,$$

the latter being a consequence of the coarea formula (24). By Lemma 1 the measure Ω_h^ℓ in (16) has an L^2-density ω_h^ℓ for almost every $h \in H$ and formula (18) holds true; furthermore, Corollary 1 tells us that ω_h can be expressed in terms of the measures $\{\nu_y\}$. Therefore

$$\begin{aligned}
\int_Y \int_X |\varphi_\ell(x)|^2 \, d\nu_y(x) \, dy &= \int_G |\langle \varphi_\ell, U_g \eta \rangle|^2 \, dg \\
&= \int_H \big(\int_Y |\omega_h^\ell(y)|^2 \, dy \big) \, \frac{dh}{\alpha(h)\beta(h)} \\
&= \int_H \big(\int_Y |\langle \varphi_\ell, \eta_y^h \rangle_{\nu_y}|^2 \, dy \big) \, \frac{dh}{\alpha(h)\beta(h)} \\
&= \int_Y \big(\int_H |\langle \varphi_\ell, \eta_y^h \rangle_{\nu_y}|^2 \big) \, \frac{dh}{\alpha(h)\beta(h)} \, dy,
\end{aligned}$$

where in the last line we have applied Fubini's theorem. Let $N_\ell \subset Y$ be the set of $y \in Y$ where the equality

$$\|\varphi_\ell\|_{\nu_y}^2 = \int_H |\langle \varphi_\ell, \eta_y^h \rangle_{\nu_y}|^2 \frac{dh}{\alpha(h)\beta(h)}. \tag{27}$$

does not hold. Reasoning as in the proof of Corollary 1, the equality of the first and last term of the above string is equivalent to saying that N_ℓ is negligible.
Put $N = \cup_\ell N_\ell$, a negligible set. For any $y \notin N$, (27) shows that $\{\varphi_\ell\} \subset \mathcal{D}_y$ and $\mathcal{W}_y$ is an isometry on this dense subset. Since $\mathcal{W}_y$ is a closed operator, it follows that

$\mathcal{D}_y = L^2(X, \nu_y)$ and (25) holds true for every $u \in L^2(X, \nu_y)$.
Now, N is the set consisting of those $y \in Y$ for which the equality (25) does not hold for at least a $u \in L^2(X, \nu_y)$. We show that N is H-invariant. Take $h \in H$ and $y \notin N$. For any $\varphi \in C_c(X)$, both φ and $\varphi^{h^{-1}}$ are in $L^2(X, \nu_y)$. Hence (25) does hold for $u = \varphi$ and $u = \varphi^{h^{-1}}$. Using (21) and (22), we obtain

$$\begin{aligned}
\int_X |\varphi(x)|^2 \, d\nu_{h[y]}(x) &= \int_X |\varphi(h.x)|^2 \, d\nu^h_{h[y]}(x) \\
&= \int_X |\varphi(h.x)|^2 \, \alpha(h)\beta(h) d\nu_y(x) \\
&= \alpha(h)\beta(h) \int_H \Big| \int_X \varphi(h.x)\bar{\eta}(k^{-1}.x) \, d\nu_y(x) \Big|^2 \frac{dk}{\alpha(k)\beta(k)} \\
(h.x = z) \quad &= \alpha(h)\beta(h) \int_H \Big| \int_X \varphi(z)\bar{\eta}((hk)^{-1}.z) \, d\nu_y^{h^{-1}}(z) \Big|^2 \frac{dk}{\alpha(k)\beta(k)} \\
(hk = s) \quad &= \alpha^2(h)\beta^2(h) \int_H \Big| \int_X \varphi(z)\bar{\eta}(s^{-1}.z) \, d\nu_y^{h^{-1}}(z) \Big|^2 \frac{ds}{\alpha(s)\beta(s)} \\
&= \int_H \Big| \int_X \varphi(z)\bar{\eta}(s^{-1}.z)\alpha(h)\beta(h) \, d\nu_y^{h^{-1}}(z) \Big|^2 \frac{ds}{\alpha(s)\beta(s)} \\
&= \int_H \Big| \int_X \varphi(z)\bar{\eta}(s^{-1}.z) \, d\nu_{h[y]}(z) \Big|^2 \frac{ds}{\alpha(s)\beta(s)} \\
&= \int_H |\langle \varphi, \eta^s \rangle_{\nu_{h[y]}}|^2 \, \frac{ds}{\alpha(s)\beta(s)},
\end{aligned}$$

that is, $h[y] \notin N$, as desired. Finally, since N is H-invariant and negligible, $\pi(\cup_\ell N_\ell)$ is λ-negligible and (ii) follows.

The fact that (ii) implies that U is reproducing is proved by reversing the argument.

□

Remark 4. Since π induces a Borel isomorphism between the orbit space Y/H and $\pi(Y)$, in the above statement and in the theorems of the following section it would be possible to avoid the space Z by considering on Y/H a σ-finite measure defined on the quotient σ-algebra, which, by Assumption 2 (Theorem 2.9 in [14]), coincides with the Borel σ-algebra induced by the quotient topology. However, this measure could fail to be finite on compact subsets.

3.4. **Disintegration formulae.** Our next result, Theorem 6, is based on some classical formulae that allow both a geometric interpretation of the integral (25) and a computational reduction that in the known examples is indeed significant. This is inspired by the irreducible case, where it is known that U is reproducing (i.e. square integrable) if and only if the H-orbit, unique by irreducibility, has full measure and the inducing representation of the stabilizer H_y is square integrable [1].

We allude to formulae that express an integral over Y as a double integral, first along the single H-orbits and then with respect to the measure λ on the space Z. Although these kinds of formulae can be traced back to Bourbaki [4] and Mackey [28], perhaps one of the most famous occurrences of such a disintegration procedure appears in the celebrated paper of Kleppner and Lipsman [23]; for a recent review see [19]. Much in the same spirit, we shall also need to decompose integrals over H by integrating along a closed subgroup H_0 first, and then over the homogeneous space H/H_0, which we identify with a suitable orbit of Y. The topological hypothesis formulated in Assumption 2 is needed in order that these decomposition formulae can be safely applied.

Recall that in the beginning of Section 3.3 we fixed a space Z that labels the orbits of Y and a measure λ on Z whose null sets are in one-to-one correspondence with the H-invariant null sets of Y.

Theorem 4. *There exists a family $\{\tau_z\}$ of measures on Y, labeled by the points of Z, with the following properties:*

(i) *τ_z is concentrated on $\pi^{-1}(z)$ for all $z \in Z$;*
(ii) *$dy = \int_Z \tau_z \, d\lambda(z)$.*

Furthermore, for almost every $z \in Z$ the measure τ_z is relatively invariant and

$$\tau_z^h = \alpha(h)^{-1} \tau_z \tag{28}$$

holds for every $h \in H$. The family $\{\tau_z\}$ is unique in the sense that if $\{\tau_z'\}$ is another family satisfying (i) *and* (ii)*, then $\tau_z' = \tau_z$ for almost every $z \in Z$.*

Proof. The content of the theorem can be found in many different papers, such as Lemmas 11.1 and 11.5 in [28] and Theorem 2.1 of [23], in slightly different contexts. The cited results are both based on Bourbaki's treatment of disintegration of measures. Here we simply adapt this theory to our setting.
Theorem 2 Ch.VI § 3.3 of [3] yields a family $\{\tau_z\}$ of measures on Y labeled by the points $z \in Z$, unique in the sense of the statement, such that

- $\tau_z \neq 0$ if and only if $z \in \pi(Y)$
- τ_z is concentrated on $\pi^{-1}(z)$
- $dy = \int_Z \tau_z d\lambda(z)$.

The proof of Lemma 11.5 in [28] shows, under the circumstances that we are considering, that for almost all $z \in Z$ (28) holds true for all $h \in H$; the density appearing in Lemma 11.4 of [28] is precisely α^{-1}. □

3.4.1. *A topological detour.* Assumption 2 is needed in order to prove Theorem 4 because we apply results on disintegration of measures that use it, as developed in [3]. The same theorem actually holds under the (weaker) conditions that are described in the proposition below. Their equivalence does not seem to be a known fact. In [19], Theorem 12, it is shown that (ii) in Lemma 2 below is a necessary condition for the disintegration in Theorem 4 to hold true. In the next statement $\hat{\pi}$ denotes the canonical projection from Y onto Y/H.

Proposition 2. *The following two conditions are equivalent:*

(i) there exists an increasing sequence of compact subset $\{K_n\}$ of Y such that the complement of $\cup K_n$ is Lebesgue negligible and $\hat{\pi}(K_n)$ endowed with the relative topology is a Hausdorff space;
(ii) there exists an H-invariant null set $N \subset Y$ such that $(Y \setminus N)/H$ is a standard Borel space with respect to the σ-algebra induced by $\hat{\pi}$.

Proof. First we show that (i) implies (ii). Denote by R the equivalence relation induced by the action of H on Y, that is, $y \sim_R y'$ if and only if $\hat{\pi}(y) = \hat{\pi}(y')$.
Claim 1: there exists a Lebesgue measurable map p from Y into a locally compact second countable space Ω with the property

$$p(y) = p(y') \qquad \Longleftrightarrow \qquad y \sim_R y'. \tag{29}$$

By assumption for each n the space $\hat{\pi}(K_n)$ is Hausdorff and, by Prop. 3 Ch.1 § 5.3 of [2], this is equivalent to the fact the quotient space K_n/R_n is Hausdorff with respect to the quotient topology, where R_n the restriction of R to $K_n \times K_n$. Since Y is σ-compact, the above property implies that R is a Lebesgue measurable equivalence relation according to the definition in Ch. VI § 3.4 of [3]. By Proposition 2 Ch. VI § 3.4 of [3] there exists a map $p : Y \to \Omega$ with the desired properties.
Claim 2: for any compact set K of Y, the set $H[K]$ is Borel measurable. Indeed, since H is σ-compact, there exists a countable family $\{H_m\}$ of compact subsets of H such that $H = \cup_m H_m$ and, hence, $H[K] = \cup_m H_m[K]$. Hence $H[K]$ is countable union of compact subsets, hence Borel measurable, since the action of H on Y is continuous and $H_m \times K$ is compact.
Claim 3: there exists an H-invariant Borel set Y_1 whose complement is Lebesgue negligible and such that the restriction $p_{|Y_1}$ is Borel measurable. The proof of Proposition 2 Ch. VI § 3.4 of [3] actually implies the claim. For completeness, however, we present a direct proof. Lusin's theorem[3] yields an increasing sequence of compact subsets $\{K'_m\}$ of Y such that the complement of $\cup K'_m$ is Lebesgue negligible and the restriction of p to each K_m is continuous. By Claim 2 the set $Y_1 = H[\cup_m K'_m]$ and its complement $N_1 = Y \setminus Y_1$ are both H-invariant Borel subsets, and N_1 is Lebesgue negligible since $N_1 \subset Y \setminus \cup_m K'_m$. To prove that $p_{|Y_1}$ is Borel measurable, for any closed subset $C \subset \Omega$

$$\begin{aligned} p_{|Y_1}^{-1}(C) &= p^{-1}(C) \cap Y_1 = \cup_m p^{-1}(C) \cap H[K'_m] \\ &= \cup_m H[C \cap K'_m] = \cup_m H[p_{|K_m}^{-1}(C)], \end{aligned}$$

since $p^{-1}(C) = H[p^{-1}(C)]$ by (29). Since $p_{|K_m}^{-1}(C)$ is compact, Claim 2 implies that $p_{|Y_1}^{-1}(C)$ is Borel measurable.
Claim 4: the quotient space Y_1/H is analytic. Since Y_1 is a Borel subset of a locally compact second countable space, it is standard and, hence, analytic. By Theorem 5.1 of [29], if a quotient space of an analytic Borel space is countably separated, then it is analytic. Hence, it is enough to exhibit a countable family $\{A_m\}$ of H-invariant Borel sets of Y_1 with the property that for any pair of points $y, y' \in Y_1$ such that $y \not\sim_R y'$, there exists A_m such that $y \in A_m$ and $y' \notin A_m$. To find such a family, choose a countable base $\{V_m\}$ for the second countable topology of Ω and define $A_m = p_{|Y_1}^{-1}(V_m)$, which is an H-invariant Borel subset of Y_1 by (29) and Claim 3. If $y \not\sim_R y'$, then $p(y) \neq p(y')$ and, since Ω is Hausdorff, there exists V_m such that $p(y) \in V_m$ and $p(y') \notin V_m$, that is, $y \in A_m$ and $y' \notin A_m$.
Claim 5: there exists an H-invariant Borel set $Y_2 \subset Y_1$ whose complement is Lebesgue negligible and Y_2/H is a standard Borel space. Since Y is second countable, there exists a finite measure on the analytic space Y_1/H, which is the pseudo-image measure of the

[3]See, for example, Theorem 5.6.23 [31] or the definition of measurable function given in [3].

Lebesgue measure of Y. By Theorem 6.1 of [29], there exists a Borel subset $E \subset Y_1/H$ whose complement is negligible and E is a standard Borel space. The set $Y_2 = \hat{\pi}^{-1}(E)$ has the desired properties.

Item (ii) is proved by setting $N = Y \setminus Y_2 = N_1 \cup (Y_1 \setminus Y_2)$ and observing that $(Y \setminus N)/H$ is Borel isomorphic to E.

We now show that (ii) implies (i). By assumption there exists a Borel H-invariant Borel set $N \subset Y$ with zero Lebesgue measure such that $(Y \setminus N)/H$ is Borel isomorphic to a Borel subset of $[0,1]$ and, hence, there exists a Borel injective map $j : (Y \setminus N)/H \to \mathbb{R}$. If $N \neq \emptyset$, fix a section $s : N/H \to N$, a point $y_0 \in Y \setminus N$, and define $p : Y \to Y \times [0,1]$ by

$$p(y) = \begin{cases} (y_0, i(\hat{\pi}(y))) & y \notin N \\ (s(\hat{\pi}(y)), 0) & y \in N. \end{cases}$$

Clearly, the map p is Lebesgue measurable and $p(y') = p(y)$ if and only if $\hat{\pi}(y) = \hat{\pi}(y')$. Lusin's theorem implies that there exists an increasing sequence of compact subsets $\{K_m\}$ such that the complement of $\cup K_m$ is Lebesgue negligible and the restriction of p to each K_m is continuous. By a standard result in topology, (see e.g. Corollary 1 of Proposition 8 § 10.6 of [2]), $\hat{\pi}(K_m)$ is homeomorphic to $p(K_m)$ which is a compact subset of a Hausdorff space, so it is Hausdorff. $\square$

In the statement of the above proposition Y can be replaced by any locally compact second countable space, the Lebesgue measure by a measure on Y and the equivalence relation induced by H by any other equivalence relation.

3.5. **The integral decomposition of U.** From now on Assumptions 1 and 2 are taken for granted. The main result here is that Theorems 2 and 4, which hold both true, yield an integral decomposition of the mock metaplectic representation in terms of induced representations of the isotropy subgroups of H. This fact, which is of independent interest, is at the root of Theorem 6, where the admissible vectors for U are characterized.

