[Math. Proc. Cambridge Philos. Soc. **138** (2005), 233-262.]
REVISED PREPRINT VERSION

Directed combinatorial homology and noncommutative tori (*)

(The breaking of symmetries in algebraic topology)

Marco Grandis

Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, 16146-Genova, Italy.
e-mail: grandis@dima.unige.it

Abstract. This is a brief study of the homology of cubical sets, with two main purposes.

First, this combinatorial structure is viewed as representing *directed spaces*, breaking the intrinsic symmetries of topological spaces. Cubical sets have a *directed homology*, consisting of preordered abelian groups where the positive cone comes from the structural cubes.

But cubical sets can also express topological facts missed by ordinary topology. This happens, for instance, in the study of group actions or foliations, where a topologically-trivial quotient (the orbit set or the set of leaves) can be enriched with a natural cubical structure whose directed cohomology agrees with Connes' analysis in noncommutative geometry. Thus, cubical sets can provide a sort of 'noncommutative topology', without the metric information of C^*-algebras.

MSC: 55U10, 55Nxx, 81R60.

Keywords: Cubical sets, combinatorial homology, noncommutative spaces, directed algebraic topology.

Introduction

A topological space T has *intrinsic symmetries*, appearing - at the lowest level - in the reversion of its paths. More generally, the set $\Delta_n T = \mathbf{Top}(\Delta^n, T)$ of its singular simplices inherits from the standard simplex Δ^n an obvious action of the symmetric group S_{n+1}, while the set $\square_n T = \mathbf{Top}([0, 1]^n, T)$ of its singular cubes has a similar action of the hyperoctahedral group (the group of symmetries of the n-cube). These combinatorial structures produce the singular homology of the space T, which can be equivalently defined as the homology of the chain complex associated to the simplicial set ΔT, or the homology of the (normalised) chain complex associated to the cubical set $\square T$. The less usual cubical approach (followed for instance in Massey's text [Ms]) has advantages, mainly due to the fact that cubes are closed under products, while products of tetrahedra have to be 'covered' with tetrahedra; thus, the proof of homotopy invariance and the study of products or fibrations [Se] are easier and more natural in the cubical setting, which we shall follow here. Here, a more specific motivation for this choice is our use of the natural order on $\mathbf{I}^n$ (cf. the last remark in 4.2).

(*) Work supported by MIUR Research Projects.

Now, *bypassing topological spaces*, an abstract cubical set X is a merely combinatorial structure, consisting of a sequence of sets X_n, with faces $\partial_i^\alpha: X_n \to X_{n-1}$ and degeneracies $e_i: X_{n-1} \to X_n$ ($\alpha = \pm$; $i = 1,..., n$)

$$(1) \qquad X_0 \; \underset{e}{\overset{\partial}{\leftleftarrows\!\!\rightarrow}} \; X_1 \; \underset{e}{\overset{\partial}{\begin{smallmatrix}\leftarrow\\ \leftarrow\\ \leftarrow\\ \leftarrow\\ \rightarrow\\ \rightarrow\end{smallmatrix}}} \; X_2 \; ...$$

satisfying the well-known cubical relations (1.2). This structure will be used in two ways: to break the symmetries considered above and to perform constructions, namely quotients, which would be useless in ordinary topology.

For the first aspect, note that an 'edge' in X_1 need not have any counterpart with reversed vertices, nor a 'square' in X_2 any counterpart with horizontal and vertical faces interchanged. Thus, our structure has 'privileged directions', in any dimension, and the (usual) combinatorial homology of X can be given a preorder, generated by taking the given cubes as positive. For instance, the obvious cubical model $\uparrow\mathbf{s}^n$ of the n-dimensional sphere, with one non-degenerate cube in dimension n, has *directed homology* $\uparrow H_n(\uparrow\mathbf{s}^n)$ consisting of the group of integers, *with the natural order*; the positive generator, of course, is the homology class of the generator of our cubical set (2.3). Direction should not be confused with orientation, as shown by the model $\uparrow\mathbf{t}^2 = \uparrow\mathbf{s}^1 \otimes \uparrow\mathbf{s}^1$ of the torus, where $\uparrow H_1(\uparrow\mathbf{t}^2) \cong \uparrow\mathbf{Z}^2$ has the *product order* (2.9). Note also that our preorder becomes trivial (chaotic, or coarse) for a 'symmetric' cubical set, like the singular cubical set of a topological space.

Secondly, it may happen that the quotient $T/\!\sim$ of a topological space has a trivial topology, while the corresponding quotient of its singular cubical set $\Box T$ keeps a relevant topological information, detected by its homology and agreeing with the interpretation of such quotients in noncommutative geometry. These links, briefly explored here, should be further clarified.

Let us start from the classical results on the homology of an orbit space T/G, for a group G acting *properly* on a space T; these results can be extended to *free* actions if we replace T with its singular cubical set and take the *quotient cubical set* $(\Box T)/G$ (Thm. 3.3). Thus, for the group $G_\vartheta = \mathbf{Z}+\vartheta\mathbf{Z}$ (ϑ irrational), the orbit space $\mathbf{R}/G_\vartheta$ has a *trivial topology* (the coarse one), but can be replaced with a non-trivial cubical set, $X = (\Box\mathbf{R})/G_\vartheta$, whose homology is the same as the homology of the group $G_\vartheta \cong \mathbf{Z}^2$, and coincides thus with the homology of the torus $\mathbf{T}^2$ (4.2.1). The same can be done for the Kronecker foliation of the torus (with slope ϑ), replacing a topologically trivial set of leaves $\mathbf{T}^2_\vartheta$ with a non-trivial cubical set, obtained as a quotient of the singular cubical set of the torus. *Algebraically*, all this is in accord with Connes' interpretation of $\mathbf{R}/G_\vartheta$ and $\mathbf{T}^2_\vartheta$ as a 'noncommutative space', i.e. a noncommutative C*-algebra A_ϑ [C1, C2, C3, R1, Bl]; however, our $\uparrow H_n(\mathbf{T}^2_\vartheta)$ has a trivial preorder, for $n > 0$.

But this similarity can be enhanced. The quotient $(\Box\mathbf{R})/G_\vartheta$ can be modified, replacing $\Box\mathbf{R}$ with the cubical set $\uparrow\mathbf{R}$ of all *order-preserving* maps $\mathbf{I}^n \to \mathbf{R}$. Algebraically, the homology groups are unchanged (and independent of ϑ), but now $\uparrow H_1(\uparrow\mathbf{R}/G_\vartheta) \cong \uparrow G_\vartheta$ as a (totally) ordered subgroup of $\mathbf{R}$ (Thm. 4.8): thus the *rotation cubical sets* $C_\vartheta = \uparrow\mathbf{R}/G_\vartheta$ have the same classification *up to isomorphism* (Thm. 4.9) as the rotation C*-algebras A_ϑ *up to strong Morita equivalence* [PV, R1] (cf. 4.1): ϑ is determined up to the action of the group $PGL(2, \mathbf{Z})$. This example shows that the ordering of directed homology can carry a relevant information. Further, comparison with the stricter classification of the algebras A_ϑ *up to isomorphism* (4.1) shows that cubical sets provide a sort of 'noncommutative topology', without the metric character of noncommutative geometry (cf. 4.2).

The reader can have a quick overview of these motivations, reading 2.9 (cubical tori) and 4.1-4.3 (rotation structures and foliations of tori); Section 4 contains other results on higher dimensional tori.

'Directed algebraic topology' is a recent field, whose present applications deal mainly with concurrency [GG, Go, Ra]; other references can be found in two previous works on directed homotopy [G4, G5]. Cubical sets are more present in the literature, if less than the simplicial ones. Cubical singular homology of topological spaces can be found in Massey [Ms] and Hilton-Wylie [HW]. It should be noted that, while the basic structure of faces and degeneracies (used here) can be sufficient for introducing their homology, 'intrinsic' homotopy theory requires more. Works by Brown-Higgins [BH1, BH2] have proved the importance of adding *compositions* and higher degeneracies, called *connections* (see also [To, ABS, An]); the interest of considering also the action of symmetries, generated by *reversions* and *interchanges*, is stressed in various works of the present author [G1, G2] and sketched here, in 1.1. Cubical sets are presheaves, on a category which depends on how much structure we want to consider [GM]. Formal cubical settings of homotopy theory go back to Kan [K1, K2] and his introduction of an abstract cylinder; see Kamps-Porter's book [KP] and its references. A Quillen structure on cubical sets has been recently studied by Jardine [Ja].

As discussed in 6.4, the cubical set $(\square \mathbf{R})/G_\vartheta$ could also be interpreted as an *equilogical space*, in D. Scott's sense [Sc], while $\uparrow\mathbf{R}/G_\vartheta$ would require a more complex setting, in this line. Finally, also *quantales* - a noncommutative version of locales - offer a notion of noncommutative space (see [MP]), which might have interesting links with the present approach.

The author acknowledges useful information from R. Brown and G. Landi.

Outline. Section 1 recalls the basic properties of cubical sets and their homotopies. Their *directed* (co)homology is introduced in Section 2, studying the interaction of preorder with: (preordered) coefficients (2.2), exact sequences (2.4, 2.6), excision (2.6), tensor products (2.7) and cohomology-multiplication (2.8, 2.9). Section 3 studies the action of groups on cubical sets; these results are applied in Section 4 to analyse the second aspect mentioned above: cubical sets related with noncommutative spaces. The last two sections deal with the directed homology of a pointed suspension and the links with our previous works on directed homotopy [G4, G5].

Terminology. **Top** denotes the category of topological spaces and continuous mappings, or *maps*. A homotopy φ between maps $f, g: X \to Y$ is written as $\varphi: f \to g: X \to Y$. A *preorder* relation is reflexive and transitive; it is a (partial) *order* if it is also anti-symmetric. The index α takes values 0, 1, also written –, + (e.g. in superscripts). $\mathbf{I} = [0, 1]$ is the standard euclidean interval. $\uparrow\mathbf{Z}$ is the *ordered* group of integers; *or* also, but exceptionally, the cubical set of the *directed integral line* (1.5). Homology is often written in a *polynomial form*, $H_*(X) = \Sigma\, \sigma^i.H_i(X)$, as explained in 2.9.

1. Cubical sets and elementary homotopy

Cubical sets and their combinatorial homotopies are briefly recalled.

1.1. Topological spaces and symmetries. Let us start considering topological spaces and the standard interval **I** = [0, 1], with a very basic structure consisting of three maps, two *faces* (δ^-, δ^+) and a *degeneracy* (ε), linking it with its 0-th cartesian power, the singleton $\mathbf{I}^0 = \{*\}$

(1) $\delta^\alpha : \{*\} \rightleftarrows \mathbf{I} : \varepsilon,$

$\delta^-(*) = 0, \qquad \delta^+(*) = 1, \qquad \varepsilon(t) = *.$

This is sufficient to produce, for every topological space T, a *cubical set* □T, with components $□_n T = \mathbf{Top}(\mathbf{I}^n, T)$, the set of *singular n-cubes* of T; faces and degeneracies arise (contravariantly) from the faces and degeneracies of the standard cubes $\mathbf{I}^n$ (for $\alpha = 0, 1$; $i = 1,..., n$)

(2) $\delta^\alpha_i = \mathbf{I}^{i-1} \times \delta^\alpha \times \mathbf{I}^{n-i} : \mathbf{I}^{n-1} \to \mathbf{I}^n, \qquad \delta^\alpha_i(t_1,..., t_{n-1}) = (t_1,..., \alpha,..., t_{n-i}),$

$\varepsilon_i = \mathbf{I}^{i-1} \times \varepsilon \times \mathbf{I}^{n-i} : \mathbf{I}^n \to \mathbf{I}^{n-1}, \qquad \varepsilon_i(t_1,..., t_n) = (t_1,..., \hat{t}_i,..., t_n).$

Abstract cubical sets are defined and studied below. But let us note that a cubical set *of the preceding type* □T has actually a much richer, relevant structure, obtained from the structure of the standard interval **I** as an *involutive lattice* in **Top**. Thus, the join and meet operations, reversion and interchange

(3) $\gamma^-: \mathbf{I}^2 \to \mathbf{I}, \qquad \gamma^-(t, t') = \max(t, t'),$

$\gamma^+: \mathbf{I}^2 \to \mathbf{I}, \qquad \gamma^+(t, t') = \min(t, t'),$

$\rho: \mathbf{I} \to \mathbf{I}, \quad \rho(t) = 1-t, \qquad \sigma: \mathbf{I}^2 \to \mathbf{I}^2, \quad \sigma(t, t') = (t', t),$

yield similar transformations between singular cubes of the space T: *connections* (or higher degeneracies), *reversions* and *interchanges* (for $\alpha = 0, 1$; $i = 1,..., n$)

(4) $g^\alpha_i: □_n T \to □_{n+1} T, \qquad r_i: □_n T \to □_n T, \qquad s_i: □_{n+1} T \to □_{n+1} T.$

The group of symmetries of the n-cube, $(\mathbf{Z}/2)^n \rtimes S_n$, acts on $□_n T$: reversions and interchanges generate, respectively, the action of the first or second factor of this semidirect product (cf. [GM]). Now, in homotopy theory, reversion (together with connections) yields reverse homotopies and inverses in homotopy groups, while interchange yields the homotopy invariance of the cylinder, cone and suspension endofunctors (cf. [G2]).

On the other hand, *not assigning this additional structure* allows us to break symmetries (reversion and interchange) which are intrinsic to topological spaces.

