Un metodo numerico per il problema di Dirichlet su domini con frontiera regolare a tratti

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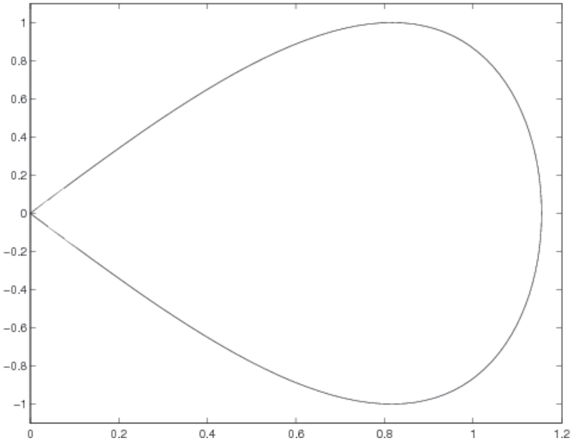
The interior Dirichlet problem

$$\begin{cases} \Delta u(P) = 0, & P \in D \\ u(P) = g(P), & P \in \Sigma \end{cases}$$

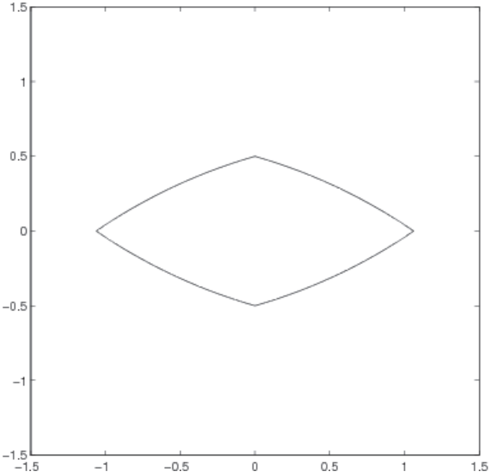
where

- D is a simply connected bounded region in the plane;
- Σ is the boundary of D. We assume that it is a simple closed **piecewise smooth curve**;
- g is a given sufficiently smooth boundary function on Σ.

Example 2



Example 3



A possible numerical approach....

$$u(A) = \int_{\Sigma} \psi(Q) \frac{\partial}{\partial \mathbf{n}_Q} [\log |A - Q|] d\Sigma_Q, \quad A \in D,$$

where

- ψ is the **double layer density function**,

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where

- ψ is the **double layer density function**,solution of the following BIE

$$(-2\pi + \Omega(P))\psi(P) + \int_{\Sigma} \psi(Q) \frac{\partial}{\partial \mathbf{n}_Q} [\log |P - Q|] d\Sigma_Q = g(P), \; P \in \Sigma,$$

with $\Omega(P)$ the interior angle to Σ at P.

A possible numerical approach...

1. Approximate the double layer density function ψ

$$(-\pi + K)\psi = g$$

where

$$K\psi(P) = (-\pi + \Omega(P))\psi(P) + \int_{\Sigma} \psi(Q) \frac{\partial}{\partial \mathbf{n}_Q} [\log |P - Q|] d\Sigma_Q, P \in \Sigma$$

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2. Approximate the solution of the initial problem

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$K : C(\Sigma) \rightarrow C(\Sigma)$ **not compact**!

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- II STEP: Apply a suitable Nyström method to the system by proving stability and convergence of the method and well-conditioning of system we solve.

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I STEP: ...a system of BIE...

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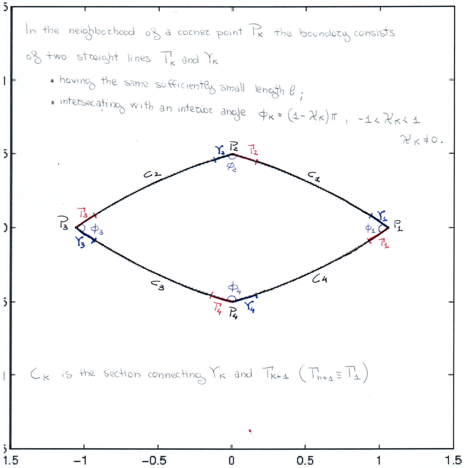

