Directed Algebraic Topology

Models of non-reversible worlds

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To

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Contents

Introduction

0.1 Aims and applications

Directed Algebraic Topology is a recent subject which arose in the 1990's, on the one hand in abstract settings for homotopy theory, like [G1], and on the other hand in investigations in the theory of concurrent processes, like [FGR1, FGR2]. Its general aim should be stated as 'modelling non-reversible phenomena'. The subject has a deep relationship with category theory.

The domain of Directed Algebraic Topology should be distinguished from the domain of classical Algebraic Topology by the principle that *directed spaces have privileged directions and directed paths therein need not be reversible*. While the classical domain of Topology and Algebraic Topology is a reversible world, where a path in a space can always be travelled backwards, the study of non-reversible phenomena requires broader worlds, where a directed space can have non-reversible paths.

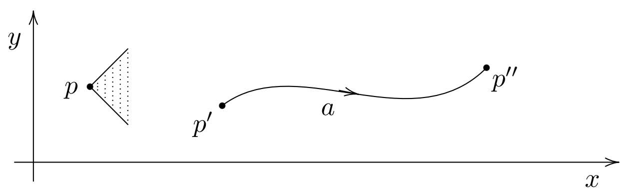
The homotopical tools of Directed Algebraic Topology, corresponding in the classical case to ordinary homotopies, the fundamental group and fundamental n-groupoids, should be similarly 'non-reversible': *directed homotopies*, the *fundamental monoid* and *fundamental n-categories*. Similarly, its homological theories will take values in 'directed' algebraic structures, like *preordered* abelian groups or abelian *monoids*. Homotopy constructions like mapping cone, cone and suspension, occur here in a directed version; this gives rise to new 'shapes', like (lower and upper) directed cones and directed spheres, whose elegance is strengthened by the fact that such constructions are determined by universal properties.

Applications will deal with domains where privileged directions appear, such as concurrent processes, rewrite systems, traffic networks, space-time models, biological systems, etc. At the time of writing, the most developed ones are concerned with concurrency: see [FGR1, FGR2, FRGH, Ga1, GG, GH, Go, Ra1, Ra2].

A recent issue of the journal 'Applied Categorical Structures', guest-edited by the author, has been devoted to 'Directed Algebraic Topology and Category Theory' (vol. 15, no. 4, 2007).

0.2 Some examples

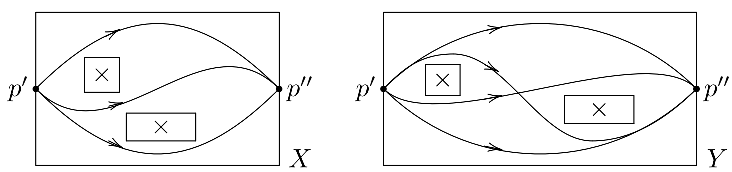
As an elementary example of the notions and applications we are going to treat, consider the following (partial) order relation in the cartesian plane



$$(x, y) \leqslant (x', y') \quad \Leftrightarrow \quad |y' - y| \leqslant x' - x. \qquad (0.1)$$

The picture shows the 'cone of the future' at a point p (i.e. the set of points which follow it) and a *directed path* from p' to p'', i.e. a continuous mapping $a\colon [0,1] \to \mathbf{R}^2$ which is (weakly) *increasing*, with respect to the natural order of the standard interval and the previous order of the plane: if $t \leqslant t'$ in $[0,1]$, then $a(t) \leqslant a(t')$ in the plane.

Take now the following (compact) subspaces X, Y of the plane, with the induced order (the cross-marked open rectangles are taken out). A directed path in X or Y satisfies the same conditions as above



(0.2)

We shall see that - as displayed in the figures above - there are, respectively, 3 or 4 'homotopy classes' of directed paths from the point p' to the point p'', in the fundamental categories $\uparrow\Pi_1(X)$, $\uparrow\Pi_1(Y)$; in both cases there are none from p'' to p', and every loop is constant.

(The prefixes $\uparrow$ and d- are used to distinguish a directed notion from the corresponding 'reversible' one.)

First, we can view each of these 'directed spaces' as a stream with two islands, and the induced order as an upper bound for the relative velocity feasible in the stream. Secondly, one can interpret the horizontal coordinate as (a measure of) time, the vertical coordinate as position in a 1-dimensional physical medium, and the order as the possibility of going from (x, y) to (x', y') with velocity $\leqslant 1$ (with respect to a 'rest frame' of the medium). The two forbidden rectangles are now linear obstacles in the medium, with a bounded duration in time. Thirdly, our figures can be viewed as execution paths of concurrent automata subject to some conflict of resources, as in [FGR2], fig. 14.