Proposition 3. *For almost every $z \in Z$ the family of measures $\{\nu_y\}$ is scalarly integrable with respect to τ_z, the measure on X*

$$\mu_z = \int_Y \nu_y \, d\tau_z(y)$$

is concentrated on the H-invariant subset $\Phi^{-1}(\pi^{-1}(z))$ and for all $h \in H$

$$\mu_z^h = \beta(h)\mu_z. \tag{30}$$

Furthermore, the family of measures $\{\mu_z\}$ is scalarly integrable with respect to λ and

$$dx = \int_Z \mu_z \, d\lambda(z). \tag{31}$$

Proof. The map $\pi \circ \Phi$ is a Lebesgue measurable map from X to Z and λ is a pseudo-image measure of the Lebesgue measure restricted to X under $\pi \circ \Phi$ by construction of λ and Assumption 1. Hence, Theorem 2 Ch. VI § 3.3 of [3] yields a family $\{\mu_z\}$ of positive measures on X such that each μ_z is concentrated on $\Phi^{-1}(\pi^{-1}(z))$ and, for all $\varphi \in C_c(X)$

$$\int_X \varphi(x)dx = \int_Z \Big(\int_X \varphi(x)d\mu_z(x)\Big)d\lambda(z). \tag{32}$$

For any fixed $\varphi \in C_c(X)$, $y \mapsto \int_X \varphi(x)d\nu_y(x)$ is Lebesgue integrable by (ii) of Theorem 2. Hence, appealing to (ii) of Theorem 4 and to (iii) of Theorem 11, we know that for almost all $z \in Z$, the map $y \mapsto \int_X \varphi(x)d\nu_y(x)$ is τ_z-integrable, the map $z \mapsto \int_Y (\int_X \varphi(x)d\nu_y(x))d\tau_z(y)$ is λ-integrable, and

$$\int_Z \Big(\int_Y \Big(\int_X \varphi(x)d\nu_y(x)\Big)d\tau_z(y)\Big)d\lambda(z) = \int_Y \Big(\int_X \varphi(x)d\nu_y(x)\Big)dy = \int_X \varphi(x)dx.$$

Comparing this with (32) we infer that for almost every $z \in Z$

$$\int_Y \Big(\int_X \varphi(x)d\nu_y(x)\Big)d\tau_z(y) = \int_X \varphi(x)d\mu_z(x). \tag{33}$$

The set N of $z \in Z$ where the above inequality does not hold is λ-negligible and, can be chosen independently of φ. Indeed, as explained in Footnote 5 we may find a countable subset $\mathcal{S}$ of $C_c(X)$ such that, for any $\varphi \in C_c(X)$, there is a sequence (φ_i) in $\mathcal{S}$ converging to φ uniformly and $|\varphi_i| \leq |\varphi_0|$ for all i. For each $\varphi \in \mathcal{S}$ there is a negligible set $N_\varphi \subset Z$ such that the map $y \mapsto \int_X \varphi(x)\, d\nu_y(x)$ is integrable with respect to τ_z for all $z \notin N_\varphi$. Denote by N the λ-negligible set $\cup_{\varphi\in\mathcal{S}} N_\varphi$. We now claim that the family $\{\nu_y\}$ is scalarly integrable with respect to τ_z for all $z \notin N$. Indeed, given $\varphi \in C_c(X)$, there is a sequence (φ_i) in $\mathcal{S}$ converging to φ uniformly and $|\varphi_i| \leq |\varphi_0|$ for all i. Write (33) for each φ_i. Since $|\varphi_i| \leq |\varphi_0|$ we may apply the dominated convergence theorem to the right hand side. As for the left hand side, for the same reason we may apply the dominated convergence theorem to the inner integral. Further, since $y \mapsto \nu_y(\operatorname{supp}\varphi_0)$ is τ_z-integrable we may apply dominated convergence to the outer integral. The claimed independence of φ is proved.

Hence for all $z \notin N$, the family $\{\nu_y\}$ is scalarly integrable with respect to τ_z and $\mu_z = \int_Y \nu_y\, d\tau_z(y)$. Finally, fix $z \notin N$ and $h \in H$. For all $\varphi \in C_c(X)$

$$\begin{aligned}
\int_X \varphi(h^{-1}.x)d\mu_z(x) &= \int_Y \left(\int_X \varphi(h^{-1}.x)\, d\nu_y(x)\right) d\tau_z(y) \\
&= \alpha(h)\beta(h) \int_Y \left(\int_X \varphi(x)\, d\nu_{h^{-1}[y]}(x)\right) d\tau_z(y) \\
&= \beta(h) \int_Y \left(\int_X \varphi(x)\, d\nu_y(x)\right) d\tau_z(y) \\
&= \beta(h) \int_X \varphi(x)d\mu_z(x)
\end{aligned}$$

where the second line is due to the change of variables $x \mapsto h.x$ and (22), and the third line to $y \mapsto h.y$ and (28). This proves that $\mu_z^h = \beta(h)\mu_z$. □

By virtue of Proposition 3 we may consider the Hilbert space $L^2(X, \mu_z)$ for almost every $z \in Z$. Whenever μ_z is not defined, we redefine $\tau_z = 0$ and $\mu_z = 0$, and set

$L^2(X, \mu_z) = \{0\}$. Proposition 6 below, or equation (73), both based on Proposition 3, will allow the following Hilbert space identifications

$$L^2(X) = \int_Z L^2(X, \mu_z)\, d\lambda(z) \qquad f = \int_Z f_z\, d\lambda(z) \tag{34}$$

$$L^2(X, \mu_z) = \int_Y L^2(X, \nu_y)\, d\tau_z(y) \qquad f_z = \int_Y f_{z,y}\, d\tau_z(y), \tag{35}$$

where $f \in L^2(X)$, $f_z \in L^2(X, \mu_z)$ for all $z \in Z$ and, fixed z, $f_{z,y} \in L^2(X, \nu_y)$ for all $y \in Y$. The integrals of Hilbert spaces are direct integrals with respect to the measurable field associated with $C_c(X)$, and the integral of functions are scalar integrals of vector valued functions taking value in $M(X)$. Indeed, as explained in the Appendix, we shall regard $L^2(X)$, $L^2(X, \mu_z)$ and $L^2(X, \nu_y)$ as subspaces of $M(X)$ in the natural way. In particular, if $f \in C_c(X)$, f_z is the restriction of f to $\Phi^{-1}(\pi^{-1}(z))$ and $f_{z,y}$ is the restriction to $\Phi^{-1}(y)$. Furthermore, for any $f \in L^2(X)$

$$\|f\|^2 = \int_Z \int_Y \|f_{z,y}\|^2_{\nu_y}\, d\tau_z(y)\, d\lambda(z). \tag{36}$$

Formula (34) induces the following decomposition of U.

Lemma 3. *The representation U is the direct integral of the family $\{U_z\}$ of representations acting on $L^2(X, \mu_z)$ by*

$$(U_{z,g} f)(x) = \beta(h)^{-\frac{1}{2}} e^{-2\pi i \langle \Phi(x), a\rangle} f(h^{-1}.x)$$

for $g = (a, h) \in G$ and $f \in L^2(X, \mu_z)$.

Proof. For each $z \in Z$, the map $g \mapsto U_{z,g}$ is a strongly continuous unitary representation of G by the same proof of Proposition 1 since μ_z and the Lebesgue measure are both relatively invariant with the same character β, (compare (7) with (30)). We now prove that $\{U_z\}$ is a λ-measurable field of representations. Indeed, for any $g \in G$ and $\varphi, \varphi' \in C_c(X)$,

$$\langle U_z \varphi, \varphi' \rangle_{\mu_z} = \int_X \beta(h)^{-\frac{1}{2}} e^{-2\pi i \langle \Phi(x), a\rangle} \varphi(h^{-1}.x) \overline{\varphi'(x)}\, d\mu_z(x).$$

Since $x \mapsto e^{-2\pi i \langle \Phi(x), a\rangle} \varphi(h^{-1}.x)\overline{\varphi'(x)}$ is a compactly supported continuous function and the family $\{\mu_z\}$ is λ-scalarly integrable, the map $x \mapsto \langle U_z \varphi, \varphi' \rangle_{\mu_z}$ is λ-integrable, hence λ-measurable.

Finally, to prove that $U = \int_Z U_z\, dz$ it is enough to test the equality on $C_c(X)$. For any $g \in G$ and $\varphi \in C_c(X)$, we regard $U_g \varphi$ and $U_{z,g}\varphi$ as elements of $M(X)$. Hence, (31) gives

$$U_g \varphi \cdot dx = \int_Z (U_g \varphi \cdot \mu_z) dz = \int_Z (U_{z,g} \varphi \cdot \mu_z) dz$$

by definition of U_z. $\square$

The next technical lemma is needed in order to prove that U_z is equivalent to an induced representation.

Lemma 4. *Fix $y \in Y$ and $h \in H$. The map $T_{y,h} : L^2(X, \nu_y) \to L^2(X, \nu_{h[y]})$ defined for $\nu_{h[y]}$-almost every $x \in X$ by*

$$(T_{y,h}f)(x) = \sqrt{\alpha(h^{-1})\beta(h^{-1})}\, f(h^{-1}.x)$$

is a unitary operator. Furthermore, for every $h, h' \in H$ and every $y \in Y$

$$T_{h[y],h'} T_{y,h} = T_{y,h'h} \tag{37}$$

$$T_{y,h}^{-1} = T_{h[y],h^{-1}}. \tag{38}$$

Proof. Given a Borel measurable function f which is square-integrable with respect to ν_y, the map $x \mapsto (T_{y,h}f)(x)$ is also Borel measurable and it is square-integrable with respect to $\nu_{h[y]}$ since

$$\alpha(h^{-1})\beta(h^{-1}) \int_X |f(h^{-1}.x)|^2 \, d\nu_{h[y]}(x) = \int_X |f(x)|^2 \, d\nu_y(x),$$

by the change of variables $x \mapsto h.x$ and (22). The above equation implies that $T_{y,h}$ is a well-defined isometry from $L^2(X, \nu_y)$ to $L^2(X, \nu_{h[y]})$. Equality (37) is clear and, as a consequence, $T_{h[y],h^{-1}} T_{y,h} = T_{y,e}$ is the identity on $L^2(X, \nu_{h[y]})$ so that $T_{y,h}$ is surjective, thereby showing (38). □

For any $z \in \pi(Y)$, we fix an origin y_0 in the orbit $\pi^{-1}(z) = H[y_0]$ and we denote by H_z the stabilizer at y_0. We denote by $\mathcal{K}_z = L^2(X, \nu_{y_0})$.

By (22) we know that ν_{y_0} is relatively invariant under H_z. It follows that it make sense to look at the quasi-regular representation Λ_z of H_z acting on $\mathcal{K}_z$, whose value at $s \in H_z$ is $\Lambda_{z,s} = T_{y_0,s}$. As usual, we extend Λ_z to a representation of $\mathbb{R}^n \rtimes H_z$ by setting $\Lambda_{z,a} = e^{-2\pi i \langle y_0, a\rangle}\,\mathrm{id}$ for all $a \in \mathbb{R}^n$. Finally, we denote by W_z the representation of G unitarily induced by Λ_z from $\mathbb{R}^n \rtimes H_z$ to G. We realize W_z as a representation acting on the space $\mathcal{H}_z$ of those functions $F : G \to \mathcal{K}_z$ that satisfy

(K1) F is dg-measurable;

(K2) For all $g \in G$ and $(a, s) \in \mathbb{R}^n \rtimes H_z$

$$F(gas) = \sqrt{\alpha(s^{-1})}\; \Lambda_{z,as}^{-1} F(g);$$

(K3) $\|F\|^2_{\mathcal{H}_z} := \displaystyle\int_Y \|F(h(y))\|^2_{\mathcal{K}_z}\, \alpha(h(y)) d\tau_z(y) < +\infty.$

Here $h(y) \in H$ is any element in H that satisfies $h(y)[y_0] = y$ for τ_z- almost all $y \in Y$. Since τ_z is concentrated on $H[y_0]$, it is enough to define $h(y)$ for $y \in H[y_0]$ and, due to the covariance property in (K2), the integral does not depend on the choice of $h(y)$ in the coset hH_z. Furthermore, (K2) implies that it is enough to know these functions on H. Two functions F and F' are identified if $\|F - F'\|^2_{\mathcal{H}_z} = 0$. The induced representation on $\mathcal{H}_z$ is defined for $g \in G$ by the equality

$$(W_{z,g}F)(g') = F(g^{-1}g')$$

valid for dg-almost every $g' \in G$.

For the sake of precision, if $z \in Z \setminus \pi(Y)$ we put $\nu_z = 0$, $\mathcal{K}_z = \{0\}$ and $H_z = \{e\}$; recall that $\tau_z = 0$ and that $\lambda(Z \setminus \pi(Y)) = 0$.

Lemma 5. *Fix $z \in Z$ such that $\tau_z \neq 0$. The map $S_z : L^2(X, \mu_z) \to \mathcal{H}_z$ whose value at $f_z = \int_Y f_{z,y}\, d\tau_z(y)$ is given by*

$$(S_z f_z)(a,h) = \sqrt{\alpha(h^{-1})}\ e^{2\pi i \langle h[y_0], a\rangle}\ T^{-1}_{y_0,h}(f_{z,h[y_0]})$$

is a unitary operator intertwining U_z with S_z.

Proof. For any $(a,h) \in G$, $f_{z,h[y_0]} \in L^2(X, \nu_{h[y_0]})$. Hence $T^{-1}_{y_0,h}(f_{z,h[y_0]}) \in \mathcal{K}_z$. In order to prove that $S_z f_z$ is dg-measurable it is enough to show that

$$h \mapsto \langle T^{-1}_{y_0,h}(f_{z,h[y_0]}), \varphi\rangle_{\mathcal{K}_z} = \sqrt{\alpha(h)\beta(h)} \int_X f_{z,h[y_0]}(h.x)\varphi(x) d\nu_{y_0}(x)$$

is dh-measurable for every $\varphi \in C_c(X)$ because $C_c(X)$ is a dense subspace of the separable Hilbert space $\mathcal{K}_z$. Since $f_z = \int_Y f_{z,y}\, d\tau_z(y)$, there exists a square-integrable function $\tilde{f} : X \to \mathbb{C}$ and a τ_z-negligible set $N \subset Y$ such that, for all $y \notin N$, $\tilde{f}$ belongs to the equivalence class of $f_{z,y} \in L^2(X, \nu_y)$. Define $N' = \{h \in H \mid h[y_0] \in N\}$, a negligible set with respect to the Haar measure dh because, by (28), τ_z is non-zero relatively invariant on the orbit $H[y_0]$. Then for all $h \notin N'$

$$\begin{aligned} h \mapsto \sqrt{\alpha(h)\beta(h)} &\int_X f_{z,h[y_0]}(h.x)\varphi(x) d\nu_{y_0}(x) \\ = \sqrt{\alpha(h)\beta(h)} &\int_X \tilde{f}(h.x)\varphi(x) d\nu_{y_0}(x), \end{aligned}$$

which is clearly dh-measurable. Next we prove the covariance property (K2). For $g = (a,h) = ah$ and $(b,s) = bs \in \mathbb{R}^n \rtimes H_z$,

$$\begin{aligned} (S_z f_z)(ahbs) &= (S_z f_z)(a + h^\dagger[b], hs) \\ &= \sqrt{\alpha(h^{-1})\alpha(s^{-1})}\ e^{2\pi i \langle hs[y_0], a + h^\dagger[b]\rangle}\ T^{-1}_{y_0,hs}(f_{z,hs[y_0]}) \\ &= \sqrt{\alpha(s^{-1})}\ e^{2\pi i \langle h[y_0], h^\dagger[b]\rangle}\ T^{-1}_{y_0,s}(S_z f_z)(a,h) \\ &= \sqrt{\alpha(s^{-1})}\ e^{2\pi i \langle y_0, b\rangle}\ \Lambda^{-1}_{y_0,s}(S_z f_z)(a,h) \end{aligned}$$

by definition of $h^\dagger$ and Λ_z. Further,

$$\begin{aligned} \int_Y \|(S_z f_z)(h(y))\|^2_{\mathcal{K}_z}\, \alpha(h(y)) d\tau_z(y) &= \int_Y \|T^{-1}_{y_0,h(y)}(f_{z,h(y)[y_0]})\|^2_{\mathcal{K}_z} d\tau_z(y) \\ &= \int_Y \|f_{z,y}\|^2_{\nu_y} d\tau_z(y) = \int_X |f(x)|^2 d\mu_z(x), \end{aligned}$$

whence (K3). This also shows that S_z is an isometry from $L^2(X, \mu_z)$ into $\mathcal{H}_z$.
Finally we prove that S_z is surjective. Given $F \in \mathcal{H}_z$, for all $h \in H$ define

$$f_{z,h} = \sqrt{\alpha(h)}\ T_{y_0,h}(F(h)) \in L^2(X, \nu_{h[y_0]}).$$

Since F satisfies (K2), it follows that $f_{z,hs} = f_{z,h}$. For $\varphi \in C_c(X)$ the map

$$h \mapsto \sqrt{\alpha(h)} \langle T_{y_0,h}(F(h)), \varphi\rangle_{\nu_{h[y_0]}} = \sqrt{\alpha(h)} \langle F(h), \varphi^h\rangle_{\mathcal{K}_z}$$

is dh-measurable since $h \mapsto F(h)$ is dh-measurable from H into $\mathcal{K}_z$ and the map $h \mapsto \sqrt{\alpha(h)}\varphi^h$ is continuous from H into $\mathcal{K}_z$. Therefore

$$\int_Y \|f_{z,h(y)}\|^2_{\nu_y}\, d\tau_z(y) = \int_Y \|F(h(y))\|^2_{\mathcal{K}_z}\, \alpha(h(y)) d\tau_z(y) < +\infty.$$