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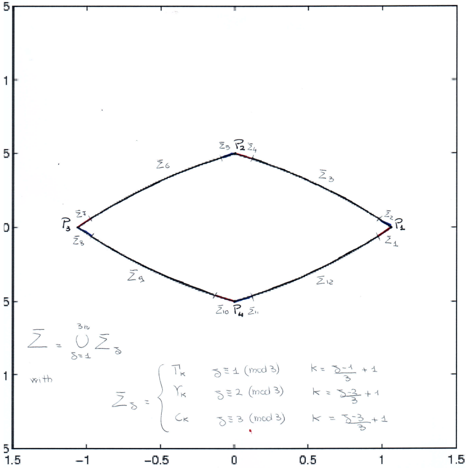

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We can now rewrite the boundary integral equation

$$(-2\pi+\Omega(P))\psi(P)+\int_{\Sigma}\psi(Q)\frac{\partial}{\partial \mathbf{n}_Q}[\log|P-Q|]d\Sigma_Q = g(P),\ P\in\Sigma,$$

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as the following system of $3n$ boundary integral equations

$$(-2\pi+\Omega(P))\psi_i(P)+\sum_{j=1}^{3n}\int_{\Sigma_j}\psi_j(Q)\frac{\partial}{\partial \mathbf{n}_Q}[\log|P-Q|]d\Sigma_Q = g_i(P),$$

$$P\in\Sigma_i$$

where

$$\psi_i=\psi|_{\Sigma_i},\quad g_i=g|_{\Sigma_i}$$

1. Approximate the double layer density function ψ

I STEP: ...a system of BIE...

Let us introduce a parametric representation for each arc Σ_i

$$\sigma_i : s \in [0,1] \rightarrow (\xi_i(s), \eta_i(s)) \in \Sigma_i,$$

with $\sigma_i \in C^{q+2}([0,1])$ and

$$|\sigma_i'(s)| \neq 0 \quad 0 \leq s \leq 1, \quad i = 1, ..., 3n$$

Without any loss of generality, we can assume that when

$$\sigma_i(0) = P_k \quad \text{if} \quad \Sigma_i = \Gamma_k \quad \text{or} \quad \Sigma_i = \Upsilon_k$$

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$$(-2\pi+\Omega_i(\sigma_i(s)))\psi_i(\sigma_i(s))+\sum_{j=1}^{3n}\int_0^1 k^{i,j}(t,s)\psi_j(\sigma_j(t))dt = g_i(\sigma_i(s)),$$

where

$$k^{i,j}(t,s) = \frac{\partial}{\partial \mathbf{n}_{\sigma_j(t)}}[\log|\sigma_j(t)-\sigma_i(s)|]\cdot|\sigma_j'(t)|$$

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- $\bar{g} = (\bar{g}_1, \bar{g}_2, ..., \bar{g}_{3n})^T, \quad \bar{g}_i(s) = g_i(\sigma_i(s))$
- $\mathcal{I} = \begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & I \end{pmatrix},$

1. Approximate the double layer density function ψ

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- $\mathcal{I}=\begin{pmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & I \end{pmatrix},$

1. Approximate the double layer density function ψ

I STEP: ...a system of integral equations on $[0,1]$...

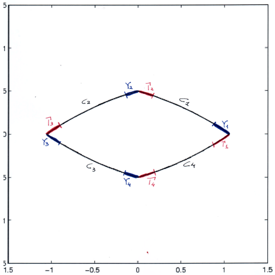
- $$\mathcal{K} = \begin{pmatrix} (-\pi+\bar{\Omega}_1)I & \mathcal{K}^{1,2} & \cdots & \mathcal{K}^{1,3n-1} & \mathcal{K}^{1,3n} \\ \mathcal{K}^{2,1} & (-\pi+\bar{\Omega}_2)I & \cdots & \mathcal{K}^{2,3n-1} & \mathcal{K}^{2,3n} \\ \mathcal{K}^{3,1} & \mathcal{K}^{3,2} & \ddots & \mathcal{K}^{3,3n-1} & \mathcal{K}^{3,3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{K}^{3n-2,1} & \mathcal{K}^{3n-2,2} & \cdots & \mathcal{K}^{3n-2,3n-1} & \mathcal{K}^{3n-2,3n} \\ \mathcal{K}^{3n-1,1} & \mathcal{K}^{3n-1,2} & \cdots & (-\pi+\bar{\Omega}_{3n-2})I & \mathcal{K}^{3n-1,3n} \\ \mathcal{K}^{3n,1} & \mathcal{K}^{3n,2} & \cdots & \mathcal{K}^{3n,3n-1} & \mathcal{K}^{3n,3n} \end{pmatrix}$$

