Directed combinatorial homology and noncommutative geometry

(The breaking of symmetries in algebraic topology)

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Abstract. We will present a brief study of the homology of cubical sets, with two main purposes.

First, this combinatorial structure is viewed as representing *directed spaces*, breaking the intrinsic symmetries of topological spaces. Cubical sets have a *directed homology*, consisting of preordered abelian groups where the positive cone comes from the structural cubes.

But cubical sets can also express topological facts missed by ordinary topology. This happens, for instance, in the study of group actions or foliations, where a topologically-trivial quotient (the orbit set or the set of leaves) can be enriched with a natural cubical structure whose directed homology agrees with Connes' analysis in noncommutative geometry [C1]. Thus, cubical sets can provide a sort of 'noncommutative topology', without the metric information of C^*-algebras [G1].

This similarity can be made stricter by introducing *normed* cubical sets and their *normed* directed homology, formed of *normed* preordered abelian groups. The normed cubical sets associated with irrational rotations have thus the same classification up to isomorphism as the well-known irrational rotation C^*-algebras [G2].

Finally, we will see that part of these results can also be obtained with a different approach, based on D. Scott's *equilogical spaces* [Sc] and developed in [G3, G4].

Comments printed in gray characters can be omitted. The index α takes values 0, 1, also written as –, + (e.g. in superscripts).

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NOTES

1. Singular homology by cubes [Ms]

1.0. Introduction

The singular homology of a topological space X can be equivalently defined as the homology of the chain complex associated to the *simplicial set* ΔX (produced by all maps $\Delta^n \to X$ defined on standard tetrahedra) or the homology of the chain complex associated to the *cubical set* $\square X$ (produced by all maps $\mathbf{I}^n \to X$ defined on standard cubes).

The less usual cubical approach, followed in Massey's text [Ms], has various advantages, mainly due to the fact that cubes are closed under products, while products of tetrahedra have to be 'covered' with tetrahedra; thus, the proof of homotopy invariance and the study of cartesian products or fibrations are easier and more natural in the cubical setting. Here, a more specific motivation for this choice is our use of the natural order on $\mathbf{I}^n$, in the sequel. The equivalence with the simplicial construction can be proved by a technique called 'acyclic models' [EM, HW].

In this section we give a brief outline of the cubical construction of singular homology, as a preparation to abstracting cubical sets and their homology.

1.1. The singular cubical set of a space

- **Top**: the category of topological spaces and continuous mappings (= maps).

- $\mathbf{I} = [0, 1]$: the *standard interval*, with euclidean topology.

- Basic structure: two *faces* (δ^0, δ^1) and a *degeneracy* (ε), linking it with the singleton $\mathbf{I}^0 = \{*\}$

(1) $\delta^\alpha \colon \{*\} \rightleftarrows \mathbf{I} : \varepsilon$ $(\alpha = 0, 1)$,

$\delta^0(*) = 0, \qquad \delta^1(*) = 1, \qquad \varepsilon(t) = *.$

- *Faces* and *degeneracies* of the standard cubes $\mathbf{I}^n$ (for $\alpha = 0, 1$; $i = 1,..., n$)

(2) $\delta^\alpha_i = \mathbf{I}^{i-1} \times \delta^\alpha \times \mathbf{I}^{n-i} \colon \mathbf{I}^{n-1} \to \mathbf{I}^n$, $\qquad \delta^\alpha_i(t_1,..., t_{n-1}) = (t_1,..., t_{i-1}, \alpha,..., t_{n-1})$,

$\varepsilon_i = \mathbf{I}^{i-1}\times\varepsilon\times\mathbf{I}^{n-i}\colon \mathbf{I}^n \to \mathbf{I}^{n-1}$, $\qquad \varepsilon_i(t_1,..., t_n) = (t_1,..., \hat{t}_i,..., t_n)$.

- They satisfy the *co-cubical* relations (where $\alpha, \beta = 0, 1$)

(3) $\delta^\beta_j\, \delta^\alpha_i = \delta^\alpha_i\, \delta^\beta_{j-1}$ $(i < j)$, $\qquad \varepsilon_i\, \varepsilon_j = \varepsilon_{j-1}\, \varepsilon_i$ $(i < j)$,
$\varepsilon_j\, \delta^\alpha_i = \delta^\alpha_{i-1}\, \varepsilon_j$ $(j < i)$, $\qquad$ *or* id $(j = i)$, $\qquad$ *or* $\delta^\alpha_i\, \varepsilon_{j-1}$ $(j > i)$.

- This produces, for every topological space X, a *cubical set* $\square X$

(4) $\square X = ((\square_n X), (\partial^\alpha_i), (e_i))$, $\qquad$ the *singular cubical set* of X,
$\square_n X = \mathbf{Top}(\mathbf{I}^n, X)$, $\qquad$ the set of *singular n-cubes* $a\colon \mathbf{I}^n \to X$ of the space X,
$\partial^\alpha_i = \partial^\alpha_{ni}\colon \square_n X \to \square_{n-1} X$, $\qquad \partial^\alpha_i(a) = a\, \delta^\alpha_i\colon \mathbf{I}^{n-1} \to X$,
$e_i = e_{ni}\colon \square_{n-1} X \to \square_n X$, $\qquad e_i(a) = a\, \varepsilon_i\colon \mathbf{I}^n \to X$, $\qquad (\alpha = 0, 1;\ i = 1,..., n)$.

- In general: a *cubical set* $K = ((K_n), (\partial^\alpha_i), (e_i))$ is a sequence of sets K_n $(n \geq 0)$, together with mappings, called *faces* (∂^α_i) and *degeneracies* (e_i)

(5) $\partial^\alpha_i = \partial^\alpha_{ni}\colon K_n \to K_{n-1}$, $\qquad e_i = e_{ni}\colon K_{n-1} \to K_n$ $\qquad (\alpha = 0, 1;\ i = 1,..., n)$.

satisfying the *cubical* relations

(6) $\partial^\alpha_i\, \partial^\beta_j = \partial^\beta_{j-1}\, \partial^\alpha_i$ $(i < j)$, $\qquad e_j\, e_i = e_i\, e_{j-1}$ $(i < j)$,
$\partial^\alpha_i\, e_j = e_j\, \partial^\alpha_{i-1}$ $(j < i)$, $\qquad$ *or* id $(j = i)$, $\qquad$ *or* $e_{j-1}\, \partial^\alpha_i$ $(j > i)$.

A *morphism* of cubical sets $f = (f_n)\colon K \to L$ is a sequence of mappings $f_n\colon K_n \to L_n$ commuting with faces and degeneracies. Cubical sets and their morphisms form a category **Cub**.

- The *singular cubical set* functor $\square\colon \mathbf{Top} \to \mathbf{Cub}$ acts as follows on the map $f\colon X \to Y$

(7) $\square f\colon \square X \to \square Y$, $\qquad (\square f)_n\colon a \mapsto f{\circ}a\colon \mathbf{I}^n \to Y$.

1.2. The singular chain complex of a space

- *Degenerate elements* of a cubical set K: all elements of type $e_i(a)$

(1) $\mathrm{Deg}_n K = \bigcup_i \mathrm{Im}(e_i\colon K_{n-1} \to K_n)$, $\qquad \mathrm{Deg}_0 K = \emptyset$.

- Because of the cubical relations, we have (for $i = 1,..., n$)

(2) $a \in \mathrm{Deg}_n K \;\Rightarrow\; (\partial^\alpha_i a \in \mathrm{Deg}_{n-1} K$ *or* $\partial^-_i a = \partial^+_i a)$, $\qquad e_i(\mathrm{Deg}_{n-1} K) \subset \mathrm{Deg}_n K$.

- The cubical set K determines a (*normalised*) *chain complex* $C_*(K)$, i.e. a sequence of abelian groups and homomorphisms (called *boundaries*, or *differentials*)

(3) $$\dots \longrightarrow C_{n+1}(K) \xrightarrow{\partial_{n+1}} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \longrightarrow \dots \longrightarrow C_1(K) \xrightarrow{\partial_1} C_0(K) \longrightarrow 0$$

with $\partial_n \partial_{n+1} = 0$, defined as follows:

(4) $C_n(K) = (\mathbf{Z}K_n)/(\mathbf{Z}\mathrm{Deg}_n K) = \mathbf{Z}\bar{K}_n$ $\qquad (\bar{K}_n = K_n \setminus \mathrm{Deg}_n K)$,
$\partial_n\colon C_n(K) \to C_{n-1}(K)$, $\qquad \partial_n(\hat{a}) = \Sigma_{i,\alpha}\, (-1)^{i+\alpha}\, (\partial^\alpha_i a)^\wedge$ $\qquad (a \in K_n)$,

(**Z**S is the free abelian group on the set S; $\hat{a}$ is the class of the n-cube a up to degenerate cubes; but we will write the normalised class $\hat{a}$ as a, identifying all degenerate cubes with 0.)

- To prove that $\partial_n\partial_{n+1} = 0$ one uses the cubical relations for faces: $\partial^\alpha_i \partial^\beta_j = \partial^\beta_{j-1} \partial^\alpha_i$ $(i < j)$.

- In general: a *chain complex* $A = ((A_n), (\partial_n))$ of abelian groups is a sequence as above, with $\partial_n\partial_{n+1} = 0$. A *morphism* $\varphi: A \to B$ of chain complexes is a sequence of homomorphisms $\varphi_n: A_n \to B_n$ commuting with differentials: $\partial_n\varphi_n = \varphi_{n-1}\partial_n$. They form the category $C_*\mathbf{Ab}$ of chain complexes of abelian groups.

- The functor $C_*: \mathbf{Cub} \to C_*\mathbf{Ab}$ acts on the morphism $f = (f_n): K \to L$ by $\mathbf{Z}$-linear extension

(5) $f_\# = C_*(f): C_*(K) \to C_*(L),$ $\qquad f_{\#n}(a) = f_n(a).$

- Composing with the functor $\square: \mathbf{Top} \to \mathbf{Cub}$, we get the *singular chain complex* of a space, or *complex of singular chains* (with integral coefficients), written again C_*

(6) $C_*: \mathbf{Top} \to C_*\mathbf{Ab},\qquad C_*(X) = C_*(\square X),\qquad f_{\#n}(a) = f\circ a \qquad (a: \mathbf{I}^n \to X).$

1.3. Singular homology of spaces

- The homology functor of chain complexes: the group of n-*cycles* modulo the group of n-*boundaries*

(1) $H_n: C_*\mathbf{Ab} \to \mathbf{Ab} \qquad (n \geq 0),$

$H_n(A) = \mathrm{Ker}\partial_n / \mathrm{Im}\partial_{n+1}, \qquad H_n(\varphi)[z] = [\varphi_n z].$

- Composing with the previous functors, we have the *singular homology* of a space (with integral coefficients)

(2) $\mathbf{Top} \xrightarrow{\square} \mathbf{Cub} \xrightarrow{C_*} C_*\mathbf{Ab} \xrightarrow{H_n} \mathbf{Ab}$

$H_n: \mathbf{Top} \to \mathbf{Ab} \qquad H_n(X) = H_n(C_*(\square X)) \qquad (n \geq 0),$

$H_n(f) = f_{*n}, \qquad f_{*n}[\Sigma_i \lambda_i a_i] = [\Sigma_i \lambda_i (fa_i)].$

1.4. Exercises

- $H_n(X) \cong \bigoplus_{i\in I} H_n(X_i)$, where $(X_i)_{i\in I}$ is the family of path-connected components of the space X.

- $H_0(\{*\}) \cong \mathbf{Z}$, $\quad H_n(\{*\}) = 0 \quad (n > 0).$

- If X is path-connected, non empty, there is an isomorphism $\varphi: H_0(X) \cong \mathbf{Z}$, $\varphi[\Sigma\, \lambda_i.x_i] = \Sigma\, \lambda_i$.

Hint. Use the *augmented* chain complex $\ldots \to C_1(X) \to C_0(X) \to \mathbf{Z}$ where $\partial_0(\Sigma\, \lambda_i.x_i) = \Sigma\, \lambda_i$; ∂_0 is surjective and $\mathrm{Ker}(\partial_0) = \mathrm{Im}(\partial_1)$. Then $\varphi: H_0(X) \to \mathbf{Z}$ is the induced iso. □

1.5. Homotopy for topological spaces

- Two maps $f_0, f_1: X \to Y$ in **Top** are *homotopic* $(f_0 \simeq f_1)$ if there is a map $F: \mathbf{I}\times X \to Y$ such that $F(\alpha, x) = f_\alpha(x)$, for all $x\in X$ $(\alpha = 0, 1)$. This relation is a congruence of categories.

- Two spaces X, Y are homotopy equivalent $(X \simeq Y)$ if there are maps $f: X \rightleftarrows Y :g$ with $gf \simeq \mathrm{id}X$, $fg \simeq \mathrm{id}Y$.

- A space is said to be *contractible* if it is homotopy equivalent to $\{*\}$.

1.6. Homotopy for chain complexes of abelian groups

- Two maps $\varphi, \psi: A \to B$ in $C_*\mathbf{Ab}$ are *homotopic* ($\varphi \simeq \psi$) if there is a sequence of homomorphisms $\Phi_n: A_n \to B_{n+1}$ ($n \geq 0$) such that $\partial_{n+1}\Phi_n + \Phi_{n-1}\partial_n = -\varphi_n + \psi_n$.

- This relation is a congruence of categories, in $C_*\mathbf{Ab}$.

Proposition (Homotopy Invariance of algebraic homology). The functors $H_n: C_*\mathbf{Ab} \to \mathbf{Ab}$ are *homotopy invariant*: if $\varphi \simeq \psi: A \to B$ then $H_n(\varphi) = H_n(\psi): H_n(A) \to H_n(B)$ (for all $n \geq 0$).

1.7. Homotopy Invariance of singular homology

Theorem. The functors $H_n: \mathbf{Top} \to \mathbf{Ab}$ are *homotopy invariant*: if $f \simeq g: X \to Y$ then $H_n(f) = H_n(g): H_n(X) \to H_n(Y)$ (for all $n \geq 0$).

Hint. Given a homotopy $F: \mathbf{I}{\times}X \to Y$ between $f, g: X \to Y$, one constructs a homotopy between the associated chain morphisms $C_*(X) \to C_*(Y)$

(1) $\Phi_n: C_n(X) \to C_{n+1}(Y), \qquad \Phi_n(a) = F.(\mathbf{I}{\times}a) \qquad (a: \mathbf{I}^n \to X),$

$\partial_{n+1}\Phi_n + \Phi_{n-1}\partial_n = -C_n(f) + C_n(g).$ □

Corollary. If the spaces X, Y are homotopy equivalent, then $H_n(X) \cong H_n(Y)$ (for all $n \geq 0$).

Corollary. If the space X is contractible, then $H_n(X) \cong H_n(\{*\})$ (for all $n \geq 0$) and X is path-connected.

2. Cubical sets [G1, Section 1; Ka; BH]

We shift now our interest *from topological spaces to cubical sets*. An abstract cubical set will generally be denoted as X and viewed as a 'virtual directed space', with privileged directions in every dimension. If X is the cubical singular set of a topological space T, then its direction is undistinguished.