It follows that $f_z = \int_Y f_{z,h(y)}\, d\tau_z(y)$ is in $\int_Y L^2(X,\nu_y)\, d\tau_z(y) = L^2(X,\mu_z)$ and, by construction, $S_z f_z = F$.
Finally, we check the intertwining property on the dense subset $C_c(X)$ of $L^2(X)$. If $g = a \in \mathbb{R}^n$, for any $\varphi \in C_c(X)$ and for almost every $h \in H$

$$\begin{aligned}(S_z(U_a\varphi))(h) &= \sqrt{\alpha(h^{-1})}\, T^{-1}_{y_0,h}\left(e^{-2\pi i\langle \Phi(\cdot),a\rangle}\varphi\right)\\ &= \sqrt{\alpha(h^{-1})}\, e^{-2\pi i\langle h[y_0],a\rangle} T^{-1}_{y_0,h}\varphi\\ &= (S_z\varphi)(-a,h) = (S_z\varphi)(a^{-1}h)\end{aligned}$$

where, in the second line, we have used $\Phi(x) = h[y_0]$ for $\nu_{h[y_0]}$-almost every $x \in X$. If $g = k \in H$,

$$\begin{aligned}(S_z(U_k\varphi))(h) &= \sqrt{\alpha(h^{-1})}\, T^{-1}_{y_0,h}\left(\sqrt{\beta(k^{-1})}\varphi^k\right)\\ &= \sqrt{\alpha(h^{-1})}\sqrt{\alpha(k)} T^{-1}_{y_0,h}(T_{k^{-1}h[y_0],k}\varphi)\\ &= \sqrt{\alpha((k^{-1}h)^{-1})}\left(T^{-1}_{k^{-1}h[y_0],k} T_{y_0,h}\right)^{-1}\varphi\\ &= \sqrt{\alpha((k^{-1}h)^{-1})}\left(T_{h[y_0],k^{-1}} T_{y_0,h}\right)^{-1}\varphi\\ &= \sqrt{\alpha((k^{-1}h)^{-1})}\left(T_{y_0,k^{-1}h}\right)^{-1}\varphi = (S_z\varphi)(k^{-1}h).\end{aligned}$$

Since two functions in $\mathcal{H}_z$ that are equal for almost every $h \in H$, are equal almost everywhere in G, the intertwining is proved. □

Recall that $L^2(X) = \int_Z L^2(X,\mu_z)\, dz$, where the direct integral is defined by the measurable structure associated with any fixed dense countable family $\{\varphi_k\}$ in $C_c(X)$. Clearly, $z \mapsto \{S_z\varphi_k\}$ is a measurable structure for the family $\{\mathcal{H}_z\}$, and we define the direct integral $\mathcal{H} = \int_Z \mathcal{H}_z\, dz$.

Theorem 5. *The map $S : L^2(X) \to \mathcal{H}$*

$$Sf = \int_Z S_z f_z\, dz \qquad f = \int_Z f_z\, dz$$

is a unitary map intertwining the mock metaplectic representation U with the unitary representation W of G acting on $\mathcal{H}$ given by

$$W = \int_Z W_z\, dz.$$

Proof. The statement follows from the definition of the measurable structure for the direct integral $\int_Z \mathcal{H}_z\, dz$, from Lemma 3 and Lemma 5. □

3.6. **Admissible vectors.** We are in a position to state our main result. We need, however, a last disintegration formula, sometimes referred to as Weil's formula (see e.g. [17]), a rather straightforward consequence of the theory of quasi-invariant measures on homogeneous spaces. The easiest way of formulating it is perhaps that for any $\varphi \in C_c(H)$ the following integral formula holds

$$\int_H \varphi(h)\alpha(h^{-1})dh = \int_Y \left(\int_{H_z} \varphi(h(y)s)ds\right) d\tau_z(y), \tag{39}$$

where ds is a suitable Haar measure on the stabilizer H_z and where as before $h(y) \in H$ is any element that satisfies $h(y)[y_0] = y$ for τ_z-almost every $y \in Y$. We interpret (39) along the same lines of thought that we have followed for the other formulae by writing

$$\alpha^{-1} \cdot dh = \int_Y (ds)^{h(y)^{-1}} d\tau_z(y) \tag{40}$$

as an equality of measures on H. This time ds is regarded as a measure on H concentrated on H_z, so that the translated measure $(ds)^{h(y)^{-1}}$ is concentrated on $h(y)H_z$. As usual, we shall extend (39) to L^1-functions by means of Theorem 11. By Theorem 2 (and the comments below) in Ch. VII § 3.5 of [4], for all $s \in H_z$ the modular functions of H and H_z are related by the formula

$$\alpha^{-1}(s) = \frac{\Delta_{H_z}(s)}{\Delta_H(s)}. \tag{41}$$

Theorem 5 establishes that U and W are equivalent. Therefore, we formulate our necessary and sufficient condition for the existence of admissible vectors of U for those of W. Thus, any admissible vector $F \in \mathcal{H}$ for W is to be thought of as the image under $S : L^2(X) \to \mathcal{H}$ of an analyzing wavelet η.

Theorem 6. *The function $F = \int F_z\, d\lambda(z)$ is an admissible vector for W if and only if for almost every $z \in Z$ and for every $u \in \mathcal{K}_z = L^2(X, \nu_{y_0})$*

$$\|u\|^2_{\mathcal{K}_z} = \int_Y \Big(\int_{H_z} |\langle u, \Lambda_{z,s}\left(F_z\Delta_G^{-1/2}\right)(h(y))\rangle_{\mathcal{K}_z}|^2\, ds\Big)\alpha(h(y))\, d\tau_z(y). \tag{42}$$

Proof. By the definition of T given in Lemma 4, for every $h \in H$ and $y_0 \in Y$

$$T_{y_0,h^{-1}}(\eta^h)_{z,y_0}(x) = \sqrt{\alpha(h)\beta(h)}(\eta^h)_{z,h^{-1}[y_0]}(h.x) = \sqrt{\alpha(h)\beta(h)}\eta_{z,h^{-1}[y_0]}(x)$$

holds for any $\eta = \int_Z \int_Y \eta_{z,y}\, d\tau_z(y)\, d\lambda(z) \in L^2(X)$ and hence

$$(\eta^h)_{z,y_0} = \sqrt{\alpha(h)\beta(h)}\,(T_{y_0,h^{-1}})^{-1}\,\eta_{z,h^{-1}[y_0]}.$$

Suppose now that η is an admissible vector for U or, equivalently, that $F = S\eta$ is such for W. By Theorem 3, what we have just established and the definition of S given in

Lemma 5, for almost every $z \in Z$ and any fixed $y_0 \in \pi^{-1}(z)$

$$
\begin{aligned}
\|u\|^2_{\mathcal{K}_z} &= \int_H |\langle u, (\eta^h)_{z,y_0}\rangle|^2 \frac{dh}{\alpha(h)\beta(h)} \\
&= \int_H |\langle u, \sqrt{\alpha(h)\beta(h)}\, (T_{y_0,h^{-1}})^{-1}\, \eta_{z,h^{-1}[y_0]}\rangle|^2 \frac{dh}{\alpha(h)\beta(h)} \\
&= \int_H |\langle u, S_z\eta_z(h^{-1})\rangle|^2 \frac{dh}{\alpha(h)} \\
&= \int_H |\langle u, F_z(h^{-1})\rangle|^2 \frac{dh}{\alpha(h)} \\
(h \mapsto h^{-1}) \quad &= \int_H |\langle u, F_z(h)\rangle|^2 \Delta_H(h^{-1})\alpha(h)\, dh.
\end{aligned} \tag{43}
$$

Hence, applying (39), the covariance property (K2), (41) and (5) we obtain

$$
\begin{aligned}
\|u\|^2_{\mathcal{K}_z} &= \int_Y \left(\int_{H_z} |\langle u, F(h(y)s)\rangle|^2 \frac{\alpha^2(h(y)s)}{\Delta_H(h(y)s))}\, ds \right) d\tau_z(y) \\
&= \int_Y \left(\int_{H_z} |\langle u, \sqrt{\alpha(s^{-1})}\Lambda_{z,s^{-1}} F(h(y))\rangle|^2 \frac{\alpha^2(h(y)s)}{\Delta_H(h(y)s))}\, ds \right) d\tau_z(y) \\
&= \int_Y \left(\int_{H_z} |\langle u, \Lambda_{z,s^{-1}} F(h(y))\rangle|^2 \frac{\alpha^2(h(y))}{\Delta_H(h(y))} \Delta_{H_z}(s^{-1})\, ds \right) d\tau_z(y) \\
(s \mapsto s^{-1}) \quad &= \int_Y \left(\int_{H_z} |\langle u, \Lambda_{z,s} F(h(y))\rangle|^2 \frac{1}{\Delta_G(h(y))}\, ds \right) \alpha(h(y)) d\tau_z(y),
\end{aligned}
$$

which is (42). Conversely, if (42) holds for some $F \in \mathcal{H}$, then reading the above strings of equalities backwards yields the first line in (43). Therefore, by Theorem 3, η is admissible for U, hence F is such for W. $\square$

Corollary 2. *Assume that U is a reproducing representation and suppose that $z \in Z$ is such that* (42) *holds true. Then:*

(i) if $\Phi^{-1}(y_0)$ is a finite set for some $y_0 \in \pi^{-1}(z)$, then the stabilizer H_y is compact for every $y \in \pi^{-1}(z)$;

(ii) if G is unimodular and the stabilizer H_y is compact, then $\Phi^{-1}(y)$ is a finite set, hence $n = d$.

Proof. Clearly, it is enough to prove (i) and (ii) for the origin y_0. Take a (countable) Hilbert basis $\{u_i\}$ of $\mathcal{K}_z$. Apply (42) to each element of the basis and sum

$$
\begin{aligned}
\dim \mathcal{K}_z &= \int_Y \Big(\int_{H_z} \sum_i |\langle u_i, \Lambda_{z,s} \left(F_z \Delta_G^{-1/2} \right)(h(y))\rangle_{\mathcal{K}_z}|^2\, ds \Big) \alpha(h(y))\, d\tau_z(y) \\
&= \int_Y \Big(\int_{H_z} \|\Lambda_{z,s} \left(F_z \Delta_G^{-1/2} \right)(h(y))\|^2_{\mathcal{K}_z}\, ds \Big) \alpha(h(y))\, d\tau_z(y) \\
&= \Big(\int_{H_z} ds \Big) \int_Y \|F_z \Delta_G^{-1/2}(h(y))\|^2_{\mathcal{K}_z} \alpha(h(y))\, d\tau_z(y).
\end{aligned} \tag{44}
$$

Now, if $\Phi^{-1}(y_0)$ is a finite set, then the left hand side is finite and strictly positive, hence so is the right hand side, so H_z has finite volume. This proves (i). If $\Delta_G = 1$ and H_z has finite volume, then the right hand side is finite and strictly positive by (K3). Hence $\Phi^{-1}(y_0)$ is a finite set and since it is a regular submanifold of dimension $d-n$, necessarily $n = d$. Thus (ii) holds. $\square$

3.7. **Compact stabilizers.** As a preliminary step, we assume that the stabilizer H_z of a given $z \in Z$ is compact, hence such is any other stabilizer in the same orbit. Later we shall assume that this is the case for almost every orbit.
The compactness of the stabilizer allows us to use Schur's orthogonality relations for computing the inner integral over H_z in (42). Indeed, since H_z is compact, the representation Λ_z is completely reducible. Hence, for each equivalence class $\hat{s}$ in the dual group $\widehat{H}_z$, we can choose a closed subspace $\mathcal{K}_{z,\hat{s}} \subset \mathcal{K}_z$ such that the restriction $\Lambda_{z,\hat{s}}$ of Λ_z to $\mathcal{K}_{z,\hat{s}}$ belongs to $\hat{s}$, and we denote by $m_{\hat{s}}$ the multiplicity of $\hat{s}$ in Λ_z (with the convention that $\mathcal{K}_{z,\hat{s}} = 0$ if $m_{\hat{s}} = 0$). The following direct decomposition in primary inequivalent representations holds true

$$\mathcal{K}_z \simeq \bigoplus_{\hat{s}\in\widehat{H}_z} \mathcal{K}_{z,\hat{s}} \otimes \mathbb{C}^{m_{\hat{s}}} \qquad \Lambda_z \simeq \bigoplus_{\hat{s}\in\widehat{H}_z} \Lambda_{z,\hat{s}} \otimes \mathrm{id}, \tag{45}$$

where we interpret $\mathbb{C}^{m_{\hat{s}}} = \ell^2$ whenever $m_{\hat{s}} = \aleph_0$. Furthermore, for any cardinal $m \in \{1, \dots, \aleph_0\}$, we denote by $\{e_j\}_{j=1}^m$ the canonical basis of $\mathbb{C}^m$.

Mackey's theorem on induced representations of semi-direct products [28] guarantees that each induced representation $\mathrm{Ind}_{\mathbb{R}^d \rtimes H_z}^G(e^{-2\pi i\langle y_0,\cdot\rangle}\,\Lambda_{z,\hat{s}})$ is irreducible on $\mathcal{H}_{z,\hat{s}}$ and gives the following direct decomposition in primary inequivalent representations for W_z:

$$\mathcal{H}_z \simeq \bigoplus_{\hat{s}\in\widehat{H}_z} \mathcal{H}_{z,\hat{s}} \otimes \mathbb{C}^{m_{\hat{s}}} \qquad W_z \simeq \bigoplus_{\hat{s}\in\widehat{H}_z} \mathrm{Ind}_{\mathbb{R}^d \rtimes H_z}^G(e^{-2\pi i\langle y_0,\cdot\rangle}\,\Lambda_{z,\hat{s}}) \otimes \mathrm{id}\,. \tag{46}$$

By (45) and (46), respectively, we have

$$F_z = \sum_{\hat{s}\in\widehat{H}_z} \sum_{i=1}^{m_{\hat{s}}} F_{z,\hat{s},i} \otimes e_i, \qquad F_z \in \mathcal{H}_z$$
$$u = \sum_{\hat{s}\in\widehat{H}_z} \sum_{i=1}^{m_{\hat{s}}} u_{\hat{s},i} \otimes e_i, \qquad u \in \mathcal{K}_z.$$

We write $\mathrm{vol}\, H_z$ for the mass of H_z relative to the unique Haar measure ds that makes formula (39) work. Note that $\mathrm{vol}\, H_z$ is not necessarily one.

Proposition 4. *Let $z \in Z$ be such that the stabilizer H_z is compact. Given $F_z \in \mathcal{H}_z$ the following facts are equivalent:*

(i) equality (42) holds true for all $u \in \mathcal{K}_z$;

(ii) for all $\hat{s} \in \widehat{H}_z$ such that $m_{\hat{s}} \neq 0$, and for all $i, j = 1, \dots, m_{\hat{s}}$

$$\int_Y \langle F_{z,\hat{s},i}(h(y)), F_{z,\hat{s},j}(h(y))\rangle_{\mathcal{K}_{z,\hat{s}}} \frac{\alpha(h(y))}{\Delta_G(h(y))} d\tau_z(y) = \frac{\dim \mathcal{K}_{z,\hat{s}}}{\mathrm{vol}\, H_z}\delta_{ij}. \tag{47}$$

Proof. Take $u \in \mathcal{K}_z$. We compute the inner integral in (42) using Schur's orthogonality relations. For τ_z-almost every $y \in Y$

$$\int_{H_z} |\langle u, \Lambda_{z,s} F_z(h(y))\rangle_{\mathcal{K}_z}|^2\, ds = \sum_{\hat{s}\in\widehat{H}_z} \sum_{i,j=1}^{m_{\hat{s}}} \langle u_{\hat{s},i}, u_{\hat{s},j}\rangle_{\mathcal{K}_{z,\hat{s}}} \times$$
$$\times \langle F_{z,\hat{s},j}(h(y)), F_{z,\hat{s},i}(h(y))\rangle_{\mathcal{K}_{z,\hat{s}}} \frac{\operatorname{vol} H_z}{\dim \mathcal{K}_{z,\hat{s}}}.$$

Choosing $u = u_{\hat{s},i}$, (42) is equivalent to

$$\int_Y \|F_{z,\hat{s},i}(h(y))\|^2_{\mathcal{K}_{z,\hat{s}}} \frac{\alpha(h(y))}{\Delta_G(h(y))} d\tau_z(y) = \frac{\dim \mathcal{K}_{z,\hat{s}}}{\operatorname{vol} H_z}. \tag{48}$$

Choose next $j \neq i$ and $u = u_{\hat{s},i} \oplus u_{\hat{s},j}$. Taking (48) into account, (42) is equivalent to

$$\int_Y \langle F_{z,\hat{s},i}(h(y)), F_{z,\hat{s},j}(h(y))\rangle_{\mathcal{K}_{z,\hat{s}}} \frac{\alpha(h(y))}{\Delta_G(h(y))} d\tau_z(y) = 0.$$

Hence (i) is equivalent to (ii). □

Equation (47) has the following interpretation in terms of the abstract theory developed by Führ [18]. Indeed, for each irreducible representation of G in (46), we can define the (possibly unbounded) operator $d_{z,\hat{s}}$ on $\mathcal{H}_{z,\hat{s}}$

$$d_{z,\hat{s}} F_{z,\hat{s}}(g) = \frac{\dim \mathcal{K}_{z,\hat{s}}}{\operatorname{vol} H_z} \Delta_G(g)\, F_{z,\hat{s}}(g), \tag{49}$$

which satisfies (K2) precisely because the stabilizer is compact. The operator $d_{z,\hat{s}}$ is a positive self-adjoint injective operator semi-invariant with weight Δ_G^{-1} [13]. Now, (47) says that $F_{z,\hat{s},i}$ is in the domain of $d_{z,\hat{s}}^{-1/2}$ and

$$\langle d_{z,\hat{s}}^{-1/2} F_{z,\hat{s},i}, d_{z,\hat{s}}^{-1/2} F_{z,\hat{s},j}\rangle_{\mathcal{H}_{z,\hat{s}}} = \delta_{ij}, \qquad i,j = 1,\ldots,m_{\hat{s}}. \tag{50}$$

One should compare this with Theorem 4.20 and equations (4.15) and (4.16) of [18].