  with

  $$(\mathcal{K}^{i,j}\psi_j)(s) = \int_0^1 k^{i,j}(t,s)\psi_j(t)dt, \quad \bar{\Omega}_i(s) = \Omega_i(\sigma_i)(s).$$

1. Approximate the double layer density function ψ

I STEP: ...a system of integral equations on $[0, 1]$...



The operators $\mathcal{K}^{i,j}$ are **compact** on the space $C([0,1])$ when

- $i \equiv 0 \pmod 3$ or $j \equiv 0 \pmod 3$
- $i, j \equiv 1, 2 \pmod 3$ and $|i - j| \neq 1$

since their kernels $k^{i,j}(t, s)$ are of class C^q for $0 \leq t, s \leq 1$.

1. Approximate the double layer density function ψ

I STEP: ...a system of integral equations on $[0,1]$...

The operators $\mathcal{K}^{i,j}$ are **not compact** on the space $C([0,1])$ when

- $i, j \equiv 1, 2 \pmod 3$ and $|i-j| = 1$

since the kernel $k^{i,j}(t,s)$ are of Mellin type, having the following form

$$k^{i,j}(t,s) = -\frac{s \sin(\chi_k \pi)}{s^2 + 2ts\cos(\chi_k \pi) + t^2}, \quad k = \frac{i-1}{3} + 1 \text{ if } i \equiv 1 \pmod 3$$

$$k = \frac{i-2}{3} + 1 \text{ if } i \equiv 2 \pmod 3$$

From now on, in these cases we set

$$L^{i,j} := k^{i,j} \qquad \mathcal{L}^{i,j} := \mathcal{K}^{i,j}.$$

1. Approximate the double layer density function ψ

I STEP: ...a system of integral equations on $[0,1]$...

$$\mathcal{W} = \begin{pmatrix} \mathcal{L}^1 & \underline{0} & & & & & \\ \underline{0}^T & 0 & & & & & \\ & & \mathcal{L}^4 & \underline{0} & & & \\ & & \underline{0}^T & 0 & & & \\ & & & & \ddots & & \\ & & & & & \mathcal{L}^{3n-2} & \underline{0} \\ & & & & & \underline{0}^T & 0 \end{pmatrix}$$

where $\underline{0} = (0,0)^T$ and

$$\mathcal{L}^i = \begin{pmatrix} (-\pi + \bar{\Omega}_i)I & \mathcal{L}^{i,i+1} \\ \mathcal{L}^{i+1,i} & (-\pi + \bar{\Omega}_{i+1})I \end{pmatrix}, \quad i \equiv 1 \pmod 3)$$

1. Approximate the double layer density function ψ

I STEP: ...a system of integral equations on $[0,1]$...

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & \mathcal{K}^{1,3} & \cdots & \mathcal{K}^{1,3n-2} & \mathcal{K}^{1,3n-1} & \mathcal{K}^{1,3n} \\ 0 & 0 & \mathcal{K}^{2,3} & \cdots & \mathcal{K}^{2,3n-2} & \mathcal{K}^{2,3n-1} & \mathcal{K}^{2,3n} \\ \mathcal{K}^{3,1} & \mathcal{K}^{3,2} & \mathcal{K}^{3,3} & \ddots & \mathcal{K}^{3,3n-2} & \mathcal{K}^{3,3n-1} & \mathcal{K}^{3,3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathcal{K}^{3n-2,1} & \mathcal{K}^{3n-2,2} & \mathcal{K}^{3n-2,3} & \cdots & 0 & 0 & \mathcal{K}^{3n-2,3n} \\ \mathcal{K}^{3n-1,1} & \mathcal{K}^{3n-1,2} & \mathcal{K}^{3n-1,3} & \cdots & 0 & 0 & \mathcal{K}^{3n-1,3n} \\ \mathcal{K}^{3n,1} & \mathcal{K}^{3n,2} & \mathcal{K}^{3n,3} & \cdots & \mathcal{K}^{3n,3n-1} & \mathcal{K}^{3n,3n-1} & \mathcal{K}^{3n,3n} \end{pmatrix}$$