In all these cases, the fundamental category distinguishes between obstructions (islands, temporary obstacles, conflict of resources) which intervene essentially together (in the earlier diagram on the left) or one after the other (on the right). On the other hand, the underlying topological spaces are homeomorphic, and topology, or algebraic topology, cannot distinguish these two situations. Notice also that, here, all the fundamental monoids $\uparrow\pi_1(X, x_0)$ are trivial: as a striking difference with the classical case, the fundamental monoids often carry a very minor part of the information of the fundamental category $\uparrow\Pi_1(X)$.

The study of the fundamental category of a directed space, via minimal models up to directed homotopy of categories, will be developed in Chapter 3.

0.3 Directed spaces and other directed structures

The framework of ordered topological spaces is a simple starting point but is too poor to develop directed homotopy theory.

We want a 'world' sufficiently rich to contain a 'directed circle' $\uparrow\mathbf{S}^1$ and higher directed spheres $\uparrow\mathbf{S}^n$ - all of them arising from the discrete two-point space under directed suspension (of pointed objects). In $\uparrow\mathbf{S}^1$, directed paths will move in a particular direction, with fundamental monoids $\uparrow\pi_1(\uparrow\mathbf{S}^1, x_0) \cong \mathbf{N}$; its directed homology will give $\uparrow H_1(\uparrow\mathbf{S}^1) \cong \uparrow\mathbf{Z}$, i.e. the group of integers *equipped with the natural order*, where the positive homology classes are generated by cycles which are directed paths (or, more generally, positive linear combinations of directed paths).

Our main structure, to fulfil this goal, will be a topological space X *equipped* with a set dX of *directed paths* $[0, 1] \to X$, closed under:

constant paths, partial increasing reparametrisation and concatenation (Section 1.4). Such objects are called *d-spaces* or *spaces with distinguished paths*, and a morphism of d-spaces $X \to Y$ is a continuos mapping which preserves directed paths. All this forms a category d**Top** where limits and colimits exist and are easily computed - as topological limits or colimits, equipped with the adequate d-structure.

Furthermore, the standard *directed interval* $\uparrow\mathbf{I} = \uparrow[0,1]$, i.e. the real interval $[0,1]$ with the natural order and the associated d-structure, is an *exponentiable* object: in other words, the (directed) cylinder $I(X) = X \times \uparrow\mathbf{I}$ determines an object of (directed) paths $P(Y) = Y^{\uparrow\mathbf{I}}$ (providing the functor right adjoint to I), so that a directed homotopy can equivalently be defined as a map of d-spaces $IX \to Y$ or $X \to PY$. The underlying set of the d-space $P(Y)$ is the set of distinguished paths dY.

Various d-spaces of interest arise from an ordinary space equipped with an order relation, as in the case of $\uparrow\mathbf{I}$, the directed line $\uparrow\mathbf{R}$ and their powers; or, more generally, from a space equipped with a *local preorder* (Sections 1.9.2 and 1.9.3), as for the directed circle $\uparrow\mathbf{S}^1$. But other d-spaces of interest, which are able to build a bridge with noncommutative geometry, cannot be defined in this way: for instance, the quotient d-space of the directed line $\uparrow\mathbf{R}$ modulo the action of a dense subgroup (see Section 6 of this Introduction).

The category **Cub** of cubical sets is also an important framework where directed homotopy can be developed. It actually has some advantages on d**Top**: in a cubical set K, after observing that an element of K_1 need not have any counterpart with reversed vertices, we can also note that an element of K_n need not have any counterpart with faces permuted (for $n \geqslant 2$). Thus, a cubical set has 'privileged directions', in any dimension. In other words, **Cub** allows us to break *both* basic symmetries of topological spaces, the reversion of paths and the transposition of variables in 2-dimensional paths, parametrised on $[0,1]^2$, while d**Top** is essentially based on a one-dimensional information and only allows us to break the symmetry of reversion. As a consequence, pointed directed homology of cubical sets is much better behaved than that of d-spaces, and yields a *perfect* directed homology theory (Section 2.6.3).

On the other hand, **Cub** presents various drawbacks, beginning with the fact that elementary paths and homotopies, based on the obvious interval, cannot be concatenated; however, higher homotopy properties of **Cub** can be studied with the geometric realisation functor **Cub** $\to$

d**Top** and the notion of *relative equivalence* which it provides (Section 5.8.6).

The *breaking of symmetries* is an essential feature which distinguishes directed algebraic topology from the classical one; a discussion of these aspects can be found in Section 1.1.5.

Directed homotopies have been studied in various structures, either because of general interests in homotopy theory, or with a purpose of modelling concurrent systems, or in both perspectives. Such structures comprise: differential graded algebras [G3], ordered or locally ordered topological spaces [FGR2, GG, Go, Kr], simplicial, precubical and cubical sets [FGR2, GG, G1, G12], inequilogical spaces [G11], small categories [G8], flows [Ga2], etc. Our main structure, d-spaces, was introduced in [G8]; it has also been studied by other authors, e.g. in [FhR, FjR, Ra2].