1.2. Cubical sets. A *cubical set* $X = ((X_n), (\partial^\alpha_i), (e_i))$ is a sequence of sets X_n ($n \geq 0$), together with mappings, called *faces* (∂^α_i) and *degeneracies* (e_i)

(1) $\partial^\alpha_i = \partial^\alpha_{ni}: X_n \to X_{n-1}, \qquad e_i = e_{ni}: X_{n-1} \to X_n \qquad (\alpha = \pm;\ i = 1,..., n).$

satisfying the *cubical* relations

(2) $\partial^\alpha_i.\partial^\beta_j = \partial^\beta_j.\partial^\alpha_{i+1} \quad (j \leq i), \qquad e_j.e_i = e_{i+1}.e_j \quad (j \leq i),$

$\partial^\alpha_i.e_j = e_j.\partial^\alpha_{i-1} \quad (j < i),$ *or* $\mathrm{id} \quad (j = i),$ *or* $e_{j-1}.\partial^\alpha_i \quad (j > i).$

Elements of X_n are called n-*cubes*; *vertices* and *edges* for n = 0 or 1, respectively. Every n-cube $x \in X_n$ has 2^n vertices: $\partial^\alpha_1 \partial^\beta_2 \partial^\gamma_3(x)$ for n = 3.

A *morphism* $f = (f_n): X \to Y$ is a sequence of mappings $f_n: X_n \to Y_n$ commuting with faces and degeneracies. All this forms a category **Cub** which has all limits and colimits and is cartesian closed. (It is the presheaf category of functors $X: \mathbb{I}^{op} \to$ **Set**, where $\mathbb{I}$ is the subcategory of **Set** consisting of the *elementary cubes* 2^n, together with the maps $2^m \to 2^n$ which delete some coordinates and insert some 0's and 1's, without modifying the order of the remaining coordinates [GM]. The cocubical set $\mathbb{I} \to$ **Set** given by the embedding will be written 2^*, since it realises the 'formal n-cube' as 2^n).

The category **Cub** has two involutions (covariant involutive endofunctors), *reflection* and *exchange* (or *transposition* [BH2])

(3) R: **Cub** $\to$ **Cub**, $RX = X^{op} = ((X_n), (\partial_i^{-\alpha}), (e_i))$ (*reflection*),

(4) S: **Cub** $\to$ **Cub**, $SX = ((X_n), (\partial^\alpha_{n+1-i}), (e_{n+1-i}))$ (*exchange*),

the first reversing the 1-dimensional direction, the second the 2-dimensional one.

We say that a cubical set X is *reflexive* if $RX \cong X$ and *symmetric* if $SX \cong X$.

1.3. Subobjects and quotients. A *cubical subset* $Y \subset X$ is a sequence of subsets $Y_n \subset X_n$, stable under faces and degeneracies. An *equivalence relation* $\mathcal{E}$ in X is a cubical subset of $X{\times}X$ whose components $\mathcal{E}_n \subset X_n{\times}X_n$ are equivalence relations; then, the *quotient* $X/\mathcal{E}$ is the sequence of quotient sets $X_n/\mathcal{E}_n$, with induced faces and degeneracies. In particular, for $Y \subset X$, the quotient X/Y has components X_n/Y_n, where all cubes $y \in Y_n$ are identified.

For a cubical set X, we define the *homotopy set*

(1) $\pi_0(X) = X_0/\simeq$,

where $\simeq$ (*connection*) is the equivalence relation in X_0 generated by being vertices of a common edge. The *connected component* of X at an equivalence class $[x] \in \pi_0(X)$ is the cubical subset formed by all cubes of X whose vertices lie in [x]; X is always the sum (or coproduct, disjoint union) of its connected components. If X is not empty, we say that it is *connected* if it has one connected component, or equivalently if $\pi_0(X)$ is a singleton.

One can easily see that the forgetful functor $(-)_0$: **Cub** $\to$ **Set** has a left adjoint, the *discrete* cubical set on a set

(2) D: **Set** $\to$ **Cub**, DS = **Set**$(1^*, S)$,

where components are constant, $(DS)_n = S$ $(n \in \mathbf{N})$, faces and degeneracies are identities. Then, the functor π_0: **Cub** $\to$ **Set** is left adjoint to D. (The forgetful functor $(-)_0$ has also a right adjoint CS = **Set**$(2^*, S)$, the *codiscrete* cubical set on S.)

1.4. Tensor product. The category **Cub** has a monoidal structure [K1, BH2]

(1) $(X \otimes Y)_n = (\Sigma_{p+q=n} X_p{\times}Y_q)/{\sim_n}$,

where $\sim_n$ is the equivalence relation generated by identifying $(e_{r+1}x, y)$ with (x, e_1y), for all $(x, y) \in X_r{\times}Y_s$ (for $r+s = n-1$). Writing $x \otimes y$ the equivalence class of (x, y), faces and degeneracies are defined as

(2) $\partial^\alpha_i(x \otimes y) = (\partial^\alpha_i x) \otimes y$ $(1 \leq i \leq p)$, $\partial^\alpha_i(x \otimes y) = x \otimes (\partial^\alpha_{i-p} y)$ $(p+1 \leq i \leq p+q)$,

(3) $e_i(x\otimes y) = (e_i x)\otimes y \quad (1 \le i \le p+1), \qquad e_i(x\otimes y) = x\otimes(e_{i-p}y) \quad (p+1 \le i \le p+q+1),$

(and $e_{p+1}(x\otimes y) = (e_{p+1}x)\otimes y = x\otimes(e_1 y)$ is well defined precisely because of the previous equivalence relation).

The identity of the tensor product is the singleton $\{*\}$, i.e. the cubical set generated by one 0-dimensional cube; it is reflexive and symmetric. *The tensor product is not symmetric*, but is linked with reversion and exchange as follows

(4) $R(X\otimes Y) = RX\otimes RY, \qquad S(X\otimes Y) \cong (SY)\otimes(SX).$

Therefore, reflexive objects are stable under tensor product while symmetric objects are stable under tensor *powers*: if $SX \cong X$, then $S(X^{\otimes n}) = (SX)^{\otimes n} \cong X^{\otimes n}$.

(The construction of the internal homs will be recalled in 1.6.7.)

1.5. Standard models. The *elementary directed interval* $\uparrow\mathbf{i} = \mathbf{2}$ is freely generated by a 1-cube, u

(1) $0 \xrightarrow{u} 1 \qquad \partial_1^-(u) = 0, \quad \partial_1^+(u) = 1;$

this cubical set is reflexive and symmetric.

The *elementary directed* n*-cube* is its n-th tensor power $\uparrow\mathbf{i}^n = \uparrow\mathbf{i}\otimes...\otimes\uparrow\mathbf{i}$ (for $n \ge 0$), freely generated by one n-cube $u^{\otimes n}$, still reflexive and symmetric. (It is the representable presheaf $y(2^n) = \mathbb{I}(-, 2^n)\colon \mathbb{I}^{op} \to \mathbf{Set}$). The *elementary directed square* $\uparrow\mathbf{i}^2 = \uparrow\mathbf{i}\otimes\uparrow\mathbf{i}$ can be represented as follows, showing the generator $u\otimes u$ and its faces

(2)
$$\begin{array}{ccc} 00 & \xrightarrow{0\otimes u} & 01 \\ {\scriptstyle u\otimes 0}\downarrow & u\otimes u & \downarrow{\scriptstyle u\otimes 1} \\ 10 & \xrightarrow[1\otimes u]{} & 11 \end{array} \qquad\qquad \begin{array}{cc} \bullet & \xrightarrow{2} \\ \downarrow{\scriptstyle 1} & \end{array}$$

where the face $\partial_1^-(u\otimes u) = 0\otimes u$ is drawn orthogonally to direction 1 (and directions are chosen so that the labelling of vertices agrees with matrix indexing). Note that, for each cubical object X, $\mathbf{Cub}(\uparrow\mathbf{i}^n, X) = X_n$.

The *directed (integral) line* $\uparrow\mathbf{Z}$ is generated by (countably many) vertices $n\in\mathbf{Z}$ and edges u_n, from $\partial_1^-(u_n) = n$ to $\partial_1^+(u_n) = n+1$. The *directed integral interval* $\uparrow[i, j]_\mathbf{Z}$ is the obvious cubical subset with vertices in the integral interval $[i, j]_\mathbf{Z}$ (and all cubes whose vertices lie there); in particular, $\uparrow\mathbf{i} = \uparrow[0, 1]_\mathbf{Z}$.

The *elementary directed circle* $\uparrow\mathbf{s}^1$ is generated by one 1-cube u with equal faces

(3) $* \xrightarrow{u} * \qquad \partial_1^-(u) = \partial_1^+(u).$

Similarly, the *elementary directed n-sphere* $\uparrow\mathbf{s}^n$ (for $n > 1$) is generated by one n-cube u all whose faces are totally degenerate (hence equal)

(4) $\partial_i^\alpha(u) = (e_1)^{n-1}(\partial_1^-)^n(u) \qquad (\alpha = \pm;\ i = 1,..., n),$

while $\uparrow\mathbf{s}^0 = \mathbf{s}^0$ is generated by two vertices: it is the discrete cubical set D{0, 1} (1.3.2). The *elementary directed n-torus* is a tensor power of $\uparrow\mathbf{s}^1$

(5) $\uparrow\mathbf{t}^n = (\uparrow\mathbf{s}^1)^{\otimes n}$.

We also consider the *ordered circle* $\uparrow\mathbf{o}^1$, generated by two edges with the same faces (the name is motivated by its realisation as a space with distinguished paths [G4])

(6) $v^- \overset{u'}{\underset{u''}{\rightrightarrows}} v^+$ $\qquad\qquad \partial_1^\alpha(u') = \partial_1^\alpha(u'')$.

and more generally the *ordered spheres* $\uparrow\mathbf{o}^n$, generated by two n-cubes u', u" with the same boundary: $\partial_i^\alpha(u') = \partial_i^\alpha(u'')$. We shall see that, starting from $\mathbf{s}^0$, the *unpointed suspension* provides all $\uparrow\mathbf{o}^n$ (1.7.5) while the *pointed suspension* provides all $\uparrow\mathbf{s}^n$ (5.2.8); of course, these models have the same geometric realisation $\mathbf{S}^n$ (as a topological space) and the same homology; but their *directed* homology is different (2.3). The models $\uparrow\mathbf{s}^n$ are more interesting: for instance, their order in directed homology is not trivial.

All these cubical sets are reflexive and symmetric.

1.6. Elementary directed homotopies. Since the tensor product is not symmetric, the elementary directed interval produces a *left (elementary) cylinder* $\uparrow\mathbf{i}\otimes X$ and a *right cylinder* $X\otimes\uparrow\mathbf{i}$. But each of these functors determines the other, using the exchange S (1.4.4) and the property $S(\uparrow\mathbf{i}) = \uparrow\mathbf{i}$

(1) I: **Cub** → **Cub**, $\qquad IX = \uparrow\mathbf{i}\otimes X$,

SIS: **Cub** → **Cub**, $\qquad SIS(X) = S(\uparrow\mathbf{i}\otimes SX) = X\otimes\uparrow\mathbf{i}$.

Let us begin considering the left cylinder, I. It has two faces and a degeneracy, the following natural transformations

(2) $\partial^\alpha: X \to IX, \qquad \partial^\alpha(x) = \alpha\otimes x \qquad\qquad (\alpha = 0, 1)$,

$e: IX \to X, \qquad e(u\otimes x) = e_1(x)$.

Moreover, I has a right adjoint, the (elementary) *left cocylinder* or *left path* functor, which shifts down all components discarding the faces and degeneracies of index 1 (which are then used to build the faces and degeneracy of P, as natural transformations)

(3) P: **Cub** → **Cub**, $\qquad PY = ((Y_{n+1}), (\partial^\alpha_{i+1}), (e_{i+1}))$,

$\partial^\alpha = \partial^\alpha_1: PY \to Y, \qquad e = e_1: Y \to PY$.

Now, an (elementary) *left homotopy* $f: f^- \to_L f^+: X \to Y$ is defined as a map $f: IX \to Y$ with $f\partial^\alpha = f^\alpha$. Or, equivalently (because of the adjunction), as a map $f: X \to PY$ with $\partial^\alpha f = f^\alpha$. This second expression leads immediately to a simple expression of f as a family of mappings

(4) $f_n: X_n \to Y_{n+1}, \qquad \partial^\alpha_{i+1} f_n = f_{n-1}\, \partial^\alpha_i, \qquad e_{i+1} f_{n-1} = f_n\, e_i$,

$\partial^\alpha_1 f_n = f^\alpha \qquad\qquad (\alpha = \pm;\ i = 1,..., n)$.

Dually, the right cylinder $SIS(X) = X\otimes\uparrow\mathbf{i}$ has a right adjoint SPS, the *right cocylinder* or *right path* functor, which discards the faces and degeneracies of highest index (used again to build the corresponding natural transformations)

(5) $\mathrm{SPS}\colon \mathbf{Cub} \to \mathbf{Cub}$, $\qquad \mathrm{SPS}(Y) \;=\; ((Y_{n+1}), (\partial_i^\alpha), (e_i))$,

$\partial^\alpha\colon \mathrm{SPS}(Y) \to Y$, $\qquad \partial^\alpha \;=\; (\partial_{n+1}^\alpha\colon Y_{n+1} \to Y_n)_{n\geq 0}$,

$e\colon Y \to \mathrm{SPS}(Y)$, $\qquad e \;=\; (e_{n+1}\colon Y_n \to Y_{n+1})_{n\geq 0}$.

An (elementary) *right homotopy* $f\colon f^- \to_R f^+\colon X \to Y$ is a map $f\colon X \to \mathrm{SPS}(Y)$ with faces $\partial^\alpha f = f^\alpha$, i.e. a family (f_n) such that

(6) $f_n\colon X_n \to Y_{n+1}$, $\qquad \partial_i^\alpha f_n \;=\; f_{n-1}\, \partial_i^\alpha$, $\qquad e_i\, f_{n-1} \;=\; f_n\, e_i$,

$\partial_{n+1}^\alpha f_n \;=\; f^\alpha$ $\qquad (\alpha = \pm;\; i = 1,..., n)$.