1. Approximate the double layer density function ψ

I STEP: ...a system of integral equations on $[0,1]$...

Then the system

$$(-\pi\mathcal{I} + \mathcal{K})\bar{\psi} = \bar{g}$$

can be rewritten as

$$(-\pi\mathcal{I} + \mathcal{W} + \mathcal{S})\bar{\psi} = \bar{g}$$

where the array of the unknowns is

$$\bar{\psi} = \left(\bar{\psi}_1, \bar{\psi}_2, ..., \bar{\psi}_{3n}\right)^T, \quad \bar{\psi}_i(s) = \psi_i(\sigma_i(s))$$

and the array of the right-hand sides is

$$\bar{g} = \left(\bar{g}_1, \bar{g}_2, ..., \bar{g}_{3n}\right)^T, \quad \bar{g}_i(s) = g_i(\sigma_i(s))$$

1. Approximate the double layer density function ψ

I STEP: ...a system of integral equations on $[0,1]$...

Let us consider the direct product

$$C([0,1])^{3n} = \underbrace{C([0,1]) \times C([0,1]) \times \ldots \times C([0,1])}_{3n}$$

which is complete with the norm

$$\|(f_1, f_2, \ldots, f_{3n})\|_\infty = \max_{i=1,2,\ldots,3n} \|f_i\|_\infty,$$

and the following complete subspace of $C([0,1])^{3n}$

$$\mathcal{X} = \left\{ (f_1, \ldots, f_{3n}) \in C([0,1])^{3n} \mid f_i(0) = f_{i+1}(0),\ i \equiv 1 \pmod 3 \right\}$$

1. Approximate the double layer density function ψ

I STEP: ...a system of integral equations on $[0,1]$...

$$(-\pi\mathcal{I} + \mathcal{W} + \mathcal{S})\bar{\psi} = \bar{g} \tag{1}$$

One can prove the following.

Theorem

Let $\mathrm{Ker}(-\pi\mathcal{I} + \mathcal{W} + \mathcal{S}) = \{0\}$ *in the Banach space* $\mathcal{X}$. *Then system* (1) *has a unique solution in* $\mathcal{X}$ *for each given right hand side* $\bar{g} \in \mathcal{X}$.

1. Approximate the double layer density function ψ

II STEP: ...Nyström method...

$$\int_0^1 f(x)dx = \sum_{k=0}^{m+1} \lambda_k f(x_k) + e_m(f)$$

where $x_0 = 0$, $x_1 < x_2 < ... < x_m$ zeros of $p_m(v^{1,1})$, $x_{m+1} = 1$.

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- For $i \equiv 1, 2 \pmod 3$ and $|i - j| = 1$

$$(\mathcal{L}_m^{i,j} \rho)(s) = \sum_{h=0}^{m+1} \lambda_h L^{i,j}(x_h, s) \rho(x_h)$$

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$$(\mathcal{L}_m^{i,j}\rho)(s) = \sum_{h=0}^{m+1} \lambda_h L^{i,j}(x_h, s)\rho(x_h)$$

- For $i \not\equiv 1, 2 \pmod 3$ or $i \equiv 1, 2 \pmod 3$ and $|i - j| \neq 1$

$$(\mathcal{K}_m^{i,j}\rho)(s) = \sum_{h=0}^{m+1} \lambda_h k^{i,j}(x_h, s)\rho(x_h)$$

1. Approximate the double layer density function ψ

II STEP: ...Nyström method...