2.1. Remarks on Directed Algebraic Topology

We shall use cubical sets as a setting for developing *directed* homology. Directed Algebraic Topology is a recent subject, whose present applications deal mainly with concurrency. Its domain should be distinguished from classical Algebraic Topology by the principle that *directed spaces have privileged directions and their paths need not be reversible*. Its homotopical and homological tools are similarly 'non-reversible': *directed homotopies, fundamental categories*, *directed homology*. Its applications can deal with domains where privileged directions appear, like concurrent processes, traffic networks, space-time models, etc. See [GX, GY] and references there.

A topological space T has *intrinsic symmetries*, appearing - at the lowest level - in the reversion of its paths. In higher dimension, the set $\square_n T = \mathbf{Top}([0, 1]^n, T)$ of its singular cubes has an obvious action of the hyperoctahedral group (the group of symmetries of the n-cube).

Now, *bypassing topological spaces*, an abstract cubical set $X = ((X_n), (\partial_i^\alpha), (e_i))$ is a merely combinatorial structure (see 1.1.5-6, or below). This structure will be used in two ways: to break the symmetries considered above and to perform constructions, namely quotients, which would be useless in ordinary topology.

(a) For the first aspect, note that an 'edge' in X_1 need not have any counterpart with reversed vertices, nor a 'square' in X_2 any counterpart with horizontal and vertical faces interchanged. Thus, our structure has 'privileged directions', in any dimension, and the (usual) combinatorial homology of X can be given a preorder, generated by taking the given cubes as positive. Now, the cubical set X has a *geometric realisation* $\mathcal{R}X$ as a topological space, obtained - loosely speaking - by pasting a copy of the standard cube $\mathbf{I}^n$ for each n-cube $x \in X_n$, along faces and degeneracies (see 3.3); but let us note from now that this construction loses any information on 'directions' we had in X: the homology groups of $\mathcal{R}X$ have no useful preorder and only coincide *algebraically* with the ones of X.

Thus, the obvious cubical model $\uparrow\mathbf{s}^n$ of the n-dimensional sphere, with one non-degenerate cube in dimension n (whose geometric realisation is the usual, topological sphere $\mathbf{S}^n$), will have *directed homology* $\uparrow H_n(\uparrow\mathbf{s}^n) \cong \uparrow\mathbf{Z}$, i.e. the group of integers *with the natural order*. Similarly, the model $\uparrow\mathbf{t}^2 = \uparrow\mathbf{s}^1 \otimes \uparrow\mathbf{s}^1$ of the torus has $\uparrow H_1(\uparrow\mathbf{t}^2) \cong \uparrow\mathbf{Z}^2$, with the *product order* (4.6) and two obvious *positive* generators (coming from each copy of $\uparrow\mathbf{s}^1$); this example also shows that *direction* should not be confused with *orientation*, which plainly cannot select privileged generators in the 1-homology group of a torus. We shall also see that our preorder on $\uparrow H_1(X)$ becomes trivial (coarse) for a 'symmetric' cubical set, like the singular cubical set of a topological space (4.1).

(b) Secondly, it may happen that a quotient $T/\sim$ of a topological space has a trivial topology, while the corresponding quotient of its singular cubical set $\square T$ keeps a relevant topological information, detected by its homology and agreeing with the interpretation of such a 'virtual space' in noncommutative geometry. This wil be dealt with below.

2.2. Cubical sets

- Recall that a *cubical set* $X = ((X_n), (\partial_i^\alpha), (e_i))$ is a sequence of sets X_n $(n \geq 0)$, together with mappings, called *faces* (∂_i^α) and *degeneracies* (e_i)

(1) $\partial_i^\alpha = \partial_{ni}^\alpha \colon X_n \to X_{n-1}$, $\qquad e_i = e_{ni} \colon X_{n-1} \to X_n \qquad (\alpha = \pm;\ i = 1,..., n)$,

satisfying the *cubical* relations

(2) $\partial_i^\alpha \partial_j^\beta = \partial_j^\beta \partial_{i+1}^\alpha \ (j \leq i)$, $\qquad e_j e_i = e_{i+1} e_j \ (j \leq i)$,

$\partial_i^\alpha e_j = e_j \partial_{i-1}^\alpha \ (j < i)$, *or* $\mathrm{id} \ (j = i)$, *or* $e_{j-1} \partial_i^\alpha \ (j > i)$.

- Elements of X_n are called n-*cubes*; *vertices* and *edges* for $n = 0$ or 1, respectively. Every n-cube $x \in X_n$ has 2^n vertices: $\partial_1^\alpha \partial_2^\beta \partial_3^\gamma(x)$ for $n = 3$.

- A *morphism* $f = (f_n) \colon X \to Y$ of cubical sets is a sequence of mappings $f_n \colon X_n \to Y_n$ which commute with faces and degeneracies. These objects and morphisms form the category **Cub**.

- **Cub** has two involutions (covariant involutive endofunctors), *reflection* and *exchange*

(3) $R \colon \mathbf{Cub} \to \mathbf{Cub}$, $\qquad RX = X^{op} = ((X_n), (\partial_i^{-\alpha}), (e_i)) \qquad$ (*reflection*),

(4) $S \colon \mathbf{Cub} \to \mathbf{Cub}$, $\qquad SX = ((X_n), (\partial_{n+1-i}^\alpha), (e_{n+1-i})) \qquad$ (*exchange*),

the first reversing the 1-dimensional direction, the second the 2-dimensional one.

- We say that a cubical set X is *reflexive* if $RX \cong X$ and *symmetric* if $SX \cong X$.

2.3. Subobjects and quotients

- **Cub** has all limits and colimits (computed componentwise) and is cartesian closed.

- *Some category-theoretical remarks*. **Cub** is a category of presheaves: its objects are the functors $X: \mathbb{I}^{op} \to$ **Set**, where $\mathbb{I}$ is the subcategory of **Set** consisting of the sets 2^n (where $2 = \{0, 1\}$) together with the maps $2^m \to 2^n$ which delete some coordinates and insert some 0's and 1's, without modifying the order of the remaining coordinates (cf. [GM]). The *representable* presheaves are given by the Yoneda embedding $y: \mathbb{I} \to$ **Cub**, $y(2^n) = \mathbb{I}(-, 2^n): \mathbb{I}^{op} \to$ **Set**; the cubical set $y(2^n)$ can also be seen as the *free cubical set generated by one element of degree* n, according to the adjunction F_n: **Set** $\rightleftarrows$ **Cub** $:(-)_n$.

- A *cubical subset* $Y \subset X$ is a sequence of subsets $Y_n \subset X_n$, stable under faces and degeneracies.

- An *equivalence relation* $\mathcal{E}$ in X is a cubical subset of $X{\times}X$ whose components $\mathcal{E}_n \subset X_n{\times}X_n$ are equivalence relations; then, the *quotient* $X/\mathcal{E}$ is the sequence of quotient sets $X_n/\mathcal{E}_n$, with induced faces and degeneracies. In particular, for $Y \subset X$, the quotient X/Y has components X_n/Y_n, where all cubes $y \in Y_n$ are identified.

- For a cubical set X, we define the *homotopy set*

(1) $\pi_0(X) = X_0/\simeq$,

where $\simeq$ is the equivalence relation in X_0 generated by being vertices of a common edge.

- The *connected component* of X at an equivalence class $[x] \in \pi_0(X)$ is the cubical subset formed by all cubes of X whose vertices lie in [x]; X is always the sum (or coproduct, disjoint union) of its connected components. If X is not empty, we say that it is *connected* if it has one connected component, or equivalently if $\pi_0(X)$ is a singleton.

- One can easily see that the forgetful functor $(-)_0$: **Cub** $\to$ **Set** has a left adjoint, the *discrete* cubical set on a set

(2) D: **Set** $\to$ **Cub**, DS = **Set**$(1^*, S)$,

where components are constant, $(DS)_n = S$ $(n \in \mathbf{N})$, faces and degeneracies are identities. Then, the functor π_0: **Cub** $\to$ **Set** is left adjoint to D. (The forgetful functor $(-)_0$ has also a right adjoint CS = **Set**$(2^*, S)$, the *codiscrete* cubical set on S.)

2.4. Tensor product of cubical sets [Ka, BH; G1]

- The category **Cub** has a monoidal structure

(1) $(X \otimes Y)_n = (\Sigma_{p+q=n} X_p{\times}Y_q)/\sim_n$,

where $\sim_n$ is the equivalence relation generated by identifying $(e_{r+1}x, y)$ with (x, e_1y), for all $(x, y) \in X_r{\times}Y_s$ (for $r+s = n-1$).

- We write $x \otimes y$ the equivalence class of (x, y). Faces and degeneracies are defined as

(2) $\partial^\alpha_i(x \otimes y) = (\partial^\alpha_i x) \otimes y$ $(1 \leq i \leq p)$, $\partial^\alpha_i(x \otimes y) = x \otimes (\partial^\alpha_{i-p} y)$ $(p+1 \leq i \leq p+q)$,

(3) $e_i(x \otimes y) = (e_i x) \otimes y$ $(1 \leq i \leq p+1)$, $e_i(x \otimes y) = x \otimes (e_{i-p} y)$ $(p+1 \leq i \leq p+q+1)$,

(and $e_{p+1}(x \otimes y) = (e_{p+1}x) \otimes y = x \otimes (e_1 y)$ is well defined because of the equivalence relation $\sim_n$).

- The identity of the tensor product is the singleton $\{*\}$, i.e. the cubical set generated by one 0-dimensional cube; it is reflexive and symmetric.

- *The tensor product is not symmetric,* but is linked with reversion and exchange as follows:

(4) $R(X\otimes Y) = RX\otimes RY, \qquad S(X\otimes Y) \cong (SY)\otimes(SX).$

- Therefore, reflexive objects are stable under tensor product while symmetric objects are stable under tensor *powers*: if $SX \cong X$, then $S(X^{\otimes n}) = (SX)^{\otimes n} \cong X^{\otimes n}$. (The construction of the internal homs related with tensor products will be recalled in 3.1.7.)

2.5. Standard models

- The *elementary directed interval* $\uparrow\mathbf{i} = \mathbf{2}$ is freely generated by a 1-cube, u

(1) $0 \xrightarrow{u} 1 \qquad \partial_1^-(u) = 0, \quad \partial_1^+(u) = 1.$

- The *elementary directed* n-*cube* is its n-th tensor power $\uparrow\mathbf{i}^n = \uparrow\mathbf{i}\otimes...\otimes\uparrow\mathbf{i}$ (for $n \geq 0$), freely generated by one n-cube $u^{\otimes n}$. (It is the representable presheaf $y(2^n) = \mathbb{I}(-, 2^n)\colon \mathbb{I}^{op} \to \mathbf{Set}$).

- The *elementary directed square* $\uparrow\mathbf{i}^2 = \uparrow\mathbf{i}\otimes\uparrow\mathbf{i}$ can be represented as follows, showing the generator $u\otimes u$ and its faces

(2)
$$\begin{array}{ccc} 00 & \xrightarrow{0\otimes u} & 01 \\ {\scriptstyle u\otimes 0}\downarrow & u\otimes u & \downarrow{\scriptstyle u\otimes 1} \\ 10 & \xrightarrow[1\otimes u]{} & 11 \end{array} \qquad\qquad \begin{array}{cc} \bullet & \xrightarrow{2} \\ \downarrow{\scriptstyle 1} & \end{array}$$

where the face $\partial_1^-(u\otimes u) = 0\otimes u$ is drawn orthogonally to direction 1 (and directions are chosen so that the labelling of vertices agrees with matrix indexing).

- Note that, for each cubical object X, $\mathbf{Cub}(\uparrow\mathbf{i}^n, X) = X_n$.

- The *directed (integral) line* $\uparrow\mathbf{Z}$ is generated by (countably many) vertices $n\in\mathbf{Z}$ and edges u_n, from $\partial_1^-(u_n) = n$ to $\partial_1^+(u_n) = n+1$. The *directed integral interval* $\uparrow[i, j]_{\mathbf{Z}}$ is the obvious cubical subset with vertices in the integral interval $[i, j]_{\mathbf{Z}}$. In particular, $\uparrow\mathbf{i} = \uparrow[0, 1]_{\mathbf{Z}}$.

- The *elementary directed circle* $\uparrow\mathbf{s}^1$ is generated by one 1-cube u with equal faces

(3) $* \xrightarrow{u} * \qquad\qquad \partial_1^-(u) = \partial_1^+(u).$

- The *elementary directed n-sphere* $\uparrow\mathbf{s}^n$ (for $n > 1$) is generated by one n-cube u all whose faces are totally degenerate (hence equal)

(4) $\partial_i^\alpha(u) = (e_1)^{n-1}(\partial_1^-)^n(u) \qquad\qquad (\alpha = \pm;\ i = 1,..., n).$

- $\uparrow\mathbf{s}^0 = \mathbf{s}^0$ is generated by two vertices: it is the discrete cubical set $D\{0, 1\}$ (2.3.2).

- The *elementary directed n-torus* is a tensor power of $\uparrow\mathbf{s}^1$

(5) $\uparrow\mathbf{t}^n = (\uparrow\mathbf{s}^1)^{\otimes n}.$

- The *ordered circle* $\uparrow\mathbf{o}^1$ is generated by two edges with the same faces

(6) $v^- \underset{u''}{\overset{u'}{\rightrightarrows}} v^+$ $\partial_1^\alpha(u') = \partial_1^\alpha(u'')$.

- More generally the *ordered spheres* $\uparrow\mathbf{o}^n$, generated by two n-cubes u', u" with the same boundary: $\partial_i^\alpha(u') = \partial_i^\alpha(u'')$.

- Starting from $\mathbf{s}^0$, the *unpointed suspension* provides all $\uparrow\mathbf{o}^n$ (3.2.5) while the *pointed suspension* provides all $\uparrow\mathbf{s}^n$; of course, these models have the same geometric realisation $\mathbf{S}^n$ (as a topological space) and the same homology; but their *directed* homology is different (4.2). The models $\uparrow\mathbf{s}^n$ are more interesting: for instance, their order in directed homology is not trivial.

- All these cubical sets are reflexive and symmetric.

3. Directed homotopy of cubical sets [G1, Section 1]

3.1. Elementary directed homotopies

- Since the tensor product of cubical sets is not symmetric, the elementary directed interval produces a *left (elementary) cylinder* $\uparrow\mathbf{i}\otimes X$ and a *right cylinder* $X\otimes\uparrow\mathbf{i}$. But each of these functors determines the other, using the exchange S (2.4.4) and the property $S(\uparrow\mathbf{i}) = \uparrow\mathbf{i}$

(1) $I: \mathbf{Cub} \to \mathbf{Cub}$, $IX = \uparrow\mathbf{i}\otimes X$,

$SIS: \mathbf{Cub} \to \mathbf{Cub}$, $SIS(X) = S(\uparrow\mathbf{i}\otimes SX) = X\otimes\uparrow\mathbf{i}$.

- The left cylinder I has two faces and a degeneracy, the following natural transformations

(2) $\partial^\alpha: X \to IX$, $\partial^\alpha(x) = \alpha\otimes x$ $(\alpha = 0, 1)$,

$e: IX \to X$, $e(u\otimes x) = e_1(x)$.