Elementary homotopies of cubical sets (without connections) are a very defective notion (like intrinsic homotopies of 'face-simplicial' sets, without degeneracies): one cannot even contract the elementary interval $\uparrow\mathbf{i}$ to a vertex (a simple computation on (4) shows that this requires a non-degenerate 2-cube $f(u)$, with the same faces as $g_1^-(u)$ or $g_1^+(u)$ - if connections exist). Moreover, to obtain 'non-elementary' paths, which can be concatenated, and a fundamental category $\uparrow\Pi_1(X)$ one should use, instead of the elementary interval $\uparrow\mathbf{i} = \uparrow[0, 1]_{\mathbf{Z}}$, the *directed integral line* $\uparrow\mathbf{Z}$ (1.5), as in [G3] for simplicial sets: paths are parametrised on $\uparrow\mathbf{Z}$, but eventually constant at left and right, so to have initial and terminal vertices. However, here we are interested in homology, where concatenation is surrogated by formal sums of cubes, and we will restrain ourselves to proving its invariance up to elementary homotopies, right and left. Also, we prefer not to rely on the geometric realisation, which would ignore the directed structure.

The category **Cub** has left and right internal homs, which we shall not need (see [BH2, Ja]). Let us only recall that the *right* internal hom $\mathrm{CUB}(A, Y)$ can be constructed with the *left* cocylinder functor P and its natural transformations (which produce a cubical object P^*Y)

(7) $-\otimes A \;\dashv\; \mathrm{CUB}(A, -)$, $\qquad \mathrm{CUB}_n(A, Y) \;=\; \mathbf{Cub}(A, P^nY)$.

1.7. Cones and suspension. The *left upper cone* C^+X is defined as the first pushout, below

(1)
$$
\begin{array}{ccc}
X & \xrightarrow{\;\partial^+\;} & IX \\
\downarrow & & \downarrow \gamma \\
\{*\} & \xrightarrow[\;v^+\;]{} & C^+X
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \longrightarrow & \{*\} \\
\partial^- \downarrow & & \downarrow v^- \\
IX & \xrightarrow[\;\gamma\;]{} & C^-X
\end{array}
$$

i.e., the quotient $(IX+\{*\})/(\partial^+X+\{*\})$, where the upper basis of the cylinder is collapsed to an upper vertex $v^+ = v^+(*)$, while the lower basis $\partial^-\colon X \to IX \to C^+X$ 'subsists'. Note that $C^+\varnothing = \{*\}$: the cone C^+X is a quotient of the cylinder IX *only* if $X \neq \varnothing$. Dually, the *left lower cone* C^-X is defined as the second pushout, above, obtained by collapsing the lower basis of IX to a lower vertex $v^- = v^-(*)$.

Analytically, we can describe C^+X saying that it is generated by (n+1)-dimensional cubes $u\otimes x \in IX$ $(x\in X_n)$ plus a vertex v^+, under the relations arising from X together with

(2) $1\otimes x \;=\; e_1^n(v^+)$ $\qquad (x\in X_n)$.

Similarly, the *left suspension* ΣX is defined as the colimit of the left diagram

$$\text{(3)}\qquad \begin{array}{ccccc} & & X & \longrightarrow & \{*\} \\ & & \downarrow{\scriptstyle \partial^+} & & \\ X & \xrightarrow{\partial^-} & IX & & \downarrow{\scriptstyle v^+} \\ \downarrow & & & \searrow{\scriptstyle \sigma} & \\ \{*\} & & \xrightarrow[v^-]{\qquad\qquad} & & \Sigma X \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\partial^-} & C^+X \\ {\scriptstyle \partial^+}\downarrow & & \downarrow{\scriptstyle j^+} \\ C^-X & \xrightarrow[j^-]{} & \Sigma X \end{array}$$

obtained by collapsing, independently, the bases of IX to a lower and an upper vertex, v^- and v^+. Equivalently, it is the right-hand pushout, above.

Thus, the suspension of $\mathbf{s}^0 = D\{0, 1\}$ yields the 'ordered circle' $\uparrow\mathbf{o}^1$ (1.5.6)

$$\text{(4)}\qquad \begin{array}{ccc} v^+ & = & v^+ \\ {\scriptstyle u'}\uparrow & & \uparrow{\scriptstyle u''} \\ v^- & = & v^- \end{array} \qquad\qquad u' = \langle 0\otimes u\rangle, \quad u'' = \langle 1\otimes u\rangle,$$

where $\langle - \rangle$ denotes equivalence classes in the pushout (3). More generally

(5) $\Sigma^n(\mathbf{s}^0) = \uparrow\mathbf{o}^n$.

But we are more interested in the *pointed suspension*, which will be studied in Section 5 (and yields the directed spheres $\uparrow\mathbf{s}^n$).

1.8. Geometric realisation. We have already recalled, in 1.1, the functor

(1) $\square : \mathbf{Top} \to \mathbf{Cub}, \qquad \square T = \mathbf{Top}(\mathbf{I}^*, T),$

which assigns to a topological space T the singular cubical set of (continuous) n-cubes $\mathbf{I}^n \to T$, produced by the *cocubical* set of standard cubes $\mathbf{I}^* = ((\mathbf{I}^n), (\delta_i^\alpha), (\varepsilon_i))$ (1.1.2). As for simplicial sets, the *geometric realisation* $\mathcal{R}X$ of a cubical set is given by the left adjoint functor

(2) $\mathbf{Cub} \underset{\square}{\overset{\mathcal{R}}{\rightleftarrows}} \mathbf{Top} \qquad\qquad \mathcal{R} \dashv \square,$

which takes a cubical set X to a topological space, by pasting a copy of the standard cube $\mathbf{I}^n$ for each n-cube $x \in X_n$, along faces and degeneracies. This pasting (formally, the coend of the functor $X.\mathbf{I}^*$: $\mathbb{I}^{op}\times\mathbb{I} \to \mathbf{Top}$) comes with a family of structural mappings, one for each cube x, coherent with faces and degeneracies (of $\mathbf{I}^*$ and X)

(3) $\hat{x}: \mathbf{I}^n \to \mathcal{R}X, \qquad \hat{x}.\delta_i^\alpha = (\partial_i^\alpha x)\hat{}, \quad \hat{x}.\varepsilon_i = (e_i x)\hat{},$

and $\mathcal{R}X$ has the finest topology making all the structural mappings continuous.

This realisation is important, since it is well known that the combinatorial homology of a cubical set X coincides with the homology of the CW-space $\mathcal{R}X$ (cf. [Mu, 4.39], for the simplicial case). But we also want a finer 'directed realisation', keeping information about the privileged cubes of X: we shall use a set equipped with a presheaf of distinguished cubes (1.9); other solutions, by distinguished paths, will be discussed in Section 6.

1.9. Sets with distinguished cubes. Let us introduce the category c**Set** of *sets with distinguished cubes*, or *c-sets*.

An object K is a set equipped with a sub-presheaf c_*K of the cubical set $\mathbf{Set}(\mathbf{I}^*, K)$, such that K is covered by all distinguished cubes. In other words, the structure of the set K consists of a sequence of sets of *distinguished cubes* $c_nK \subset \mathbf{Set}(\mathbf{I}^n, K)$, preserved by faces and degeneracies (of the cocubical set $\mathbf{I}^*$) and satisfying the *covering condition* $K = \bigcup \mathrm{Im}(x)$ (for x varying in the set of all distinguished cubes); the latter amounts to saying that the canonical mapping $p_K: \mathcal{R}(c_*K) \to K$ is surjective. A morphism $f: K \to K'$ is a mapping of sets which preserves distinguished cubes: if $x: \mathbf{I}^n \to K$ is distinguished, also $fx: \mathbf{I}^n \to K'$ is.

Now, the adjunction $\mathcal{R} \dashv \Box$ of geometric realisation (1.8.2) can be factored through c**Set**

$$(1) \quad \mathbf{Cub} \underset{c_*}{\overset{\uparrow\mathcal{R}}{\rightleftarrows}} \mathrm{c}\mathbf{Set} \underset{(-)_\Box}{\overset{t}{\rightleftarrows}} \mathbf{Top}, \qquad\qquad \uparrow\mathcal{R} \dashv c_*, \quad t \dashv (-)_\Box .$$

First, if T is a topological space, its cubes cover the underlying set. Thus, we factor the functor $\Box: \mathbf{Top} \to \mathbf{Cub}$ letting $T_\Box$ be the *set* T with structural presheaf $\Box T \subset \mathbf{Set}(\mathbf{I}^*, T)$, and letting c_* be the forgetful functor assigning to a c-set K its structural presheaf c_*K. Note that c_* is faithful (because $p_K: \mathcal{R}(c_*K) \to K$ is surjective).

Then, the left adjoint of c_* yields the *directed realisation* $\uparrow\mathcal{R}(X)$ of a cubical set: it is the set R underlying the geometric realisation $\mathcal{R}X$, without topology but equipped with the distinguished cubes produced by the n-cubes $x \in X_n$, via the associated mappings $\hat{x}: \mathbf{I}^n \to R$ (1.8.3), which are closed under faces and degeneracies

$$(2) \quad c_nR = \{\hat{x} \mid x \in X_n\} \subset \mathbf{Set}(\mathbf{I}^n, R);$$

the bijection $(\uparrow\mathcal{R}(X), K) = (X, c_*K)$ is easy to construct: given $f: \uparrow\mathcal{R}(X) \to K$, define $f_n: X_n \to c_nK$ letting $f_n(x) = f\hat{x}$; given $g: X \to c_*K$, take $f = p_K.\mathcal{R}f = (\mathcal{R}X \to \mathcal{R}(c_*K) \to K)$.

Finally, the functor $t: \mathrm{c}\mathbf{Set} \to \mathbf{Top}$ (left adjoint of $(-)_\Box$), acting on a c-set K, gives the underlying set $t(K)$ equipped with the *cubical topology*, i.e. the finest topology making all distinguished cubes $\mathbf{I}^n \to K$ continuous. The bijection $(t(K), T) = (K, \Box T)$ is obvious: a mapping $K \to T$ is continuous for the cubical topology of K if and only if it is continuous on each distinguished n-cube $x: \mathbf{I}^n \to K$, if and only if each composite $f \circ x$ is an n-cube of $\Box T$.

We end with some comments on the category c**Set**. Given a c-set $K = (K, c_*K)$, a *c-subset* $H = (H, c_*H)$ will be a c-set with $c_*H \subset c_*K$; in other words, we are considering a subset $H \subset K$ equipped with a sub-presheaf $c_*H \subset c_*K \cap \mathbf{Set}(\mathbf{I}^*, H)$ satisfying the covering condition on H. It is a *regular subobject* if $c_*H = c_*K \cap \mathbf{Set}(\mathbf{I}^*, H)$, that is if the distinguished cubes of H are precisely the ones of K whose image is contained in H; a regular subobject amounts thus to a subset $H \subset K$ which is a union of images of cubes of K (equipped with the restricted structure).

The *quotient* $K/\!\sim$ of a c-set modulo an equivalence relation (on the *set* K) will be the set-theoretical quotient, equipped with the projections $\mathbf{I}^n \to K \to K/\!\sim$ of the distinguished cubes of K (plainly stable under the faces and degeneracies of $\mathbf{I}^*$). This easy description of quotients will be exploited in Section 4, as an advantage of c-sets with respect to cubical sets: *one has just to assign an equivalence relation on the underlying set.*

2. Homology and cohomology of cubical sets

Combinatorial homology of cubical sets is a simple theory with evident proofs. We study its enrichment with a natural preorder, showing that it is preserved and reflected by excision (2.6), preserved by tensor product (2.7), but not preserved by the differentials of the usual exact sequences (2.4, 2.6) nor by multiplication, in cohomology (2.8, 2.9).

2.1. Directed homology. Every cubical set X determines a collection $\mathrm{Deg}_n X = \bigcup_i \mathrm{Im}(e_i: X_{n-1} \to X_n)$ of subsets of *degenerate elements* (with $\mathrm{Deg}_0 X = \emptyset$); this collection is not a cubical subset (unless X is empty), but satisfies weaker conditions (for all $i = 1,..., n$)

(1) $x \in \mathrm{Deg}_n X \;\Rightarrow\; (\partial_i^\alpha x \in \mathrm{Deg}_{n-1} X$ *or* $\partial_i^- x = \partial_i^+ x)$, $\qquad e_i(\mathrm{Deg}_{n-1} X) \subset \mathrm{Deg}_n X.$

The cubical set X determines a (*normalised*) *chain complex* of free abelian groups

(2) $C_n(X) = (\mathbf{Z}X_n)/(\mathbf{Z}\mathrm{Deg}_n X) = \mathbf{Z}\overline{X}_n \qquad (\overline{X}_n = X_n \setminus \mathrm{Deg}_n X),$

$\partial_n(\hat{x}) = \Sigma_{i,\alpha}\, (-1)^{i+\alpha}\, (\partial_i^\alpha x)\hat{} \qquad (x \in X_n),$

where $\mathbf{Z}S$ is the free abelian group on the set S, and $\hat{x}$ is the class of the n-cube x up to degenerate cubes; but we shall generally write the normalised class $\hat{x}$ as x, identifying all degenerate cubes with 0.

Now, each component can be preordered by the positive cone of *positive chains* $\mathbf{N}\overline{X}_n$, and will be written as $\uparrow C_n(X)$ when thus enriched; note that *the positive cone is not preserved by the differential* $\partial_n: \uparrow C_n(X) \to \uparrow C_{n-1}(X)$, which is just a homomorphism of the underlying abelian groups (as stressed by marking its arrow with a dot). On the other hand, a morphism of cubical sets $f: X \to Y$ induces a sequence of *preorder-preserving* homomorphisms $\uparrow C_n(X) \to \uparrow C_n(Y)$. We have defined a covariant functor

(3) $\uparrow C_*: \mathbf{Cub} \to \mathrm{d}C_*\mathbf{Ab},$

with values in the category $\mathrm{d}C_*\mathbf{Ab}$ of *directed* chain complexes of abelian groups (directed referring to the preorder of components, preserved by chain homomorphisms). This produces the *directed homology* of a cubical set, as a sequence of preordered abelian groups

(4) $\uparrow H_n: \mathbf{Cub} \to \mathrm{d}\mathbf{Ab}, \qquad \uparrow H_n(X) = \uparrow H_n(\uparrow C_* X),$

where the *directed homology* $\uparrow H_n(\uparrow C_*)$ of a directed chain complex is its ordinary homology equipped with the preorder induced on the subquotient $\mathrm{Ker}\partial_n/\mathrm{Im}\partial_{n+1}$.