0.4 Formal foundations for directed algebraic topology

We will use settings based on an abstract *cylinder functor* $I(X)$ and natural transformations between its powers, like faces, degeneracy, connections,... Or, dually, on a *cocylinder functor* $P(Y)$, representing the object of (directed) paths of an object Y. Or also, on an adjunction $I \dashv P$ which allows one to see directed homotopies as morphisms $I(X) \to Y$ or equivalently $X \to P(Y)$, as mentioned above for d-spaces.

As a crucial aspect, such a formal structure is based on endofunctors and 'operations' on them (natural transformations between their powers). In other words, it is 'categorically algebraic', in much the same way as the theory of monads, a classical tool of category theory (Section A4, in the Appendix). This is why such structures can generally be lifted from a ground category to categorical constructions on the latter, like categories of diagrams, or sheaves, or algebras for a monad (Chapter 5).

After a basic version in Chapter 1, which covers all the frameworks we are interested in, we develop stronger settings in Chapter 4. *Relative settings*, in Section 5.8, deal with a basic world, satisfying the basic axioms of Chapter 1, which is equipped with a forgetful functor with values in a strong framework; such a situation has already been mentioned above, for the category **Cub** of cubical sets and the (directed) geometric realisation functor **Cub** $\to$ d**Top**.

A peculiar fact of all 'directed worlds' (categories of 'directed objects') is the presence of an involutive covariant endofunctor R, called *reversor*, which turns a directed object into the *opposite* one, $R(X) = X^{\mathrm{op}}$; its action on preordered spaces, d-spaces and (small) categories is obvious; for cubical sets, one interchanges lower and upper faces. Then, the ordinary reversion of paths is replaced with a *reflection* in the opposite directed object. Notice that the classical *reversible* case is a *particular instance* of the directed one, where R is the identity functor,

In the classical case, settings based on the cylinder (or path) endofunctor go back to Kan's well-known series on 'Abstract Homotopy', and in particular to [Ka2] (1956); the book [KP], by Kamps and Porter, is a general reference for such settings. In the directed case, the first occurrence of such a system, containing a reversor, is probably a 1993 paper of the present author [G1].

Quillen model structures [Qn] seem to be less suited to formalise directed homotopy. But, *in the reversible case*, we prove (in Theorem 4.9.6) that our strong setting based on the cylinder determines a structure of 'cofibration category', a non selfdual version of Quillen's model categories introduced by Baues [Ba].

0.5 Interactions with category theory

On the one hand, category theory intervenes in directed algebraic topology through the fundamental category of a directed space, viewed as a sort of algebraic model of the space itself. On the other hand, directed algebraic topology can be of help in providing a sort of geometric intuition for category theory, in a sharper way than classical algebraic topology - the latter can rather provide intuition for the theory of groupoids, a reversible version of categories.

The interested reader can see, in 1.8.9, how the pasting of comma squares of categories only works up to convenient notions of 'directed homotopy equivalence' of categories - in the same way as, in **Top**, the pasting of homotopy pullbacks leads to homotopy equivalent spaces.

The relationship of directed algebraic topology and category theory is even stronger in 'higher dimension'. It consists of higher fundamental categories for directed spaces, on the one hand, and geometric intuition for the - very complex - theory of higher dimensional categories, on the other hand. Such aspects are still under research and will not be treated in this book. The interested reader is referred to [G15, G16, G17] and references therein.

Finally, we should note that category theory has also been of help in fixing the structures which we explore here, according to general principles discussed in the Appendix, A1.6.

0.6 Interactions with non-commutative geometry

While studying the directed homology of cubical sets, in Chapter 2, we also show that cubical sets (and d-spaces) can express topological facts missed by ordinary topology and already investigated within non-commutative geometry. In this sense, they provide a sort of 'noncommutative topology', without the metric information of C*-algebras.

This happens, for instance, in the study of group actions or foliations, where a topologically-trivial quotient (the orbit set or the set of leaves) can be enriched with a natural cubical structure (or a d-structure) whose directed homology agrees with Connes' analysis in noncommutative geometry.