- I has a right adjoint, the (elementary) *left cocylinder* or *left path* functor P, which shifts down all components discarding the faces and degeneracies of index 1 (which are then used to build the faces and degeneracy of P, as natural transformations)

(3) $P: \mathbf{Cub} \to \mathbf{Cub}$, $PY = ((Y_{n+1}), (\partial_{i+1}^\alpha), (e_{i+1}))$,

$\partial^\alpha = \partial_1^\alpha: PY \to Y$, $e = e_1: Y \to PY$.

- An (elementary) *left homotopy* $f: f^- \to_L f^+: X \to Y$ is defined as a map $f: IX \to Y$ with $f\partial^\alpha = f^\alpha$. Or, equivalently (because of the adjunction), as a map $f: X \to PY$ with $\partial^\alpha f = f^\alpha$. This second expression leads immediately to a simple expression of f as a family of mappings

(4) $f_n: X_n \to Y_{n+1}$, $\partial_{i+1}^\alpha f_n = f_{n-1}\, \partial_i^\alpha$, $e_{i+1} f_{n-1} = f_n e_i$,

$\partial_1^\alpha f_n = f^\alpha$ $(\alpha = \pm;\ i = 1,..., n)$.

- Dually, the right cylinder $SIS(X) = X\otimes\uparrow\mathbf{i}$ has a right adjoint SPS, the *right cocylinder* or *right path* functor, which discards the faces and degeneracies of highest index (used again to build the corresponding natural transformations)

(5) $SPS: \mathbf{Cub} \to \mathbf{Cub}$, $SPS(Y) = ((Y_{n+1}), (\partial_i^\alpha), (e_i))$,

$\partial^\alpha: SPS(Y) \to Y$, $\partial^\alpha = (\partial_{n+1}^\alpha: Y_{n+1} \to Y_n)_{n\geq 0}$,

$$e\colon Y \to SPS(Y), \qquad\qquad e = (e_{n+1}\colon Y_n \to Y_{n+1})_{n\geq 0}.$$

- An (elementary) *right homotopy* $f\colon f^- \to_R f^+\colon X \to Y$ is a map $f\colon X \to SPS(Y)$ with faces $\partial^\alpha f = f^\alpha$, i.e. a family (f_n) such that

$$(6)\quad f_n\colon X_n \to Y_{n+1}, \qquad \partial_i^\alpha f_n = f_{n-1}\,\partial_i^\alpha, \qquad e_i\, f_{n-1} = f_n\, e_i,$$
$$\partial_{n+1}^\alpha f_n = f^\alpha \qquad\qquad (\alpha = \pm;\ i = 1,..., n).$$

- Elementary homotopies of cubical sets (without connections) are a very defective notion: one cannot even contract the elementary interval $\uparrow\mathbf{i}$ to a vertex.

- Moreover, to obtain 'non-elementary' paths which can be concatenated and a fundamental category $\uparrow\Pi_1(X)$, one should use - instead of the elementary interval $\uparrow\mathbf{i} = \uparrow[0, 1]_{\mathbf{Z}}$ - the *directed integral line* $\uparrow\mathbf{Z}$ (2.5), as in [GX] for simplicial sets: paths are parametrised on $\uparrow\mathbf{Z}$ and eventually constant.

- But here we are interested in homology, where concatenation is surrogated by formal sums of cubes, and we will restrain ourselves to proving its invariance up to elementary homotopies, right and left. Also, we prefer not to rely on the geometric realisation, which would ignore the directed structure.

- The category **Cub** has left and right internal homs, which we shall not need (see [BH]). Let us only recall that the *right* internal hom $CUB(A, Y)$ can be constructed with the *left* cocylinder functor P and its natural transformations (which produce a cubical object P^*Y)

$$(7)\quad -\otimes A \dashv CUB(A, -), \qquad\qquad CUB_n(A, Y) = \mathbf{Cub}(A, P^nY).$$

3.2. Cones and suspension

- The *left upper cone* C^+X is defined as the first pushout, below

$$(1)\qquad \begin{array}{ccc} X & \xrightarrow{\partial^+} & IX \\ \downarrow & & \downarrow \gamma \\ \{*\} & \xrightarrow[v^+]{} & C^+X \end{array} \qquad\qquad \begin{array}{ccc} X & \longrightarrow & \{*\} \\ \partial^- \downarrow & & \downarrow v^- \\ IX & \xrightarrow[\gamma]{} & C^-X \end{array}$$

i.e., the quotient $(IX+\{*\})/(\partial^+X+\{*\})$, where the upper basis of the cylinder is collapsed to an upper vertex $v^+ = v^+(*)$, while the lower basis $\partial^-\colon X \to IX \to C^+X$ 'subsists'. Dually, the *left lower cone* C^-X is defined as the second pushout, above, obtained by collapsing the lower basis of IX to a lower vertex $v^- = v^-(*)$.

- Analytically, we can describe C^+X saying that it is generated by (n+1)-dimensional cubes $u\otimes x \in IX$ $(x\in X_n)$ plus a vertex v^+, under the relations arising from X together with

$$(2)\quad 1\otimes x = e_1^n(v^+) \qquad\qquad (x\in X_n).$$

- Similarly, the *left suspension* ΣX is defined as the colimit of the left diagram

$$\begin{array}{ccccc} & & X & \longrightarrow & \{*\} \\ & & \downarrow \partial^+ & & \\ X & \xrightarrow{\partial^-} & IX & & \downarrow v^+ \\ \downarrow & & & \searrow \sigma & \\ \{*\} & & \xrightarrow[v^-]{\hspace{3em}} & & \Sigma X \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\partial^-} & C^+X \\ \partial^+ \downarrow & & \downarrow j^+ \\ C^-X & \xrightarrow[j^-]{} & \Sigma X \end{array} \qquad (3)$$

obtained by collapsing, independently, the bases of IX to a lower and an upper vertex, v^- and v^+. Equivalently, it is the right-hand pushout, above.

- Thus, the suspension of $\mathbf{s}^0 = D\{0, 1\}$ yields the 'ordered circle' $\uparrow\mathbf{o}^1$ (2.5.6)

$$\begin{array}{ccc} v^+ & = & v^+ \\ u' \uparrow & & \uparrow u'' \\ v^- & = & v^- \end{array} \qquad\qquad u' = \langle 0 \otimes u \rangle, \quad u'' = \langle 1 \otimes u \rangle, \qquad (4)$$

where $\langle - \rangle$ denotes equivalence classes in the pushout (3). More generally

(5) $\Sigma^n(\mathbf{s}^0) = \uparrow\mathbf{o}^n$.

- The *pointed suspension*, studied in [G1, Section 5], yields the directed spheres $\uparrow\mathbf{s}^n$.

3.3. Geometric realisation

- We have already recalled, in 1.1, the functor

(1) $\square : \mathbf{Top} \to \mathbf{Cub}, \qquad \square T = \mathbf{Top}(\mathbf{I}^*, T),$

which assigns to a topological space T the singular cubical set of (continuous) n-cubes $\mathbf{I}^n \to T$, produced by the *cocubical* set of standard cubes $\mathbf{I}^* = ((\mathbf{I}^n), (\delta_i^\alpha), (\varepsilon_i))$ (1.1.2). As for simplicial sets, the *geometric realisation* $\mathcal{R}X$ of a cubical set is given by the left adjoint functor $\mathcal{R} \dashv \square$

(2) $\mathbf{Cub} \underset{\square}{\overset{\mathcal{R}}{\rightleftarrows}} \mathbf{Top} \qquad \mathcal{R}(X) = (\Sigma_x \mathbf{I}^{n(x)})/\sim,$

which takes a cubical set X to a topological space, by pasting a copy of the standard cube $\mathbf{I}^{n(x)}$ for each cube x (of dimension $n(x)$), *along faces and degeneracies*. More precisely, the equivalence relation $\sim$ is generated by the pair of points which corresponds themselves, along the mappings induced by faces (δ_i^α) and degeneracies (ε_i)

(3) $\delta_i^\alpha : \mathbf{I}^{n(y)} \to \mathbf{I}^{n(x)}$ (for $y = \partial_i^\alpha x$), $\qquad \varepsilon_i : \mathbf{I}^{n(x)} \to \mathbf{I}^{n(y)}$ (for $x = e_i y$).

- This pasting (formally, the *coend* of the functor $X{\bullet}\mathbf{I}^* : \mathbb{I}^{op} \times \mathbb{I} \to \mathbf{Top}$) comes thus with a family of *structural mappings*, one for each cube x, coherent with faces and degeneracies (of $\mathbf{I}^*$ and X)

(4) $\hat{x} : \mathbf{I}^{n(x)} \to \mathcal{R}X, \qquad \hat{x} \circ \delta_i^\alpha = (\partial_i^\alpha x)\hat{}, \quad \hat{x} \circ \varepsilon_i = (e_i x)\hat{},$

and $\mathcal{R}X$ has the finest topology making all the structural mappings continuous.

- This realisation is important, since it is well known that the combinatorial homology of a cubical set X coincides with the homology of the CW-space $\mathcal{R}X$ (cf. [Mu, 4.39], for the simplicial case). But there are finer 'directed realisations', keeping information about the privileged cubes of X (see 8.11 or [G1, 1.9]).

4. Directed homology of cubical sets [G1, Section 2]

Combinatorial homology of cubical sets is a simple theory with evident proofs. We study its enrichment with a natural preorder, showing that it is preserved and reflected by excision (4.4) and tensor product (4.5), but not preserved by the differentials of the usual exact sequences (cf. 4.4).

4.1. Directed homology

- Every cubical set X determines a collection $\mathrm{Deg}_n X = \cup_i \mathrm{Im}(e_i\colon X_{n-1} \to X_n)$ of subsets of *degenerate elements* (with $\mathrm{Deg}_0 X = \emptyset$); this collection is not a cubical subset (unless X is empty), but satisfies weaker conditions (for all $i = 1,..., n$)

(1) $x \in \mathrm{Deg}_n X \;\Rightarrow\; (\partial_i^\alpha x \in \mathrm{Deg}_{n-1} X$ *or* $\partial_i^- x = \partial_i^+ x)$, $\qquad e_i(\mathrm{Deg}_{n-1} X) \subset \mathrm{Deg}_n X.$

- The cubical set X determines a (*normalised*) *chain complex* of free abelian groups

(2) $C_n(X) = (\mathbf{Z}X_n)/(\mathbf{Z}\mathrm{Deg}_n X) = \mathbf{Z}\overline{X}_n \qquad (\overline{X}_n = X_n \setminus \mathrm{Deg}_n X),$

$\partial_n(\hat{x}) = \Sigma_{i,\alpha} (-1)^{i+\alpha} (\partial_i^\alpha x)\hat{} \qquad (x \in X_n),$

where $\mathbf{Z}S$ is the free abelian group on the set S and $\hat{x}$ is the class of the n-cube x. We often write the normalised class $\hat{x}$ as x, identifying all degenerate cubes with 0.

- Each component can be preordered by the positive cone of *positive chains* $\mathbf{N}\overline{X}_n$, and will be written as $\uparrow C_n(X)$ when thus enriched.

- *The positive cone is not preserved by the differential* $\partial_n\colon \uparrow C_n(X) \to \uparrow C_{n-1}(X)$, which is just a homomorphism of the underlying abelian groups (as stressed by marking its arrow with a dot).

- A morphism of cubical sets $f\colon X \to Y$ induces a sequence of *preorder-preserving* homomorphisms $\uparrow C_n(X) \to \uparrow C_n(Y)$. We have defined a covariant functor

(3) $\uparrow C_*\colon \mathbf{Cub} \to \mathrm{d}C_*\mathbf{Ab},$

with values in the category $\mathrm{d}C_*\mathbf{Ab}$ of *directed* chain complexes of abelian groups (directed referring to the preorder of components, preserved by chain homomorphisms).

- The *directed homology* of a cubical set is a sequence of preordered abelian groups

(4) $\uparrow H_n\colon \mathbf{Cub} \to \mathrm{d}\mathbf{Ab}, \qquad \uparrow H_n(X) = \uparrow H_n(\uparrow C_* X),$

where the *directed homology* $\uparrow H_n(\uparrow C_*)$ of a directed chain complex is its ordinary homology equipped with the preorder induced on the subquotient $\mathrm{Ker}\partial_n/\mathrm{Im}\partial_{n+1}$.

- When we forget preorders, the usual chain and homology functors will be written as usual

(5) $C_*\colon \mathbf{Cub} \to C_*\mathbf{Ab}, \qquad H_n\colon \mathbf{Cub} \to \mathbf{Ab}.$

- If T is a topological space, we have $H_n(T) = H_n(\square T)$. Here we are not likely losing any essential information with respect to $\uparrow H_n(\square T)$. In fact, $\uparrow H_0(\square T)$ has an obvious order generated by the homology classes of points (4.2.1), while *the preorder of* $\uparrow H_1(\square T)$ *is easily seen to be chaotic*:

every homology class belongs to the positive cone (for every 1-cube $a: \mathbf{I} \to T$, the reversed cube $a\rho$ produced by the reversion $\rho: \mathbf{I} \to \mathbf{I}$ is equivalent to $-a$, modulo boundaries).

4.2. Elementary computations

- The homology of a sum $X = \Sigma X_i$ is a direct sum $\uparrow H_n X = \oplus_i \uparrow H_n X_i$ (and every cubical set is the sum of its connected components).

- Also here (as for spaces) it is easy to see that, if X is connected (non empty), then $\uparrow H_0(X) \cong \uparrow \mathbf{Z}$ (via the augmentation $\partial_0: \uparrow C_0 X = \uparrow \mathbf{Z} X_0 \to \uparrow \mathbf{Z}$ taking each vertex $x \in X_0$ to $1 \in \mathbf{Z}$). Thus, for every cubical set X

(1) $\uparrow H_0(X) = \uparrow \mathbf{Z}.\pi_0 X,$

is the free ordered abelian group generated by the homotopy set $\pi_0 X$ (2.3).

- In particular, $\uparrow H_0(\uparrow \mathbf{s}^0) = \uparrow \mathbf{Z}^2$. Now, it is easy to see that, for $n > 0$

(2) $\uparrow H_n(\uparrow \mathbf{s}^n) = \uparrow \mathbf{Z},$

is the group of integers with the natural order: a normalised n-chain ku (notation of 2.5) is positive if and only if $k \geq 0$ (and is always a cycle).

- On the other hand, $\uparrow H_n(\uparrow \mathbf{o}^n) = \uparrow_d \mathbf{Z}$ has the discrete order: the positive cone is reduced to 0. In fact, a normalised n-chain $hu' + ku''$ (notation of 2.5) is a cycle when $h+k = 0$, and a positive chain for $h \geq 0$, $k \geq 0$. The directed homology of the elementary directed torus $\uparrow \mathbf{t}^2$ is easy to determine; but we shall compute it for all $\uparrow \mathbf{t}^n$ (4.6.2).

4.3. Invariance Theorem

- The homology functor $\uparrow H_n$: **Cub** $\to$ d**Ab** is invariant for left (or right) immediate homotopies: given $f: f^- \to_L f^+: X \to Y$, then $\uparrow H_n(f^-) = \uparrow H_n(f^+)$.