When we forget preorders, the usual chain and homology functors will be written as usual

(5) $C_*: \mathbf{Cub} \to C_*\mathbf{Ab}, \qquad H_n: \mathbf{Cub} \to \mathbf{Ab}.$

If T is a topological space, it is well known that its singular homology can be defined by the singular cubical set $\square T$

(6) $H_n(T) = H_n(\square T);$

(the equivalence with the simplicial definition is proved by acyclic models, cf. [HW]). Notice that - here - we are not likely losing any essential information with respect to $\uparrow H_n(\square T)$. In fact, $\uparrow H_0(\square T)$ has an obvious order generated by the homology classes of points (cf. 2.3.1), while - for instance - *the* *preorder of* $\uparrow H_1(\square T)$ *is easily seen to be chaotic*: every homology class belongs to the positive cone (for every 1-cube $\mathbf{I} \to T$, the reversed cube obtained by precomposing with the reversion $\rho: \mathbf{I} \to \mathbf{I}$ is equivalent to the opposite of the original one, modulo boundaries).

Finally, we shall feel free of applying the functors $\uparrow C_*$ and $\uparrow H_n$ to a *c-set* $K = (K, c_*K)$ (1.9); obviously, this means to let them act on the cubical set c_*K of distinguished cubes of K

(7) $\uparrow H_n(K) = \uparrow H_n(c_*K)$.

2.2. Preordered coefficients. Implicitly, we have introduced the category d**Ab** of preordered abelian groups: an object $\uparrow L$ is an abelian group equipped with a preorder $\lambda \leq \lambda'$ preserved by the sum, or equivalently with a submonoid, the positive cone $L^+ = \{\lambda \in L \mid \lambda \geq 0\}$. A morphism is a preorder-preserving homomorphism.

Plainly, it is an additive category with all limits and colimits, computed as in **Ab** and equipped with a suitable preorder. It is not an abelian category, since a bijective morphism (mono and epi) need not be an isomorphism. But the symmetric monoidal structure of abelian groups can be easily lifted to d**Ab**: the positive cone of $\uparrow L \otimes \uparrow M$ is the submonoid generated by the tensors $\lambda \otimes \mu$, for $\lambda \in L^+$, $\mu \in M^+$, while $\mathrm{Hom}(\uparrow M, \uparrow N)$ is the abelian group $\mathrm{Hom}(M, N)$ of *all* algebraic homomorphisms, with positive cone given by the increasing ones

(1) $(\mathrm{Hom}(\uparrow M, \uparrow N))^+ = \mathrm{d}\mathbf{Ab}(\uparrow M, \uparrow N) = \{f \in \mathrm{Hom}(M, N) \mid f(M^+) \subset N^+\}$.

The unit of the tensor product is the ordered group of integers, $\uparrow \mathbf{Z}$. The forgetful functor d**Ab** $\to$ **Ab**, written $\uparrow L \mapsto L$, has left adjoint $\uparrow_d A$ and right adjoint $\uparrow_c A$, respectively giving to an abelian group A its discrete preorder $(A^+ = \{0\})$ or the chaotic one $(A^+ = A)$ - the latter can also be called coarse, or codiscrete. On the other hand, the forgetful functor d**Ab** $\to$ **Set** has (only) a left adjoint associating to a set S the *free ordered abelian group* $\uparrow \mathbf{Z}.S$: the usual free abelian group $\mathbf{Z}S$, equipped with the submonoid $\mathbf{N}S$ generated by S.

We have also introduced the category $\mathrm{d}C_*\mathbf{Ab}$ of directed chain complexes of abelian groups (and their directed homology). Recall that their components are preordered abelian groups, differentials are *not* assumed to preserve the preorder, *but chain morphisms are*. It is again an additive category with all limits and colimits. Similarly, we have the category of directed *co*chain complexes of abelian groups, $\mathrm{d}C^*\mathbf{Ab}$.

Now, we can consider directed combinatorial homology and cohomology of cubical sets, *with coefficients in a preordered abelian group* $\uparrow L$

(2) $\uparrow C_*(-; \uparrow L): \mathbf{Cub} \to \mathrm{d}C_*\mathbf{Ab}$, $\qquad \uparrow C_*(X; \uparrow L) = \uparrow C_*(X) \otimes \uparrow L$,

$\quad \uparrow H_n(-; \uparrow L): \mathbf{Cub} \to \mathrm{d}\mathbf{Ab}$, $\qquad \uparrow H_n(X; \uparrow L) = \uparrow H_n(\uparrow C_*(X; \uparrow L))$,

(3) $\uparrow C^*(-; \uparrow L): \mathbf{Cub}^{\mathrm{op}} \to \mathrm{d}C^*\mathbf{Ab}$, $\qquad \uparrow C^*(X; \uparrow L) = \mathrm{Hom}(\uparrow C_*(X), \uparrow L)$,

$\quad \uparrow H^n(-; \uparrow L): \mathbf{Cub}^{\mathrm{op}} \to \mathrm{d}\mathbf{Ab}$, $\qquad \uparrow H^n(X; \uparrow L) = \uparrow H^n(\uparrow C^*(X; \uparrow L))$,

where the components $\uparrow C_n(X) \otimes \uparrow L$ and $\mathrm{Hom}(\uparrow C_n(X), \uparrow L)$ are defined as above. Of course, $\uparrow H_n(X) = \uparrow H_n(X; \uparrow \mathbf{Z})$, with *ordered integral coefficients*; below, we generally consider this case, but the extension is easy.

The algebraic part of the universal coefficient theorems holds, with the usual proof; the preorder aspect should be examined, but we shall restrict to considering *rational* and *real* coefficients (also because a preorder on a torsion group cannot be of much interest). First, it is easy to verify that, for the ordered group of rationals $\uparrow\mathbf{Q}$, the canonical algebraic isomorphism

(4) $\uparrow H_n(X) \otimes \uparrow\mathbf{Q} \to \uparrow H_n(X; \uparrow\mathbf{Q}),$ $\qquad [z]\otimes\lambda \mapsto [z\otimes\lambda],$

which obviously preserves preorder, *also reflects it*. In fact, a positive chain in $\uparrow C_n(X; \uparrow\mathbf{Q})$ can plainly be written as $c = \lambda.c'$ where $\lambda > 0$ is rational and c' is a positive chain with integral coefficients; further, if c is a cycle, also c' is, and $[c] = [c']\otimes\lambda$ belongs to the positive cone of $\uparrow H_n(X)\otimes\uparrow\mathbf{Q}$.

As a consequence, the same property holds for the ordered group $\uparrow\mathbf{R}$: it suffices to take a positive basis of the reals on the rationals. More elementarily: a positive chain in $\uparrow C_n(X; \uparrow\mathbf{R})$ can be rewritten as a finite linear combination $c = \Sigma\, \lambda_i c_i$ where the $\lambda_i > 0$ are real numbers, linearly independent on the rationals, and all c_i are positive chains with integral coefficients; since each boundary $\lambda_i(\partial c_i)$ still has coefficients in $\lambda_i\mathbf{Q}$, one concludes as before: if c is a cycle, so are all c_i and $[c] = \Sigma\, [c_i]\otimes\lambda_i$ belongs to the positive cone of $\uparrow H_n(X)\otimes\uparrow\mathbf{R}$.

2.3. Elementary computations. The homology of a sum $X = \Sigma\, X_i$ is a direct sum $\uparrow H_n X = \oplus_i \uparrow H_n X_i$ (and every cubical set is the sum of its connected components, 1.3). It is also easy to see that, if X is connected (non empty), then $\uparrow H_0(X) \cong \uparrow\mathbf{Z}$ (via the augmentation $\partial_0\colon \uparrow C_0 X = \uparrow\mathbf{Z}X_0 \to \uparrow\mathbf{Z}$ taking each vertex $x\in X_0$ to $1\in\mathbf{Z}$). Thus, for every cubical set X

(1) $\uparrow H_0(X) = \uparrow\mathbf{Z}.\pi_0 X,$

is the free ordered abelian group generated by the homotopy set $\pi_0 X$ (1.3).

In particular, $\uparrow H_0(\uparrow\mathbf{s}^0) = \uparrow\mathbf{Z}^2$. Now, it is easy to see that, for $n > 0$

(2) $\uparrow H_n(\uparrow\mathbf{s}^n) = \uparrow\mathbf{Z},$

is the group of integers with the natural order: a normalised n-chain ku (notation of 1.5) is positive if and only if $k \geq 0$ (and is always a cycle).

On the other hand, $\uparrow H_n(\uparrow\mathbf{o}^n) = \uparrow_d\mathbf{Z}$ has the discrete order: the positive cone is reduced to 0. In fact, a normalised n-chain $hu' + ku''$ (notation of 1.5) is a cycle when $h+k = 0$, and a positive chain for $h \geq 0$, $k \geq 0$. The directed homology of the elementary directed torus $\uparrow\mathbf{t}^2$ is easy to determine; but we shall compute it for all $\uparrow\mathbf{t}^n$ (2.9.2).

2.4. Relative directed homology. Relative homology is defined in the usual way. A *cubical pair* (X, A) consists of a cubical subset $i\colon A \to X$; a *morphism* $f\colon (X, A) \to (Y, B)$ is a map $f\colon X \to Y$ whose restriction $A \to B$ is also a map.

The induced map on directed chain complexes $i_*\colon \uparrow C_* A \to \uparrow C_* X$ is injective as well (a cube in A is degenerate in X if and only if it is already so in A). We obtain the *relative directed chains* of (X, A) by the usual short exact sequence of (directed) chain complexes

(1) $\qquad 0 \longrightarrow \uparrow C_* A \longrightarrow \uparrow C_* X \longrightarrow \uparrow C_*(X, A) \longrightarrow 0$

and the *relative directed homology* as the homology of the quotient

(2) $\uparrow H_n(X, A) = \uparrow H_n(\uparrow C_*(X, A)).$

The exact sequence of the pair (X, A) comes from the exact homology sequence of (1), with differential $\Delta_n[c] = [\partial_n c]$; *the latter does not preserve the preorder* (its arrow is dot-marked)

(3) $\quad ... \twoheadrightarrow \uparrow H_nA \to \uparrow H_nX \to \uparrow H_n(X, A) \overset{\Delta}{\twoheadrightarrow} \uparrow H_{n-1}A \to ...$

$$... \twoheadrightarrow \uparrow H_0A \to \uparrow H_0X \to \uparrow H_0(X, A) \to 0.$$

Plainly, $\uparrow C_*(X, Ø) = \uparrow C_*(X)$ and $\uparrow H_n(X, Ø) = \uparrow H_n(X)$. More generally, given a *cubical triple* (X, A, B), consisting of cubical subsets $B \to A \to X$, the snake lemma gives a short exact sequence of chain complexes $\uparrow C_*(A, B) \rightarrowtail \uparrow C_*(X, B) \twoheadrightarrow \uparrow C_*(X, A)$, providing the exact homology sequence of the triple.

Tensoring by $\uparrow L$ our chain complexes (with free preordered components), one gets - as usual - the analogous results with arbitrary coefficients.

2.5. Invariance Theorem. The homology functor $\uparrow H_n$: **Cub** $\to$ d**Ab** is invariant for left (or right) immediate homotopies: given $f: f^- \to_L f^+: X \to Y$, then $\uparrow H_n(f^-) = \uparrow H_n(f^+)$. Similarly for relative homology.

Proof. We can forget about preorders. By 1.6.4, the homotopy $f: f^- \to_L f^+: X \to Y$ has

(1) $\quad f_n: X_n \to Y_{n+1}, \qquad \partial^\alpha_{i+1} f_n = f_{n-1} \partial^\alpha_i, \quad \partial^\alpha_1 f_n = f^\alpha, \quad f_n e_i = e_{i+1} f_{n-1} \qquad (1 \leq i \leq n),$

and produces a homotopy of the associated (normalised) chain complexes

(2) $\quad f_n: C_nX \to C_{n+1}Y, \qquad f_n(Deg_nX) \subset Deg_{n+1}Y,$

$$\partial_{n+1}f_n = \partial^+_1 f_n - \partial^-_1 f_n - \Sigma_{i\alpha} (-1)^{i+\alpha} \partial^\alpha_{i+1} f_n = f^+_n - f^-_n - f_{n-1} \partial_n.$$

It will be useful to note that the thesis also holds for a *generalised left homotopy*, replacing the condition $f_n e_i = e_{i+1} f_{n-1}$ with $f_n(Deg_nX) \subset Deg_{n+1}Y$. □

2.6. Mayer-Vietoris and excision. Given two cubical subsets $U, V \subset X$, their union $U \cup V$ (resp. intersection $U \cap V$) just consists of the union (resp. intersection) of all components. Therefore, $\uparrow C_*$ takes subobjects of X to directed chain subcomplexes of $\uparrow C_*X$, preserving joins and meets

(1) $\quad \uparrow C_*(U \cup V) = \uparrow C_*U + \uparrow C_*V, \qquad \uparrow C_*(U \cap V) = \uparrow C_*U \cap \uparrow C_*V.$

These facts have two important consequences

(a) *The Mayer-Vietoris sequence*. Let the cubical set X be *covered by its subobjects* U, V, i.e. $X = U \cup V$. Then we have an exact sequence

(2) $\quad ... \longrightarrow \uparrow H_n(U \cap V) \xrightarrow{(i_*, j_*)} (\uparrow H_nU) \oplus (\uparrow H_nV) \xrightarrow{[u_*, -v_*]} \uparrow H_n(X) \overset{\Delta}{\twoheadrightarrow} \uparrow H_{n-1}(U \cap V) \longrightarrow ...$

with the obvious meaning of brackets; the maps $u: U \to X$, $v: V \to X$, $i: U \cap V \to U$, $j: U \cap V \to X$ are inclusions and the *connective* Δ (which does not preserve preorder!) is:

(3) $\quad \Delta[c] = [\partial_na], \qquad c = a + b \qquad (a \in \uparrow C_n(U), \ b \in \uparrow C_n(V)).$

The sequence is natural, for a cubical map $f: X \to X' = U' \cup V'$, which restricts to $U \to U'$, $V \to V'$.