Corollary 3. *Let $z \in Z$ be such that the stabilizer H_z is compact. The following are equivalent:*

(i) *there exists $F_z \in \mathcal{H}_z$ such that equality* (42) *holds true for all $u \in \mathcal{K}_z$;*
(ii) $m_{\hat{s}} \leq \dim(\mathcal{H}_{z,\hat{s}})$ *for all $\hat{s} \in \widehat{H_z}$.*

If G is non-unimodular, this last condition is always satisfied.

Proof. Fix $\hat{s} \in \widehat{H_z}$ such that $m_{\hat{s}} \neq 0$. If G is unimodular, $d_{z,\hat{s}}$ is the identity up to a multiplicative constant, so that the families $\{F_{z,\hat{s},i}\}_{i=1}^{m_{\hat{s}}}$ satisfying (50) are precisely the orthogonal families in $\mathcal{H}_{z,\hat{s}}$ with square norm equal to $\dim \mathcal{K}_{z,\hat{s}}/\operatorname{vol} H_z$, whose existence is equivalent to $m_{\hat{s}} \leq \dim(\mathcal{H}_{z,\hat{s}})$. If G is non-unimodular, $d_{z,\hat{s}}$ is a semi-invariant operator with weight Δ_G^{-1}. Therefore its spectrum is unbounded (see formula (2) of [13]), so that $\dim \mathcal{H}_{z,\hat{s}} = +\infty$, provided that $m_{\hat{s}} \neq 0$. Hence the families $\{F_{z,\hat{s},i}\}_{i=1}^{m_{\hat{s}}}$ satisfying (50) are the families in the domain of $d_{z,\hat{s}}^{-1/2}$ that are orthonormal with respect to the inner product induced by $d_{z,\hat{s}}^{-1/2}$. □

If G is unimodular, (ii) of Corollary 2 implies that $\mathcal{K}_z$ is finite-dimensional, so that $m_{\hat{s}} = 0$ for all but finitely many $\hat{s} \in \widehat{H}_z$ for which $m_{\hat{s}}$ is finite. Furthermore, the orbit $\pi^{-1}(z)$ is often infinite, so that $\dim \mathcal{H}_{z,\hat{s}} = +\infty$ and the requirement $m_{\hat{s}} \leq \dim \mathcal{H}_{z,\hat{s}}$ is trivially satisfied for every $\hat{s} \in \widehat{H}_z$.

From now on we assume that almost every stabilizer H_z is compact. For each z we can thus apply Proposition 4. Theorem 7 provides an explicit decomposition of the representation W, hence of U, as a direct integral of its irreducible components, each of which is realized as induced representation of the restriction of Λ_z to a suitable (irreducible) subspace. The result does not depend on the fact that U is reproducing. To state the theorem, we fix a Borel (hence λ) measurable section $o : \pi(Y) \to Y$ whose existence is ensured by Assumption 2 and by Theorem 2.9 in [14], thereby choosing $o(z)$ as the origin of the orbit $\pi^{-1}(z)$. We then extend $o : Z \to Y$ measurably. Thus, for all $z \in Z$, we have $\mathcal{K}_z = L^2(X, \nu_{o(z)})$.

Lemma 6. *The field of Hilbert spaces $z \mapsto \mathcal{K}_z$ is λ-measurable with respect to the measurable structure induced by $C_c(X) \subset K_z$ and the corresponding direct integral $\mathcal{K} = \int_Z \mathcal{K}_z \, d\lambda(z)$ is a separable Hilbert space.*

Proof. For any $\varphi, \varphi' \in C_c(X)$ the map $z \mapsto \int_X \varphi(x)\overline{\varphi(x)'} d\nu_{o(z)}(x)$ is λ-measurable because $y \mapsto \int_X \varphi(x)\overline{\varphi(x)'} d\nu_y(x)$ is continuous (see (iii) of Theorem 2) and o is Borel measurable. Since Z is second countable, Corollary of Proposition 6 Ch. II § 1.5 in [11] implies that $\mathcal{K}$ is separable. □

Theorem 7. *Assume that for λ-almost every $z \in Z$ the stabilizer H_z is compact. There exist a countable family $\{z \mapsto \mathcal{K}_z^n\}_{n \in \mathcal{N}}$ of λ-measurable fields of Hilbert subspaces $\mathcal{K}_z^n$ of $\mathcal{K}_z$, and a family of cardinals $\{m_n\}_{n \in \mathcal{N}} \subset \{1, \ldots, \aleph_0\}$ such that, for almost every $z \in Z$,*

$$\mathcal{K}_z = \bigoplus_{n \in \mathcal{N}} \mathcal{K}_z^n \otimes \mathbb{C}^{m_n}, \tag{51}$$

$$\Lambda_z = \bigoplus_{n \in \mathcal{N}} \Lambda_z^n \otimes \mathrm{id}, \tag{52}$$

where (52) *is the decomposition of Λ_z into irreducibles.*

Before the proof, some remarks are in order.

Remark 5. In (51) it is understood that, for each $n \in \mathcal{N}$ and $j = 1, \ldots, m_n$, the field of Hilbert subspaces $z \mapsto \mathcal{K}_z^n \otimes \mathbb{C}\{e_j\}$ is λ-measurable.

Remark 6. For each $n \in \mathcal{N}$ and for almost every $z \in Z$ we denote by $\mathcal{H}_z^n$ the Hilbert space carrying the induced representation $\mathrm{Ind}_{\mathbb{R}^d \rtimes H_z}(e^{-2\pi i \langle o(z), \cdot \rangle} \Lambda_z^n)$. Reasoning as in the proof of Theorem 10.1 of [28], for each $n \in \mathcal{N}$, $z \mapsto \mathcal{H}_z^n$ is a λ-measurable field of

Hilbert subspaces, $\mathcal{H}_z^n \subset \mathcal{H}_z$, and

$$\mathcal{H} = \bigoplus_{n\in\mathcal{N}} \int_Z \mathcal{H}_z^n \, d\lambda(z) \otimes \mathbb{C}^{m_n} \tag{53}$$

$$W = \bigoplus_{n\in\mathcal{N}} \int_Z \mathrm{Ind}_{\mathbb{R}^d \rtimes H_z}(e^{-2\pi i\langle o(z),\cdot\rangle}\,\Lambda_z^n) \, d\lambda(z) \otimes \mathrm{id} \tag{54}$$

where, by Theorem 14.1 of [28], each component $\mathrm{Ind}_{\mathbb{R}^d \rtimes H_z}(e^{-2\pi i\langle o(z),\cdot\rangle}\,\Lambda_z^n)$ is irreducible and two of them are inequivalent provided that they are different from zero (see the next remark).

Remark 7. In the statement of Theorem 7, given $n \in \mathcal{N}$, it is possible that for some $z \in Z$ the Hilbert space $\mathcal{K}_z^n$ reduces to zero as well as $\mathcal{H}_z^n$. If this is the case, then clearly Λ_z^n and $\mathrm{Ind}_{\mathbb{R}^d \rtimes H_z}(e^{-2\pi i\langle o(z),\cdot\rangle}\,\Lambda_z^n)$ can be removed from the corresponding integral decompositions of Λ_z and W.

Remark 8. Fix z and compare (45) with (52). The set $\mathcal{N}$ is a parametrization of the relevant elements in the dual group $\widehat{H_z}$ defined by the direct decomposition of Λ_z into its irreducible components Λ_z^n. In other words, for each $n \in \mathcal{N}$ for which $\mathcal{K}_z^n \neq 0$ there exists $\hat{s}_n \in \widehat{H_z}$ such that $\Lambda_z^n = \Lambda_{z,\hat{s}_n}$ and $m_n = m_{\hat{s}}$ is its multiplicity, which is independent of z by its very construction.

Remark 9. As a consequence of Theorem 7 and general results on direct integrals, for each $n \in \mathcal{N}$ there exists a λ-measurable field $\{z \mapsto \varepsilon_{z,\ell}^n\}_{\ell\geq 1}$ of Hilbert bases for each field $z \mapsto \mathcal{H}_z^n$ and, for any $F \in \mathcal{H}$,

$$F = \sum_{n\in\mathcal{N}} \sum_{j=1}^{m_n} \int_Z F_{z,j}^n \, d\lambda(z) \otimes e_j \tag{55}$$

$$F_{z,j}^n = \sum_{\ell\geq 1} f_{j,\ell}^n(z)\, \varepsilon_{z,\ell}^n$$

where $z \mapsto f_{j,\ell}^n(z)$ is a λ-measurable complex function and

$$\|F\|_{\mathcal{H}}^2 = \sum_{n\in\mathcal{N}} \sum_{j=1}^{m_n} \int_Z \|F_{z,j}^n\|_{\mathcal{H}_z^n}^2 d\lambda(z) = \sum_{n\in\mathcal{N}} \sum_{j=1}^{m_n} \sum_{\ell\geq 1} \int_Z |f_{j,\ell}^n(z)|^2 d\lambda(z). \tag{56}$$

Conversely, if $\{z \mapsto f_{j,\ell}^n(z)\}_{n,j,\ell}$ is a family of λ-measurable complex functions such that

$$\sum_{n\in\mathcal{N}} \sum_{j=1}^{m_n} \sum_{\ell\geq 1} \int_Z |f_{j,\ell}^n(z)|^2 d\lambda(z) < +\infty,$$

then (55) defines an element $F \in \mathcal{H}$.

of Theorem 7. We claim that there exists a sequence of Borel measurable functions $\xi_k : Y \to H$ such that, for any $y \in Y$, the set $\{\xi_k(y)\}_{k\in\mathbb{N}}$ is dense in H_y. To this end, define $\Xi : Y \times H \to Y \times Y$ by $\Xi(y,h) = (h[y], y)$, a continuous map, hence Borel measurable. Now, the diagonal $D = \{(y,y) \mid y \in Y\}$ is a Borel set and $H_y = \{h \in H \mid \Xi(y,h) \in D\}$

for any $y \in Y$. By Aumann's measurable selection principle (see e.g. Theorem III.23 of [6]) the desired sequence exists.

For all $z \in Z$, let $M_z \subset \mathcal{L}(\mathcal{K}_z)$ denote the von Neumann algebra on $\mathcal{K}_z$ generated by the representation $\Lambda_{o(z)}$ of $H_{o(z)}$. We show that $z \mapsto M_z$ is a λ-measurable field of von Neumann algebras. For each $z \in Z$ the continuity of $s \mapsto \Lambda_{z,s}$ implies that the family $\{\Lambda_{z,\xi_k(o(z))}\}_{k\in\mathbb{N}}$ generates M_z. Hence, it is enough to prove that for any $k \in \mathbb{N}$ the field of operators $z \mapsto \Lambda_{z,\xi_k(o(z))}$ is λ-measurable. This means that for any $\varphi, \varphi' \in C_c(X)$, the map

$$z \mapsto \int_X \sqrt{\alpha(\xi_k(o(z))^{-1})\beta(\xi_k(o(z))^{-1})}\varphi(\xi_k(o(z))^{-1}.x)\overline{\varphi'(x)}d\nu_{o(z)}(x)$$

is λ-measurable. First we claim that

$$(y,h) \mapsto \int_X \varphi(h^{-1}.x)\overline{\varphi'(x)}d\nu_y(x)$$

is continuous on $Y \times H$. Fix $(y_0, h_0) \in Y \times H$ and $\varepsilon > 0$. By (iii) of Theorem 2 applied to $\varphi^{h_0}\overline{\varphi'} \in C_c(X)$ there exists a compact neighbourhood U of y_0 such that for all $y \in U$

$$|\int_X \varphi(h_0^{-1}.x)\overline{\varphi'(x)}d\nu_y(x) - \int_X \varphi(h_0^{-1}.x)\overline{\varphi'(x)}d\nu_{y_0}(x)| \leq \varepsilon/2.$$

Choose a compact neighbourhood V of h_0 e define $K = V.\operatorname{supp}\varphi$, which is a compact subset of X. The map $y \mapsto (\nu_y)_K$ is continuous from U to $M(K) = C(K)^*$ with respect to the weak* topology, so that $\sup_{y\in U}\nu_y(K)$ is bounded (Corollary II.4 of [5]). Now, the map $h \mapsto \varphi^h$ is uniformly continuous. Hence there is a compact neighbourhood $V' \subset V$ of h_0 such that, for all $h \in V'$, $\varphi(h^{-1}.x) = \varphi(h_0^{-1}.x) = 0$ if $x \notin K$ and

$$\sup_{x\in X}|\varphi(h^{-1}.x) - \varphi(h_0^{-1}.x)| \leq \frac{\varepsilon}{2(1 + \sup_{x\in X}|\varphi'(x)|\sup_{y\in U}\nu_y(K))}.$$

The triangular inequality gives that for all $(y,h) \in U \times V$

$$|\int_X \varphi(h^{-1}.x)\overline{\varphi'(x)}d\nu_y(x) - \int_X \varphi(h_0^{-1}.x)\overline{\varphi'(x)}d\nu_{y_0}(x)| \leq \varepsilon,$$

so that the claim is proved. Since $h \mapsto \sqrt{\alpha(h)^{-1}\beta(h^{-1})}$ is continuous and $z \mapsto (o(z), \xi_k(o(z)))$ is Borel measurable from Z to $Y \times H$, it follows that $z \mapsto \Lambda_{z,\xi_k(o(z))}$ is a Borel measurable field of operators and, hence, λ-measurable.

Proposition 1, Ch II § 3.2 of [11] shows that $M := \int_Z M_z \, d\lambda(z)$ is a von Neumann algebra acting on $\mathcal{K}$. Since Z is second countable, Theorem 4, Ch II § 3.3 of [11] implies that

$$M' = \int_Z {M_z}' \, d\lambda(z),$$

$$M \cap M' = \int_Z M_z \cap {M_z}' \, d\lambda(z).$$

Further, both M and M' are type I von Neumann algebras. Indeed, for almost every $z \in Z$, H_z is a group of type I, hence Λ_z is a representation of type I, that is, M_z is a type I von Neumann algebra. Corollary 2, Ch II § 3.5 of [11] implies that M is of type

I , again because Z is second countable. Finally, M' is of type I by Theorem 1 Ch. 1 § 8 of [11].