Proof. We can forget about preorders. By 3.1.4, the homotopy $f: f^- \to_L f^+: X \to Y$ has

(1) $f_n: X_n \to Y_{n+1}, \qquad \partial^\alpha_{i+1} f_n = f_{n-1} \partial^\alpha_i, \quad \partial^\alpha_1 f_n = f^\alpha, \quad f_n e_i = e_{i+1} f_{n-1} \qquad (1 \leq i \leq n),$

and produces a homotopy of the associated (normalised) chain complexes

(2) $f_n: C_n X \to C_{n+1} Y, \qquad f_n(\mathrm{Deg}_n X) \subset \mathrm{Deg}_{n+1} Y,$

$\partial_{n+1} f_n = \partial^+_1 f_n - \partial^-_1 f_n - \Sigma_{i\alpha} (-1)^{i+\alpha} \partial^\alpha_{i+1} f_n = f^+_n - f^-_n - f_{n-1} \partial_n.$ □

4.4. Mayer-Vietoris and Excision

- Given two cubical subsets $U, V \subset X$, their union $U \cup V$ (resp. intersection $U \cap V$) just consists of the union (resp. intersection) of all components. Therefore, $\uparrow C_*$ takes subobjects of X to directed chain subcomplexes of $\uparrow C_* X$, preserving joins and meets

(1) $\uparrow C_*(U \cup V) = \uparrow C_* U + \uparrow C_* V, \qquad \uparrow C_*(U \cap V) = \uparrow C_* U \cap \uparrow C_* V.$

These facts have two important consequences

Theorem (The Mayer-Vietoris sequence). Let the cubical set X be *covered by its subobjects* U, V, i.e. $X = U \cup V$. Then we have an exact sequence

$$\text{(2)}\quad \dots \longrightarrow \uparrow H_n(U\cap V) \xrightarrow{(i_*,\, j_*)} (\uparrow H_nU)\oplus(\uparrow H_nV) \xrightarrow{[u_*,\, -v_*]} \uparrow H_n(X) \xrightarrow{\Delta} \uparrow H_{n-1}(U\cap V) \longrightarrow \dots$$

with the obvious meaning of brackets; the maps $u\colon U \to X$, $v\colon V \to X$, $i\colon U\cap V \to U$, $j\colon U\cap V \to X$ are inclusions and the *connective* Δ (which does not preserve preorder!) is:

(3) $\Delta[c] = [\partial_n a]$, $\qquad c = a + b \qquad (a \in \uparrow C_n(U),\ b \in \uparrow C_n(V))$.

The sequence is natural, in an obvious sense.

Theorem (Excision). Let a cubical set X be given, with subobjects $B \subset Y\cap A$. The inclusion map $i\colon (Y, B) \to (X, A)$ is said to be *excisive* whenever $Y_n \setminus B_n = X_n \setminus A_n$, for all n (or equivalently: $Y\cup A = X$, $Y\cap A = B$, *in the lattice of subobjects of* X). Then i induces isomorphisms in homology, *preserving and reflecting preorder*.

Hints. The proof is similar to the topological one, simplified by the fact that here no subdivision is needed. For Mayer-Vietoris, it is sufficient to apply the algebraic theorem of the exact homology sequence to the following sequence of directed chain complexes

$$\text{(4)}\qquad 0 \longrightarrow \uparrow C_*(U\cap V) \xrightarrow{(i_*,\, j_*)} (\uparrow C_*U)\oplus(\uparrow C_*V) \xrightarrow{[u_*,\, -v_*]} \uparrow C_*(X) \longrightarrow 0$$

whose exactness needs one non-trivial verification. Take $a \in \uparrow C_nU$, $b \in \uparrow C_nV$ and assume that $u_*(a) = v_*(b)$; therefore, each cube really appearing in a (and b) belongs to $U\cap V$; globally, there is (one) normalised chain $c \in \uparrow C_n(U\cap V)$ such that $i_*(c) = a$, $i_*(c) = b$.

- For Excision, the proof reduces to a Noether isomorphism for directed chain complexes

(5) $\uparrow C_*(Y, B) = (\uparrow C_*Y)/(C_*(Y\cap A)) = (\uparrow C_*Y)/(C_*Y \cap C_*A)$

$= (\uparrow C_*Y + \uparrow C_*A) / (C_*A) = (\uparrow C_*(Y\cup A)) / (C_*A) = \uparrow C_*(X, A)$. □

4.5. Theorem (Tensor products)

- Given two cubical sets X, Y, there is a natural isomorphism and a natural monomorphism

(1) $\uparrow C_*(X\otimes Y) = \uparrow C_*(X) \otimes \uparrow C_*(Y)$, $\qquad \uparrow H_*(X) \otimes \uparrow H_*(Y) \rightarrowtail \uparrow H_*(X\otimes Y)$.

Proof. It suffices to prove the first part, and apply the Künneth formula. First, the canonical (positive) basis of the preordered abelian group $\uparrow C_p(X)\otimes\uparrow C_q(Y)$ is $\bar{X}_p\times\bar{Y}_q$ (as in 4.1, $\bar{X}_p = X_p \setminus \mathrm{Deg}_pX$).

- Recall now that the set $(X\otimes Y)_n$ is a quotient of $\Sigma_{p+q=n} X_p\times Y_q$ modulo an equivalence relation which only identifies pairs where a term is degenerate (2.4.1); moreover, a class $x\otimes y$ is degenerate if and only if x or y is degenerate (2.4.3). Therefore, the canonical positive basis of $\uparrow C_n(X\otimes Y)$ is precisely the sum (disjoint union) of the preceding sets $\bar{X}_p\times\bar{Y}_q$, for $p+q = n$. We can identify the preordered abelian groups

(2) $\uparrow C_n(X\otimes Y) = \bigoplus_{p+q=n} \uparrow C_p(X) \otimes \uparrow C_q(Y)$,

respecting the canonical positive bases. Finally, the differential of an element $x\otimes y$, with $(x, y) \in \bar{X}_p\times\bar{Y}_q$, is the same in both chain complexes

(3) $\Sigma_{i\alpha} (-1)^{i+\alpha} \partial^{\alpha}_{i}(x \otimes y) = \Sigma_{i \le p,\alpha} (-1)^{i+\alpha} (\partial^{\alpha}_{i} x) \otimes y + \Sigma_{j \le q,\alpha} (-1)^{p+j+\alpha} x \otimes (\partial^{\alpha}_{j} y)$

$= (\partial_p x) \otimes y + (-1)^p x \otimes (\partial_q y).$ □

4.6. Elementary cubical tori

- The graded preordered abelian group of a cubical set X will be written as a *formal polynomial*

(1) $\uparrow H_*(X) = \Sigma_i \sigma^i . \uparrow H_i(X),$

whose coefficients are preordered abelian group, while σ^i shows the homology degree.

- One can think of σ^i as a power of the suspension operator of chain complexes (acting on a preordered abelian group, embedded in dC_***Ab** in degree 0): then the expression (1) is a direct sum of graded preordered abelian groups; and the direct sum of such objects amounts to the sum of the corresponding polynomials (computed by means of the direct sum of the coefficients, in the obvious way).

- Using 4.5, it is easy to see that the directed homology of the elementary torus $\uparrow \mathbf{t}^n = (\uparrow \mathbf{s}^1)^{\otimes n}$ is:

(2) $\uparrow H_*(\uparrow \mathbf{t}^n) = (\uparrow \mathbf{Z} + \sigma . \uparrow \mathbf{Z})^{\otimes n} = \uparrow \mathbf{Z} + \sigma . \uparrow \mathbf{Z}^{\binom{n}{1}} + \sigma^2 . \uparrow \mathbf{Z}^{\binom{n}{2}} + ... + \sigma^n . \uparrow \mathbf{Z},$

where, of course, a power $\uparrow \mathbf{Z}^k$ has the product order.

- In cohomology, one can show that multiplication need not preserve the positive cone [G1, 2.9].

4.7. Some hints at pointed homology and pointed suspension [G1, Section 5]

- A *pointed cubical set* (X, x_0) is a cubical set with a distinguished vertex $x_0 \in X_0$; together with the *pointed* morphisms (preserving the base-points), they form a category $\mathbf{Cub}_*$. One defines in the obvious way *pointed homotopies*, *the pointed suspension* and *pointed directed homology* [G1, Section 5]. (The latter only differs from the ordinary directed homology in degree zero; it is a sort of reduced homology, better suited for ordering.)

- As proved in [G1, Thm. 5.4], there is a natural isomorphism of preordered abelian groups

(1) $\uparrow H_n(X, x_0) \to \uparrow H_{n+1}(\Sigma(X, x_0)), \qquad [\Sigma \lambda_k x_k] \mapsto [\Sigma \lambda_k \langle u \otimes x_k \rangle] \qquad (n \ge 0),$

where $< - >$ denotes equivalence classes in $\Sigma(X, x_0)$ as a quotient of $I(X, x_0)$, and u is the generator of the elementary interval $\uparrow \mathbf{i}$.

5. Action of groups on cubical sets [G1, Section 3]

The classical theory of *proper* actions on topological spaces, as developed in [Ma, IV.11], is extended to *free* actions on cubical sets. G is a group, always written in additive notation (independently of commutativity); the action of an operator $g \in G$ on an element x is written as x+g.

5.1. Basics

- Take a cubical set X and a group G acting on it, on the right: we have an action x+g ($x \in X_n$, $g \in G$) on each component, consistently with faces and degeneracies (or, equivalently, a cubical object in the category of G-sets).

- The *cubical set of orbits* X/G has components X_n/G and the induced structure; there is a natural projection p: X → X/G.

- Say that the action is *free* if G acts freely on each component: if x = x+g, for some $x \in X_n$ and $g \in G$, then g = 0. This is equivalent to saying that G acts freely on the set of vertices X_0 (because x = x+g implies that their first vertices coincide).

- We will extend to *free actions on cubical sets* the classical results of actions of groups on topological spaces [Ma, IV.11], which hold for groups acting *properly* on a space, a much stronger condition (every point has an open neighbourhood U such that all subsets U+g are disjoint). But note that all results below which involve the homology of G *ignore preorder*, necessarily (6.5).

5.2. Lemma (Free actions)

(a) If G acts freely on the cubical set X, then $\uparrow C_*(X)$ is a complex of free right G-modules, with a (positive) basis $B_n \subset X_n$ which projects bijectively onto X_n/G, the canonical basis of $\uparrow C_n(X/G)$.

(b) The canonical projection p: X → X/G induces an isomorphism of directed chain complexes, and hence an isomorphism in homology ($\uparrow\mathbf{Z}$ is viewed as a trivial G-module)

(1) $p_*\colon \uparrow C_*(X) \otimes_G \uparrow\mathbf{Z} \to \uparrow C_*(X/G),$ $\qquad p_{*n}\colon H_n(\uparrow C_*(X) \otimes_G \uparrow\mathbf{Z}) \to \uparrow H_n(X/G).$

Proof. (This Lemma adapts [Ma, IV.11.2-4]). It is sufficient to prove (a), which plainly implies (b). The action of G on X_n extends to a right action on the free abelian group $\mathbf{Z}X_n$, consistent with faces and degeneracies and preserving the canonical basis; it induces thus an obvious action on $\uparrow C_n(X) = \uparrow\mathbf{Z}\overline{X}_n$, consistent with the positive cone and the differential

(2) $(\Sigma\, \lambda_i x_i) + g \;=\; \Sigma\, \lambda_i (x_i + g),$ $\qquad \partial(\Sigma\, \lambda_i x_i) + g \;=\; \partial(\Sigma\, \lambda_i x_i + g).$

Thus $\uparrow C_n(X)$ is a complex of G-modules, whose components are preordered G-modules. Take now a subset $B_0 \subset X_0$ choosing exactly one point in each orbit; then B_0 is a G-basis of $\uparrow C_0(X)$. Letting $B_n \subset X_n$ be the subset of those non-degenerate n-cubes x whose 'initial vertex' $\partial_1^- \ldots \partial_n^- x$ belongs to B_0, we have more generally a G-basis of $\uparrow C_n(X)$ which satisfies our requirements. □

5.3. Theorem (Free actions on acyclic cubical sets)

Let X be an *acyclic* (connected) cubical set and G a group acting freely on it. Then, *forgetting* preorder in combinatorial homology

(1) $H_*(X/G) \;\cong\; H_*(G).$

Proof. As in [Ma, IV.11.5], the augmented sequence

(2) $\ldots \to C_1(X) \to C_0(X) \to \mathbf{Z} \to 0$

is exact, since X is acyclic (has the homology of the point). By 5.2a, this sequence forms a G-free resolution of the G-trivial module $\mathbf{Z}$. Therefore, applying the definition of $H_n(G)$ and the isomorphism 5.2.1, we get the thesis for homology (and cohomology as well)

(3) $H_n(G) \;=\; H_n(C_*(X) \otimes_G \mathbf{Z}) \;\cong\; H_n(X/G).$ □

5.4. Corollary (Free actions on acyclic spaces)

Let T be an acyclic (path connected) topological space and G a group acting freely on it. Then $H_*((\square T)/G) \cong H_*(G)$, and $\uparrow H_1((\square T)/G)$ has a chaotic preorder. (The same holds in cohomology.)

Proof. It suffices to apply the preceding theorem to the singular cubical set $\square T$ of continuous cubes of T. This cubical set has the same homology as T, and G acts obviously on it, by $(x+g)(t) = x(t) + g$ (for $t \in \mathbf{I}^n$). Moreover, the action is free because so it is on the set of vertices, T. Finally, the remark on preorder is proved as for $\uparrow H_1(\square T)$, in 4.1. □

6. Noncommutative tori, Kronecker foliations and cubical sets [G1, Section 4]

We compute the directed homology of various cubical sets, related with 'virtual spaces' of noncommutative geometry: irrational rotation algebras and noncommutative tori of dimension ≥ 2; ϑ and ζ will always denote irrational real numbers.

6.1. Rotation algebras

- Let us begin recalling some well-known 'noncommutative spaces'.

- First, take the line $\mathbf{R}$ and its (dense) additive subgroup $G_\vartheta = \mathbf{Z} + \vartheta\mathbf{Z}$, acting on the former by translations. In **Top**, the orbit space $\mathbf{R}/G_\vartheta = \mathbf{S}^1/\vartheta\mathbf{Z}$ is trivial: an uncountable set with the coarse topology.

- Second, consider the *Kronecker foliation* F' of the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, with slope ϑ (recalled in 6.3), and the set $\mathbf{T}^2_\vartheta = \mathbf{T}^2/\!\equiv_{F'}$ of its leaves. It is well known, and easy to see, that the sets $\mathbf{R}/G_\vartheta$ and $\mathbf{T}^2_\vartheta$ are in bijection (cf. 6.3). Again, ordinary topology gives no information on $\mathbf{T}^2_\vartheta$, since the quotient $\mathbf{T}^2/\!\equiv_{F'}$ in **Top** is coarse (every leaf being dense).

- In noncommutative geometry, both these sets are 'interpreted' as the (noncommutative) C*-algebra A_ϑ, generated by two unitary elements u, v under the relation $vu = \exp(2\pi i\vartheta).uv$, and called the *irrational rotation algebra* associated with ϑ, or also a *noncommutative torus* [C1, C2, Ri, Bl]. Both its complex K-theory groups are two-dimensional.