(b) *Excision*. Let a cubical set X be given, with subobjects $B \subset Y \cap A$. The inclusion map $i: (Y, B) \to (X, A)$ is said to be *excisive* whenever $Y_n \setminus B_n = X_n \setminus A_n$, for all n (or equivalently: $Y \cup A = X$,

$Y \cap A = B$, *in the lattice of subobjects of* X). Then i induces isomorphisms in homology, *preserving and reflecting preorder*.

Proof. The proof is similar to the topological one, simplified by the fact that here no subdivision is needed. For (a), it is sufficient to apply the algebraic theorem of the exact homology sequence to the following sequence of directed chain complexes

$$(4) \qquad 0 \longrightarrow {\uparrow}C_*(U \cap V) \xrightarrow{(i_*,\, j_*)} ({\uparrow}C_*U) \oplus ({\uparrow}C_*V) \xrightarrow{[u_*,\, -v_*]} {\uparrow}C_*(X) \longrightarrow 0$$

whose exactness needs one non-trivial verification. Take $a \in {\uparrow}C_nU$, $b \in {\uparrow}C_nV$ and assume that $u_*(a) = v_*(b)$; therefore, each cube really appearing in a (and b) belongs to $U \cap V$; globally, there is (one) normalised chain $c \in {\uparrow}C_n(U \cap V)$ such that $i_*(c) = a$, $i_*(c) = b$.

For (b), the proof reduces to a Noether isomorphism for directed chain complexes

$$(5) \quad {\uparrow}C_*(Y, B) = ({\uparrow}C_*Y)/(C_*(Y \cap A)) = ({\uparrow}C_*Y)/(C_*Y \cap C_*A)$$
$$= ({\uparrow}C_*Y + {\uparrow}C_*A) / (C_*A) = ({\uparrow}C_*(Y \cup A)) / (C_*A) = {\uparrow}C_*(X, A). \qquad \square$$

2.7. Theorem [Tensor products]. Given two cubical sets X, Y, there is a natural isomorphism and a natural monomorphism

$$(1) \quad {\uparrow}C_*(X \otimes Y) = {\uparrow}C_*(X) \otimes {\uparrow}C_*(Y), \qquad {\uparrow}H_*(X) \otimes {\uparrow}H_*(Y) \rightarrowtail {\uparrow}H_*(X \otimes Y).$$

Proof. It suffices to prove the first part, and apply the Künneth formula.

First, the canonical (positive) basis of the preordered abelian group ${\uparrow}C_p(X) \otimes {\uparrow}C_q(Y)$ is $\overline{X}_p \times \overline{Y}_q$ (as in 2.1, $\overline{X}_p = X_p \setminus Deg_pX$). Recall now that the set $(X \otimes Y)_n$ is a quotient of $\Sigma_{p+q=n} X_p \times Y_q$ modulo an equivalence relation which only identifies pairs where a term is degenerate (1.4.1); moreover, a class $x \otimes y$ is degenerate if and only if x or y is degenerate (1.4.3). Therefore, the canonical positive basis of ${\uparrow}C_n(X \otimes Y)$ is precisely the sum (disjoint union) of the preceding sets $\overline{X}_p \times \overline{Y}_q$, for $p+q = n$. We can identify the preordered abelian groups

$$(2) \quad {\uparrow}C_n(X \otimes Y) = \textstyle\bigoplus_{p+q=n} {\uparrow}C_p(X) \otimes {\uparrow}C_q(Y),$$

respecting the canonical positive bases. Finally, the differential of an element $x \otimes y$, with $(x, y) \in \overline{X}_p \times \overline{Y}_q$, is the same in both chain complexes

$$(3) \quad \Sigma_{i\alpha} (-1)^{i+\alpha}\, \partial_i^\alpha(x \otimes y) = \Sigma_{i \leq p, \alpha} (-1)^{i+\alpha} (\partial_i^\alpha x) \otimes y + \Sigma_{j \leq q, \alpha} (-1)^{p+j+\alpha}\, x \otimes (\partial_j^\alpha y)$$
$$= (\partial_p x) \otimes y + (-1)^p\, x \otimes (\partial_q y). \qquad \square$$

2.8. Cohomology. The (normalised) *cochain complex* ${\uparrow}C^*(X; {\uparrow}L) = Hom({\uparrow}C_*(X); {\uparrow}L)$, of a cubical set X, with coefficients in a preordered abelian group ${\uparrow}L$ (2.3) has a simple description

$$(1) \quad C^n(X; {\uparrow}L) = \{\lambda: X_n \to L \mid \lambda(Deg_nX) = 0\},$$
$$(d_n\lambda)(a) = \Sigma_{i,\alpha} (-1)^{i+\alpha}\, \lambda(\partial_i^\alpha a) \qquad (a \in X_{n+1}),$$

with components preordered by the cones of *positive cochains*, $\lambda: X_n \to L^+$, again not preserved by the differential.

Forgetting preorders and *assuming that* L *is a ring*, the cochain complex $C^*(X; L)$ has a natural structure of differential graded coalgebra, by the cup product (cf. [HW, 9.3])

(2) $(\lambda \cup \mu)(a) = \Sigma_{HK} (-1)^{\rho(HK)} \lambda(\partial^-_H a).\mu(\partial^+_K a)$ $\qquad (\lambda \in C^p(X; L),\ \mu \in C^q(X; L),\ a \in X_{p+q}),$

where (H, K) varies among all partitions of {1,..., n} in two complementary subsets of p and q elements, respectively, $\rho(HK)$ is the class of this permutation, $\partial^-_H a$ is the *lower H-face* of a and $\partial^+_K a$ its *upper K-face*. Thus, $H^*(X; L)$ is a graded algebra, isomorphic to $H^*(\mathcal{R}X; L)$ (and graded commutative).

Plainly, *the product of positive cochains need not be positive*. Graded commutativity of $H^*(X; L)$ (for a commutative ring L) says that this preservation property can hardly work for cohomology classes; an actual counterexample is given below (2.9.3).

2.9. Elementary cubical tori. The graded preordered abelian group of a cubical set X will be written as a *formal polynomial*

(1) $\uparrow H_*(X) = \Sigma_i\, \sigma^i.\uparrow H_i(X),$

whose coefficients are preordered abelian group, while the indeterminate σ shows the homology degree. One can think of σ^i as a power of the suspension operator of chain complexes (acting on a preordered abelian group, embedded in $dC_*\mathbf{Ab}$ in degree 0): then the expression (1) is a direct sum of graded preordered abelian groups; and the direct sum of such objects amounts to the sum of the corresponding polynomials (the latter is computed by means of the direct sum of the coefficients, in the obvious way).

It is easy to see (also using 2.7) that the directed homology of the elementary torus $\uparrow\mathbf{t}^n = (\uparrow\mathbf{s}^1)^{\otimes n}$

(2) $\uparrow H_*(\uparrow\mathbf{t}^n) = (\uparrow\mathbf{Z} + \sigma.\uparrow\mathbf{Z})^{\otimes n} = \uparrow\mathbf{Z} + \sigma.\uparrow\mathbf{Z}^{\binom{n}{1}} + \sigma^2.\uparrow\mathbf{Z}^{\binom{n}{2}} + ... + \sigma^n.\uparrow\mathbf{Z},$

where, of course, a power $\uparrow\mathbf{Z}^k$ has the product order.

Finally, to show that the cohomology multiplication (2.8) with coefficients in $\uparrow\mathbf{Z}$ need not preserve the positive cone, we use graded commutativity in odd degree, $[\lambda]\cup[\mu] = -[\mu]\cup[\lambda]$, looking for a case where cohomology is *ordered* (not just preordered) and $[\lambda], [\mu], [\lambda]\cup[\mu]$ are strictly positive (whence $[\mu]\cup[\lambda]$ is not).

The torus $\uparrow\mathbf{t}^2 = \uparrow\mathbf{s}^1\otimes\uparrow\mathbf{s}^1$ has one 0-cube $(*)$, two non degenerate 1-cubes $(u\otimes *, *\otimes u)$ and one non degenerate 2-cube $(u\otimes u)$, which also provide the positive generators of $\uparrow H_*(\uparrow\mathbf{t}^2)$. Similarly, in cohomology, we have an *ordered* object

(3) $\uparrow H^*(\uparrow\mathbf{t}^2) = \uparrow\mathbf{Z} + \sigma.\uparrow\mathbf{Z}^2 + \sigma^2.\uparrow\mathbf{Z},$

and the positive generators in degree 1, 2 come from the following cocycles (zero elsewhere)

(4) $\lambda(u\otimes *) = 1, \qquad \mu(*\otimes u) = 1, \qquad (\lambda\cup\mu)(u\otimes u) = 1.$

3. Group actions

The classical theory of *proper* actions on topological spaces, up to the spectral sequence, is extended to *free* actions on cubical sets. G is a group, always written in additive notation (independently of commutativity); the action of an operator $g \in G$ on an element x is written as x+g.

3.1. Basics. Take a cubical set X and a group G acting on it, on the right: we have an action x+g ($x \in X_n$, $g \in G$) on each component, consistently with faces and degeneracies (or, equivalently, a cubical object in the category of G-sets). Plainly, there is a cubical set of orbits X/G, with components X_n/G and induced structure; and a natural projection $p: X \to X/G$.

Say that the action is *free* if G acts freely on each component: if $x = x+g$, for some $x \in X_n$ and $g \in G$, then $g = 0$. This is equivalent to saying that G acts freely on the set of vertices X_0 (because $x = x+g$ implies that their first vertices coincide).

It is now easy to extend to *free actions on cubical sets* the classical results of actions of groups on topological spaces [Ma, IV.11], which hold for groups acting *properly* on a space, a much stronger condition (every point has an open neighbourhood U such that all subsets U+g are disjoint). But note that all results below which involve the homology of G *ignore preorder*, necessarily (4.6).

Of course, an *action of* G *on a c-set* (X, c_*X) (1.9) is defined to be an action on the set X coherent with the structural presheaf c_*X: for every distinguished cube $x: \mathbf{I}^n \to X$, all mappings x+g are also distinguished. Thus, for a topological space T, a G-action on the space gives an action on the c-set $T_{\square} = (T, \square T)$ and on the cubical set $\square T$.

3.2. Lemma [Free actions]. (a) If G acts freely on the cubical set X, then $\uparrow C_*(X)$ is a complex of free right G-modules, with a (positive) basis $B_n \subset X_n$ which projects bijectively onto $\overline{X}_n/G$, the canonical basis of $\uparrow C_n(X/G)$.

(b) Moreover, if $\uparrow L$ is a preordered abelian group, viewed as a trivial G-module, then the canonical projection $p: X \to X/G$ induces an isomorphism of directed (co)chain complexes, and hence an isomorphism in (co)homology

(1) $p_*: \uparrow C_*(X) \otimes_G \uparrow L \to \uparrow C_*(X/G; \uparrow L)$, $\qquad p_{*n}: H_n(\uparrow C_*(X) \otimes_G \uparrow L) \to \uparrow H_n(X/G; \uparrow L)$,

$p^*: \uparrow C^*(X/G; \uparrow L) \to \mathrm{Hom}_G(\uparrow C_*(X), \uparrow L)$, $\qquad p^{*n}: \uparrow H^n(X/G; \uparrow L) \to H_n(\mathrm{Hom}_G(\uparrow C_*(X), \uparrow L)$.

Proof. (This Lemma adapts [Ma, IV.11.2-4]). It is sufficient to prove (a), which plainly implies (b). The action of G on X_n extends to a right action on the free abelian group $\mathbf{Z}X_n$, consistent with faces and degeneracies and preserving the canonical basis; it induces thus an obvious action on $\uparrow C_n(X) = \uparrow \mathbf{Z}\overline{X}_n$, consistent with the positive cone and the differential

(2) $(\Sigma\, \lambda_i x_i) + g \;=\; \Sigma\, \lambda_i(x_i + g)$, $\qquad \partial(\Sigma\, \lambda_i x_i) + g \;=\; \partial(\Sigma\, \lambda_i x_i + g)$.

Thus $\uparrow C_n(X)$ is a complex of G-modules, whose components are preordered G-modules. Take now a subset $B_0 \subset X_0$ choosing exactly one point in each orbit; then B_0 is a G-basis of $\uparrow C_0(X)$. Letting $B_n \subset X_n$ be the subset of those non-degenerate n-cubes x whose 'initial vertex' $\partial_1^- ... \partial_n^- x$ belongs to B_0, we have more generally a G-basis of $\uparrow C_n(X)$ which satisfies our requirements. $\square$

3.3. Theorem [Free actions on acyclic cubical sets]. Let X be an *acyclic* (connected) cubical set and G a group acting freely on it. Then, for an abelian group L with trivial G-structure, and *forgetting* preorder in combinatorial (co)homology (cf. 4.6)

(1) $H_*(X/G; L) \cong H_*(G; L)$, $\qquad H^*(X/G; L) \cong H^*(G; L)$.