By applying twice (A50) of [12] we infer that there exists a countable family $\{P^i\}_{i\in I}$ of non-zero pairwise orthogonal projections in $M\cap M'$ with sum the identity such that both the reduced algebra[4] $M^i := M_{P^i}$ and the reduced algebra $(M^i)' = (M')_{P^i}$ are homogeneous. Since M and M' are decomposable, for each $i \in I$ the decomposition $P^i = \int_Z P^i_z \, d\lambda(z)$ holds, where, for all $z \in Z$, P^i_z is a projection in $M_z \cap M'_z$ and $z \mapsto P^i_z$ is a λ-measurable field of operators. Proposition 6 Ch. II § 3.5 of [11] implies that

$$M^i = \int_Z M^i_z \, d\lambda(z) \qquad M^{i'} = \int_Z M^{i'}_z \, d\lambda(z)$$

where, for all $z \in Z$, M^i_z is the reduced algebra associated with P^i_z.
Furthermore, for almost every $z \in Z$, the family $\{P^i_z\}_{i\in I}$ is pairwise orthogonal with sum the identity. Indeed, given $i, j \in I$ with $i \neq j$, Proposition 3 Ch. 2 § 2.3 in [11] gives $0 = P_i P_j = \int P^i_z P^j_z d\lambda(z)$, hence the Corollary of the cited section ensures that $P^i_z P^j_z = 0$ for almost all z. Since I is countable, then the above equality holds almost everywhere for all $i, j \in I$. Given such a z, $\{P^i_z\}_{i\in I}$ is a family of pairwise orthogonal projections, so that $\sum_i P^i_z$ converges to a projection P_z with respect to the strong operator topology, and so does $\sum_i P^i$ converge to the identity. Proposition 4 Ch. 2 in [11] § 2.3 and the uniqueness of the limit imply that $P_z = \mathrm{id}$ for almost every z.

Fix $i \in I$. Since M^i and $M^{i'}$ are homogeneous, the very definition of homogeneous von Neumann algebra (see Ch. 3, § 3.1 in [11]) and the canonical isomorphism given by Proposition 5 Ch. 1, § 2.4 in [11], give

$$\begin{aligned} P^i\mathcal{K} &= \mathbb{C}^{d_i} \otimes \mathbb{C}^{m_i} \otimes \mathcal{T}^i \\ M^i &= \mathcal{L}(\mathbb{C}^{d_i}) \otimes \mathbb{C}\,\mathrm{id}_{\mathbb{C}^{m_i}} \otimes \mathcal{A}^i \\ M^{i'} &= \mathbb{C}\,\mathrm{id}_{\mathbb{C}^{d_i}} \otimes \mathcal{L}(\mathbb{C}^{m_i}) \otimes \mathcal{A}^i, \end{aligned}$$

where $\mathcal{A}^i$ is a maximal abelian algebra acting on a suitable closed subspace $\mathcal{T}^i \subset \mathcal{K}$. Denote by Q the orthogonal projection onto $\mathbb{C}e_1 \otimes \mathbb{C}^{m_i} \otimes \mathcal{T}^i \in M^i$, where e_1 is the first element of the canonical basis of any $\mathbb{C}^p$. The corresponding reduced algebra of M^i is $\mathrm{id}_{\mathbb{C}^{d_i}} \otimes \mathcal{A}^i$. Furthermore, if $\widehat{Q}$ is the orthogonal projection onto $\mathbb{C}e_1 \otimes \mathcal{T}^i \in (M^i_Q)'$, the corresponding reduced algebra of $(M^i_Q)'$ is $\mathcal{A}^i$. Hence, reasoning as before, $\mathcal{T}^i$ is a direct integral of a λ-measurable field $z \mapsto \mathcal{T}^i_z$ of Hilbert subspaces of $P^i_z\mathcal{K}_z$ and $\mathcal{A}^i$ is a decomposable algebra, so that

$$\mathcal{A}^i = \int_Z \mathcal{A}^i_z \, d\lambda(z)$$

where, for almost every z, $\mathcal{A}^i_z$ is a maximal von Neumann algebra on $\mathcal{T}^i_z$. Proposition 3 Ch. 2 § 3.4 in [11] gives

$$M^i = \int_Z \mathcal{L}(\mathbb{C}^{d_i}) \otimes \mathbb{C}\,\mathrm{id}_{\mathbb{C}^{d_i}} \otimes \mathcal{A}^i_z \, d\lambda(z),$$

[4]The algebra of operators obtained by restricting to the subspace $W_i = P_i\mathcal{K}$ and then projecting back to W_i, hence a von Neumann algebra on W_i.

so that (ii) of Proposition 1 Ch. 2 § 3.4 implies, for almost every z,

$$P^i_z\mathcal{K}_z = \mathbb{C}^{d_i}\otimes\mathbb{C}^{m_i}\otimes\mathcal{T}^i_z \qquad M^i_z = \mathcal{L}(\mathbb{C}^{d_i})\otimes\mathbb{C}\,\mathrm{id}_{\mathbb{C}^{m_i}}\otimes\mathcal{A}^i_z. \tag{57}$$

Hence, Proposition 3 Ch. III § 3.2 and Theorems 1 and 2 of Ch. 1 §. 7.3 give the existence of a unitary operator J^i_z from $P^i_z\mathcal{K}_z$ onto $\mathbb{C}^{d_i}\otimes\mathbb{C}^{m_i}\otimes L^2(\Omega^i_z,\omega^i_z)$ such that

$$J^i_z\mathcal{M}^i_z{J^i_z}^{-1} = \mathcal{L}(\mathbb{C}^{d_i})\otimes\mathbb{C}\,\mathrm{id}_{\mathbb{C}^{m_i}}\otimes L^\infty(\Omega^i_z,\omega^i_z) \tag{58}$$

where Ω^i_z is a locally compact second countable space and ω^i_z is a measure with support Ω^i_z.

The previous arguments and the compactness assumption imply that there exists a negligible set $N\subset Z$ such that for all $z\in Z\setminus N$, (57) holds true for any $i\in I$ and H_z is compact. Hence, with the notation used in (45), for $z\notin N$ we put

$$n^i_z = \operatorname{card}\{\hat{s}\in\widehat{H_z} : \dim\mathcal{K}_{z,\hat{s}} = d_i,\ m_{\hat{s}} = m_i\}.$$

Hence in (58) the set Ω^i_z can be chosen as $\{1,\ldots,n^i_z\}$ if $1\le n^i_z<+\infty$, $\mathbb{N}$ if $n^i_z=\aleph_0$, $\emptyset$ if $n^i_z=0$, and the measure ω^i_z as the corresponding counting measure. By construction, $z\mapsto\mathcal{T}^i_z$ is a measurable field of Hilbert spaces and Proposition 1 Ch. II, § 1.4 of [11] implies that for any cardinal p the set $Z^{ip}=\{z\in Z\setminus N : n^i_z = p\}$ is λ-measurable and, for each $i\in I$

$$\bigcup_p Z^{ip} = Z\setminus N. \tag{59}$$

Clearly for all $z\in Z^{ip}$ we have $\Omega^i_z=\Omega^{ip}$, where

$$\Omega^{ip} := \begin{cases}\{1,\ldots,p\} & 1\le p<+\infty\\ \mathbb{N} & p=\aleph_0\\ \emptyset & p=0,\end{cases}$$

so that the von Neumann algebra $L^\infty(\Omega^i_z,\omega^i_z)$ is equal to $\ell^\infty(\Omega^{ip})$, independently of z. Lemma 2 Ch. 2 § 3. of [11] implies that the unitary operator $J^i_z:\mathbb{C}^{d_i}\otimes\mathbb{C}^{m_i}\otimes\mathbb{C}^p\to P^i_z\mathcal{K}_z$ can be chosen in such a way that $z\mapsto J^i_z$ is λ-measurable. The previous arguments show that the relevant indices $n=(i,p,k)$ run on a countable set that will be denoted $\mathcal{N}$. Define $m_n=m_i$ and

$$\mathcal{K}^n_z = \begin{cases}J^i_z\big(\mathbb{C}^{d_i}\otimes\mathbb{C}\{e_1\}\otimes\mathbb{C}\{e_k\}\big) & z\in Z^{ip},\ 1\le k\le p,\ p>0\\ \{0\} & \text{otherwise.}\end{cases}$$

Summarizing, we finally obtain the following facts, which entail the result.

i) For each $n\in\mathcal{N}$, the map $z\mapsto\mathcal{K}^n_z$ is λ-measurable since $z\mapsto J^i_z$ is λ-measurable field of operators.

ii) For almost all $z\in Z$ and for each $n\in\mathcal{N}$ the Hilbert space $\mathcal{K}^n_z$ is invariant with respect to Λ_z, the corresponding restriction is irreducible and the restriction to $J^i_z\big(\mathbb{C}^{d_i}\otimes\mathbb{C}^{m_i}\otimes\mathbb{C}\{e_k\}\big)$ is a factor representation, see (58). Thus

$$J^i_z\big(\mathbb{C}^{d_i}\otimes\mathbb{C}^{m_i}\otimes\mathbb{C}\{e_k\}\big) = \mathcal{K}^n_z\otimes\mathbb{C}^{m_n} \qquad \Lambda_{|\mathcal{K}^n_z\otimes\mathbb{C}^{m_n}} = \Lambda_{|\mathcal{K}^n_z}\otimes\mathrm{id}\,.$$

In particular, for each $j=1,\ldots,m_n$ the field $z\mapsto J^i_z\big(\mathbb{C}^{d_i}\otimes\mathbb{C}\{e_j\}\otimes\mathbb{C}\{e_k\}\big)$ is λ-measurable.

iii) For almost all $z \in Z$, for each $n \neq n'$ the restriction of Λ_z to $\mathcal{K}_z^n$ and $\mathcal{K}_z^{n'}$ are inequivalent, provided that both spaces are different from zero (58).
iv) For almost all $z \in Z$, $\{\mathcal{K}_z^n \otimes \mathbb{C}^{m_n}\}_{n\in\mathcal{N}}$ is a family of pairwise orthogonal closed subspace with sum $\mathcal{K}_z$, by (59) and the definition of $\mathcal{K}_z^n$.

□

By means of the intertwining operator S given by Theorem 5, the direct decomposition (54) gives rise to a corresponding decomposition of the mock-metaplectic representation U. Hence, the abstract theory of [18] applies and one can characterize the admissible vectors for U. However, we can apply directly Corollary 4. We need a last technical lemma concerning the measurability of the map $z \mapsto \mathrm{vol}(H_z)$ (compare with Lemma 18 of [19]).

Lemma 7. *Assume that for almost every $y_0 \in Y$ the stabilizer H_{y_0} is compact and define*

$$\mathrm{vol}(H_{y_0}) = \int_{H_{y_0}} ds$$

where ds is the unique Haar measure of H_{y_0} such that

$$\int_H \varphi(h)\alpha(h^{-1})dh = \int_Y \left(\int_{H_{y_0}} \varphi(h(y)s)ds\right) d\tau_{\pi(y_0)}(y) \qquad \varphi \in C_c(Y).$$

Then:

(i) for all y_0 and $h \in H$, $\mathrm{vol}(H_{h[y_0]}) = \Delta_G(h^{-1})\,\mathrm{vol}(H_{y_0})$;
(ii) the map $y_0 \mapsto \mathrm{vol}(H_{y_0})$ is Lebesgue measurable.

Furthermore, given a Borel measurable section $o : Z \to Y$, the map

$$z \mapsto \frac{\dim \mathcal{K}_z^n}{\mathrm{vol}(H_z)}$$

is λ-measurable; if G is unimodular, it is independent of the choice of o.

Proof. Fix a continuous $f \in L^1(Y)$ such that $f(y) > 0$ for all $y \in Y$. The definition of τ_z (see Theorem 4) and (iii) of Theorem 11 imply that f is τ_z-integrable for λ-almost every $z \in Z$. Clearly, the function $(y_0, h) \mapsto f(h[y_0])\alpha(h^{-1})$ is continuous on $Y \times H$. Given $y_0 \in Y$, let $z = \pi(y_0)$. Hence we can choose y_0 as the origin of $\pi^{-1}(z)$ and define ds as the unique Haar measure of $H_{y_0} = H_z$ for which (39) holds true. By (ii) of Theorem 11, for almost all $y_0 \in Y$,

$$\begin{aligned} 0 < \int_H f(h[y_0])\alpha(h^{-1})\,dh &= \int_Y \left(\int_{H_{y_0}} f(h_y s[y_0])\,ds\right) d\tau_{\pi(y_0)}(y) \\ &= \mathrm{vol}(H_{y_0}) \int_Y f(y) d\tau_{\pi(y_0)}(y) < +\infty \end{aligned} \tag{60}$$

since $h_y s[y_0] = y$; the first inequality is due to the fact $f > 0$ and the last follows from $f \in L^1(Y)$. Clearly $y_0 \mapsto \int_H f(h[y_0])\alpha(h^{-1})\,dh$ is Lebesgue-measurable as well as $y_0 \mapsto \int_Y f(y)d\tau_{\pi(y_0)}(y)$ is Lebesgue measurable and strictly positive, so that $y_0 \mapsto \mathrm{vol}(H_{y_0})$ is λ-measurable, too. The fact that the map $z \mapsto \dim \mathcal{K}_z^n$ is λ-measurable for all $n \in \mathcal{N}$

is a consequence of Proposition 1, Ch. 2 § 1.4 of [11].
If $y_1 = \ell[y_0]$ for some $\ell \in H$, whence $\pi(y_0) = \pi(y_1)$, then by (60)

$$\begin{aligned}
\operatorname{vol}(H_{y_1}) \int_Y f(y) d\tau_{\pi(y_0)}(y) &= \int_H f(h[y_1])\alpha(h^{-1})\, dh \\
(\ h \mapsto h\ell^{-1}\ ) &= \Delta_H(\ell^{-1})\alpha(\ell) \int_H f(h[y_0])\alpha(h^{-1})\, dh \\
&= \Delta_G(\ell^{-1}) \operatorname{vol}(H_{y_0}) \int_Y f(y) d\tau_{\pi(y_0)}(y).
\end{aligned}$$

The second half of the lemma is clear. □

We are ready to state our main result on the admissible vectors of G. We distinguish according as to weather G is unimodular or not. We consider first the unimodular case, compare with Eq. (4.14) of Theorem 4.22 in [18].

Theorem 8. *Assume that G is unimodular and that for almost every $z \in Z$ the stabilizer H_z is compact. The representation U is reproducing if and only if the following two conditions hold true:*

(i) the integral

$$\int_Z \frac{\operatorname{card} \Phi^{-1}(o(z))}{\operatorname{vol} H_z}\, d\lambda(z) \tag{61}$$

is finite;

(ii) for all $n \in \mathcal{N}$ and for almost every $z \in Z$ for which $\mathcal{K}_z^n \neq 0$

$$m_n \leq \dim \mathcal{H}_z^n \tag{62}$$

where the notation is as in (53) and (54).

Under the above equivalent conditions, η is an admissible vector for U if and only if

$$S\eta = \sum_{n \in \mathcal{N}} \sum_{j=1}^{m_n} \int_Z \sqrt{\frac{\dim \mathcal{K}_{z,n}}{\operatorname{vol} H_z}} \varepsilon_{z,j}^n \, d\lambda(z) \otimes e_j,$$

where $\{z \mapsto \varepsilon_{z,j}^n\}_{j \geq 1}$ is any measurable field of Hilbert bases for $z \mapsto \mathcal{H}_z^n$.