Remarks

- Any sequence of operators $\left\{\mathcal{K}_m^{i,j}\right\}_m$ is pointwise convergent to the operator $\mathcal{K}^{i,j}$ in the space $C([0,1])$

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- Unfortunately we are not able to prove a similar result for the sequences $\left\{\mathcal{L}_m^{i,j}\right\}_m$!

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- Unfortunately we are not able to prove a similar result for the sequences $\left\{\mathcal{L}_m^{i,j}\right\}_m$!
- For $\rho \in C([0,1])$, each sequence of functions $\left\{\mathcal{L}_m^{i,j}\rho\right\}_m$ converges uniformly to $\mathcal{L}^{i,j}\rho$ in any interval of the type $[\frac{c}{m^{2-2\epsilon}}, 1]$, for some constant $c > 0$ and arbitrarily small $\epsilon > 0$.

1. Approximate the double layer density function ψ

II STEP: ...Nyström method...

To overcome this problem in place of the following matrices 2×2 of operators

$$\mathcal{L}_m^i = \begin{pmatrix} (-\pi + \bar{\Omega}_i)I & \mathcal{L}_m^{i,i+1} \\ \mathcal{L}_m^{i+1,i} & (-\pi + \bar{\Omega}_{i+1})I \end{pmatrix}$$

for $i = 1, \ldots, 3n$ and $i \equiv 1 \pmod 3$

G. Mastroianni, G. Monegato,
Nyström interpolants based on the zeros of Legendre polynomials for a non-compact integral operator equation
IMA J. Numer. Anal., **14** (1993), 81-95

1. Approximate the double layer density function ψ

II STEP: ...Nyström method...

We consider **a slight modification** $\tilde{\mathcal{L}}_m^i$

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II STEP: ...Nyström method...

We consider **a slight modification** $\tilde{\mathcal{L}}^i_m$ of them defined as follows for a fixed constant $c > 0$ and a fixed arbitrarily small real number $\epsilon > 0$ and

- For $\frac{c}{m^{2-2\epsilon}} \leq s \leq 1$

$$(\tilde{\mathcal{L}}^i_m \varrho)(s) = (\mathcal{L}^i_m \varrho)(s)$$

1. Approximate the double layer density function ψ

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- For $\frac{c}{m^{2-2\epsilon}} \leq s \leq 1$

$$(\tilde{\mathcal{L}}^i_m \varrho)(s) = (\mathcal{L}^i_m \varrho)(s)$$

- For $0 \leq s < \frac{c}{m^{2-2\epsilon}}$

$$(\tilde{\mathcal{L}}^i_m \varrho)(s) = \frac{m^{2-2\epsilon}}{c} \left[s(\mathcal{L}^i_m \varrho)\left(\frac{c}{m^{2-2\epsilon}}\right) + \left(\frac{c}{m^{2-2\epsilon}} - s\right)(\mathcal{L}^i \varrho)(0) \right]$$

with $\varrho = (\varrho_1, \varrho_2)^T \in C([0,1])^2$.

1. Approximate the double layer density function ψ

II STEP: ...Nyström method...

$$\mathcal{W}_m = \begin{pmatrix} \tilde{\mathcal{L}}_m^1 & \underline{0} & & & & & \\ \underline{0}^T & 0 & & & & & \\ & & \tilde{\mathcal{L}}_m^4 & 0 & & & \\ & & \underline{0} & 0 & & & \\ & & & & \ddots & & \\ & & & & & \tilde{\mathcal{L}}_m^{3n-2} & \underline{0} \\ & & & & & \underline{0}^T & 0 \end{pmatrix}$$

Theorem

The operators $\mathcal{W}_m : \mathcal{X} \to \mathcal{X}$ are linear maps such that,

$$\lim_{m\to\infty} \|\mathcal{W}_m\| < \pi$$

$$\lim_{m\to\infty} \|(\mathcal{W}_m - \mathcal{W})\rho\|_\infty = 0, \quad \forall\ \rho \in \mathcal{X}.$$

1. Approximate the double layer density function ψ

II STEP: ...Nyström method...