Proof. As in [Ma, IV.11.5], the augmented sequence

(2) $... \to C_1(X) \to C_0(X) \to \mathbf{Z} \to 0$

is exact, since X is acyclic (has the homology of the point). By 3.2a, this sequence forms a G-free resolution of the G-trivial module $\mathbf{Z}$. Therefore, applying the definition of $H_n(G; L)$ and the isomorphism 3.2.1, we get the thesis for homology (and cohomology as well)

(3) $H_n(G; L) = H_n(C_*(X)\otimes_G L) \cong H_n(X/G; L)$. □

3.4. Corollary [Free actions on acyclic spaces]. Let T be an acyclic (path connected) topological space and G a group acting freely on it. Then $H_*((\square T)/G) \cong H_*(G)$, and $\uparrow H_1((\square T)/G)$ has a chaotic preorder. The same holds in cohomology.

Proof. It suffices to apply the preceding theorem to the singular cubical set $\square T$ of continuous cubes of T. This cubical set has the same homology as T, and G acts obviously on it, by $(x+g)(t) = x(t) +g$ (for $t \in \mathbf{I}^n$). Moreover, the action is free because so it is on the set of vertices, T. Finally, the remark on preorder is proved as for $\uparrow H_1(\square T)$, in 2.1. □

3.5. Theorem [The spectral sequence of a G-free cubical set]. Let X be a connected cubical set, G a group acting freely on it and L a G-module. Then there is a spectral sequence

(1) $E^2_{p,q} = H_p(G; H_q(X; L)) \Rightarrow_p H_n(X/G; L)$.

Proof. This result extends Corollary 3.4, without assuming X acyclic. The proof is the same as in [Ma, XI.7.1], where X is a path-connected topological space with a *proper* G-action. The argument is based on computing the terms $E^2_{p,q}$ of the two spectral sequences of the double complex

(2) $K_{pq} = L \otimes C_p(X) \otimes_G B_q(G)$,

$B_*(G)$ being a G-free resolution of $\mathbf{Z}$ as a trivial G-module. And it only depends on the fact that $C_*(X)$ is a chain complex of free G-modules with $C_*(X)\otimes_G L \cong C_*(X/G; L)$, which is also true in our case (Lemma 3.2). □

4. Rotation structures and noncommutative tori

We compute the directed homology of various cubical sets, related with 'virtual spaces' of noncommutative geometry: irrational rotation algebras and noncommutative tori of dimension ≥ 2; ϑ is always an irrational real number.

4.1. Rotation algebras. Let us begin recalling some well-known 'noncommutative spaces'.

First, take the line $\mathbf{R}$ and its (dense) additive subgroup $G_\vartheta = \mathbf{Z}+\vartheta\mathbf{Z}$, acting on the former by translations. In **Top**, the orbit space $\mathbf{R}/G_\vartheta = \mathbf{S}^1/\vartheta\mathbf{Z}$ is trivial: an uncountable set with the coarse topology.

Second, consider the *Kronecker foliation* F' of the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, with slope ϑ (recalled in 4.3), and the set $\mathbf{T}^2_\vartheta = \mathbf{T}^2/{\equiv_{F'}}$ of its leaves. It is well known, and easy to see, that the sets $\mathbf{R}/G_\vartheta$ and $\mathbf{T}^2_\vartheta$ are in bijection (cf. 4.3). Again, ordinary topology gives no information on $\mathbf{T}^2_\vartheta$, since the quotient $\mathbf{T}^2/{\equiv_{F'}}$ in **Top** is coarse.

In noncommutative geometry, both these sets are 'interpreted' as the (noncommutative) C*-algebra A_ϑ, generated by two unitary elements u, v under the relation $vu = \exp(2\pi i\vartheta).uv$, and called the *irrational rotation algebra* associated with ϑ, or also a *noncommutative torus* [C1, C2, C3, R1, Bl]. Both its complex K-theory groups are two-dimensional.

A relevant achievement of K-theory [PV, R1] classifies these algebras, by proving that $K_0(A_\vartheta) \cong \mathbf{Z}+\vartheta\mathbf{Z}$ *as an ordered subgroup of* $\mathbf{R}$; more precisely, the traces of the projections of A_ϑ cover the set $G_\vartheta\cap[0, 1]$. It follows that A_ϑ and $A_{\vartheta'}$ are *isomorphic* if and only if $\vartheta' \in \pm\,\vartheta + \mathbf{Z}$ [R1, Thm. 2] and *strongly Morita equivalent* if and only if ϑ and ϑ' are equivalent modulo the *fractional* action (on the irrationals) of the group $GL(2, \mathbf{Z})$ of invertible integral 2×2 matrices [R1, Thm. 4]

(1) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}.t \;=\; \dfrac{at+b}{ct+d}$ $\qquad$ $(a, b, c, d \in \mathbf{Z};\ ad - bc = \pm 1)$,

(or the action of the projective general linear group $PGL(2, \mathbf{Z})$ on the projective line). Since $GL(2, \mathbf{Z})$ is generated by the matrices

(2) $R \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\qquad$ $T \;=\; \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$,

the orbit of ϑ is its closure $\{\vartheta\}_{RT}$ under the transformations $R(t) = t^{-1}$ and $T^{\pm1}(t) = t\pm1$ (on $\mathbf{R}\backslash\mathbf{Q}$)

A similar result, based on the 1-cohomology of an associated etale topos, can be found in [Ta].

We show now how one can obtain similar results with cubical sets naturally arising from the previous situations: the point is to replace a topologically-trivial orbit space T/G with the corresponding quotient of the singular cubical set $\square T$, identifying the cubes $\mathbf{I}^n \to T$ modulo the action of G.

4.2. Irrational rotation structures. (a) Now, instead of considering the trivial quotient $\mathbf{R}/G_\vartheta$ of topological spaces, we replace $\mathbf{R}$ with the singular cubical set $\square\mathbf{R}$ (on which G_ϑ acts freely) and consider the cubical set $(\square\mathbf{R})/G_\vartheta$. Or, equivalently, we replace $\mathbf{R}$ with the c-set $\mathbf{R}_\square = (\mathbf{R}, \square\mathbf{R})$ and take the quotient $\mathbf{R}_\square/G_\vartheta$, i.e. the set $\mathbf{R}/G_\vartheta$ equipped with the projections of the (continuous) cubes of $\mathbf{R}$. (In fact, if the cubes $x, y\colon \mathbf{I}^n \to \mathbf{R}$ coincide when projected to $\mathbf{R}/G_\vartheta$, their difference $g = x - y\colon \mathbf{I}^n \to \mathbf{R}$ takes values in the totally disconnected subset $G_\vartheta \subset \mathbf{R}$, and is constant; therefore, x and y also coincide in $(\square\mathbf{R})/G_\vartheta$).

Then, applying Corollary 3.4, we find that the c-set $\mathbf{R}_\square/G_\vartheta$ (or $(\square\mathbf{R})/G_\vartheta$) has the same homology as the group $G_\vartheta \cong \mathbf{Z}^2$, which coincides with the ordinary homology of the torus $\mathbf{T}^2$

(1) $H_*(\mathbf{R}_\square/G_\vartheta) \;=\; H_*(G_\vartheta) \;=\; H_*(\mathbf{T}^2) \;=\; \mathbf{Z} + \sigma.\mathbf{Z}^2 + \sigma^2.\mathbf{Z}$;

(the last fact follows, for instance, from the classical version of Theorem 3.3 [Ma, IV.11.5], applied to the proper action of the group $\mathbf{Z}^2$ on the acyclic space $\mathbf{R}^2$). We also know that directed homology only gives the chaotic preorder on $\uparrow H_1(\mathbf{R}_{\square}/G_\vartheta)$ (again by 3.4).

In cohomology, we have the same graded group. *Algebraically*, this is in accord with the K-theory of the rotation algebra A_ϑ, since both $H^{even}(\mathbf{R}_{\square}/G_\vartheta)$ and $H^{odd}(\mathbf{R}_{\square}/G_\vartheta)$ are two-dimensional.

(b) A much more interesting result (and accord) can be obtained with the c-structure $\uparrow\mathbf{R}$ of the line produced by topology *and* natural order: $c_n\uparrow\mathbf{R}$ is the set of continuous order-preserving mappings $\mathbf{I}^n \to \mathbf{R}$. The quotient $C_\vartheta = \uparrow\mathbf{R}/G_\vartheta = \uparrow\mathbf{S}^1/\vartheta\mathbf{Z}$ will be called an *irrational rotation c-set* (on the directed circle $\uparrow\mathbf{S}^1 = \uparrow\mathbf{R}/\mathbf{Z}$), and we want to classify its isomorphism classes, for $\vartheta \notin \mathbf{Q}$.

We prove below (Theorems 4.8, 4.9) that $\uparrow H_1(\uparrow\mathbf{R}/G_\vartheta) \cong \uparrow G_\vartheta$, *as an ordered subgroup of the line* and that the c-sets C_ϑ have the same classification *up to isomorphism* as the rotation algebras A_ϑ *up to strong Morita equivalence*: while the algebraic homology of C_ϑ is the same as in (a), independent of ϑ, the (pre)order of directed homology determines ϑ up to the equivalence relation $\uparrow G_\vartheta \cong \uparrow G_{\vartheta'}$, which amounts to ϑ and ϑ' being conjugate under the action of the group $GL(2, \mathbf{Z})$.

Note that the stronger classification of rotation algebras up to isomorphism (recalled in 4.1) has no analogue here: *cubical sets lack the 'metric information' contained in* C^**-algebras*.

Note also the role of the ordered cube $\mathbf{I}^n$ (with its faces and degeneracies) for defining $\uparrow\mathbf{R}$. Presumably, *this cannot be easily transferred to a simplicial approach*: the standard realisations of Δ^n in $\mathbf{R}^{n+1}$ or $\mathbf{R}^n$ are of no use, since the former inherits the discrete order while the latter has a 'diagonal' face not consistent with ordering; other realisations in $\mathbf{R}^n$ have complicated faces.

4.3. The noncommutative two-dimensional torus. Consider now the *Kronecker foliation* F' of the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, with irrational slope ϑ, and the set $\mathbf{T}^2_\vartheta = \mathbf{T}^2/\equiv_{F'}$ of its leaves. F' and $\equiv_{F'}$ are induced, respectively, from the following foliation $F = (F_\lambda)$ and equivalence relation $\equiv$ on the plane

(1) $F_\lambda = \{(x, y) \in \mathbf{R}^2 \mid y = \vartheta x + \lambda\}$ $\quad(\lambda \in \mathbf{R})$,

$(x, y) \equiv (x', y') \Leftrightarrow y + k - \vartheta(x+h) = y' + k' - \vartheta(x'+h')$ $\quad$(for some $h, k, h', k' \in \mathbf{Z}$).

Now, we interpret $\mathbf{T}^2_\vartheta$ as the quotient c-set $\mathbf{T}^2_{\square}/\equiv_{F'}$, i.e. the set $\mathbf{T}^2_\vartheta$ equipped with the projection of the cubes of the torus (or of the plane). This is proved below to be isomorphic to the previous c-set $K = \mathbf{R}_{\square}/G_\vartheta$ (4.2a), whose directed (co)homology has been computed above, in accord (*algebraically*) with the complex K-theory groups of A_ϑ.

Now, the isomorphism we want can be realised with two inverse c-maps $i'\colon K \to \mathbf{T}^2_\vartheta$ and $p'\colon \mathbf{T}^2_\vartheta \to K$, respectively induced by the following maps (in **Top**):

(2) $i\colon \mathbf{R} \to \mathbf{R}^2, \qquad i(t) = (0, t),$

$p\colon \mathbf{R}^2 \to \mathbf{R}, \qquad p(x, y) = y - \vartheta x.$

First, the induction on quotients is legitimate because, for $t \equiv t + h + k\vartheta$ in $\mathbf{R}$ and $(x, y) \equiv (x', y')$ in $\mathbf{R}^2$ (as in (1))

(3) $i(t + h + k\vartheta) = (0, t + h + k\vartheta) \equiv (1, t + \vartheta) \equiv (0, t) = i(t),$

$p(x, y) - p(x', y') = (y - \vartheta x) - (y' - \vartheta x') = k' - \vartheta h' - k + \vartheta h \in \mathbf{Z} + \vartheta\mathbf{Z}.$

Second, pi is the identity, and $i'p'$ as well, because:

(4) $ip(x, y) = (0, y - \vartheta x) \equiv (x, y)$ $\qquad\qquad (y - \vartheta x - \vartheta.0 = y - \vartheta x)$.

Finally, it is obvious that distinguished cubes are preserved by i', p', since they are by i and p

$$(5) \qquad \begin{array}{ccccccc} \mathbf{I}^n & \xrightarrow{x} & \mathbf{R} & \overset{i}{\underset{p}{\rightleftarrows}} & \mathbf{R}^2 & \xleftarrow{y} & \mathbf{I}^n \\ & & \downarrow & & \downarrow & & \\ & & K = \mathbf{R}/G_\vartheta & \rightleftarrows & \mathbf{R}^2/\!\equiv\; = \mathbf{T}^2_\vartheta & & \end{array}$$

4.4. Higher foliations of codimension 1. (a) Extending 4.2a and 4.3, take an n-tuple of real numbers $\vartheta = (\vartheta_1,..., \vartheta_n)$, linearly independent on the rationals, and consider the additive subgroup $G_\vartheta = \Sigma_j \vartheta_j\mathbf{Z} \cong \mathbf{Z}^n$, acting freely on $\mathbf{R}$. (The previous case corresponds to the pair $(1, \vartheta)$.)

Now, the c-set $\mathbf{R}_\square/G_\vartheta$ has the homology (or cohomology) of the n-dimensional torus $\mathbf{T}^n$ (notation as in 2.9)

(1) $H_*(\mathbf{R}_\square/G_\vartheta) = H_*(G_\vartheta) = H_*(\mathbf{T}^n) = \mathbf{Z} + \sigma.\mathbf{Z}^{\binom{n}{1}} + \sigma^2.\mathbf{Z}^{\binom{n}{2}} + ... + \sigma^n.\mathbf{Z}$.

