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PROJECTIVE AND AFFINE ASPECTS IN CATEGORIES AND HOMOLOGICAL ALGEBRA

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• 0. Introduction

• [C, CG] give characterisations of:

– 'categories of affine spaces'
(defined as slice categories of additive categories with kernels),

– 'categories of projective spaces'
(defined as abelian categories modulo a canonical congruence).

[C] A. Carboni, Categories of affine spaces, JPAA 61 (1989), 243-250.

[CG] A. Carboni and M. Grandis, Categories of projective spaces, JPAA 110 (1996), 241-258.

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• This will be the starting point for
a tentative discussion of projective and affine aspects
in categories and homological algebra.

• 1. Categories of affine spaces [C] 1989

• **S. Schanuel:** equivalence (for a field k)

$$k\mathsf{Vct}/k \to k\mathsf{Aff}, \qquad (\varphi\colon V \to k) \mapsto \varphi^{-1}\{1_k\}.$$

• **[C]: characterises the slice categories $\mathsf{E} = \mathsf{A}/X$**
for A an additive category with kernels

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• **modular category:**
a category with finite limits and finite sums, satisfying two axioms related to modular lattices and distributive categories

– **Theorem:**

– if A is an additive category with kernels, all slices A/X are modular,

– if E is modular, then its category of points $\mathsf{A} = \mathrm{Pt}(\mathsf{E}) = E \setminus T$ is additive with kernels and $E \simeq A/U$

($U = i_2\colon T \to T + T$; T = terminal).

• 1A. Modular categories [C]

– Definition:

a category E with finite limits and finite sums, satisfying:

(i) for each slice category E/U, for each arrow $f\colon X \to Z$ of E/U
and each object Y of E/U, the following canonical morphism is invertible

$$m\colon X + (Y \times Z) \to (X + Y) \times Z, \qquad mi_1 = \langle i_1, f \rangle, \qquad mi_2 = i_2 \times Z.$$

(all injections of sums are denoted as i_1, i_2)

(ii) for each arrow $f\colon X \to U$ of E the following commutative square is a pullback

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & U \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle i_1} \\ X + U & \xrightarrow[f+U]{} & U + U \end{array}$$

(Modular lattices do not satisfy property (ii))

• 2. Categories of projective spaces [CG] 1996

– Aurelio was in Genoa, in 1993-96. We gave in [CG]:

a characterisation of 'categories of projective spaces'.

• [G1] (1984): $\mathsf{E} \mapsto \mathrm{Pr}(\mathsf{E}) = \mathsf{E}/\mathrm{R}$ ('projective category' associated to E Puppe-exact)

– fRg: f, g have the same direct (or inverse) images of subobjects

(If $\mathsf{E} = k\mathsf{Vct}$, $fRg \Leftrightarrow \exists\lambda \in k^*\colon f = \lambda g$. $\mathrm{Pr}(\mathsf{E}) \cong$ category of k-projective spaces)

– $\mathrm{Pr}(\mathsf{E})$ is a Puppe-exact category [Pu, Mt, FS], also for E abelian

– $\mathrm{Pr}(\mathsf{E})$ is 'projective' (its congruence R is trivial).

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• [CG] Theorem

– A abelian: $\mathrm{Pr}(\mathsf{A})$ is projective Puppe-exact with *projective biproducts*

– A is determined by $\mathrm{Pr}(\mathsf{A})$

'Projective biproducts: the 'trace' left on P by the biproducts of A:
a zero-preserving functor $\oplus\colon \mathrm{Matr}(\mathrm{Ob}\mathsf{P}) \to \mathsf{P}$ satisfying a list of properties [CG]
$\mathrm{Matr}(\mathrm{Ob}\mathsf{P})$: the free additive category on $\mathrm{Ob}\mathsf{P}$.

• 3. Projective and affine aspects in categories and homological algebra

– All this leads us to distinguish, in category theory:

– projective aspects
about lattices of subobjects and quotients, linked by kernels and cokernels

– natural framework: Puppe-exact categories and variations

– affine aspects
about finite limits and finite sums

– natural framework: regular categories and variations

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• Abelian categories are a common ground for both aspects.

• 4. Basic projective aspects [G1, G3]

• Calculus of subobjects in E (Puppe-exact category)

$\mathrm{Sub}_{\mathsf{E}} \colon \mathsf{E} \to \mathsf{Mlc}$ *transfer functor* for subobjects

$A \mapsto \mathrm{Sub}(A), \qquad (f \colon A \to B) \mapsto (f_* \colon \mathrm{Sub}A \rightleftarrows \mathrm{Sub}B : f^*)$

$f^*f_*(x) = x \vee f^*0 \geq x, \qquad f_*f^*(y) = y \wedge f_*1 \leq y \qquad$ (modular conditions).