Proof. We use the same notation as in Remark 9. Theorem 6 and Corollary 4 with $\Delta_G(h_y) = 1$ give that $\eta \in L^2(X)$ is an admissible vector for U if and only if $F = W\eta \in \mathcal{H}$ satisfies the condition that follows. Given $n \in \mathcal{N}$, for almost every $z \in Z$ for which $\mathcal{K}_z^n \neq \{0\}$ (see Remarks 7 and 8), for all $i, j = 1, \dots, m_n$

$$\langle F_{z,i}^n, F_{z,j}^n \rangle_{\mathcal{H}_z^n} = \delta_{i,j} \frac{\dim \mathcal{K}_{z,n}}{\operatorname{vol} H_z},$$

that is, the family $\{F_{z,i}^n\}_{i=1}^{m_n}$ is orthogonal in $\mathcal{H}_z^n$ and normalized with square norm equal to $\dim \mathcal{K}_{z,n} / \operatorname{vol} H_z$.
As a consequence, if η is an admissible vector, then clearly (62) holds true and, by (56), we have that

$$\|F\|_{\mathcal{H}}^2 = \int_Z \left(\sum_{n \in \mathcal{N}} \sum_{i=1}^{m_n} \frac{\dim \mathcal{K}_{z,n}}{\operatorname{vol} H_z} \right) d\lambda(z) = \int_Z \frac{\operatorname{card} \Phi^{-1}(y_0)}{\operatorname{vol} H_z}\, d\lambda(z),$$

and (61) follows. Conversely, define $F \in \mathcal{H}$ such that, for all $j = 1, \dots, m_n$ and $\ell \geq 1$

$$f^n_{j,\ell}(z) = \delta_{j,\ell}\sqrt{\frac{\dim \mathcal{K}_{z,n}}{\operatorname{vol} H_z}} \qquad \text{a.e.} z \in Z,$$

which is possible due to (62). All the functions $f^n_{j,\ell}$ are λ-measurable by Lemma 7. Finally, (61) and the last string of equalities imply $\|F\|^2_{\mathcal{H}} < +\infty$. □

We now consider the non-unimodular case. For all $n \in \mathcal{N}$ and for almost every $z \in Z$ we define the positive self-adjoint injective operator $d_{z,n}$ acting on $\mathcal{H}^n_z$ by multiplication as in (49), namely

$$(d_{z,n}F_{z,n})(g) = \frac{\dim \mathcal{K}_{z,n}}{\operatorname{vol} H_z}\Delta_G(g)\, F_{z,n}(g) \qquad g \in G.$$

Theorem 9. *Assume that G is non-unimodular and that the stabilizer H_z is compact for almost every $z \in Z$. Then U is reproducing and $\eta \in L^2(X)$ is an admissible vector for U if and only if $S\eta = \sum_{n\in\mathcal{N}}\sum_{j=1}^{m_n} \int\limits_Z F^n_{z,j}\, d\lambda(z) \otimes e_j$ is such that*

- *(i) for all $n \in \mathcal{N}$ and $i = 1, \dots, m_n$, the map $z \mapsto F_{z,n,i}$ is a measurable field of vectors for $\{\mathcal{H}^n_z\}$;*
- *(ii) for all$n \in \mathcal{N}$ and for almost all $z \in Z$ for which $\mathcal{K}^n_z \neq 0$*

$$\langle d^{-1/2}_{z,n} F^n_{z,i}, d^{-1/2}_{z,n} F^n_{z,j}\rangle_{\mathcal{H}_{z,n}} = \delta_{ij} \qquad i, j = 1, \dots, m_n$$

(iii) $\displaystyle\sum_{n\in\mathcal{N}}\sum_{j=1}^{m_n}\int_Z \|F^n_{z.j}\|^2_{\mathcal{H}^n_z} d\lambda(z) < +\infty.$

Proof. The fact that η is admissible if and only if (i), (ii) and (iii) hold true is similar to the proof of Theorem 8. The non-trivial part is the existence of an admissible vector. This fact is a consequence of Theorem 4.23 of [18], whose proof can be repeated in our setting. We report the main ideas.
Fix a strictly positive sequence such that $\sum_{n\in\mathcal{N}}\sum_{i=1}^{m_n} a_{n,i} < +\infty$. For almost every $z \in Z$ the stability subgroup H_z is compact, hence the modular function Δ_G defines a continuous surjective $\hat{\Delta}_z : \pi^{-1}(z) \to (0, +\infty)$ by $\hat{\Delta}_z(y) = \Delta_G(h(y))$, where $h(y)[o(z)] = y$. Therefore there exists a subset $Y_{z,n,i}$ of $\pi^{-1}(z)$ with strictly positive τ_z-measure such that for all $y \in Y_{z,n,i}$

$$\sup_{y\in Y_{z,n,i}} \hat{\Delta}_z(y) \leq \frac{a_{n,i} \operatorname{vol} H_z}{\dim \mathcal{K}_{z,n}}.$$

By Lemma 7 we may select a family of λ-measurable fields $\{z \mapsto F^n_{z,j}\}^{m_n}_{j=1}$ of vectors in $\operatorname{dom} d^{-1/2}_{z,n}$, that are orthonormal with respect to the scalar product induced by $d^{-1/2}_{z,n}$ with the property that the support with respect to τ_z of the map $y \mapsto \|F^n_{z,j}(h(y))\|^2_{\mathcal{K}_{z,n}}$ is contained in $Y_{z,n,i}$. Thus, (iii) is satisfied because

$$\|F^n_{z,j}\|^2_{\mathcal{H}^n_z} \leq \sup_{y\in O_{z,n,i}} \frac{\dim \mathcal{K}_{z,n}\Delta_G(h(y))}{\operatorname{vol} H_z} \leq a_{n,i}.$$

Finally, (i) and (ii) are true by construction. □

4. Examples

We now discuss the examples introduced in Section 2.

4.1. **Example 1.** Here the map Φ is the identity so that the set of critical points reduces to the empty set and Assumption 1 is satisfied with the choice $X = Y = \mathbb{R}^d$ (recall that $n = d$) and $\alpha(h)\beta(h) = 1$ for all $h \in H$. Assumption 2 is the fact that the semi-direct product $\mathbb{R}^d \rtimes H$ is regular. In general, nothing more specific can be said on the parameter space Z and the measure λ on it, other than what was said in the comments following Assumption 2. Clearly, for all $y \in \mathbb{R}^d$, $\Phi^{-1}(y)$ is a singleton, the corresponding measure ν_y is trivial, so that Theorem 3 states that $\eta \in L^2(X)$ is admissible for U, for λ-almost $z \in Z$ if and only if

$$\int_H |\eta(h^{-1}[y_0])|^2 \, dh = 1$$

where y_0 is a fixed origin in $\pi^{-1}(z)$. Since the above equation holds true for any other point in $\pi^{-1}(z)$, it follows that η is a weak admissible vector in the sense of Definition 7 of [19]. Theorem 6 of the cited paper proves that Assumption 2 is essentially necessary to have weak admissible vectors, (see the comment at the end of Section 3.4). Corollary 2 guarantees that the stabilizers H_z are compact for almost every $z \in Z$. Hence the results of Section 3.7 hold true. Clearly, for almost every $\mathcal{K}_z = \mathbb{C}$, $\mathcal{N}$ is a singleton and $m_n = \dim(\mathcal{K}_z^n) = 1$, so that U is always reproducing if G is non-unimodular. Otherwise, it is such if and only if $\int_Z (\operatorname{vol} H_z)^{-1} d\lambda(z)$ is finite, which is precisely the content of Theorem 19 of [19]. See also Section 5 of [18]. The presence of $\operatorname{vol} H_z$ is due to a different normalization of the Haar measures on the stabilizers.

4.2. **Example 2.** In this example $n = 2$ and $d = 1$ so that U is not reproducing. This fact is well known since G has a non-compact center and U is irreducible.

4.3. **Example 3.** The main result here is about groups of the form (11) with $n = d$, namely:

Theorem 10. *Let $n = d$. If the H-orbits of $\Phi(\mathbb{R}^d)$ are locally closed, the restriction of the metaplectic representation to G is reproducing if and only if G is non-unimodular and H_y is compact for almost every $y \in \Phi(\mathbb{R}^d)$.*

In order to prove Theorem 10, which could be stated under the slightly more general hypothesis that Φ is a homogeneous polynomial without referring to the symplectic group, we need an auxiliary result which is of some interest by itself and whose main idea goes back to [27].

Proposition 5. *Let $n \leq d$. Assume that Φ is a homogeneous map of degree $p > 0$ and that the action on $\mathbb{R}^d$ is linear. If U is a reproducing representation, then G is non-unimodular.*

The proof is based on the following lemma.

Lemma 8. *Let $n \leq d$. Assume that Φ is a homogeneous map of degree $p > 0$ and that the action on $\mathbb{R}^d$ is linear. If η is an admissible vector for U, then for any $\delta \in \mathbb{R}_+$, the dilated vector $\sqrt{\delta^{np-d}}\eta^\delta$ is also admissible.*

Proof. Put $q = np - d$. The assumption of Φ implies that for all $x \in \mathbb{R}^d$, $a \in \mathbb{R}^n$ and $\delta \in \mathbb{R}_+$,

$$\langle \Phi(\delta x), \delta^{-p} a \rangle = \langle \Phi(x), a \rangle. \tag{63}$$

Clearly, $\sqrt{\delta^q}\eta^\delta \in L^2(X)$ and, for all $f \in L^2(X)$, the linearity of $x \mapsto h.x$ gives

$$\begin{aligned}
\int_G |\langle f, U_g \sqrt{\delta^q}\eta^\delta \rangle|^2 \, dg &= \delta^q \int_H \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} f(x)\beta(h)^{-\frac{1}{2}} \times \\
&\qquad \times e^{2\pi i \langle \Phi(x), a\rangle} \overline{\eta}(h^{-1}.(\delta^{-1}x)) dx \Big|^2 \frac{dadh}{\alpha(h)} \\
(x \mapsto \delta x,\, a \mapsto \delta^{-p}a, (63)) &= \delta^{q+2d-np} \int_G |\langle f^{\delta^{-1}}, U_g \eta \rangle|^2 dg \\
(\text{reproducing formula}) &= \delta^{q+2d-np} \int_{\mathbb{R}^d} |f(\delta x)|^2 dx \\
(x \mapsto \delta^{-1}x) &= \delta^{q+d-np} \, \|f\|^2 = \|f\|^2,
\end{aligned}$$

so that $\sqrt{\delta^q}\eta^\delta$ is an admissible vector for U. □

Proof of Proposition 5. By contradiction, assume that G is unimodular. Fix $\delta \in \mathbb{R}_+$. Choose an admissible vector $\eta \in L^2(X)$. Then

$$\begin{aligned}
\int_X |\eta(x)|^2 dx &= \delta^{-d} \int_X |\eta^\delta(x)|^2 dx \\
(\text{reproducing formula for } \eta\,) &= \delta^{-d} \int_H \int_A |\langle \eta^\delta, U_{ah}\eta \rangle|^2 \frac{dadh}{\alpha(h)} \\
(a \mapsto -a,\, h \mapsto h^{-1}) &= \delta^{-d} \int_H \int_A |\langle U_{(h^\dagger[a],h)}\eta^\delta, \eta \rangle|^2 \frac{\alpha(h)\, dadh}{\Delta_H(h)} \\
(a \mapsto (h^\dagger)^{-1}[a]) &= \delta^{-d} \int_H \int_A |\langle U_{(a,h)}\eta^\delta, \eta \rangle|^2 \Delta_G(h^{-1}) \frac{dadh}{\alpha(h)} \\
(q = np - d) &= \delta^{-q-d} \int_G |\langle \eta, U_g \sqrt{\delta^q}\eta^\delta \rangle|^2 dg \\
(\text{reproducing formula for } \sqrt{\delta^q}\eta) &= \delta^{-np} \int_X |\eta(x)|^2 dx = \delta^{-np}\|\eta\|^2.
\end{aligned}$$

Since $\|\eta\| \neq 0$ and $np \neq 0$, this is a contradiction. □

of Theorem 10. Clearly Assumption 2 is satisfied. Suppose that U is reproducing. Since Φ is quadratic, Proposition 5 implies that G is non-unimodular and Theorem 1 gives that the set $\mathcal{C}$ of critical points is negligible. The Jacobian criterion implies that for all $y \in \Phi(\mathcal{R})$ the fiber $\Phi^{-1}(y) \cap \mathbb{R}$ is finite (see Appendix B). Theorem 6 implies that for almost all $y \in \Phi(\mathcal{R})$ equality (42) holds true and, as a consequence of (i) of Corollary 2, the corresponding stabilizer H_y is compact. Conversely, if G is non-unimodular and almost every stabilizer is compact, the set of critical points is a proper Zariski closed subset of $\mathbb{R}^d$, so that it is negligible. Theorem 9 implies that U is reproducing. □

Theorem 9 characterizes the admissible vectors. However, one can also apply directly Theorem 3, taking into account that $\Phi^{-1}(y)$ is a finite set.

Corollary 4. *A function $\eta \in L^2(X)$ is an admissible vector for U if and only if for λ-almost every $z \in Z$, there exists $y \in \pi^{-1}(z)$ such that for all points $x_1, \dots x_M \in \Phi^{-1}(y)$*

$$\int_H \eta(h^{-1}.x_i)\overline{\eta(h^{-1}.x_j)}\frac{dh}{\alpha(h)\beta(h)} = (J\Phi)(x_i)\ \delta_{ij} \qquad i,j = 1,\dots M.$$

If the above equation is satisfied for a pair $x_i, x_j \in \Phi^{-1}(y)$, then it holds true for any pair $s.x_i, s.x_j \in \Phi^{-1}(y)$ with $s \in H_y$.

Proof. We apply Theorem 3. Given $z \in Z$ and $y \in \pi^{-1}(z)$ for which (25) holds true, formula (75) gives that

$$\nu_y = \sum_{i=1}^{M} \frac{\delta_{x_i}}{(J\Phi)(x_i)}.$$

Arguing as in the proof of Proposition 4, (25) is equivalent to

$$\int_H \eta(h^{-1}.x_i)\overline{\eta(h^{-1}.x_j)}\frac{dh}{\alpha(h)\beta(h)} = (J\Phi)(x_i)\delta_{ij} \qquad i,j = 1,\dots N_y.$$

The last claim is clear because H_y is compact so that for all $s \in H_z$ we have $\alpha(s) = \beta(s) = 1$ and hence the equality

$$(J\Phi)(h.x) = (J\Phi)(x)\alpha(h)^{-1}\beta(h)^{-1} \qquad h \in H.$$

□

As an example, we apply the above corollary to the metaplectic representation restricted to the shearlet group $G = TDS(2)$. Notice that

$$(J\Phi)(x_1, x_2) = x_1^2/2 \qquad \alpha(\ell, t) = t^{1+\gamma} \qquad \beta(\ell, t) = t^{-\gamma}.$$

We set $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \neq 0\}$, which is an H-invariant open set with full Lebesgue measure and $Y = \Phi(X) = \mathbb{R}_- \times \mathbb{R}$, is a transitive free H-space. We choose as origin the point $y_0 = (-1/2, 0)$ so that $\Phi^{-1}(y_0) = \{(\pm 1, 0)\}$. Since for any $h = (\ell, t) \in H$

$$h^{-1}.(1,0) = (t^{\frac{1}{2}}, t^{\gamma-\frac{1}{2}}\ell),$$

a function $\eta \in L^2(X)$ is an admissible vector if and only if

$$\int_{(0,+\infty)\times\mathbb{R}} |\eta(t^{\frac{1}{2}}, t^{\gamma-\frac{1}{2}}\ell)|^2 \frac{dt d\ell}{t^{3-\gamma}} = \frac{1}{2} \tag{64}$$

$$\int_{(0,+\infty)\times\mathbb{R}} |\eta(-t^{\frac{1}{2}}, -t^{\gamma-\frac{1}{2}}\ell)|^2 \frac{dt d\ell}{t^{3-\gamma}} = \frac{1}{2} \tag{65}$$

$$\int_{(0,+\infty)\times\mathbb{R}} \eta(t^{\frac{1}{2}}, t^{\gamma-\frac{1}{2}}\ell)\overline{\eta(-t^{\frac{1}{2}}, -t^{\gamma-\frac{1}{2}}\ell)} \frac{dt d\ell}{t^{3-\gamma}} = 0. \tag{66}$$

To recover the usual admissibility condition, put $X_\pm = \{(x_1, x_2) : \pm x_1 > 0\}$ and define the unitary operator $R_\pm : L^2(Y) \to L^2(X_\pm)$

$$(R_\pm \hat{f})(x_1, x_2) = \hat{f}(\Phi(x_1, x_2))|J\Phi(x_1, x_2)|^{\frac{1}{2}},$$

so that

$$R_\pm^{-1} U_{(a;\ell,t)} R_\pm \hat{f}(y) = t^{(1+\gamma)/2} e^{-2\pi i\langle y,a\rangle} \hat{f}(ty_1, t^\gamma(\ell y_1 + y_2)),$$

which clarifies the connection with the shearlet representation, see [20]. Denote by $\hat{\eta}_\pm = R_\pm^{-1}\eta_{|X_\pm}$, equations (64) and (65) become

$$\int_Y |\hat{\eta}_\pm(-\frac{1}{2}t, -\frac{1}{2}t^\gamma \ell)|^2 \frac{t^{\gamma-2}}{2} dt d\ell = \frac{1}{2}.$$

With the change of variables $\omega_1 = -\frac{1}{2}t$ and $\omega_2 = -\frac{1}{2}t^\gamma\ell$, whose Jacobian is $\frac{1}{4}t^\gamma$, they become

$$\int_{\mathbb{R}_+\times\mathbb{R}} |\hat{\eta}_\pm(\omega_1,\omega_2)|^2 \, \frac{d\omega_1 d\omega_2}{\omega_1^2} = 1.$$

Similarly, (66) becomes

$$\int_{\mathbb{R}_+\times\mathbb{R}} \hat{\eta}_+(\omega_1,\omega_2)\overline{\eta_-(\omega_1,\omega_2)} \, \frac{d\omega_1 d\omega_2}{\omega_1^2} = 0.$$

One should compare this with formula (2.1) in [25]. Note that U is equivalent to two copies of the irreducible representation $\operatorname{Ind}_{\mathbb{R}^2}^G(\chi)$, where χ is the character of $\mathbb{R}^2$ $(a_1, a_2) \mapsto e^{\pi i a_1}$.