- A relevant achievement of K-theory [PV, Ri] classifies these algebras, by proving that $K_0(A_\vartheta) \cong \mathbf{Z}+\vartheta\mathbf{Z}$ *as an ordered subgroup of* $\mathbf{R}$; more precisely, the traces of the projections of A_ϑ cover the set $G_\vartheta \cap [0, 1]$. Therefore [Ri, Thm. 2 and Thm. 4]:

- (a) A_ϑ and A_ζ are *isomorphic* if and only if $\zeta \in \pm\vartheta + \mathbf{Z}$;

- (b) A_ϑ and A_ζ are *strongly Morita equivalent* if and only if ϑ and ζ are equivalent modulo the *fractional* action (on the irrationals) of the group $GL(2, \mathbf{Z})$ of invertible integral 2×2 matrices

(1) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.t = \frac{at+b}{ct+d} \qquad (a, b, c, d \in \mathbf{Z};\ ad - bc = \pm 1),$$

(or the action of the projective general linear group $PGL(2, \mathbf{Z})$ on the projective line).

- Since $GL(2, \mathbf{Z})$ is generated by the matrices

(2) $$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

the orbit of ϑ is its closure $\{\vartheta\}_{RT}$ under the transformations $R(t) = t^{-1}$ and $T^{\pm1}(t) = t \pm 1$ (on $\mathbf{R}\setminus\mathbf{Q}$)

- We show now how one can obtain similar results with cubical sets naturally arising from the previous situations: the point is to replace a topologically-trivial orbit space T/G with the corresponding quotient of the singular cubical set $\square T$, identifying the cubes $\mathbf{I}^n \to T$ modulo the action of G.

6.2. Irrational rotation structures

(a) Now, instead of considering the trivial quotient $\mathbf{R}/G_\vartheta$ of topological spaces, we replace $\mathbf{R}$ with the singular cubical set $\square\mathbf{R}$ (on which G_ϑ acts freely) and consider the cubical set $(\square\mathbf{R})/G_\vartheta$.

- Applying Corollary 5.4, we find that the cubical set $(\square\mathbf{R})/G_\vartheta$ has the same homology as the group $G_\vartheta \cong \mathbf{Z}^2$, which coincides with the ordinary homology of the torus $\mathbf{T}^2$

(1) $H_*((\square\mathbf{R})/G_\vartheta) = H_*(G_\vartheta) = H_*(\mathbf{T}^2) = \mathbf{Z} + \sigma.\mathbf{Z}^2 + \sigma^2.\mathbf{Z}$;

(the last fact follows, for instance, from the classical version of Theorem 5.3 [Ma, IV.11.5], applied to the proper action of the group $\mathbf{Z}^2$ on the acyclic space $\mathbf{R}^2$). We also know that directed homology only gives the chaotic preorder on $\uparrow H_1((\square\mathbf{R})/G_\vartheta)$ (again by 5.4).

- In cohomology, we have the same graded group. *Algebraically*, this is in accord with the K-theory of the rotation algebra A_ϑ, since both $H^{even}((\square\mathbf{R})/G_\vartheta)$ and $H^{odd}((\square\mathbf{R})/G_\vartheta)$ are two-dimensional.

(b) A much more interesting result (and accord) can be obtained from the cubical sets

(2) $\square\uparrow\mathbf{R} \subset \square\mathbf{R}$, *the cubical set of the ordered line,*

$\square_n\uparrow\mathbf{R}$ = the set of continuous order-preserving mappings $\mathbf{I}^n \to \mathbf{R}$;

(3) $\square\uparrow\mathbf{S}^1 = (\square\uparrow\mathbf{R})/\mathbf{Z}$, *the cubical set of the directed circle*.

- We want to classify the isomorphism classes of the cubical sets

(4) $C_\vartheta = (\square\uparrow\mathbf{R})/G_\vartheta = (\square\uparrow\mathbf{S}^1)/\vartheta\mathbf{Z}$, the *irrational rotation cubical set* (associated to $\vartheta \notin \mathbf{Q}$).

- We prove below (Theorems 6.7, 6.8) that $\uparrow H_1(C_\vartheta) \cong \uparrow G_\vartheta$, *as an ordered subgroup of the line* and that the cubical sets C_ϑ have the same classification *up to isomorphism* as the rotation algebras A_ϑ *up to strong Morita equivalence*: while the algebraic homology of C_ϑ is the same as in (a), independent of ϑ, the (pre)order of directed homology determines ϑ up to the equivalence relation $\uparrow G_\vartheta \cong \uparrow G_\zeta$, which amounts to ϑ and ζ being conjugate under the action of the group $GL(2, \mathbf{Z})$.

- Note that the stronger classification of rotation algebras up to isomorphism (recalled in 6.1) has no analogue here: *cubical sets lack the 'metric information' contained in* C*-*algebras*.

- Note also the role of the ordered cube $\mathbf{I}^n$ (with its faces and degeneracies) for defining $\square\uparrow\mathbf{R}$. Presumably, *this cannot be easily transferred to a simplicial approach*: the standard realisations of Δ^n in $\mathbf{R}^{n+1}$ or $\mathbf{R}^n$ are of no use (the former inherits the discrete order while the latter has a 'diagonal' face not consistent with ordering). Other realisations in $\mathbf{R}^n$ have complicated faces. Not to mention the problem of having a *subdivision consistent with ordering* (which has an obvious solution in $\mathbf{I}^n$).

6.3. The noncommutative two-dimensional torus

Consider now the *Kronecker foliation* F' of the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, with irrational slope ϑ, and the set $\mathbf{T}^2_\vartheta = \mathbf{T}^2/\equiv_{F'}$ of its leaves. F' and $\equiv_{F'}$ are induced, respectively, from the following foliation $F = (F_\lambda)$ and equivalence relation $\equiv$ on the plane

(1) $F_\lambda \; = \; \{(x, y) \in \mathbf{R}^2 \;|\; y = \vartheta x + \lambda\}$ $\qquad (\lambda \in \mathbf{R})$,

$(x, y) \equiv (x', y') \;\Leftrightarrow\; y + k - \vartheta(x+h) \;=\; y' + k' - \vartheta(x'+h')$ $\qquad$ (for some $h, k, h', k' \in \mathbf{Z}$).

Now, we interpret $\mathbf{T}^2_\vartheta$ as the quotient cubical set $(\square\,\mathbf{T}^2)/\!\equiv_{F'}$, i.e. the cubical set of the torus (or of the plane) modulo the equivalence relation induced by projecting cubes modulo $\equiv_{F'}$ (or modulo $\equiv$). This can be proved to be isomorphic to the previous cubical set $K = (\square\,\mathbf{R})/G_\vartheta$ [G1, 4.3]; the isomorphism is induced by the following maps:

(2) $i: \mathbf{R} \to \mathbf{R}^2, \;\; i(t) \;=\; (0, t),$ $\qquad p: \mathbf{R}^2 \to \mathbf{R}, \;\; p(x, y) \;=\; y - \vartheta x.$

6.4. Higher foliations of codimension 1

(a) Extending 6.2a and 6.3, take an n-tuple of real numbers $\vartheta = (\vartheta_1,..., \vartheta_n)$, linearly independent on the rationals, and consider the additive subgroup $G_\vartheta = \Sigma_j\, \vartheta_j\mathbf{Z} \cong \mathbf{Z}^n$, acting freely on $\mathbf{R}$. (The previous case corresponds to the pair $(1, \vartheta)$.)

- Now, the cubical set $(\square\,\mathbf{R})/G_\vartheta$ has the homology (or cohomology) of the n-dimensional torus $\mathbf{T}^n$ (notation as in 4.6)

(1) $H_*((\square\,\mathbf{R})/G_\vartheta) \;=\; H_*(G_\vartheta) \;=\; H_*(\mathbf{T}^n) \;=\; \mathbf{Z} + \sigma.\mathbf{Z}^{\binom{n}{1}} + \sigma^2.\mathbf{Z}^{\binom{n}{2}} + ... + \sigma^n.\mathbf{Z}.$

- Again, this coincides with the homology of a cubical set arising from the foliation F' of the n-dimensional torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ induced by the hyperplanes $\Sigma_j\, \vartheta_j x_j = \lambda$ of $\mathbf{R}^n$.

(b) Extending now 6.2b (and Theorem 6.7), the cubical set $(\square\uparrow\!\mathbf{R})/G_\vartheta$ has a more interesting directed homology, with a relevant total order in degree 1:

(2) $\uparrow\!H_1((\square\uparrow\!\mathbf{R})/G_\vartheta) \;=\; \uparrow\!G_\vartheta \;=\; \uparrow\!(\Sigma_j\, \vartheta_j\mathbf{Z})$ $\qquad (G^+_\vartheta = G_\vartheta \cap \mathbf{R}^+)$.

6.5. Remarks

- The previous results show also that *it is not possible to preorder group-homology* so that the isomorphism $H_*(G) \cong H_*(X/G)$ (5.3.1) be extended to $\uparrow\!H_*(X/G)$: a group G can act *freely* on two *acyclic* cubical sets X_i producing *different preorders* on some $\uparrow\!H_n(X_i/G_\vartheta)$.

- In fact, it is sufficient to take $G_\vartheta = \mathbf{Z}+\vartheta\mathbf{Z}$, as above, and recall that $\uparrow\!H_1((\square\,\mathbf{R})/G_\vartheta)$ has a *chaotic preorder* (5.4) while $\uparrow\!H_1((\square\uparrow\!\mathbf{R})/G_\vartheta) = \uparrow\!G_\vartheta$ is totally ordered (6.7).

6.6. Lemma

Let ϑ, ζ be irrationals. Then $G_\vartheta = G_\zeta$, as subsets of $\mathbf{R}$, if and only if $\zeta \in \pm\,\vartheta + \mathbf{Z}$. Moreover the following conditions are equivalent

(a) $\uparrow\!G_\vartheta \cong \uparrow\!G_\zeta$ as ordered groups,

(b) ϑ and ζ are conjugate under the action of $GL(2, \mathbf{Z})$ (6.1.1),

(c) ζ belongs to the closure $\{\vartheta\}_{RT}$ of $\{\vartheta\}$ under the transformations $R(t) = t^{-1}$ and $T^{\pm1}(t) = t\pm1$.

Further, these conditions imply the following one (which will be proved to be equivalent in 6.7)

(d) $(\square\uparrow\!\mathbf{R})/G_\vartheta \cong (\square\uparrow\!\mathbf{R})/G_\zeta$ as cubical sets.

6.7. Theorem (Directed homology of the irrational rotation cubical sets)

The cubical set $\square\uparrow\mathbf{R}$ (6.2b) is acyclic. The directed homology of $C_\vartheta = (\square\uparrow\mathbf{R})/G_\vartheta$ is the homology of $\mathbf{T}^2$, with a total order on $\uparrow H_1$ and a chaotic preorder on $\uparrow H_2$

(1) $\uparrow H_1(C_\vartheta) = \uparrow G_\vartheta = \uparrow(\mathbf{Z} + \vartheta\mathbf{Z})$ $\qquad (G^+_\vartheta = G_\vartheta \cap \mathbf{R}^+)$,

$\uparrow H_2(C_\vartheta) = \uparrow_c\mathbf{Z}$,

and obviously $\uparrow H_0(C_\vartheta) = \uparrow\mathbf{Z}$. The first isomorphism above has a simple description *on the positive cone* $G_\vartheta \cap \mathbf{R}^+$ ($p\colon \square\uparrow\mathbf{R} \to C_\vartheta$ is the canonical projection)

(2) $\varphi\colon \uparrow G_\vartheta \to \uparrow H_1(C_\vartheta)$, $\qquad \varphi(\rho) = [pa_\rho]$ $\qquad (\rho \in G_\vartheta \cap \mathbf{R}^+)$,

$a_\rho\colon \mathbf{I} \to \mathbf{R}$, $\qquad a_\rho(t) = \rho t$.

6.8. Classification Theorem (For the cubical sets of irrational rotation)

The cubical sets $(\square\uparrow\mathbf{R})/G_\vartheta$ and $(\square\uparrow\mathbf{R})/G_\zeta$ are isomorphic if and only if the ordered groups $\uparrow G_\vartheta$ and $\uparrow G_\zeta$ are isomorphic, if and only if ϑ and ζ are conjugate under the action of $GL(2, \mathbf{Z})$ (6.1.1), if and only if ζ belongs to the closure $\{\vartheta\}_{RT}$ (6.1.2).

Proof. From Lemma 6.6 and Theorem 6.7. □

7. Metric aspects by normed cubical sets [G2]

7.0. Introduction

- Enriching cubical sets and their homology groups with a *norm*, we get stronger results.

- First, let us note that this homology norm can distinguish between metrically-different realisations of the same homotopy type, the one of the circle. Thus, applying the normed directed 1-homology group $\mathsf{N}\uparrow H_1$ to the standard normed directed circle $\mathsf{N}\square\uparrow\mathbf{S}^1$, where the length of a homology generator is 2π, we get $2\pi.\uparrow\mathbf{Z}$ as a *normed ordered* subgroup of the line. Similarly, the normed directed 1-torus $\mathsf{N}\square\uparrow\mathbf{T} = (\square\uparrow\mathbf{R})/\mathbf{Z}$ gives the group of integers $\uparrow\mathbf{Z}$ with natural norm and order, since now the length of a homology generator is 1. Finally, the (naturally normed) singular cubical set of the punctured plane $\mathbf{R}^2\setminus\{0\}$ assigns to the group $\mathbf{Z}$ the coarse preorder and the *zero* (semi)norm, making manifest the existence of (reversible) 1-cycles of arbitrarily small length (7.6).

- These rather obvious aspects become of interest for the cubical set, $C_\vartheta = (\square\uparrow\mathbf{R})/G_\vartheta$. We have seen (6.8) that the classification of the cubical sets C_ϑ up to isomorphism coincides with that of the algebras A_ϑ up to strong Morita equivalence. The stricter classification of the latter *up to isomorphism* suggests that cubical sets provide a sort of 'noncommutative topology', without the metric character of noncommutative geometry.

- To account for this character, we enrich C_ϑ with a natural *normed* structure $\mathsf{N}C_\vartheta$, essentially produced by the length of (increasing) paths $\mathbf{I} \to \mathbf{R}$ (7.3). Now, *normed* directed 1-homology gives $\mathsf{N}\uparrow H_1(\mathsf{N}C_\vartheta) \cong \uparrow G_\vartheta$ as a *normed* ordered subgroup of $\mathbf{R}$ (Thm. 7.7). It follows easily that the normed cubical sets $\mathsf{N}C_\vartheta$ have precisely the same classification up to isomorphism as the C*-algebras A_ϑ (Thm. 7.8).

- We end this introduction with some technical remarks. Norms for cubical sets (7.1) and abelian groups (7.4) will take values in $[0, +\infty]$, so that these categories have all products (and some useful left adjoints); morphisms in these categories are always assumed to be (weakly) contracting, so that isomorphisms are isometrical. Moreover, in an abelian group, $||x|| = 0$ will *not* imply $x = 0$: this assumption would annihilate useful information, as for the punctured plane recalled above.

- Preorder of homology groups does not play a relevant role here, since the metric information is sufficient for our main goals; however, preorder is an independent aspect, which might be of use in other cases. It is also interesting to note that, in the present proofs, the arguments concerning norms are similar to the ones concerning preorders in the preceding sections, if more complicated; this is likely related with the fact that preorder is a simplified, two-valued generalised metric [G2, 1.5].