$$\mathcal{S}_m = \begin{pmatrix} 0 & 0 & \mathcal{K}_m^{1,3} & \cdots & \mathcal{K}_m^{1,3n-2} & \mathcal{K}_m^{1,3n-1} & \mathcal{K}_m^{1,3n} \\ 0 & 0 & \mathcal{K}_m^{2,3} & \cdots & \mathcal{K}_m^{2,3n-2} & \mathcal{K}_m^{2,3n-1} & \mathcal{K}_m^{2,3n} \\ \mathcal{K}_m^{3,1} & \mathcal{K}_m^{3,2} & \mathcal{K}_m^{3,3} & \ddots & \mathcal{K}_m^{3,3n-2} & \mathcal{K}_m^{3,3n-1} & \mathcal{K}_m^{3,3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathcal{K}_m^{3n-2,1} & \mathcal{K}_m^{3n-2,2} & \mathcal{K}_m^{3n-2,3} & \cdots & 0 & 0 & \mathcal{K}_m^{3n-2,3n} \\ \mathcal{K}_m^{3n-1,1} & \mathcal{K}_m^{3n-1,2} & \mathcal{K}_m^{3n-1,3} & \cdots & 0 & 0 & \mathcal{K}_m^{3n-1,3n} \\ \mathcal{K}_m^{3n,1} & \mathcal{K}_m^{3n,2} & \mathcal{K}_m^{3n,3} & \cdots & \mathcal{K}_m^{3n,3n-1} & \mathcal{K}_m^{3n,3n-1} & \mathcal{K}_m^{3n,3n} \end{pmatrix}$$

Theorem

The operators $\mathcal{S}_m : \mathcal{X} \to \mathcal{X}$ are linear maps such that the set $\{\mathcal{S}_m\}_m$ is collectively compact and

$$\lim_{m\to\infty} \|(\mathcal{S}_m - \mathcal{S})\rho\|_\infty = 0, \quad \forall\ \rho \in \mathcal{X}.$$

1. Approximate the double layer density function ψ

II STEP: ...Nyström method...

Now in place of the following system

$$(-\pi\mathcal{I} + \mathcal{W} + \mathcal{S})\bar{\psi} = \bar{g}$$

we consider the approximating one

$$(-\pi\mathcal{I} + \mathcal{W}_m + \mathcal{S}_m)\bar{\psi}_m = \bar{g},$$

whose unknown is the array of functions

$$\bar{\psi}_m = (\bar{\psi}_{m,1}, ..., \bar{\psi}_{m,3n})^T.$$

By collocating the equations at the quadrature nodes x_l, we get

$$A_m\psi = \mathbf{g}$$

1. Approximate the double layer density function ψ

II STEP: ...Nyström method...

Theorem

Let us assume

- $\Sigma \setminus \{P_1, \ldots, P_n\}$ *of class* C^{q+2}*, for some* $q \geq 2$.
- $\bar{g} \in \mathcal{X} \cap C^p([0,1])^{3n}$ *with* p *large enough.*
- $\mathrm{Ker}\{-\pi\mathcal{I} + \mathcal{W} + \mathcal{S}\} = \{0\}$ *in the Banach space* $\mathcal{X}$.

Then, for sufficiently large m*, say* $m \geq m_0$,

- $\exists\,(-\pi\mathcal{I} + \mathcal{W}_m + \mathcal{S}_m)^{-1}$*, uniformly bounded on* $\mathcal{X}$.
- *The solutions* $\bar{\psi}$ *and* $\bar{\psi}_m$ *satisfy the following* **error estimates**

$$\|(\bar{\psi} - \bar{\psi}_m)(s)\|_\infty \leq \begin{cases} \mathcal{C} \max\left\{ \left(\dfrac{1}{m^{2-2\epsilon}} \right)^{\beta}, \dfrac{1}{m^{1+\epsilon}} \right\}, & s \in \left[0, \dfrac{c}{m^{2-2\epsilon}}\right] \\ \dfrac{\mathcal{C}}{m^2} \dfrac{1}{s^{\frac{1}{2}}}, & s \in \left[\dfrac{c}{m^{2-2\epsilon}}, 1\right] \end{cases}$$

where $\beta = \min_{k=1,\ldots,n} \beta_k$*, with* $\frac{1}{2} < \beta_k < 1$.