4.4. **Example 4.** With the choice $X = \mathbb{R}^2 \setminus \{0\}$ and $Y = \Phi(X) = (0, +\infty)$ Assumption 1 is satisfied because X is an H-invariant open set whose complement has zero Lebesgue measure. The group H acts freely on Y so that Assumption 2 holds true and Z reduces to a singleton. We choose $y_0 = 1$ as the origin of the orbit, whose stabilizer is the compact group $H_1 = \mathbb{T}$. Since G is non-unimodular, U is reproducing by Theorem 9. In order to characterize its admissible vectors note that in Theorem 4 the relatively invariant measure on Y is $\tau_1 = dy$. Furthermore, the map $\xi \mapsto (\cos\xi, \sin\xi)$ is diffeomorphism of S^1 onto the Riemannian submanifold $\Phi^{-1}(1) = \{x_1^2 + x_2^2 = 1\}$. The Riemannian measure on S^1 is $d\xi$ so that, for all $\varphi \in C_c(X)$

$$\int_X \varphi(x_1,x_2) d\nu_1(x_1,x_2) = \int_0^{2\pi} \varphi(\cos\xi, \sin\xi)\frac{d\xi}{2}.$$

Put $h(y) = (\sqrt{y}, 0)$ so that $h(y)[1] = y$. Then (39) says that the Haar measure on $\mathbb{T}$ is $d\theta/4\pi$ because

$$\int_H \varphi(t,\theta) t dt \frac{d\theta}{2\pi} = \int_0^{+\infty} \left(\int_0^{2\pi} \varphi(\sqrt{y},\theta)\frac{d\theta}{4\pi} \right) dy,$$

so that $\operatorname{vol}\mathbb{T} = \frac{1}{2}$.

The representation Λ_1 of $\mathbb{T}$ on $L^2(X,\nu_1) \simeq L^2(S^1, d\xi/2)$ is the regular representation, and

$$L^2(X,\nu_1) \simeq \bigoplus_{n\in\mathbb{Z}} \mathbb{C}\{e^{in\xi}\}$$

$$\Lambda_{1,\theta} \simeq \bigoplus_{n\in\mathbb{Z}} e^{-in\theta},$$

where each component is irreducible and any two of them are inequivalent.

Since any $g = (a, t, \theta)$ can be written as $g = (0, t, 0)(t^2 a, 0, \theta)$, any function $F \in \mathcal{H}$ can be identified with its restriction to $\mathbb{R}_+$ due to (K2). Further, (K3) becomes

$$\int_0^\infty |F(\sqrt{y})|^2 y^{-1} dy = \int_0^\infty |F(t)|^2 2t^{-1} dt < +\infty.$$

Hence we have the following unitary identifications

$$\mathcal{H} \simeq L^2(\mathbb{R}_+, 2t^{-1}dt, L^2(S^1, d\xi/2)) \simeq L^2(\mathbb{R}_+ \times S^1, t^{-1}dtd\xi).$$

The unitary map $S : L^2(X) \to \mathcal{H}$ is given explicitly by

$$(Sf)(t, \xi) = t\, T_{t^2,t^{-1}}(f_{1,t^2})(\xi) = tf(t\cos\xi, t\sin\xi).$$

For $n \in \mathbb{Z}$, the space $\mathcal{H}_n$ carrying the representation induced by $e^{-2\pi i a - in\theta}$ is

$$\mathcal{H}_n = \{F \in L^2(\mathbb{R}_+ \times \mathbb{T}, t^{-1}dtd\xi) \mid F(t, \xi) = F_n(t)e^{in\xi}, F_n \in L^2(\mathbb{R}_+, t^{-1}dt)\}.$$

If $\eta \in L^2(X)$, then $S\eta = \sum_{n\in\mathbb{Z}} F_n e^{in\xi}$ with $F_n \in L^2(\mathbb{R}_+, t^{-1}dt)$. It follows that η is an admissible vector if and only if, for any $n \in \mathbb{Z}$,

$$\int_0^{+\infty} \left(\int_{S^1} |F_n(\sqrt{y})e^{in\xi}|^2 \frac{d\xi}{2} \right) y^{-2} dy = \frac{\dim \mathcal{K}_n}{\operatorname{vol} \mathbb{T}} = 2,$$

since $\dim \mathcal{K}_n = 1$. By the change of variable $t = \sqrt{y}$, this is equivalent to

$$\int_0^{+\infty} |F_n(t)|^2 t^{-3} dt = \frac{1}{\pi}.$$

Finally, since

$$F_n(t) = \frac{1}{2\pi} \int_0^{2\pi} t\eta(t\cos\xi, t\sin\xi)e^{-in\xi} d\xi =: t\hat{\eta}(t, n),$$

the set of admissible vectors consists of the Lebesgue measurable functions $\eta : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\sum_{n\in\mathbb{Z}} \int_0^{+\infty} |\hat{\eta}(t, n)|^2 t dt < +\infty \iff \eta \in L^2(\mathbb{R}^2)$$

$$\int_0^{+\infty} |\hat{\eta}(t, n)|^2 t^{-1} dt = \frac{1}{\pi} \qquad \forall n \in \mathbb{Z}.$$

4.5. **Example 5.** In this example Assumption 1 is satisfied with the choice $X = \mathbb{R}^2 \setminus \{x_2 = 0\}$ and $Y = \Phi(X) = \mathbb{R} \setminus \{0\}$, because X is a H-invariant open set whose complement has zero Lebesgue measure. The group H acts freely on Y so that Assumption 2 holds true and Z reduces to a singleton. We choose $y_0 = 1$ as the origin of the orbit so that the corresponding stabilizer is the non-compact group $H_1 = \mathbb{R}^*$. To prove that G a reproducing group, we use Theorem 6. In Theorem 4 the relatively invariant measure on Y is $\tau_1 = dy$. Furthermore, the map $\xi \mapsto (\xi, 1)$ is a diffeomorphism of $\mathbb{R}$ onto the Riemannian submanifold $\Phi^{-1}(1) = \{x_2 = 1\}$. The Riemannian measure on $\mathbb{R}$ is $d\xi$ and $(J\Phi)(x) = 1$, so that (75) gives for all $\varphi \in C_c(X)$

$$\int_X \varphi(x_1, x_2) d\nu_1(x_1, x_2) = \int_{\mathbb{R}} \varphi(\xi, 1) d\xi.$$

Put $h(y) = (y, 0)$ so that $h(y)[1] = y$. Then (39) says that the Haar measure on $\mathbb{R}$ is db because

$$\int_H \varphi(t,b)|t|\frac{dt}{|t|}db = \int_{\mathbb{R}^*}\left(\int_{\mathbb{R}} \varphi(y,b)\, db\right) dy.$$

The representation Λ_1 of $\mathbb{R}$ on $L^2(X, \nu_1) \simeq L^2(\mathbb{R}, d\xi)$ is the regular representation, and

$$L^2(X,\nu_1) \simeq \int_{\mathbb{R}} \mathbb{C}\, d\omega$$
$$\Lambda_{1,b} \simeq \int_{\mathbb{R}} e^{-2\pi i\omega b}\, d\omega$$

where each component is irreducible, any two of them are inequivalent and the intertwining operator is given by the Fourier transform.

Since any $g = (a, t, b) \in G$ can be written as $g = (0, t, 0)(ta, 0, b)$, any function $F \in \mathcal{H}$ can be identified with its restriction to $\mathbb{R}^*$ due to (K2) and we have the following unitary identifications

$$\mathcal{H} \simeq L^2(\mathbb{R}^*, t^{-1}dt, L^2(\Phi^{-1}(1), \nu_1)) \simeq L^2(\mathbb{R}^2, y^{-1}dyd\xi).$$

The unitary map $S : L^2(X) \to L^2(\mathbb{R}^2, y^{-1}dyd\xi)$ is given explicitly by

$$(Sf)(y,\xi) = |t|^{\frac{1}{2}} f(y,\xi).$$

Theorem 6 implies that $\eta \in L^2(X)$ is an admissible vector if and only if for all $u \in L^2(\mathbb{R}, d\xi)$

$$\int_{\mathbb{R}} |u(\xi)|^2\, d\xi = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} |\langle u, |y|^{-\frac{1}{2}}\Lambda_{1,b}(S\eta)(y,\cdot)\rangle|^2\, db\right) |y|^{-1}dy$$
$$= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} |\hat{u}(\omega)|^2 |\hat{\eta}(y,\omega)|^2 d\omega\right) |y|^{-1}dy$$

where we use that $\Delta(h(y)) = \alpha(h(y))^{-1} = |y|$ and where $\hat{}$ denotes the Fourier transform with respect to ξ. It follows that the set of admissible vectors is the set of Lebesgue measurable functions $\eta : \mathbb{R}^2 \to \mathbb{C}$ such that

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |\hat{\eta}(y,\omega)|^2 d\omega\right) dy < +\infty \iff \eta \in L^2(\mathbb{R}^2)$$
$$\int_{\mathbb{R}} |\hat{\eta}(y,\omega)|^2 |y|^{-1} dy = 1 \qquad \text{for almost every } \omega \in \mathbb{R}.$$

This set is clearly non empty: take for example any strictly positive continuous function $\sigma \in L^1(\mathbb{R})$ and define

$$\widehat{\eta}(y,\omega) = \left(\frac{1}{\sqrt{2\pi}\sigma(\omega)}|y|e^{-\frac{y^2}{2\sigma(\omega)^2}}\right)^{\frac{1}{2}}.$$

Appendix A. Appendix: some measure theory revisited

In this Appendix we review some known facts that are somehow hard to locate in the literature in a way that is both easily accessible and stated under the assumptions that we are making. The spaces X and Y are as in Section 3 and are regarded as measure spaces with respect to the Lebesgue measure, denoted dx and dy respectively.

A.1. Disintegration of measures.

We start by adapting to our setting some facts from integration theory on general locally compact spaces. The main reference for the issues at hand is [3]. Hereafter, $C_c(X)$ denotes the space of compactly supported continuous functions on X, endowed with the usual locally convex (separable) inductive limit topology, for which a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $C_c(X)$ converges to zero if there exists a compact set K such that $\operatorname{supp}\varphi_n \subset K$ for all n and $\lim_{n\to\infty}\sup_{x\in K}|\varphi_n(x)| = 0$. We denote by $M(X)$ the topological dual of $C_c(X)$; when equipped with the $\sigma(M(X), C_c(X))$-topology, the topological dual of $M(X)$ is again $C_c(X)$ ([30], Th. IV.20). Since X is second countable, the Riesz-Markov representation theorem uniquely identifies the measures with the positive elements of $M(X)$. By the word measure on a locally compact second countable topological space, we mean a positive measure defined on the Borel σ-algebra, which is finite on compact subsets.

The following theorem, in some sense a version of Fubini's theorem, summarizes the main properties of the kind of disintegration of measures we are concerned with. The main point here, though, is the possibility of extending the disintegration from C_c to L^1. We state it for X and Y, but it also holds *verbatim* if we replace X and Y with two arbitrary locally compact second countable topological spaces.

Theorem 11. *Suppose that ω is a measure on X and ρ a measure on Y and let $\Psi : X \to Y$ be a ω-measurable map. Assume further that $\{\omega_y\}$ is a family of measures on X such that*

(a) *ω_y is concentrated on $\Psi^{-1}(y)$ for all $y \in Y$;*

(b) $\displaystyle\int_X \varphi(x)d\omega(x) = \int_Y \left(\int_X \varphi(x)d\omega_y(x)\right) d\rho(y)$ *for all $\varphi \in C_c(X)$.*

Then, for any ω-measurable function $f : X \to \mathbb{C}$ the following facts hold true:

(i) *f is ω_y-measurable for almost every $y \in Y$;*

(ii) *f is ω-integrable if and only if* $\displaystyle\int_Y \left(\int_X |f(x)|d\omega_y(x)\right) d\rho(y)$ *is finite;*

(iii) *if f is ω-integrable, then f is ω_y-integrable for ρ-almost every $y \in Y$, the function (defined almost everywhere) $y \mapsto \int_X f(x)d\omega_y(x)$ is ρ-integrable, and*

$$\int_X f(x)d\omega(x) = \int_Y \left(\int_X f(x)d\omega_y(x)\right) d\rho(y); \tag{67}$$

(iv) *if $\{\omega'_y\}$ is another family of measures on X satisfying (a) and (b), then $\omega'_y = \omega_y$ for ρ-almost all $y \in Y$.*

Proof. The theorem is essentially contained in [3], scattered in several statements. For the proof of (i), (ii) and (iii) we quote from Chapter 5, and for the proof of (iv) from Chapter 6.

Statement (i) is the content of a) Prop. 4, § 3.2, taking into account that, since it is second countable, X is σ-compact and, a fortiori, ω-moderated (a subset is ω-moderated if it is contained into the union of a countable sequence of compact subsets and a ω-negligible set).

As for (ii), since X is second countable, Prop. 2, § 3.1, guarantees that the family $\int_X \varphi(x) d\omega_y(x)$ is ρ-adequate in the sense of Def. 1, § 3.1. The equivalence of the two conditions in (ii) is then the content of the Corollary at the end of § 3.2.

As for (iii), it is just Th. 1, § 3.3, observing that any function is ω-moderated since X is ω-moderated (a function is ω-moderated if it is null on the complement of a ω-moderated subset).

Finally, for (iv), by assumption $\int_Y \omega_y d\rho(y) = \int_Y \omega'_y d\rho(y)$, where the integral is a scalar integral of vector valued functions taking values in $M(X)$. Now Lemma 1, § 3.1 ensures that $C_c(X)$ has a countable subset which is dense[5] in $C_c(X)$ with respect to the $\sigma(C_c(X), M(X))$ topology, so that, by Remark 2 in §1.1, it is enough to show that for any $\varphi \in C_c(X)$ and for ρ-almost every $y \in Y$

$$\int_X \varphi(x) d\omega_y(x) = \int_X \varphi(x) d\omega'_y(x).$$

This is in turn equivalent to proving that

$$\int_Y \left(\int_X \varphi(x) d\omega_y(x) \right) \xi(y) d\rho(y) = \int_Y \left(\int_X \varphi(x) d\omega'_y(x) \right) \xi(y) d\rho(y) \tag{68}$$

holds for all $\varphi \in C_c(X)$ and $\xi \in C_c(Y)$. Fix then $\varphi \in C_c(X)$ and $\xi \in C_c(Y)$, and put $f(x) = \xi(\Psi(x))\varphi(x)$. This function is ω-measurable since Ψ is ω-measurable and ξ and φ are continuous, it is bounded since both ξ and φ are bounded, and it has a compact support since φ is compactly supported. Hence f is ω- integrable. Applying twice (67) we get

$$\int_Y \left(\int_X \xi(\Psi(x))\varphi(x) d\omega_y(x) \right) d\rho(y) = \int_Y \left(\int_X \xi(\Psi(x))\varphi(x) d\omega'_y(x) \right) d\rho(y). \tag{69}$$

Given $y \in Y$, (a) implies that $\xi(\Psi(x)) = \xi(y)$ for ω_y-almost all $x \in X$, so that

$$\int_Y \left(\int_X \xi(\Psi(x))\varphi(x) d\omega_y(x) \right) d\rho(y) = \int_Y \left(\int_X \varphi(x) d\omega_y(x) \right) \xi(y) d\rho(y),$$

and similarly for the right hand side of (69). Hence (68) is true and the claim is proved. □

The integral formula (b) will be written for short

$$d\omega = \int_Y \omega_y \, d\rho(y). \tag{70}$$

[5] It is proved there that there exists a countable subset $S \subset C_c(X)$ such that for every $\varphi \in C_c(X)$ there is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in S converging to φ uniformly and $|\varphi_n| \leq |\varphi_0|$.