7.1. Normed cubical sets [G2, 1.1]

- A *normed cubical set* will be a cubical set X equipped with a sequence of 'norms' which annihilate on degenerate elements

(1) $|| - ||: X_n \to [0, +\infty]$, $\qquad ||e_i(a)|| = 0 \qquad$ (for all $a \in X_n$).

- We do *not* require any coherence condition for faces, nor any restriction on the norm of a point; for instance, a degenerate edge must have norm zero, but its vertices can have any norm. The category N**Cub** of normed cubical sets has, for *morphisms*, the (weakly) contracting morphisms of cubical sets $f: X \to Y$, with $||f_n(x)|| \leq ||x||$, for all $x \in X_n$.

7.2. Elementary models [G2, 1.2]

- A *normal* cubical set has norm 1 on all *non-degenerate entries* (and 0 on the degenerate ones). All the 'elementary' cubical sets considered in 2.5 will be equipped with this normal norm and denoted with the same symbols (in this section).

- $\uparrow$**i** = **2**: the *normal directed elementary interval*, freely gen. (as a normal cubical set) by a 1-cube u

(1) $0 \xrightarrow{u} 1 \qquad \partial_1^-(u) = 0, \quad \partial_1^+(u) = 1, \quad ||u|| = ||0|| = ||1|| = 1.$

- The *normed directed elementary* n-*cube* $\uparrow\mathbf{i}^n$: the normal object generated by one n-cube, for $n \geq 0$.

- The *normed directed elementary circle* $\uparrow\mathbf{s}^1$: the normal object gen. by a 1-cube u with equal faces

(2) $* \xrightarrow{u} * \qquad \partial_1^-(u) = \partial_1^+(u), \quad ||u|| = ||*|| = 1.$

- The *normed directed elementary n-sphere* $\uparrow\mathbf{s}^n$ $(n > 1)$: the normal object generated by an n-cube u, all whose faces are totally degenerate (hence equal)

(3) $\partial_i^\alpha(u) = (e_1)^{n-1}(\partial_1^-)^n(u), \qquad ||u|| = ||*|| = 1 \qquad (\alpha = \pm;\ i = 1,..., n),$

- $\uparrow\mathbf{s}^0 = \mathbf{s}^0$: the normal object generated by two vertices.

- The n-dimensional torus $\uparrow\mathbf{t}^n$ can be defined as a tensor power of $\uparrow\mathbf{s}^1$ [G2, 2.3].

- The *normed ordered circle* $\uparrow\mathbf{o}^1$: the normal object generated by two edges with the same faces

(4) $v^- \underset{u''}{\overset{u'}{\rightrightarrows}} v^+ \qquad \partial_1^\alpha(u') = \partial_1^\alpha(u''), \qquad ||u'|| = ||u''|| = ||v^-|| = ||v^+|| = 1.$

- More generally, the *normed ordered sphere* $\uparrow\mathbf{o}^n$ is the normal object generated by two n-cubes u', u" with the same boundary: $\partial_i^\alpha(u') = \partial_i^\alpha(u'')$.

- For the links of these objects with suspension, pointed or not, see 3.2 and [G1, 5.2].

7.3. Normed circles and irrational rotation structures [G2, 1.3-1.4]

- Here, we distinguish between the *standard circle* $\mathbf{S}^1$, equipped with the natural geodetic metric, and the *standard 1-torus* $\mathbf{T}$, with the metric induced by the line

(1) $\mathbf{S}^1 \cong \mathbf{R}/2\pi\mathbf{Z}$, $\qquad\qquad \mathbf{T} = \mathbf{R}/\mathbf{Z}$,

so that a simple loop has, respectively, a length of 2π and 1.

- The *normed directed line* $\mathsf{N}\square\uparrow\mathbf{R}$ will be the cubical *directed line* $\square\uparrow\mathbf{R}$ (6.2.2), with the following, obvious norm on the n-cube $a: \mathbf{I}^n \to \mathbf{R}$

(2) $n = 0$: $||a|| = 1$, $\qquad n = 1$: $||a|| = a(1) - a(0)$, $\qquad n > 1$: $||a|| = 0$;

note that, in degree 1, a is an *increasing* path and $||a||$ is its length.

- Now, the groups $\mathbf{Z}$ and $2\pi\mathbf{Z}$ act (isometrically) on the line, by translations, as well as on $\mathsf{N}\square\uparrow\mathbf{R}$. The quotient cubical sets are, by definition

(3) $\mathsf{N}\square\uparrow\mathbf{S}^1 = (\mathsf{N}\square\uparrow\mathbf{R})/(2\pi\mathbf{Z})$, $\qquad$ the *normed directed circle,*

$\mathsf{N}\square\uparrow\mathbf{T} = (\mathsf{N}\square\uparrow\mathbf{R})/\mathbf{Z}$, $\qquad$ the *normed directed 1-torus* ,

the quotient norm is obviously $||\,[a]\,|| = ||a||$. (For quotient norms see [G2, 2.1].)

- Similarly, to enrich the cubical set $C_\vartheta = (\square\uparrow\mathbf{R})/G_\vartheta$ with a norm, it suffices to replace the cubical set $\square\uparrow\mathbf{R}$ with the normed analogue $\mathsf{N}\square\uparrow\mathbf{R}$. The group $G_\vartheta = \mathbf{Z}+\vartheta\mathbf{Z}$ acts isometrically on it, and

(4) $\mathsf{N}C_\vartheta = (\mathsf{N}\square\uparrow\mathbf{R})/G_\vartheta$, $\qquad ||\,[a]\,|| = ||a||$.

7.4. Normed abelian groups and chain complexes [G2, 3.1-3.2]

- Normed directed homology will take values in normed preordered abelian groups, a 'metric' version of the category d**Ab** of preordered abelian groups.

- Here, a *normed* abelian group L is equipped with a *norm* $||\lambda|| \in [0, \infty]$ such that

(1) $||0|| = 0$, $\qquad ||-\lambda|| = ||\lambda||$, $\qquad ||\lambda + \mu|| \leq ||\lambda|| + ||\mu||$.

- Note that, for $n\in\mathbf{N}$, we only have $||n.\lambda|| \leq n.||\lambda||$ (requiring equality would make quotients difficult to handle).

- For a *normed preordered abelian group* $\uparrow L$, no coherence conditions between preorder and norm are required. In the category Nd**Ab** of such objects, a *morphism* is a contracting homomorphism ($||f(\lambda)|| \leq ||\lambda||$) which respects preorder. But also the purely algebraic homomorphisms of the underlying abelian groups will intervene, denoted by arrows with a dot, $\dashrightarrow$.

- Nd**Ab** has all limits and colimits, computed as in **Ab** and equipped with a suitable norm and preorder. The tensor product $\uparrow L\otimes\uparrow M$ of d**Ab** (with positive cone generated by the tensors of positive elements) can be lifted to Nd**Ab**, with a norm

(2) $||\xi|| = \inf\{\Sigma_i ||\lambda_i||.||\mu_i|| \mid \xi = \Sigma_i \lambda_i\otimes\mu_i\}$ $\qquad (\xi \in \uparrow L\otimes\uparrow M)$,

which solves the universal problem for preorder-preserving bi-homomorphisms $\varphi: \uparrow L \times \uparrow M \to \uparrow N$ such that $||\varphi(\lambda, \mu)|| \leq ||\lambda||.||\mu||$.

- This makes a *closed* symmetric monoidal structure: the internal hom $\uparrow\mathrm{Hom}(\uparrow M, \uparrow N)$ is the abelian group of *all* homomorphisms of the underlying abelian groups, with the positive cone of preorder preserving homomorphisms and the Lipschitz norm [G2, 2.2].

- The unit of the tensor product is the ordered group of integers $\uparrow\mathbf{Z}$ with the natural norm, $|k|$. Again, the representable functor $\mathsf{Nd}\mathbf{Ab}(\uparrow\mathbf{Z}, -)$, applied to the internal Hom, gives back the set of morphisms

(3) $\mathsf{Nd}\mathbf{Ab}(\uparrow\mathbf{Z}, \uparrow L) = B_1(L^+)$, $\qquad B_1(\mathrm{Hom}^+(\uparrow M, \uparrow N)) = \mathsf{Nd}\mathbf{Ab}(\uparrow M, \uparrow N)$.

- The forgetful functor $\mathsf{Nd}\mathbf{Ab} \to \mathrm{d}\mathbf{Ab}$ has a left adjoint $\mathsf{N}_\infty\uparrow L$ and right adjoint $\mathsf{N}_0\uparrow L$, respectively giving to a preordered abelian group $\uparrow L$ its discrete ∞-norm ($||\lambda|| = \infty$ for $\lambda \neq 0$) or the coarse one ($||\lambda|| = 0$).

- The forgetful functor $\mathsf{Nd}\mathbf{Ab} \to \mathsf{N}\mathbf{Set}$ has a left adjoint, associating to a normed set S the *free normed ordered abelian group* $\uparrow\mathbf{Z}S$, which is the free abelian group generated by the underlying set, equipped with the obvious norm

(4) $||\Sigma_x k_x.x|| = \Sigma_x |k_x|.||x||$,

($(k_x)_{x\in S}$ is a quasi-null family of integers) and with the order whose positive cone is the monoid $\mathbf{N}S$ of positive combinations, with $k_x \in \mathbf{N}$.

- $\mathsf{Nd}C_*\mathbf{Ab}$ will denote the category of *normed directed chain complexes*: their components are normed preordered abelian groups, differentials are *not* assumed to respect norms or preorders, *but chain morphisms are*: they must be *contracting* and preorder-preserving.

- The *normed directed homology* of such a complex $\uparrow C_*$ is a sequence of normed preordered abelian groups, consisting of the ordinary homology subquotients

(5) $\mathsf{N}\uparrow H_n: \mathsf{Nd}C_*\mathbf{Ab} \to \mathsf{Nd}\mathbf{Ab}$, $\qquad \mathsf{N}\uparrow H_n(\uparrow C_*) = \mathrm{Ker}\partial_n/\mathrm{Im}\partial_{n+1}$,

with the induced norm and preorder. (To forget about preorder, we take out the prefixes d, $\uparrow$.)

7.5. Normed directed homology [G2, 3.3]

- The normed cubical set X determines a *chain complex* of free normed ordered abelian groups

(1) $\mathsf{N}\uparrow C_n(X) = (\uparrow\mathbf{Z}X_n)/(\uparrow\mathbf{Z}\mathrm{Deg}_nX) = \uparrow\mathbf{Z}\overline{X}_n \qquad (\overline{X}_n = X_n \setminus \mathrm{Deg}_nX)$,

$\partial_n(\hat{x}) = \Sigma_{i,\alpha} (-1)^{i+\alpha} (\partial_i^\alpha x)\hat{} \qquad (x \in X_n)$.

- Again, we shall write the class $\hat{x}$ as x, identifying all degenerate cubes with 0. This is consistent with the norm, since all degenerate chains have norm 0 and all representatives of $\hat{x}$ have the same norm in $\uparrow\mathbf{Z}X_n$. (For this chain complex, we shall avoid the usual term 'normalised', which might give rise to confusion with norms.)

- Also here (cf. 4.1), *the positive cone and the norm are not respected by the differential* $\partial_n: \mathsf{N}\uparrow C_n(X) \,\dot{\to}\, \mathsf{N}\uparrow C_{n-1}(X)$, which is just a homomorphism of the underlying abelian groups, as stressed by marking its arrow *with a dot*.

- On the other hand, a morphism of normed cubical sets $f: X \to Y$ induces a sequence of *morphisms* $N{\uparrow}C_n(X) \to N{\uparrow}C_n(Y)$, which do preserve preorder and respect norms. We have defined a functor

(2) $N{\uparrow}C_*: N\mathbf{Cub} \to NdC_*\mathbf{Ab}$,

with values in the category $NdC_*\mathbf{Ab}$ of normed directed chain complexes of abelian groups (7.4).

- This produces the *normed directed homology* of a cubical set

(3) $N{\uparrow}H_n: N\mathbf{Cub} \to Nd\mathbf{Ab}$, $\qquad N{\uparrow}H_n(X) = N{\uparrow}H_n(N{\uparrow}C_*X)$,

given by the ordinary homology subquotient, with the induced preorder and norm. When we forget preorder, the normed chain and homology functors will be written as NC_*X and NH_*X.

7.6. Normed homology of circles and tori [G2, 3.5-3.7]

- The normed directed 1-homology group of the normed directed circle $N\square{\uparrow}\mathbf{S}^1$ and 1-torus $N\square{\uparrow}\mathbf{T}$ (7.3) are easy to compute, taking into account the length of the standard generating 1-cycle:

(1) $N{\uparrow}H_1(N\square{\uparrow}\mathbf{S}^1) = 2\pi.{\uparrow}\mathbf{Z}$, $\qquad N{\uparrow}H_1(N\square{\uparrow}\mathbf{T}) = {\uparrow}\mathbf{Z}$,

with the natural norm and order.

- The punctured plane $\mathbf{R}^2 \setminus \{0\}$ (with the euclidean metric) gets the coarse preorder and the zero 'norm', since the homology generator contains arbitrarily small cycles

(2) $N{\uparrow}H_1(N\square(\mathbf{R}^2 \setminus \{0\})) = N_0{\uparrow}_c\mathbf{Z}$.

(Of course, in all these cases, $N{\uparrow}H_0$ is the normed ordered abelian group ${\uparrow}\mathbf{Z}$.)

- Also because of a theorem on the tensor product of normed cubical sets [G2, 3.6], the normed directed homology of the normed torus ${\uparrow}\mathbf{t}^n = ({\uparrow}\mathbf{s}^1)^{\otimes n}$ is expressed as in 4.6

(3) $N{\uparrow}H_k({\uparrow}\mathbf{t}^n) = {\uparrow}\mathbf{Z}^{\binom{n}{k}}$ $\qquad (0 \leq k \leq n)$,

but now ${\uparrow}\mathbf{Z}$ is the *normed* ordered abelian group of integers.

7.7. Theorem (Normed homology of the normed cubical sets of irrational rotation) [G2, 4.1]

For any irrational number ϑ, the normed homology groups of $NC_\vartheta = (N\square{\uparrow}\mathbf{R})/G_\vartheta$ (7.3) are:

(1) $NH_1(NC_\vartheta) = G_\vartheta = \mathbf{Z}+\vartheta\mathbf{Z} \subset \mathbf{R}$,

$NH_0(NC_\vartheta) = \mathbf{Z} \subset \mathbf{R}$, $\qquad NH_2(NC_\vartheta) = N_0\mathbf{Z}$,

(with the norm induced by the reals in degrees 0 and 1; and null in degree 2). The first isomorphism acts on the positive cone $G_\vartheta \cap \mathbf{R}^+$ as in 6.7.2.

7.8. Classification Theorem (For the normed cubical sets of irrational rotation) [G2, 4.2]

The normed cubical sets $(N\square{\uparrow}\mathbf{R})/G_\vartheta$ and $(N\square{\uparrow}\mathbf{R})/G_\zeta$ are (isometrically) isomorphic if and only if $G_\vartheta = G_\zeta$ as subsets of $\mathbf{R}$, if and only if $\zeta \in \pm\vartheta + \mathbf{Z}$.