1. Approximate the double layer density function ψ

II STEP: ...Nyström method...

Theorem

Denoting by $\mathrm{cond}(-\pi\mathcal{I} + \mathcal{W}_m + \mathcal{S}_m)$ *the condition number of the operator* $-\pi\mathcal{I} + \mathcal{W}_m + \mathcal{S}_m : \mathcal{X} \to \mathcal{X}$ *and by* $\mathrm{cond}(A_m)$ *the condition number of the matrix in infinity norm, we have that, for any* $m \geq m_0$*,*

$$\mathrm{cond}(A_m) \leq \mathrm{cond}(-\pi\mathcal{I} + \mathcal{W}_m + \mathcal{S}_m).$$

2. Evaluation of the double layer potential

The **double layer potential** u defined by

$$u(A) = \int_{\Sigma} \psi(Q) \frac{\partial}{\partial \mathbf{n}_Q} [\log |A - Q|] d\Sigma_Q, \quad A \in D,$$

2. Evaluation of the double layer potential

The **double layer potential** u defined by

$$u(A) = \int_{\Sigma} \psi(Q) \frac{\partial}{\partial \mathbf{n}_Q} [\log |A - Q|] d\Sigma_Q, \quad A \in D,$$

can be rewritten as

$$u(x,y) = \sum_{i=1}^{3n} \int_0^1 M_i(x,y,t) \bar{\psi}_i(t) dt, \quad \forall\ (x,y) \in D$$

where $\bar{\psi}_i = \psi_i \circ \sigma_i$, $\psi_i = \psi|_{\Sigma_i}$ and

$$M_i(x,y,t) = \frac{\eta_i'(t)[x - \xi_i(t)] - \xi_i'(t)[y - \eta_i(t)]}{[x - \xi_i(t)]^2 + [y - \eta_i(t)]^2}$$

2. Evaluation of the double layer potential

We propose to approximate

$$u(x,y) = \sum_{i=1}^{3n} \int_0^1 M_i(x,y,t)\bar{\psi}_i(t)dt, \quad \forall\ (x,y) \in D$$

by means of the following function

$$u_m(x,y) = \sum_{i=1}^{3n} \sum_{h=0}^{m+1} \lambda_h M_i(x,y,x_h)\bar{\psi}_{m,i}(x_h), \tag{2}$$

Let us observe that the values $\bar{\psi}_{m,i}(x_h)$ involved in the formula (2) are just the solutions of the linear system.

2. Evaluation of the double layer potential

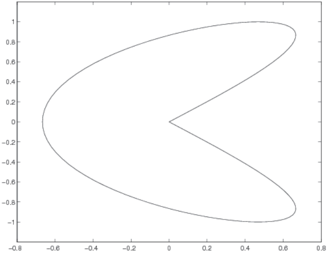
Theorem

For any $(x, y) \in D$*, the double layer potential* u *solution of the Dirichlet problem satisfies the following pointwise error estimate*

$$|u(x,y) - u_m(x,y)| \leq \frac{\mathcal{C}}{\delta^4 m^3} + \frac{\mathcal{C}'}{\delta} \left\| \bar{\psi} - \bar{\psi}_m \right\|_\infty ,$$

where $\delta = \min\limits_{i=1,\dots,3n} \delta_i$ *with* $\delta_i = \min\limits_{0 \leq t \leq 1} |(x,y) - (\xi_i(t), \eta_i(t))|$ *and* $\mathcal{C}$*,* $\mathcal{C}'$ *positive constants independent of* (x, y) *and* m*.*

Numerical Example 1



$$\begin{cases} \Delta u(P) = 0, & P \in D \\ u(P) = g(P), & P \in \Sigma \end{cases} \qquad \Sigma : \quad \sigma(t) = \left(\tfrac{2}{3}\sin(2\pi t), \sin(2\pi t)\right), \quad t \in [0,1]$$