Let us only recall here that, if ϑ is an irrational number, $G_\vartheta = \mathbf{Z} + \vartheta\mathbf{Z}$ is a dense subgroup of the additive group $\mathbf{R}$, and the topological quotient $\mathbf{R}/G_\vartheta$ is trivial (has the indiscrete topology). Noncommutative geometry 'replaces' this quotient with the well-known irrational rotation C*-algebra A_ϑ (Section 2.5.1). Here we replace it with the cubical set $C_\vartheta = (\Box{\uparrow}\mathbf{R})/G_\vartheta$, a quotient of the singular cubical set of the directed line (or the quotient d-space $D_\vartheta = {\uparrow}\mathbf{R}/G_\vartheta$, cf. 2.5.2). Computing its directed homology, we prove that the (pre)ordered group ${\uparrow}H_1(C_\vartheta)$ is isomorphic to the totally ordered group ${\uparrow}G_\vartheta \subset \mathbf{R}$. It follows that the classification up to isomorphism of the family C_ϑ (or D_ϑ) coincides with the classification of the family A_ϑ up to strong Morita equivalence. Notice that, *algebraically* (i.e. forgetting order), we only get $H_1(C_\vartheta) \cong \mathbf{Z}^2$, which gives no information on ϑ: here, the information content provided by the ordering is much finer than that provided by the algebraic structure.

0.7 From directed to weighted algebraic topology

In Chapter 6 we end this study by investigating 'spaces' where paths have a 'weight', or 'cost', expressing length or duration, price, energy, etc. The general aim is now: measuring the cost of (possibly non-reversible) phenomena.

The weight function takes values in $[0, \infty]$ and is *not* assumed to be invariant up to path-reversion. Thus, 'weighted algebraic topology' can be developed as an enriched version of directed algebraic topology, where illicit paths are penalised with an infinite cost, and the licit ones are measured. Its algebraic counterpart will be 'weighted algebraic structures', equipped with a sort of directed seminorm.

A generalised metric space in the sense of Lawvere [Lw1] yields a prime structure for this purpose. For such a space we define a *fundamental weighted category*, by providing each homotopy class of paths with a weight, or seminorm, which is subadditive with respect to composition.

We also study a more general framework, *w-spaces* or *spaces with weighted paths* (a natural enrichment of d-spaces), *whose relationship with noncommutative geometry also takes into account the metric aspects* - in contrast with cubical sets and d-spaces. Here, the irrational rotation C*-algebra A_ϑ corresponds to the w-space $W_\vartheta = \mathrm{w}\mathbf{R}/G_\vartheta$, a quotient of the standard weighted line, whose classification up to isometric isomorphism (resp. Lipschitz isomorphism) is the same as the classification of A_ϑ up to isomorphism (resp. strong Morita equivalence).

0.8 Terminology and notation

The reader is assumed to be acquainted with the basic notions of topology, algebraic topology and category theory. However, most of the notions and results of category theory which are used here are recalled in the Appendix, Chapter A.

In a category **A**, the set of morphisms (or maps, or arrows) $X \to Y$, between two given objects, is written as $\mathbf{A}(X, Y)$. A natural transformation between the functors $F, G\colon \mathbf{A} \to \mathbf{B}$ is written as $\varphi\colon F \to G\colon \mathbf{A} \to \mathbf{B}$, or $\varphi\colon F \to G$.

Top denotes the category of topological spaces and continuous mappings. A homotopy φ between maps $f, g\colon X \to Y$ is written as $\varphi\colon f \to g\colon X \to Y$, or $\varphi\colon f \to g$. $\mathbf{R}$ is the euclidean line and $\mathbf{I} = [0, 1]$ is the standard euclidean interval. The concatenation of paths and homotopies is written in additive notation: $a + b$ and $\varphi + \psi$; trivial paths and homotopies are written as $0_x, 0_f$. **Gp** (resp. **Ab**) denotes the category of groups (resp. abelian groups) and their homomorphisms.

Cat denotes the 2-category of small categories, functors and natural transformations. In a small category, the composition of two consecutive arrows $a\colon x \to x'$, $b\colon x' \to x''$ is either written in the usual notation ba or in additive notation $a + b$. In the first case, the identity of the object x is written as $\mathrm{id}\, x$ or 1_x, in the second as 0_x. Loosely speaking, we tend to use additive notation in the fundamental category of some directed object, or in a small category which is itself 'viewed' as a directed object; on the other hand, we follow the usual notation when we are applying the standard techniques of category theory, which would look unfamiliar in additive notation.

A *preorder* relation, generally written as $x \prec y$, is assumed to be reflexive and transitive; an *order*, often written as $x \leqslant y$, is also assumed to be anti-symmetric (and need *not* be total). A mapping which preserves preorders is said to be *increasing* (always used in the weak sense). As usual, a preordered set X will be *identified* with the (small) category whose objects are the elements of X, with precisely one arrow $x \to x'$ when $x \prec x'$ and none otherwise. We shall distinguish between the ordered real line $\mathbf{r}$ and the ordered topological space $\uparrow\mathbf{R}$ (the euclidean line with the natural order), whose fundamental category is $\mathbf{r}$. $\uparrow\mathbf{Z}$ is the *ordered* group of integers, while $\mathbf{z}$ is the underlying ordered set.

The index α takes values $0, 1$; these are often written as $-, +$, e.g. in superscripts.

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