– Mlc: the (projective) Puppe-exact category of modular lattices and modular connections

– maps: $u = (u_\bullet, u^\bullet) \colon X \to Y$ $(u_\bullet \dashv u^\bullet)$ satisfying the modular conditions

– Mlc has NO products and NO exact embedding into an abelian category

– The functor $\mathrm{Sub}_{\mathsf{E}}$ is exact (i.e. preserves exact sequences)

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• E is a *projective Puppe-exact category* when $\mathrm{Sub}_{\mathsf{E}}$ is faithful

• 'making it faithful' gives the projective Puppe-exact category $\mathrm{Pr}(\mathsf{E}) = E/R$

$$(fRg \Leftrightarrow f_* = g_* \Leftrightarrow f^* = g^*)$$

• 4A. Hints at other projective aspects [G3]

– Relationship with double categories

– $F\colon E \to \mathsf{E}'$ (exact functor of Puppe-exact categories) gives

$$\mathrm{Sub}_F\colon \mathrm{Sub}_{\mathsf{E}} \to \mathrm{Sub}_{\mathsf{E}'}F\colon \mathsf{E} \to \mathsf{Mlhc} \qquad \text{hor. transformation of vert. functors}$$

$$\mathrm{Sub}_F A\colon \mathrm{Sub}_{\mathsf{E}}(A) \to \mathrm{Sub}_{\mathsf{E}'}(FA), \qquad x \mapsto F(x)$$

– Mlhc: the double category of modular lattices, homomorphisms and modular connections.

– Duality between subobjects and quotients

– A 'Coherence Theorem' for homological algebra

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– Universal models for homological systems spectral sequences

(e.g. filtered complexes, double complexes, Eilenberg's exact systems, Massey's exact couples).

• 5. Projective aspects in strongly non-abelian settings [G2, G4]

– A much wider extension of this 'projective approach':
semiexact and homological categories (1990 Como Conference)
including the 'categories of pairs', like Top_2 and Gp_2
- Previous work on 'categories of pairs': R. Lavendhomme [La], 1965
(kernels and cokernels with respect to an assigned ideal of 'null morphisms')

– Original motivation:
to include the exact/spectral sequences of (unstable) homotopy theory

– Calculus of normal subobjects in E (semiexact/homological category)

$$\mathrm{Nsb}_{\mathrm{E}}\colon \mathsf{E} \to \mathsf{Ltc} \qquad \textit{transfer functor} \text{ for normal subobjects}$$
$$A \mapsto \mathrm{Nsb(A)}, \qquad (\mathrm{f}\colon \mathrm{A} \to \mathrm{B}) \mapsto (\mathrm{f}_*\colon \mathrm{NsbA} \rightleftarrows \mathrm{NsbB} : \mathrm{f}^*)$$

– Ltc: the (projective) pointed homological category of lattices and Galois connections

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– Recent applications to radical theory and closure operators:
MG, G. Janelidze and L. Márki [GJM], to appear (JAMS).

• 6. Affine aspects

– Natural frameworks: regular and Barr-exact categories and variations:

– protomodular categories: Bourn [Bou], 1990 Como Conference
(following Aurelio's modular categories [C])

– semiabelian categories: Janelidze, Márki and Tholen [JMT] 2002, [Bo]

– Borceux-Bourn homological categories
Borceux and Bourn [BoB] 2004

(= pointed regular protomodular)

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• Here I would just point out some bridges with the projective approach

– C regular cat. with finite sums $\mapsto$ C_2 homological cat. 'of pairs'

– E.g. Gp_2. But similar facts hold much more generally: for instance Top_2

– Recent results by Zurab Janelidze [Jz]
on 'projective aspects' of regular protomodular and semiabelian categories.

• 6A. A bridge from affine to projective aspects: [Jz] (APCS, to appear)

• Semiabelian and regular protomodular categories are characterised

among regular categories C, via:

– $\varphi\colon \mathsf{C} \to \mathsf{Gls}$: the 'form of subobjects' of C (or transfer functor)
Gls: the category of (possibly large) ordered sets and Galois connections.

• Theorem 1. A regular category C is protomodular if and only if:
the image of the right adjoint $f^*\colon \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ is up-closed ($\forall\, f\colon X \to Y$ in C).

– Corollary 1. A regular category is protomodular if and only if its form φ is cartesian.

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• Theorem 2. A regular category is semiabelian if and only if:
it is pointed, has binary sums and its form φ is right cartesian and right stable.

– Corollary 2. A regular category is semiabelian if and only if:
it is pointed, has binary sums and a stable cartesian form of subobjects.

– φ is *right (left) cartesian:*
the unit (counit) of the adjunction $\varphi(f)\colon \mathrm{Sub}(X) \to \mathrm{Sub}(Y)$ is cocartesian (cartesian) ($\forall\, f$ in C)

these conditions amount to the 'modular conditions':

$$f^*f_*(x) = x \vee f^*0, \qquad f_*f^*(y) = y \wedge f_*1 \qquad (x \in X,\ y \in Y).$$

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