Proof. By Theorem 7.7, if our normed cubical sets are isomorphic, also their normed groups NH_1 are, and $G_\vartheta \cong G_\zeta$ (isometrically). Since the values of the norm $||-||: G_\vartheta \to \mathbf{R}$ form the set

$G_\vartheta \cap \mathbf{R}^+$, it follows that G_ϑ coincides with G_ζ. Finally, if this is the case, then $\vartheta = a + b\zeta$ and $\zeta = c + d\vartheta$, whence $\vartheta = a + bc + bd\vartheta$ and $d = \pm 1$. □

7.9. An extension [G2, 4.3]

- Extending the previous case (and enriching 6.4b), take an n-tuple of real numbers $\vartheta = (\vartheta_1, ..., \vartheta_n)$, linearly independent on the rationals, and consider the normed additive subgroup $G_\vartheta = \Sigma_j \vartheta_j \mathbf{Z} \subset \mathbf{R}$, acting freely and isometrically on the line. (The previous case corresponds to the pair $(1, \vartheta)$.)

- Again, the normed cubical set $(\mathsf{N} \,\Box\, \uparrow\mathbf{R})/G_\vartheta$ has a normed directed homology, isomorphic to the normed ordered abelian group $\uparrow G_\vartheta$

(1) $\mathsf{N}\uparrow H_1((\mathsf{N} \,\Box\, \uparrow\mathbf{R})/G_\vartheta) = \uparrow G_\vartheta = \uparrow(\Sigma_j \vartheta_j \mathbf{Z})$ $\qquad (G_\vartheta^+ = G_\vartheta \cap \mathbf{R}^+)$.

8. Similar models by equilogical and inequilogical spaces [G3, G4]

After introducing singular homology for D. Scott's *equilogical spaces* [Sc], we show how these structures can express 'formal quotients' of topological spaces, which do not exist as ordinary spaces and are related with well-known noncommutative C*-algebras. This study also uses a wider notion of *local maps* between equilogical spaces, which might be of interest for the general theory of the latter.

8.1. Equilogical spaces [Sc; G3, 1.1-1.2]

- Equilogical spaces are sort of 'formal quotients' of topological spaces.

- An *equilogical space* $X = (X^\#, \sim_X)$ will be a topological space $X^\#$ provided with an equivalence relation, written $\sim_X$ or $\sim$. The space $X^\#$ will be called the *support* of X, while the quotient $|X| = X^\#/\!\sim$ is the *underlying space* (or *set*, according to convenience). One can think of the object X *as a set* $|X|$ *covered with a chart* $p: X^\# \to |X|$ containing the topological information.

- A *map* of equilogical spaces $f: X \to Y$ (also called an *equivariant mapping* [Sc]) is a mapping $f: |X| \to |Y|$ which admits *some* continuous lifting $f': X^\# \to Y^\#$. It can also be defined as an equivalence class of continuous mappings $f': X^\# \to Y^\#$ *coherent* with the equivalence relations

(1) $\forall\, x, x' \in X: \quad x \sim_X x' \;.\!\Rightarrow.\; f'(x) \sim_Y f'(x')$,

under the associated *pointwise* equivalence relation

(2) $f' \sim f''$ if $(\forall\, x \in X,\; f'(x) \sim_Y f''(x))$.

- The category **Eql** thus obtained contains **Top** as a full subcategory, identifying the space X with the obvious pair $(X, =_X)$. An equilogical space X is isomorphic to a topological space A if and only if A is a retract of X, with a retraction $p: X \to A$ whose equivalence relation is precisely $\sim_X$. We shall see that the new category has relevant new objects.

- The terminal object of **Eql** is the singleton space $\{*\}$. Therefore, a *point* $x: \{*\} \to X$ is an element of the underlying set $|X| = X^\#/\!\sim$ (*not* an element of the support $X^\#$). The (faithful) forgetful functor, with values in **Top** (or in **Set**, when convenient)

(3) $|-|: \mathbf{Eql} \to \mathbf{Top}$, $\qquad |X| = X^\#/\!\sim$,

sends $f: X \to Y$ to the underlying mapping $f: |X| \to |Y|$ (also written $|f|$, to be more precise).

- The 'function' $X \mapsto X^{\#}$ is *not* part of a functor, as it does not preserve isomorphic objects.

Remarks. Equilogical spaces have been introduced in [Sc] using T_0-spaces as supports, so that they can be viewed as subspaces of algebraic lattices with the Scott topology (which is always T_0). The category so obtained - a full subcategory of the category **Eql** we are using here - is generally written as **Equ**. As a relevant, non obvious fact, **Equ** is *cartesian closed* (while **Top** is not): one can define an 'internal hom' Z^Y satisfying the exponential law **Equ**$(X{\times}Y, Z) =$ **Equ**(X, Z^Y); this has been proved in [BBS]; for other references, see [G3].

- Here, we prefer to drop the condition T_0, *so that every topological space be an equilogical one*. The category **Eql** can be obtained from **Top** by a general construction, as its *regular completion* $\mathbf{Top}_{\mathrm{reg}}$ [CV]. This fact can be used to prove that also **Eql** is cartesian closed [Rs, p. 161].

- Cartesian closedness is crucial in the theory of data types, where equilogical spaces originated; but here it only plays a marginal role: we are essentially interested in the (easy) fact that the path space $X^{\mathbf{I}}$ exists in **Eql**, and coincides with the topological one when X is in **Top** [G3, 1.5].

8.2. Limits [BBS; G3, 1.3]

- The category **Eql** has all limits and colimits.

- The construction of products and sums is obvious: a product ΠX_i is the product of the supports $X_i^{\#}$, equipped with the product of all equivalence relations; a sum (or coproduct) ΣX_i is the sum of the spaces $X_i^{\#}$, with the sum of their equivalences.

- Now, take two maps $f, g \colon X \to Y$. For their equaliser $E = (E^{\#}, \sim)$, take first the (set-theoretical or topological) equaliser E_0 of the underlying mappings $f, g \colon |X| \to |Y|$; then, the space $E^{\#}$ is the counterimage of E_0 in $X^{\#}$, with the restricted topology and equivalence relation; the map $E \to X$ is induced by the inclusion $E^{\#} \to X^{\#}$.

- For the coequaliser C of the same maps, let us begin forming the set-theoretical coequaliser of the underlying mappings $f, g \colon |X| \to |Y|$: it is a quotient $|Y|/R$, which can be rewritten as $Y^{\#}/{\sim_C}$ by a suitable equivalence relation coarser than $\sim_Y$ (namely, the counterimage of R along the projection $Y^{\#} \to |Y| = Y/{\sim_Y}$). Then $C = (Y^{\#}, \sim_C)$, with the map $Y \to C$ induced by the identity of $Y^{\#}$ (and represented by the canonical projection $|Y| \to |C|$). Notice that coequalisers in **Top** (whose disagreement with products precludes cartesian closedness) are not used.

- An (equilogical) *subspace*, or regular subobject $A = (A^{\#}, \sim)$ of X is a topological subspace $A^{\#} \subset X^{\#}$ *saturated* with respect to $\sim_X$, with the restricted equivalence relation. The order relation $A \subset B$ (of regular subobjects) amounts to $A^{\#} \subset B^{\#}$, or equivalently to $|A| \subset |B|$. We say that the equilogical subspace A is *open* (resp. *closed*) in X if $A^{\#}$ is open (resp. closed) in $X^{\#}$; or, equivalently, if the underlying set $|A|$ is open (resp. closed) in $|X|$.

- An (equilogical) *quotient*, or regular quotient of X is the space $X^{\#}$ itself, equipped with a *coarser* equivalence relation. A map $f \colon X \to Y$ has a canonical factorisation through its *coimage* (a quotient of X) and its *image* (a subspace of Y)

(1) $\quad X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y,$

where $\mathrm{Coim}(f) = (X^{\#}, R)$ is determined by the equivalence relation associated to the composed mapping $X^{\#} \to |Y|$, while $(\mathrm{Im}(f))^{\#}$ is the counterimage of $f(|X|)$ in $Y^{\#}$.

- The (faithful) forgetful functor $|-|$: **Eql** → **Top** (8.1.3) is left adjoint to the (full) embedding **Top** ⊂ **Eql**

(2) $\mathbf{Top}(|X|, T) = \mathbf{Eql}(X, T)$ (X in **Eql**; T in **Top**),

since every map $|X| \to T$ can be lifted to $X^{\#}$. The left adjoint $|-|$ preserves colimits (obviously) and equalisers, but not products, while the embedding **Top** ⊂ **Eql** preserves limits (obviously) and sums, but not coequalisers (see 8.3).

8.3. Equilogical circles and spheres [G3, 1.4]

- The category **Eql** has various (non isomorphic) *models of the circle*, i.e., objects whose associated space is homeomorphic to $\mathbf{S}^1$. Similar facts happen with other structures of common use in algebraic topology: simplicial complexes, simplicial sets, cubical sets. We will see that the models we consider here are equivalent up to 'local homotopy' (8.4).

- First, we have the *topological circle* itself: $\mathbf{S}^1$ is the coequaliser *in* **Top** of the faces of $\mathbf{I} = [0, 1]$

(1) $\partial^\alpha: \{*\} \rightrightarrows \mathbf{I}, \qquad \partial^\alpha(*) = \alpha$ ($\alpha = 0, 1$),

and represents loops in **Top** (as maps $\mathbf{S}^1 \to X$); it also lives in **Eql**.

- But the coequaliser *in* **Eql** of the faces of the interval is produced by the equivalence relation $R_{\partial \mathbf{I}}$ which identifies the endpoints

(2) $\mathbf{S}^1_e = (\mathbf{I}, R_{\partial \mathbf{I}})$, the *standard equilogical circle*;

(R_A will often denote the equivalence relation which identifies the points of a subset A).

- A third model is the orbit quotient of the action of the group **Z** on the line, in **Eql**

(3) $\bar{\mathbf{S}}^1_e = (\mathbf{R}, \equiv_{\mathbf{Z}})$.

- Finally, we consider a sequence of models

(4) $C_k = (k\mathbf{I}, R_k)$, the k-*gonal equilogical circle*,

where $k\mathbf{I} = \mathbf{I} + ... + \mathbf{I}$ (the sum of k copies) and R_k is the equivalence relation identifying the terminal point of any addendum with the initial point of the following one, circularly. *This can be pictured as a polygon* for $k \geq 3$; but the definition makes sense for $k \geq 1$, and $C_1 = \mathbf{S}^1_e$.

- There are obvious maps

(5) $... \to C_{k+1} \to C_k \to ... \to C_2 \to C_1 = \mathbf{S}^1_e \to \bar{\mathbf{S}}^1_e \to \mathbf{S}^1$

(where $C_{k+1} \to C_k$ collapses the last 'edge'); their underlying map is (at least) a homotopy equivalence. But it is easy to see that any morphism in the opposite direction has an underlying map which is homotopically trivial. This situation will be further analysed below (8.4).

- Similarly, in dimension $n > 0$, we have the topological n-sphere $\mathbf{S}^n$ and

(6) $\mathbf{S}^n_e = (\mathbf{I}^n, R_{\partial \mathbf{I}^n})$ (the *standard equilogical n-sphere*),

(7) $\bar{\mathbf{S}}^n_e = (\mathbf{R}^n, \sim_n)$,

where the equivalence relation $\sim_n$ is generated by the congruence modulo $\mathbf{Z}^n$ and by identifying all points $(t_1,..., t_n)$ where at least one coordinate belongs to **Z**.

- Of course, $\mathbf{S}^0 = \mathbf{S}^0_e = (\{0, 1\}, =)$ has the discrete topology. All the standard equilogical spheres are pointed suspensions of $\mathbf{S}^0$ [G3, 1.6].

8.4. Local maps [G3; 2.1, 2.3, 2.4]

- An important feature of topology is the *local character* of continuity: a mapping between two spaces is continuous if and only if it is on a convenient neighbourhood of every point. This local character fails in **Eql**: for instance, the canonical map $(\mathbf{R}, \equiv_\mathbf{Z}) \to \mathbf{S}^1$ has a *topological inverse* $\mathbf{S}^1 \to \mathbf{R}/\equiv_\mathbf{Z}$ which cannot be lifted to a map $\mathbf{S}^1 \to \mathbf{R}$, even though it can be *locally lifted*.

- This suggests us to extend **Eql** to the category **EqL** of equilogical spaces and *locally liftable mappings*, or *local maps*. A *local map* f: X ⇸ Y (the arrow is marked with a dot) is a mapping f: $|X| \to |Y|$ between the underlying sets which admits an *open saturated cover* $(U_i)_{i\in I}$ of the space $X^\#$ (by open subsets, saturated for $\sim_X$), so that the mapping f has a partial (continuous) lifting f_i: $U_i \to Y^\#$, for all i

(1) $f[x] = [f_i(x)]$, for $x \in U_i$ and $i \in I$.

- Equivalently, for every point $[x] \in |X|$, the mapping f restricts to a map of equilogical spaces on a suitable saturated neighbourhood U of x in $X^\#$. The previous remark on the local character of continuity in **Top** has two consequences: the embedding **Top** ⊂ **EqL** is (still) *full* and *reflective*, with reflector (left adjoint) $|-|$: **EqL** → **Top**.

- *Finite* limits and *arbitrary* colimits of **Eql** (as constructed in 8.2) still 'work' in the extension.

- A *local isomorphism* will be an isomorphism of **EqL**; a *local path* will be a local map **I** ⇸ X; a *local homotopy* will be a local map X×**I** ⇸ Y, etc. Items of **Eql** will be called *global* (or *elementary*, for paths) when we want to distinguish them from the corresponding local ones.

- Coming back to our models of the circle (8.3), the canonical map $\bar{\mathbf{S}}^1_e = (\mathbf{R}, \equiv_\mathbf{Z}) \to \mathbf{S}^1$ is easily seen to be *locally invertible*, and these models are locally isomorphic. This is not true, in the strict sense, of the canonical map p: $\mathbf{S}^1_e \to \bar{\mathbf{S}}^1_e$: the topological inverse $\mathbf{R}/\mathbf{Z} \to \mathbf{I}/\partial\mathbf{I}$ cannot be locally lifted at [0]; but we see below that a *local inverse up to homotopy* exists (8.4).

- The fact that $\mathbf{S}^1_e$ and $\mathbf{S}^1$ be not locally isomorphic *can be interpreted* viewing $\mathbf{S}^1_e = (\mathbf{I}, R_{\partial\mathbf{I}})$ as a 'circle with a corner point' (at [0]), which elementary paths are not allowed to cross; similarly, C_k would have k corner points. Thus, elementary paths are able to capture properties of equilogical spaces which can be of interest, but are missed by local paths, fundamental groups [G3, 2.6] and singular homology [G3, 3.7], as well as by any functor invariant up to local homotopy.

- Also in higher dimension, the canonical map $(\mathbf{R}^n, \sim_n) \to \mathbf{S}^n$ is locally invertible, while this is not true, in the strict sense, for $(\mathbf{I}^n, R_{\partial\mathbf{I}^n}) \to \mathbf{S}^n$ $(n > 0)$.