And again, this coincides with the homology of a c-set $\mathbf{T}^n_\square/\!\equiv_{F'}$ arising from the foliation F' of the n-dimensional torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ induced by the hyperplanes $\Sigma_j \vartheta_j x_j = \lambda$ of $\mathbf{R}^n$. (In the previous proof, one can replace the maps i, p (4.3.2) with $i(t) = (t/\vartheta_1, 0,..., 0)$ and $p(x_1,..., x_n) = \Sigma_j \vartheta_j x_j$.)

(b) Extending now 4.2b (and Theorem 4.8), the c-set $\uparrow\mathbf{R}/G_\vartheta$ has a more interesting directed homology, with a relevant total order in degree 1:

(2) $\uparrow H_1(\uparrow\mathbf{R}/G_\vartheta) = \uparrow G_\vartheta = \uparrow(\Sigma_j \vartheta_j\mathbf{Z})$ $\qquad\qquad (G^+_\vartheta = G_\vartheta \cap \mathbf{R}^+)$.

4.5. Higher foliations. More generally, consider a linear subspace $H \subset \mathbf{R}^n$ of codimension k $(0 < k < n)$ and such that $H \cap \mathbf{Z}^n = \{0\}$. (In case (a), H is the hyperplane $\Sigma_j \vartheta_j x_j = 0$.)

Let F be the foliation of $\mathbf{R}^n$ whose leaves are the (n–k)-dimensional planes H+x, parallel to H. These can be parametrised letting x vary in some convenient k-dimensional subspace transverse to H; equivalently, choose a projector $e: \mathbf{R}^n \to \mathbf{R}^n$ with H = Ker(e) and an epi-mono (linear) factorisation of the latter through $\mathbf{R}^k$

(1) $\mathbf{R}^n \xrightarrow{p} \mathbf{R}^k \xrightarrow{i} \mathbf{R}^n$ $\qquad\qquad ip = e, \quad pi = id,$

so that the leaves of F are bijectively parametrised on $\mathbf{R}^k$

(2) $F_\lambda = \{x \in \mathbf{R}^n \mid p(x) = \lambda\}$ $\qquad\qquad (\lambda \in \mathbf{R}^k)$.

The projection $\mathbf{R}^n \to \mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ is injective on each leaf F_λ (because $\mathrm{Ker}(p) \cap \mathbf{Z}^n = H \cap \mathbf{Z}^n = \{0\}$). Therefore, F induces a foliation F' of $\mathbf{T}^n$ with codimension k, and an equivalence relation $\equiv_{F'}$ (to belong to the same leaf). The set of leaves $\mathbf{T}^n/\!\equiv_{F'}$ can be identified with the quotient $\mathbf{R}^n/\!\equiv$, modulo the equivalence relation $\equiv$ generated by the translations of $\mathbf{Z}^n$ and the equivalence relation $x \equiv_F y$ of the original foliation (i.e., $p(x) = p(y)$):

(3) $x \equiv x'$ in $\mathbf{R}^n$ if and only if $p(x) - p(x') \in p(\mathbf{Z}^n)$.

Note that $G_p = p(\mathbf{Z}^n)$ is an additive subgroup of $\mathbf{R}^k$ *isomorphic to* $\mathbf{Z}^n$, by $\mathrm{Ker}(p) \cap \mathbf{Z}^n = \{0\}$ again. Now, we are interested in the c-set $\mathbf{T}^n_\square/{\equiv_{F'}}$, isomorphic to $\mathbf{R}^n_\square/{\equiv}$. Because of (3), the maps p, i in (1) induce a bijection of sets

(4) $$\mathbf{R}^n/{\equiv} \xrightarrow{p'} \mathbf{R}^k/G_p \xrightarrow{i'} \mathbf{R}^n/{\equiv},$$

and an isomorphism of c-sets

(5) $$\mathbf{T}^n_\square/{\equiv_{F'}} \;\cong\; \mathbf{R}^n_\square/{\equiv} \;\cong\; \mathbf{R}^k_\square/G_p.$$

Since the cubical set $\square\,\mathbf{R}^k$ is acyclic and $G_p \cong \mathbf{Z}^n$, we conclude by 3.3 (and its classical version) that the homology of $\mathbf{T}^n_\square/{\equiv_{F'}}$ is the same as the ordinary homology of the torus $\mathbf{T}^n$ (cf. 2.9)

(6) $$H_*(\mathbf{T}^n_\square/{\equiv_{F'}}) = H_*(\mathbf{R}^k_\square/G_p) = H_*(G_p) = H_*(\mathbf{T}^n).$$

It should be interesting to study the relations of the above with the general n-*dimensional noncommutative torus* A_Θ [R2]. This is the C*-algebra generated by n unitary elements $u_1,..., u_n$ under the relations $u_k u_h = \exp(2\pi i \vartheta_{hk}).u_h u_k$ produced by an antisymmetric matrix $\Theta = (\vartheta_{hk})$; it has the same K-groups as $\mathbf{T}^n$.

4.6. Remarks. The previous results show also that *it is not possible to preorder group-homology* so that the isomorphism $H_*(G) \cong H_*(X/G)$ (3.3.1) be extended to $\uparrow H_*(X/G)$: a group G can act *freely* on two *acyclic* cubical sets X_i producing *different preorders* on some $\uparrow H_n(X_i/G_\vartheta)$.

In fact, it is sufficient to take $G_\vartheta = \mathbf{Z} + \vartheta\mathbf{Z}$, as above, and recall that $\uparrow H_1(\mathbf{R}_\square/G_\vartheta)$ has a *chaotic preorder* (3.4) while $\uparrow H_1(\uparrow\mathbf{R}/G_\vartheta) = \uparrow G_\vartheta$ is totally ordered (4.8).

Another example comes from a different c-structure $\mathbf{R}_\vartheta$ on the real line, defined by the sub-presheaf $X = c_*\mathbf{R}_\vartheta \subset \mathbf{Set}(\mathbf{I}^*, \mathbf{R})$ having the following non-degenerate n-cubes (stable under the action of G_ϑ on $\mathbf{R}$):

(1) $x: \{*\} \to \mathbf{R}$, $\quad$ (n = 0, $x \in G_\vartheta \subset \mathbf{R}$),

$c_{1x}, c_{2x}: \mathbf{I} \to \mathbf{R}$, $\quad c_{1x}(t) = x + t$, $\quad c_{2x}(t) = x + t\vartheta$ $\quad$ (n = 1, $x \in G_\vartheta$),

$a_x: \mathbf{I}^2 \to \mathbf{R}$, $\quad a_x(t, t') = x + t\vartheta + t'$ $\quad$ (n = 2, $x \in G_\vartheta$),

$\partial^\alpha_1(a_x) = c_{1,x+\alpha\vartheta}$, $\quad \partial^\alpha_2(a_x) = c_{2,x+\alpha}$.

Now, $(c_*\mathbf{R}_\vartheta)/G_\vartheta$ has precisely four non-degenerate cubes ([0], [c_{10}], [c_{20}], [a_0]) and is plainly isomorphic to the cubical set $\uparrow\mathbf{t}^2$. Thus, all the homology groups $\uparrow H_n(\mathbf{R}_\vartheta/G_\vartheta) \cong \uparrow H_n(\uparrow\mathbf{t}^2)$ are *ordered* (2.9.2). And it is not difficult to show that the cubical set X itself is indeed acyclic: in degree 2, take a chain $z = \Sigma_x \lambda_x a_x$ and let $y \in G_\vartheta$ be the lowest index with non-zero coefficient (if any); then the lower faces $\partial^-_i(a_y) = c_{iy}$ are distinct, and different from all faces of the other summands in z; we conclude that the only 2-cycle is 0. A similar argument shows that the only 1-cycle is 0.

We end this section by proving the main results on the directed homology of the rotation c-set $C_\vartheta = \uparrow\mathbf{R}/G_\vartheta$, already announced in 4.2.

4.7. Lemma. Let ϑ, ϑ' be irrationals. Then $G_\vartheta = G_{\vartheta'}$, as subsets of $\mathbf{R}$, if and only if $\vartheta' \in \pm\vartheta + \mathbf{Z}$. Moreover the following conditions are equivalent

(a) $\uparrow G_\vartheta \cong \uparrow G_{\vartheta'}$ as ordered groups,

(b) ϑ and ϑ' are conjugate under the action of GL(2, **Z**) (4.1),

(c) ϑ' belongs to the closure $\{\vartheta\}_{RT}$ of $\{\vartheta\}$ under the transformations $R(t) = t^{-1}$ and $T^{\pm 1}(t) = t \pm 1$.

Further, these conditions imply the following one (which will be proved to be equivalent in 4.8)

(d) $\uparrow\mathbf{R}/G_\vartheta \cong \uparrow\mathbf{R}/G_{\vartheta'}$ as c-sets.

Proof. First, if $G_\vartheta = G_{\vartheta'}$, then $\vartheta = a + b\vartheta'$ and $\vartheta' = c + d\vartheta$, whence $\vartheta = a + bc + bd\vartheta$ and $d = \pm 1$; the converse is obvious.

We have already seen, in 4.1, that (b) and (c) are equivalent, because the group GL(2, **Z**) is generated by the matrices R, T (4.1.2), which give the transformations $R(t) = t^{-1}$ and $T^k(t) = t+k$ (on $\mathbf{R}\backslash\mathbf{Q}$; for $k \in \mathbf{Z}$). To prove that (c) implies (a) and (d), it suffices to consider the cases $\vartheta' = \vartheta + k$ and $\vartheta' = \vartheta^{-1}$. In the first case, $\uparrow G_\vartheta$ and $\uparrow G_{\vartheta'}$ coincide (as well as their action on $\uparrow\mathbf{R}$); in the second, the isomorphism of c-sets

(1) $f: \uparrow\mathbf{R} \to \uparrow\mathbf{R}, \qquad f(t) = |\vartheta|.t,$

restricts to an isomorphism $f': \uparrow G_\vartheta \to \uparrow G_{\vartheta'}$, obviously consistent with the actions $(f(t + g) = f(t) + f'(g))$, and induces an isomorphism $\uparrow\mathbf{R}/G_\vartheta \to \uparrow\mathbf{R}/G_{\vartheta'}$.

We are left with proving that (a) implies (c). Let us begin noting that any irrational ϑ defines an algebraic isomorphism $\mathbf{Z}^2 \cong G_\vartheta$, which becomes an order isomorphism for the structure $\uparrow_\vartheta\mathbf{Z}^2$

(2) $\uparrow_\vartheta\mathbf{Z}^2 \to G_\vartheta, \qquad (a, b) \mapsto a + b\vartheta,$

$(a, b) >_\vartheta 0 \quad \Leftrightarrow \quad a + b\vartheta > 0,$

and the number ϑ is (completely) determined by this order, as an upper bound in **R**

(3) $\vartheta = \sup\{ -a/b \mid a, b \in \mathbf{Z},\ b > 0,\ (a, b) >_\vartheta 0\}.$

Take now an algebraic isomorphism $f: \mathbf{Z}^2 \to \mathbf{Z}^2$. Since GL(2, **Z**) is generated by R and T, this isomorphism can be factored as $f = f_n...f_1$, with factors f_R, f_T^k

(4) $f_R(a, b) = (b, a), \qquad f_T^k(a, b) = (a + kb, b).$

Now, take $\uparrow_\vartheta\mathbf{Z}^2$ and replace ϑ with a positive representative in $\{\vartheta\}_{RT}$. Then f_R (resp. f_T^k) is an *order isomorphism* $\uparrow_\vartheta\mathbf{Z}^2 \to \uparrow_\zeta\mathbf{Z}^2$ with $\zeta = R(\vartheta)$ (resp. $\zeta = T^{-k}(\vartheta)$), still belonging to $\{\vartheta\}_{RT}$

(5) $(a, b) >_\vartheta 0 \ \Leftrightarrow\ a + b\vartheta > 0 \ \Leftrightarrow\ b + a\vartheta^{-1} > 0 \ \Leftrightarrow\ (b, a) >_\zeta 0 \qquad (\zeta = \vartheta^{-1}),$

$(a, b) >_\vartheta 0 \ \Leftrightarrow\ a + b\vartheta > 0 \ \Leftrightarrow\ a + kb + b(\vartheta - k) > 0 \ \Leftrightarrow\ (a + kb, b) >_\zeta 0 \qquad (\zeta = \vartheta - k).$

Thus, $f = f_n...f_1$ can be viewed as an isomorphism $\uparrow_\vartheta\mathbf{Z}^2 \to \uparrow_\zeta\mathbf{Z}^2$ where ζ belongs to the closure $\{\vartheta\}_{RT}$. Finally, given ϑ, ϑ', an isomorphism $\uparrow G_\vartheta \cong \uparrow G_{\vartheta'}$ yields an iso $\uparrow_\vartheta\mathbf{Z}^2 \to \uparrow_{\vartheta'}\mathbf{Z}^2$; but we have seen that the *same* algebraic isomorphism is an order isomorphism $\uparrow_\vartheta\mathbf{Z}^2 \to \uparrow_\zeta\mathbf{Z}^2$ where ζ belongs to the closure of $\{\vartheta\}$; by (3), $\vartheta' = \zeta$ and the thesis holds. □

4.8. Theorem. The c-set $\uparrow\mathbf{R}$ (4.2b) is acyclic. The directed homology of $\uparrow\mathbf{R}/G_\vartheta$ is the homology of $\mathbf{T}^2$, with a total order on $\uparrow H_1$ and a chaotic preorder on $\uparrow H_2$

(1) $\uparrow H_1(\uparrow\mathbf{R}/G_\vartheta) = \uparrow G_\vartheta = \uparrow(\mathbf{Z} + \vartheta\mathbf{Z}) \qquad (G_\vartheta^+ = G_\vartheta \cap \mathbf{R}^+),$

$\uparrow H_2(\uparrow\mathbf{R}/G_\vartheta) = \uparrow_c\mathbf{Z},$

and obviously $\uparrow H_0(\uparrow\mathbf{R}/G_\vartheta) = \uparrow\mathbf{Z}$. The first isomorphism above has a simple description *on the positive cone* $G_\vartheta \cap \mathbf{R}^+$

$$(2)\quad \varphi\colon \uparrow G_\vartheta \to \uparrow H_1(\uparrow\mathbf{R}/G_\vartheta), \qquad \varphi(\rho) = [pa_\rho] \qquad (\rho \in G_\vartheta \cap \mathbf{R}^+),$$
$$a_\rho\colon \mathbf{I} \to \mathbf{R}, \qquad a_\rho(t) = \rho t,$$

where $p\colon \uparrow\mathbf{R} \to \uparrow\mathbf{R}/G_\vartheta$ is the canonical projection.