Corner point $P = (0,0)$, interior angle $\phi = \frac{3}{2}\pi$

$$u(x,y) = r^{\frac{2}{3}} \cos \tfrac{2}{3}\theta$$

Numerical Example 1

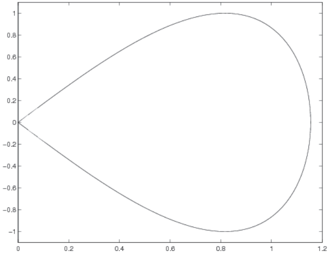
$$\varepsilon_m(x,y) = |u(x,y) - u_m(x,y))|$$

$$\ell = 8.88e-03, \quad \epsilon = 10^{-6}, \quad c = 1$$

m	$\varepsilon_m(-0.01,0)$	$\varepsilon_m(0,0.1)$	$\varepsilon_m(-0.4,0.4)$	$\varepsilon_m(0.4,0.8)$
32	2.04e-02	1.74e-03	1.76e-02	2.69e-03
64	7.70e-05	3.70e-06	8.54e-04	7.56e-06
128	1.07e-05	1.05e-06	1.46e-05	1.07e-08
256	3.96e-06	3.76e-07	5.10e-08	3.79e-09
512	1.23e-06	1.03e-07	1.44e-08	1.05e-09
1024	1.40e-07	5.65e-09	3.96e-10	3.60e-11

$$\text{cond}(A_m) \simeq 16.93$$

Numerical Example 2



$$\begin{cases} \Delta u(P) = 0, & P \in D \\ u(P) = g(P), & P \in \Sigma \end{cases} \qquad \Sigma: \quad \sigma(t) = \left(\frac{2}{\sqrt{3}}\sin \pi t, -\sin 2\pi t\right) \quad t \in [0,1]$$

Corner point $P = (0,0)$, interior angle $\phi = \frac{2}{3}\pi$

$$u(x,y) = r^{\frac{3}{2}} \cos \frac{3}{2}\theta$$

Numerical Example 2

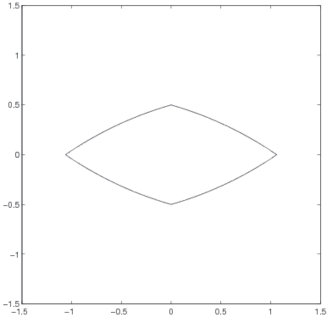
$$\varepsilon_m(x,y) = |u(x,y) - u_m(x,y))|$$

$$\ell = 7.25e-04, \quad \epsilon = 10^{-6}, \quad c = 1$$

m	$\varepsilon_m(0.01,0)$	$\varepsilon_m(0.1,0)$	$\varepsilon_m(0.8,0.6)$	$\varepsilon_m(0.9,0.8)$
32	1.87e-01	9.84e-04	2.22e-05	8.98e-03
64	8.78e-03	6.59e-05	6.84e-07	4.78e-05
128	6.66e-05	1.06e-06	8.47e-09	1.59e-09
256	5.24e-08	1.72e-09	1.37e-11	3.08e-12
512	5.84e-11	1.84e-12	1.28e-14	7.99e-15

$$\text{cond}(A_m) \simeq 4.18$$

Numerical Example 3



$$\begin{cases} \Delta u(P) = 0, & P \in D \\ u(P) = g(P), & P \in \Sigma \end{cases}$$

$$u(x, y) = \exp(x)\cos(y)$$

Numerical Example 3

$$\varepsilon_m(x, y) = |u(x, y) - u_m(x, y))|$$

$$\ell = 1.17e - 04, \quad \epsilon = 10^{-6}, \quad c = 1$$

m	$\varepsilon_m(0.9, 0)$	$\varepsilon_m(-0.4, -0.3)$	$\varepsilon_m(0.2, 0.2)$	$\varepsilon_m(0.1, -0.2)$
16	4.07e-04	6.21e-03	4.46e-04	6.38e-04
32	1.35e-06	3.59e-04	4.16e-05	5.95e-05
64	1.43e-08	1.14e-06	2.18e-06	3.06e-06
128	4.84e-09	7.33e-09	4.44e-08	6.28e-08
256	4.62e-09	6.62e-12	5.80e-11	6.57e-11

$$\text{cond}(A_m) \leq 135.31$$

THANKS FOR THE ATTENTION !