Proposition [Local homotopy equivalences of spheres]. All the canonical maps linking the models of the circle (8.3.5)

(2) $\quad \ldots \to C_{k+1} \to C_k \to \ldots \to C_2 \to C_1 = \mathbf{S}^1_e \to \bar{\mathbf{S}}^1_e \to \mathbf{S}^1$

are *local homotopy equivalences*. The same holds for the higher spheres

(3) $\bar{\mathbf{S}}^n_e \to \mathbf{S}^n_e \to \mathbf{S}^n$ $\qquad (\bar{\mathbf{S}}^n_e = (\mathbf{R}^n, \sim_n), \quad \mathbf{S}^n_e = (\mathbf{I}^n, R_{\partial\mathbf{I}^n}))$.

8.5. Singular homology of equilogical spaces [G3, Section 3]

- Singular homology can be easily extended to equilogical spaces and used to study the new objects. Less trivially, we prove that singular homology can be equivalently computed by *local cubes* and deduce that it is also invariant under *local* homotopy equivalence.

- An *equilogical* space X has a *cubical set of singular cubes* $\square X$, constructed in the category **Eql**

(1) $\square X = ((\square_n X), (\partial_i^\alpha), (e_i)),$ $\square_n X = \mathbf{Eql}(\mathbf{I}^n, X) = (\square_n X^\#)/\sim_n.$

- Therefore, a cube $\mathbf{I}^n \to X$ is a mapping $\mathbf{I}^n \to |X|$ which can be (continuously) lifted to $X^\#$; or also an equivalence class in the quotient of the set $\square_n(X^\#) = \mathbf{Top}(\mathbf{I}^n, X^\#)$ (the n-cubes of the support $X^\#$), modulo the associated equivalence relation $\sim_n$ obtained by projecting such cubes along the canonical projection $X^\# \to |X| = X^\#/\sim$.

- We have defined in (1) a canonical embedding $\square: \mathbf{Eql} \to \mathbf{Cub}$, acting on a map $f: X \to Y$ of equilogical spaces in the obvious way

(2) $(\square_n f): \square_n X \to \square_n Y,$ $(\square_n f)(a) = f \circ a$ (for $a: \mathbf{I}^n \to X$).

- This embedding produces the (normalised) *singular chain complex* of equilogical spaces and their *singular homology*:

(3) $C_*: \mathbf{Eql} \to C_*\mathbf{Ab},$ $C_*(X) = C_*(\square X),$

$H_n: \mathbf{Eql} \to \mathbf{Ab},$ $H_n(X) = H_n(\square X) = H_n(C_*(X)),$

which extends the singular homology of topological spaces; but we shall see that $H_n(X)$ does *not* reduce to the homology of the underlying space, $H_n(|X|)$.

- Using the wider category **EqL** of local maps (8.4), we have the *local cubes* $a: \mathbf{I}^n \twoheadrightarrow X$, the complex of *local chains* $CL_*(X)$ and the *local homology* groups $HL_n(X)$

(4) $\square L_n X = \mathbf{EqL}(\mathbf{I}^n, X),$ $CL_*(X) = C_*(\square LX),$

$HL_n: \mathbf{EqL} \to \mathbf{Ab},$ $HL_n(X) = H_n(CL_*(X)).$

- One can prove that $HL_n(X)$ *always coincides with the global homology* $H_n(X)$ [G3, Thm. 3.5]. Then, the classical (cubical) proof of homotopy invariance (1.7) can be extended to show that:

Theorem [Homotopy Invariance]

Homotopic maps of equilogical spaces induce the same homomorphisms in homology. The same holds for *local homotopy* and *local homology*. Because of the coincidence previously recalled, *global homology is also invariant for local homotopy*.

8.6. Actions of groups [G3, 4.1-4.3]

- Let X be a topological space and G a group acting on it. Under appropriate hypotheses, the *orbit cubical set* $(\square X)/G$ used above can be replaced with the *orbit equilogical space* $(X, \equiv_G)$.

- We say that G acts *pathwise freely* on X if, whenever two paths $a, b: \mathbf{I} \to X$ have the same projection to the orbit space X/G, there is precisely one $g \in G$ such that $a = b + g$. Then, the same works for all pairs of n-cubes $a, b: \mathbf{I}^n \to X$, and the 0-dimensional case shows that the action is free. On the other hand, a proper action is always pathwise free.

- If the action is pathwise free and the space X is acyclic, then the canonical surjection $(\Box X)/G \to \Box(X, \equiv_G) = (\Box X)/(\equiv_G)_*$ is a bijection and (applying 5.4) we have:

(1) $H_*(X, \equiv_G) = H_*((\Box X)/G) \cong H_*(G)$.

(a) Various *pathwise free* (non proper) actions will be obtained as follows: the space X is an (additive) topological group and G is a *totally disconnected* subgroup, acting on X by translations x+g. Indeed, if the paths $a, b: \mathbf{I} \to X$ have the same projection to X/G, their difference $a - b: \mathbf{I} \to G$ must be constant. (And the action is proper if and only if G is discrete.)

(b) As an example of *a free action which is not pathwise free*, take a (non trivial) group G acting on its underlying set X, equipped with the coarse topology.

8.7. Equilogical spaces and irrational rotations [G3, 4.5]

- The group $G_\vartheta \subset \mathbf{R}$ is totally disconnected, so that its action on the line is also *pathwise free* (8.6) and the homology of the orbit equilogical space $(\mathbf{R}, \equiv_{G_\vartheta})$ is

(1) $H_*(\mathbf{R}, \equiv_{G_\vartheta}) = H_*((\Box \mathbf{R})/G_\vartheta) \cong H_*(\mathbf{T}^2)$.

- Two generators $[a], [b] \in H_1(\mathbf{R}, \equiv_{G_\vartheta}) \cong \mathbf{Z}^2$ and a generator $[A] \in H_2(\mathbf{R}, \equiv_{G_\vartheta}) \cong \mathbf{Z}$ are given by the following cycles (as it follows from 6.7, or from its proof for the second case)

(2) $a, b: \mathbf{I} \to \mathbf{R}$, $\qquad a(t) = t, \quad b(t) = \vartheta t$,

(3) $A: [0, 1]^2 \to \mathbf{R}/G_\vartheta$, $\qquad A(t, t') = t\vartheta + t'$;

this yields a sort of 'homological correspondence' between the virtual space $(\mathbf{R}, \equiv_{G_\vartheta})$ and the torus $\mathbf{T}^2$, together with some geometric intuition of the former.

- *Algebraically*, all this is in accord with the 'interpretation' of $\mathbf{R}/G_\vartheta$ as the C*-algebra A_ϑ, which has the same K-theory groups as the torus; but note that here we lose the order information, and we cannot recover ϑ, at any extent. This can be obtained *enriching equilogical spaces with preorders*.

8.8. Inequilogical spaces [G4]

- The new category is built on the category p**Top** of preordered topological spaces (and preorder-preserving maps), in the same way as the category of equilogical spaces is built on **Top**.

- An *inequilogical space*, or *preordered equilogical space* $X = (X^\#, \sim_X)$ will be a *preordered* topological space $X^\#$ endowed with an equivalence relation $\sim_X$ (or $\sim$); the preorder relation will generally be written as $\prec_X$. The quotient $|X| = X^\#/\sim$ will be viewed as a preordered topological space (with the induced preorder and topology), or a topological space, or a set, as convenient. A *map* $f: X \to Y$ 'is' a mapping $f: |X| \to |Y|$ which admits some *continuous preorder-preserving* lifting $f': X^\# \to Y^\#$. Equivalently, a map is an equivalence class of maps f' in p**Top** which respect the equivalence relations (8.1.1), under the equivalence relation $f' \sim f''$ (8.1.2). Note that there are *no mutual conditions* among topology, preorder and equivalence relation.

- This category will be denoted as p**Eql**. The forgetful functor

(1) $|-|: \mathrm{p}\mathbf{Eql} \to \mathrm{p}\mathbf{Top}$, $\qquad |X| = X^\#/\sim_X$,

with values in preordered topological spaces (or spaces, or sets, *when convenient*) sends the map f: $X \to Y$ to the underlying mapping $f: |X| \to |Y|$ (also written $|f|$). A *point* $x: \{*\} \to X$ is an element of the *underlying space* $|X|$.

- The following embeddings will be viewed as *inclusions* ($\approx$ is again the chaotic preorder on a set)

$$(2) \qquad \begin{array}{ccc} \mathbf{Top} & \xrightarrow{J_2} & \mathrm{p}\mathbf{Top} \\ {\scriptstyle J_1}\downarrow & & \downarrow{\scriptstyle J_3} \\ \mathbf{Eql} & \xrightarrow[J_4]{} & \mathrm{p}\mathbf{Eql} \end{array} \qquad \begin{array}{l} J_1(T) = (T, =_T), \qquad J_2(T) = (T, \approx_T), \\ J_3(T, \prec) = (T, \prec, =_T), \\ J_4(T, \sim) = (T, \approx_T, \sim). \end{array}$$

8.9. Directed homology of inequilogical spaces [G4, 3.2]

- Now, an inequilogical space X (on a *preordered* space $X^\# = (T, \prec)$) has a *singular cubical set*

(1) $\quad \square: \mathrm{p}\mathbf{Eql} \to \mathbf{Cub}, \qquad\qquad \square_n X = \mathrm{p}\mathbf{Eql}(\uparrow\mathbf{I}^n, X) = (\square_n X^\#)/\sim_n,$

whose n-component 'is' the quotient of $\square_n X^\# = \mathrm{p}\mathbf{Top}(\uparrow\mathbf{I}^n, X^\#)$ modulo the equivalence relation $\sim_n$ obtained by projecting cubes along the canonical projection $X^\# \to |X| = X^\#/\sim$. Notice that $\square X$ is a subobject of the cubical set of the underlying *equi*logical space $(T, \sim)$

(2) $\quad \square_n X \subset \square_n(T, \sim) = \mathbf{Eql}(\mathbf{I}^n, (T, \sim)).$

- This canonical embedding of p**Eql** in **Cub** defines the singular homology of an inequilogical space, again as a sequence of *preordered* abelian groups:

(3) $\quad \uparrow H_n: \mathrm{p}\mathbf{Eql} \to \mathrm{d}\mathbf{Ab}, \qquad\qquad \uparrow H_n(X) = \uparrow H_n(\square X),$

and a p**Eql**-map induces preorder-preserving homomorphisms. This functor is homotopy invariant.

- If X is an *equi*logical space (with the coarse preorder), the cubical set $\square X$ is precisely the one already considered in 8.5, and the singular homology groups are - algebraically - the same, while their preorder is likely of no interest.

- In the general case, the groups $\uparrow H_n(X)$ *can differ* - even algebraically - from the groups $H_n(T, \sim)$ of the underlying equilogical space; as a trivial example, if the preorder $\prec_X$ is discrete (the equality), all directed cubes $\uparrow\mathbf{I}^n \to X$ are constant and $\uparrow H_n(X) = 0$ for $n > 0$.

8.10. Classification of the inequilogical spaces of irrational rotation [G4, Section 4]

- The *irrational rotation* inequilogical space:

(1) $\quad C'_\vartheta = (\uparrow\mathbf{R}, \equiv_{G_\vartheta}) = (\mathbf{R}, \leq, \equiv_{G_\vartheta}),$

- Since G_ϑ is a totally disconnected subgroup of $\mathbf{R}$, and its action on the (ordered) line is pathwise free, the directed homolgy of C'_ϑ coincides with the one of the cubical set $C_\vartheta = (\square\uparrow\mathbf{R})/G_\vartheta$

(2) $\quad \uparrow H_1(C'_\vartheta) = \uparrow H_1((\square\uparrow\mathbf{R})/G_\vartheta) \cong \uparrow G_\vartheta.$

- It follows easily:

Classification Theorem (For the inequilogical spaces of irrational rotation)

The inequilogical spaces C'_ϑ have the same classification up to isomorphism as the cubical sets C_ϑ (6.8). $\square$

- More generally, let us take (as in 6.4 and 7.9) an n-tuple of real numbers $\vartheta = (\vartheta_1,..., \vartheta_n)$, linearly independent on **Q**, and consider the additive subgroup $G_\vartheta = \Sigma_j \vartheta_j\mathbf{Z} \cong \mathbf{Z}^n$ of the line. Again, the (totally disconnected) group G_ϑ acts pathwise freely on $\uparrow\mathbf{R}$ and on the cubical set $\Box\uparrow\mathbf{R}$, whence

(3) $\uparrow H_1(\uparrow\mathbf{R}, \equiv_{G_\vartheta}) = \uparrow H_1((\Box\uparrow\mathbf{R})/G_\vartheta) = \uparrow G_\vartheta = \uparrow(\Sigma_j \vartheta_j\mathbf{Z})$.

- The classification theorem can also be extended to this case.

8.11. Inequilogical realisation [G4, 3.7]

- We end with remarking that a cubical set K can be given a realisation in p**Eql** which, in contrast with the classical geometric realisation $\mathcal{R}(K)$ in **Top** (recalled in 3.3), does not lose priviliged directions. The *inequilogical realisation* is the functor

(1) $\uparrow\mathcal{E}$: **Cub** → p**Eql**, $\uparrow\mathcal{E}(K) = (\Sigma_x \uparrow\mathbf{I}^{n(x)}, \sim)$,

left adjoint to $\Box$: p**Eql** → **Cub** (8.9.1). As in the geometric realisation $\mathcal{R}(K) = (\Sigma_x \mathbf{I}^{n(x)})/\sim$ (3.3), the sum is indexed on the cubes x of K, and the equivalence relation ~ is the same (see 3.3.3); thus, the geometric realisation is precisely the topological space underlying the inequilogical realisation.

- It is easy to prove that $\uparrow\mathcal{E}(K\otimes L) \cong \uparrow\mathcal{E}(K)\times\uparrow\mathcal{E}(L)$, putting together the following facts:

(a) this is obviously true when $K = \uparrow\mathbf{i}^m$, $L = \uparrow\mathbf{i}^n$ and $K\otimes L = \uparrow\mathbf{i}^{m+n}$ (representable presheaves),

(b) each cubical set is a colimit of representable ones (say $K = \mathrm{colim}_x \uparrow\mathbf{i}^{n(x)}$, $L = \mathrm{colim}_y \uparrow\mathbf{i}^{n(y)}$),

(c) left adjoints preserve colimits,

(3) $\uparrow\mathcal{E}(K\otimes L) = \uparrow\mathcal{E}((\mathrm{colim}_x \uparrow\mathbf{i}^{n(x)}) \otimes (\mathrm{colim}_y \uparrow\mathbf{i}^{n(y)})) = \uparrow\mathcal{E}(\mathrm{colim}_{xy} (\uparrow\mathbf{i}^{n(x)}\otimes\uparrow\mathbf{i}^{n(y)}))$
$= \mathrm{colim}_{xy} (\uparrow\mathcal{E}(\uparrow\mathbf{i}^{n(x)}\otimes\uparrow\mathbf{i}^{n(y)})) = \mathrm{colim}_{xy} (\uparrow\mathcal{E}(\uparrow\mathbf{i}^{n(x)})\times\uparrow\mathcal{E}(\uparrow\mathbf{i}^{n(y)}))$
$= (\mathrm{colim}_x \uparrow\mathcal{E}(\uparrow\mathbf{i}^{n(x)})) \times (\mathrm{colim}_y \uparrow\mathcal{E}(\uparrow\mathbf{i}^{n(y)})) = \uparrow\mathcal{E}(K)\times\uparrow\mathcal{E}(L)$. □

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