A.2. **Direct integrals.** Next we recall the definition of direct integral, following [17]. Hereafter we assume that the hypotheses of Theorem 11 are satisfied. Fix a countable family $\{\varphi_k\}_{k\in\mathbb{N}}$ dense in $C_c(X)$, and hence also in every $L^2(X,\omega_y)$, with $y \in Y$. The map $y \mapsto \langle \varphi_k, \varphi_\ell\rangle_{\omega_y}$ is ρ-measurable since it is ρ-integrable by hypothesis (b) of Theorem 11. Under these circumstances, $\{\varphi_k\}_{k\in\mathbb{N}}$ is called a ρ-measurable structure for the family of Hilbert spaces $\{L^2(X,\omega_y)\}$. The direct integral $\int_Y L^2(X,\omega_y)dy$ is defined as the set consisting of all the families $\{f_y\}$ satisfying:

(D1) $f_y \in L^2(X,\omega_y)$ for all $y \in Y$;

(D2) $\displaystyle\int_Y \|f_y\|^2_{\omega_y} d\rho(y) < +\infty$;

(D3) $y \mapsto \langle f_y, \varphi_k\rangle_{\omega_y}$ is ρ-measurable for all $k \in \mathbb{N}$.

Two families $\mathcal{F} = \{f_y\}$ and $\mathcal{G} = \{g_y\}$ are identified if for almost every $y \in Y$ $f_y = g_y$ as elements in $L^2(X,\omega_y)$. The space $\int_Y L^2(X,\omega_y)d\rho(y)$ is a Hilbert space under

$$\langle \mathcal{F}, \mathcal{G}\rangle = \int_Y \langle f_y, g_y\rangle_{\omega_y} d\rho(y).$$

Since $C_c(X)$ has a dense countable subset, see Footnote 5, (D3) is equivalent to

(D3') $y \mapsto \langle f_y, \varphi\rangle_{\omega_y}$ is ρ-measurable for all $\varphi \in C_c(X)$,

so that, as long as we choose the functions of $\{\varphi_k\}_{k\in\mathbb{N}}$ in $C_c(X)$, the measurable structure is independent of the choice of the particular family.

Proposition 6. *Given $f \in L^2(X,\omega)$, there exists a unique family $\{f_y\}$ in the Hilbert space direct integral $\int_Y L^2(X,\omega_y)d\rho(y)$ such that, for almost every $y \in Y$, the equality $f_y(x) = f(x)$ holds for ω_y-almost every $x \in X$. Furthermore, the map $f \mapsto \{f_y\}$ is a unitary operator from $L^2(X,\omega)$ onto $\int_Y L^2(X,\omega_y)d\rho(y)$.*

Proof. By hypothesis (b) of Theorem 11, for every $\varphi \in C_c(X)$ we have

$$\int_X \varphi(x)d\omega(x) = \int_Y \left(\int_X \varphi(x)d\omega_y\right) d\rho(y).$$

Given a function[6] $f : X \to \mathbb{C}$ which is square-integrable with respect to ω, hence in particular ω-measurable, (i) of Theorem 11 implies that f is ω_y-measurable for almost every $y \in Y$. Further, since $|f|^2$ is integrable with respect to ω, (iii) of the same theorem ensures that $|f|^2$ is ω_y-integrable for almost all $y \in Y$, the map $y \mapsto \int_X |f(x)|^2 d\omega_y(x)$ is integrable, and

$$\int_X |f(x)|^2 \, d\omega(x) = \int_Y \left(\int_X |f(x)|^2 d\omega_y(x)\right) d\rho(y). \tag{71}$$

Hence there is a ρ-full set $Y' \subset Y$ such that, if $y \in Y'$, f is square-integrable with respect to ω_y. For $y \in Y'$ define f_y to be the equivalence class of f in $L^2(X,\omega_y)$ and, for $y \notin Y'$, put $f_y = 0$.

[6]Here it is important that f is a function, and not an equivalence class modulo a.e. equality.

We claim that $\mathcal{F} = \{f_y\}$ is in $\int_Y L^2(X, \omega_y) d\rho(y)$. By (71), conditions (D1) and (D2) are clearly satisfied. To prove (D3'), take $\varphi \in C_c(X)$. Clearly, $f\overline{\varphi}$ is ω-integrable and hence, by (iii) of Theorem 11, it is ω_y-integrable for almost every $y \in Y$ and

$$y \mapsto \int_X f(x)\overline{\varphi(x)} d\omega_y(x) = \langle f_y, \varphi \rangle_{\omega_y}$$

is integrable, hence measurable. Therefore $f \mapsto \mathcal{F}$ is a well defined map from the space of square-integrable functions on X to $\int_Y L^2(X, \omega_y) d\rho(y)$, it is linear and, by (71),

$$\int_X |f(x)|^2 d\omega(x) = \int_Y \|f_y\|^2_{\omega_y} d\rho(y). \tag{72}$$

Hence, it defines an isometry from $L^2(X, \omega)$ into $\int_Y L^2(X, \omega_y) d\rho(y)$ and, by construction, for almost every $y \in Y$, the equality $f_y(x) = f(x)$ holds for ω_y-almost every $x \in X$.

We claim that the isometry $f \mapsto \mathcal{F}$ is surjective. It is enough to prove that for any family $\mathcal{F}$ whose members f_y are positive, there exists a positive $f \in L^2(X, \omega)$ such that, for almost every $y \in Y$, the equality $f_y(x) = f(x)$ holds for ω_y-almost every $x \in X$. Take then such an $\mathcal{F}$. First of all, we show that the family of measures $\{f_y \cdot \omega_y\}$ is scalarly integrable with respect to ρ. This is equivalent to saying that for all $\varphi \in C_c(X)$ the function $y \mapsto F_\varphi(y) = \int_X \varphi(x) f_y(x) d\omega_y(x)$, certainly well defined because (D1) implies that φf_y is ω_y-integrable for every $y \in Y$, is ρ-integrable. Indeed, (D3') says that F_φ is ρ-measurable, whereas Hölder's inequality and Cauchy-Schwartz give

$$\begin{aligned} \int_Y |F_\varphi(y)| d\rho(y) &\leq \int_Y \|\varphi\|_{\omega_y} \|f\|_{\omega_y} d\rho(y) \\ &\leq \left(\int_Y \|\varphi\|^2_{\omega_y} d\rho(y) \right)^{1/2} \left(\int_Y \|f\|^2_{\omega_y} d\rho(y) \right)^{1/2} \end{aligned}$$

so that by (D2) and (72) applied to φ yield

$$\int_Y |F_\varphi(y)| d\rho(y) \leq C \|\varphi\| < +\infty.$$

Hence the claim is proved and $\mu = \int_Y (f_y \cdot \omega_y) d\rho(y)$ defines a measure. We show next that μ is a measure with base[7] ω. This will produce the required f that maps to $\mathcal{F}$. The Lebesgue-Nikodym theorem (see Th. 2 ,§ 5.5, Ch. 5 of [3]) ensures that it is enough to prove that any compact subset $K \subset X$ for which $\omega(K) = 0$ satisfies $\mu(K) = 0$. Take such a K. Item (iii) of Theorem 11 applied to the characteristic function χ_K gives that for almost every $y \in Y$, K is ω_y-negligible and, *a fortiori*, $f_y \cdot \omega_y$-negligible. Thus, (67) with $\omega = \mu$, $\omega_y = f \cdot \omega_y$ and $f = \chi_K$ yields

$$\mu(K) = \int_Y \left(\int_K f_y(x) d\omega_y(x) \right) d\rho(y) = 0.$$

Hence there exists a locally integrable positive function f such that $f \cdot \omega = \mu$. Moreover, if $\varphi \in C_c(X)$, φf is integrable, so that again (iii) of Theorem 11 tells us that, for almost

[7] A measure which is the product $\psi \cdot \mathcal{L}$ of a measure $\mathcal{L}$ by a locally $\mathcal{L}$-integrable positive function ψ is called a measure with base $\mathcal{L}$ (see Def. 2, § 5.2, Ch. V in [3]).

every $y \in Y$, φf is ω_y-integrable, the map $y \mapsto \int_X \varphi(x) f(x)\, d\omega_y(x)$ is integrable and by definition of μ

$$\begin{aligned}\int_Y \left(\int_X \varphi(x) f_y(x)\, d\omega_y(x) \right) d\rho(y) &= \int_X \varphi(x)\, d\mu(x) \\ &= \int_Y \left(\int_X \varphi(x) f(x)\, d\omega_y(x) \right) d\rho(y).\end{aligned}$$

By the above equality, (iv) of Theorem 11 may be applied to infer that for almost every $y \in Y$ the equality $f = f_y$ holds ω_y-almost everywhere. Finally, (D2) gives

$$\int_Y \left(\int_X |f(x)|^2 d\omega_y(x) \right) d\rho(y) = \int_Y \left(\int_X |f_y(x)|^2 d\omega_y(x) \right) d\rho(y) < +\infty.$$

Hence (iii) of Theorem 11 implies that f is square integrable. The equivalence class of f in $L^2(X, \omega)$ is then the element required to prove surjectivity. □

Both $L^2(X, \omega)$ and each of the spaces $L^2(X, \omega_y)$ can be identified with subspaces of $M(X)$ simply by viewing their elements as continuous linear functionals on $C_c(X)$ via integration with respect to ω and ω_y, respectively. Further, (iv) of Theorem 11 implies that saying that for almost every $y \in Y$ the equality $f_y(x) = f(x)$ holds for ω_y-almost every $x \in X$ is equivalent to

$$f \cdot \omega = \int_Y (f_y \cdot \omega_y)\, d\rho(y),$$

in the sense that the map $Y \to M(X)$, $y \mapsto f_y \cdot \omega_y$ is ρ-scalarly-integrable. These remarks together with Proposition 6 imply that

$$L^2(X, \omega) = \int_Y L^2(X, \omega_y) d\rho(y) \tag{73}$$

by means of the equality in $M(X)$

$$f = \int_Y f_y d\rho(y), \tag{74}$$

where the integral is a scalar integral.

A.3. **The coarea formula for submersions.** Below we give a simple proof of the Coarea Formula for submersions; the general case is due to Federer [16]. Suppose that $n \leq d$ and let $X \subset \mathbb{R}^d$ be an open set. Recall that a C^1-map $\Phi : X \to \mathbb{R}^n$ is called a submersion if its differential Φ_{*x} is surjective for all $x \in X$. For every $y \in Y = \Phi(X)$, let $dv^y(x)$ denote the volume element of the Riemannian submanifold $\Phi^{-1}(y)$ and by $J\Phi$ the Jacobian. We introduce the measure ν_y on X by

$$\nu_y(E) = \int_{\Phi^{-1}(y) \cap E} \frac{dv^y(x)}{(J\Phi)(x)}, \qquad E \in \mathcal{B}(X). \tag{75}$$

It is worth observing that ν_y is finite on compact sets and concentrated on $\Phi^{-1}(y)$.

Theorem 12 (Coarea formula for submersions)**.** *Suppose that $\Phi : X \to \mathbb{R}^n$ is a submersion. Then*

$$dx = \int_Y d\nu_y \, dy, \tag{76}$$

where dx and dy are the Lebesgue measures on $\mathbb{R}^d$ and $\mathbb{R}^n$, respectively.

Proof. We must show that

$$\int_X f(x)\, dx = \int_Y \left(\int_X f(x) \frac{dv^y(x)}{(J\Phi)(x)} \right) dy$$

holds for every $f \in C_c(X)$. Fix $x_0 \in X$. Since Φ_{*x_0} is surjective, the Inverse Mapping Theorem implies (Corollary 5.8 in [26]) that there exists a diffeomorphism $\Psi : U \times V \mapsto W$ such that

$$\Phi(\Psi(z, y)) = y \qquad z \in U,\, y \in V, \tag{77}$$

where U is an open subset of $\mathbb{R}^{d-n}$, V is an open subset of $\mathbb{R}^n$ and W is an open neighborhood of x_0.

Take $f \in C_c(X)$. For any such f, since $\operatorname{supp} f$ is compact, by choosing a suitable finite covering if necessary, we can always assume that $\operatorname{supp} f \subset W$. The change of variables formula and Fubini's Theorem give

$$\int_W f(x)\, dx = \int_V \left(\int_U f(\Psi(z, y))(J\Psi)(z, y)\, dz \right) dy. \tag{78}$$

To obtain the coarea formula we simply compute the Jacobian $J\Psi$. Observe that for any given $y \in V$, $\Psi^y = \Psi(\cdot, y)$ is a diffeomorphism from U onto $W \cap \Phi^{-1}(y)$, regarded as a submanifold. In particular, using this local chart, the volume element at the point $x = \Psi(z, y)$ is given by

$$dv^y(x) = \sqrt{\det\left[{}^t(\Psi^y)_{*z}(\Psi^y)_{*z}\right]}\, dz. \tag{79}$$

Taking the derivatives of(77) with respect to z and y separately, we obtain

$$\Phi_{*\Psi(z,y)}\, D_1\Psi_{(z,y)} = 0, \qquad \Phi_{*\Psi(z,y)}\, D_2\Psi_{(z,y)} = I_{n\times n}. \tag{80}$$

Fix $(z, y) \in U \times V$ and let P_1 denote the orthogonal projection from $\mathbb{R}^d$ onto $\ker \Phi_{*\Psi(z,y)}$, and $P_2 = I - P_1$ the orthogonal projection onto $[\ker \Phi_{*\Psi(z,y)}]^\perp$, which is a subspace of dimension n because Φ is a submersion. From (80) it follows that

$$P_2(D_1\Psi)_{(z,y)} = 0, \qquad P_2(D_2\Psi)_{(z,y)} = (\Phi_{*\Psi(z,y)} \circ \iota)^{-1}, \tag{81}$$

where $\iota : [\ker \Phi_{*\Psi(z,y)}]^\perp \to \mathbb{R}^d$ is the natural injection. Let $R \in \mathrm{O}(d)$ be the rotation that takes $\ker \Phi_{*\Psi(z,y)}$ onto the z-hyperplane (first $d-n$ coordinates) and its orthogonal complement onto the y-hyperplane (last n coordinates), so that $RP_1(z, y) = z$ and $RP_2(z, y) = y$. Then (81) imply

$$R\Psi_{*(z,y)} = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

where $A = R(D_1\Psi)_{(z,y)}$, $B = RP_1(D_2\Psi)_{(z,y)}$ and $C = RP_2(D_2\Psi)_{(z,y)}$. Therefore

$$(J\Psi)(z,y) = |\det R\Psi_{*(z,y)}| = |\det A|\,|\det C| = \frac{\sqrt{\det\left[{}^t(\Psi^y)_{*z}(\Psi^y)_{*z}\right]}}{\sqrt{\det\left[\Phi_{*\Psi(z,y)}{}^t\Phi_{*\Psi(z,y)}\right]}},$$

where we have used (81). Taking (79) into account, for $x = \Psi(z,y)$ we have

$$(J\Psi)(z,y)\,dz = \frac{dv^y(x)}{(J\Phi)(x)}\,,$$

which inserted in (78) yields the result. □

Appendix B. The Jacobian criterion

We show below that Theorem 16.19 in [15] implies that for all $y \in \Phi(\mathcal{R})$ the fiber $\Phi^{-1}(y)$ is finite. First of all, we can view Φ as a polynomial map from $\mathbb{C}^d$ into itself, so we write $\Phi = (f_1, \dots, f_d)$. Without loss of generality we assume further that $y = 0$. Following [15], we write $S = \mathbb{C}[X_1, \dots, X_d]$ and we denote by I the ideal in S generated by $f_1, \dots, f_d$. We are interested in its radical $\sqrt{I}$, which decomposes as an intersection, unique up to order, of prime ideals $\sqrt{I} = P_1 \cap \cdots \cap P_s$. Hence $V(I) = \{w \in \mathbb{C}^d : f_1(w) = \cdots = f_d(w) = 0\} = V_1 \cup \cdots \cup V_s$, the corresponding decomposition into irreducible components, namely $V_i = Z(P_i)$. Under the present circumstances, $\dim V_i = d - \operatorname{codim}(P_i)$, where the latter is the Krull codimension of P_i. Clearly, $\operatorname{codim}(P_i) = d$ if and only if V_i is a singleton. Suppose that $\dim V_j > 0$ for some j. We will show that at the points $w \in V_j$ the Jacobian determinant

$$J\Phi(w) = \det\left(\frac{\partial f_i}{\partial w_j}(w)\right)$$

vanishes. Suppose by contradiction that $J\Phi(w) \neq 0$. Now, the codimension of I_{P_j} in S_{P_j} is equal to $\operatorname{codim}(P_j)$ because P_j is a minimal prime of I. By assumption, this is strictly smaller than d. By the Jacobian criterion, the Jacobian matrix taken modulo P_j has rank strictly less than d. This means that $J\Phi \in P_j$. But $w \in V_j$, which implies that $J\Phi(w) = 0$, a contradiction. Therefore $\Phi^{-1}(0) \cap \mathcal{R}$ does not intersect irreducible components with positive dimension, hence it is a finite set.

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