Proof. First, let us consider the cubical subset $\uparrow[x, +\infty[$ $(x \in \mathbf{R})$ of $\uparrow\mathbf{R}$ and the following left homotopy of cubical sets (1.6.4; noting that it does preserve directed cubes)

$$(3)\quad f_n\colon c_n(\uparrow[x, +\infty[) \to c_{n+1}(\uparrow[x, +\infty[),$$
$$f_n(a)\colon (t_1,..., t_{n+1}) \mapsto x + t_1.(a(t_2,..., t_{n+1}) - x),$$
$$\partial^\alpha_{i+1} f_n = f_{n-1}\, \partial^\alpha_i, \qquad f_n e_i = e_{i+1} f_{n-1}.$$

Computing its faces ∂^α_1, f is a homotopy between the identity $f^+ = (\partial^+_1 f_n)$ and the map $f^- = (\partial^-_1 f_n)$, constant at x; therefore, every $\uparrow[x, +\infty[$ is contractible (to its minimum x). Since cubes of $\uparrow\mathbf{R}$ have a compact image in the line, it follows easily that also $\uparrow\mathbf{R}$ is acyclic.

Now, Theorem 3.3 proves that the cubical homology of $\uparrow\mathbf{R}/G_\vartheta$ coincides, algebraically, with the homology of the group G_ϑ, or of the space $\mathbf{T}^2$. It also proves that $H_1(\uparrow\mathbf{R}/G_\vartheta)$ is generated by the homology classes $[pa_1]$ and $[pa_\vartheta]$. Since $[pa_{\rho+\rho'}] = [pa_\rho] + [pa_{\rho'}]$, the mapping φ in (2) is an algebraic isomorphism. By construction, it preserves preorders, and we still have to prove that it reflects it.

To simplify the argument, a 1-chain z of $\uparrow\mathbf{R}$ which projects to a cycle $p_*(z)$ in $\uparrow\mathbf{R}/G_\vartheta$, or a boundary, will be called a *pre-cycle* or a *pre-boundary*, respectively. (Note that, since p_* is surjective, the homology of $\uparrow\mathbf{R}/G_\vartheta$ is isomorphic to the quotient of pre-cycles modulo pre-boundaries.) Let $z = \Sigma_i \lambda_i a_i$ be a positive pre-cycle, with all $\lambda_i > 0$; let us call $\lambda = \Sigma_i \lambda_i$ its *weight*. We have to prove that z is equivalent to a positive combination of pre-cycles of type a_ρ $(\rho \in G^+_\vartheta)$, modulo pre-boundaries.

Let $z = z'+z''$, putting in z' all the summands $\lambda_i a_i$ which are pre-cycles themselves, and replace any such a_i, up to pre-boundaries, with a_{ρ_i}, where $\rho_i = \partial^+ a_i - \partial^- a_i \in G^+_\vartheta$. If $z'' = 0$ we are done, otherwise $z'' = z - z'$ is still a pre-cycle; let us act on it. Reorder its paths a_i so that a_1 has a minimal coefficient λ_1 (strictly positive); since $\partial^+ a_1$ has to annihilate in $\partial p_*(z')$, there is some a_i $(i > 1)$ with $\partial^+ a_1 - \partial^- a_i \in G_\vartheta$. By a G_ϑ-translation of a_i (leaving pa_i unaffected), we can assume that $\partial^- a_i = \partial^+ a_1$, and then replace (modulo pre-boundaries) $\lambda_1 a_1 + \lambda_i a_i$ with $\lambda_1 \hat{a}_1 + (\lambda_i - \lambda_1) a_i$ where $\hat{a}_1 = a_1 * a_i$ is the concatenation (and $\lambda_i - \lambda_1 \geq 0$). Now, the new weight is $\lambda - \lambda_1 < \lambda$, strictly less than the previous one.

Continuing this way, the procedure ends in a finite number of steps; this means that, modulo pre-boundaries, we have changed z into a positive combination of pre-cycles of the required form, a_ρ.

Finally, it is easy to see that $H_2(\uparrow\mathbf{R}/G_\vartheta) = \mathbf{Z}$ gets the chaotic preorder. In fact, we already know that the 2-cycle

$$(4)\quad pa\colon [0, 1]^2 \to \mathbf{R}/G_\vartheta, \qquad a(t, t') = t\vartheta + t',$$

gives a (positive) generator of $\uparrow H_2$. But the interchange $s\colon [0, 1]^2 \to [0, 1]^2$ preserves the natural order of the square, whence $pa{\circ}s$ is also a positive cycle and $[pas] = -[pa]$ is also (weakly) positive.

□

4.9. Theorem. The c-sets $\uparrow\mathbf{R}/G_\vartheta$ and $\uparrow\mathbf{R}/G_{\vartheta'}$ are isomorphic if and only if the ordered groups $\uparrow G_\vartheta$ and $\uparrow G_{\vartheta'}$ are isomorphic, if and only if ϑ and ϑ' are conjugate under the action of $GL(2, \mathbf{Z})$ (4.1.1), if and only if ϑ' belongs to the closure $\{\vartheta\}_{RT}$ (4.1.2).

Proof. Follows immediately from Lemma 4.7 and Theorem 4.8, which gives the missing implication of the Lemma: if our c-sets are isomorphic, also their ordered groups $\uparrow H_1$ are, and $\uparrow G_\vartheta \cong \uparrow G_{\vartheta'}$ □

5. Pointed suspension and homology

Pointed suspension is well linked with *directed pointed homology*; the latter can also be viewed as a form of reduced homology, well adapted to preorder.

5.1. Pointed cubical sets. *Unpointed* and *pointed* suspension produce different results on the discrete two-point cubical set $\mathbf{s}^0 = \{0, 1\}$, since $\Sigma\mathbf{s}^0 = \uparrow\mathbf{o}^1$ (1.7.4) while, plainly, $\Sigma(\mathbf{s}^0, 0) = \uparrow\mathbf{s}^1$ (cf. 5.2); these cubical sets have different directed homology (2.3).

Since we are more interested in the spheres $\uparrow\mathbf{s}^n$, we shall consider the suspension (and homology) of *pointed* cubical sets. The latter form the category $\mathbf{Cub}_*$: an object (X, x_0) is a cubical set with a base point $x_0 \in X_0$; morphisms $f: (X, x_0) \to (Y, y_0)$ preserve the base points.

Again, limits and colimits are obvious: *limits* and *quotients* are computed as in **Cub** and pointed in the obvious way, whereas *sums* are quotients of the corresponding unpointed sums, under identification of the base points (as for pointed sets).

5.2. Pointed homotopies. The *pointed left (elementary) cylinder* is

(1) $I: \mathbf{Cub}_* \to \mathbf{Cub}_*,$ $\qquad I(X, x_0) = (IX/I\{x_0\}, [0\otimes x_0]),$

(2) $\partial^\alpha: (X, x_0) \to I(X, x_0),$ $\qquad \partial^\alpha(x) = [\alpha\otimes x],$

$e: I(X, x_0) \to (X, x_0),$ $\qquad e[u\otimes x] = e_1(x).$

Its right adjoint, the *pointed left (elementary) cocylinder*, is

(3) $P: \mathbf{Cub}_* \to \mathbf{Cub}_*,$ $\qquad P(Y, y_0) = (PY, \omega_0),$ $\qquad \omega_0 = e_1(y_0) \in Y_1.$

Again, an (immediate) *pointed left homotopy* $f: f^- \to_L f^+: (X, x_0) \to (Y, y_0)$ is defined as a map $f: I(X, x_0) \to (Y, y_0)$ with $f\partial^\alpha = f^\alpha$. Or, equivalently (because of the adjunction), as a map $f: (X, x_0) \to P(Y, y_0)$ with $\partial^\alpha f = f^\alpha$, which amounts to a family

(4) $f_n: X_n \to Y_{n+1},$ $\qquad \partial^\alpha_{i+1} f_n = f_{n-1}\, \partial^\alpha_i,$ $\qquad \partial^\alpha_1 f_n = f^\alpha,$

$e_{i+1} f_{n-1} = f_n\, e_i,$ $\qquad f_0(x_0) = \omega_0$ $\qquad (\alpha = \pm;\ i = 1,..., n).$

The pointed left upper cone $C^+(X, x_0)$ is a quotient of the pointed cylinder

(5) $$\begin{array}{ccc} (X, x_0) & \xrightarrow{\partial^+} & I(X, x_0) \\ \downarrow & & \downarrow \gamma \\ \{*\} & \xrightarrow[v^+]{} & C^+(X, x_0) \end{array} \qquad C^+(X, x_0) = (IX)/(I\{x_0\}\cup\partial^+X).$$

The pointed left suspension is the quotient

(6) $\Sigma(X, x_0) = (IX)/(\partial^-X \cup I\{x_0\} \cup \partial^+X)$.

Thus, the pointed suspension of $(\mathbf{s}^0, 0)$ yields the elementary directed circle $\uparrow\mathbf{s}^1$ (1.5.3)

(7)
$$\begin{array}{ccc} * & = & * \\ \| & & \uparrow \langle 1\otimes u\rangle \\ * & = & * \end{array}$$

and, more generally

(8) $(\uparrow\mathbf{s}^n, *) = \Sigma^n(\mathbf{s}^0, 0)$.

5.3. Pointed homology. A pointed cubical set (X, x_0) produces naturally a chain complex $\uparrow C_*(X, x_0)$ where the 0-component is the free preordered abelian group generated by the *pointed* set (X, x_0), so that the base point is annihilated

(1) $\uparrow C_0(X, x_0) = \uparrow\mathbf{Z}(X_0, x_0) = (\uparrow\mathbf{Z}X_0)/(\mathbf{Z}x_0)$,

(the functor $\uparrow\mathbf{Z}(–, –)$ being left adjoint to the forgetful functor $\mathrm{d}\mathbf{Ab} \to \mathbf{Set}_*$, $A \mapsto (A^+, 0)$).

We have thus the *pointed directed homology* of a pointed cubical set

(2) $\uparrow H_n\colon \mathbf{Cub}_* \to \mathrm{d}\mathbf{Ab}$, $\quad \uparrow H_n(X, x_0) = \uparrow H_n(\uparrow C_*(X, x_0))$,

which only differs from the unpointed one in degree zero, where $\uparrow H_0(X, x_0)$ is the free ordered abelian group generated by the *pointed set* of connected components of (X, x_0), or equivalently by the *set* of components different from the one of the base point.

Algebraically, $H_0(X, x_0)$ is plainly isomorphic to the *reduced homology* $\tilde{H}_0(X)$ of the underlying cubical set (defined as the kernel of the natural homomorphism $H_0(X) \to \mathbf{Z}$). But this is not true for preorders: this kernel inherits from $\uparrow H_0(X)$ a trivial (discrete) preorder, since the trace of the positive cone $\Sigma\, \lambda_i[x_i]$ $(\lambda_i \in \mathbf{N})$ on this kernel is $\{0\}$. One can also note that, independently of directions or preorders, and also for topological spaces, pointed homology (of pointed objects) *preserves sums* while reduced homology (of unpointed objects) does not.

Our next result gives again the ordered homology $\uparrow H_n(\uparrow\mathbf{s}^n) = \uparrow\mathbf{Z}$ ($n > 0$; cf. 2.3).

5.4. Theorem [Homology of suspension]. There is a natural isomorphism of preordered abelian groups (where $\langle - \rangle$ denotes equivalence classes in $\Sigma(X, x_0)$ as a quotient of $I(X, x_0)$, and u is the generator of the elementary interval $\uparrow\mathbf{i}$)

(1) $\uparrow H_n(X, x_0) \to \uparrow H_{n+1}(\Sigma(X, x_0))$, $\quad [\Sigma\, \lambda_k x_k] \mapsto [\Sigma\, \lambda_k \langle u\otimes x_k\rangle]$ $\qquad (n \geq 0)$.

Proof. First, let us note that, for $x \in X_n$, we have the following relation in $\uparrow C_{n+1}(\Sigma(X, x_0))$

(2) $\partial\langle u\otimes x\rangle = \langle 1\otimes x - 0\otimes x\rangle - \Sigma_{i,\alpha}\, (-1)^{i+\alpha} \langle u\otimes\partial_i^\alpha x\rangle = -\langle u\otimes\partial x\rangle$.

Now, the isomorphism is induced by the following inverse isomorphisms of preordered abelian groups, which *anti*-commute with differentials

(3) $f_n\colon \uparrow C_n(X, x_0) \to \uparrow C_{n+1}(\Sigma(X, x_0))$, $\quad f(x) = \langle u\otimes x\rangle$ $\qquad (x \in X_n)$,

$\partial f(x) = \partial\langle u\otimes x\rangle = -\langle u\otimes \partial x\rangle = -f(\partial x), \qquad f(x_0) = \langle u\otimes x_0\rangle = 0,$

$f(e_k y) = \langle u\otimes e_k y\rangle = \langle e_{k+1}(u\otimes y)\rangle = 0 \qquad\qquad (\text{for } n > 0,\ y\in X_{n-1});$

(4) $g_n\colon \uparrow C_{n+1}(\Sigma(X, x_0)) \to \uparrow C_n(X, x_0), \qquad g\langle u\otimes x\rangle = x,$

$g\partial\langle u\otimes x\rangle = -g\langle u\otimes \partial x\rangle = -\partial x = -\partial g\langle u\otimes x\rangle.$ □

6. Comparisons with other structures

Cubical sets have other directed realisations, besides $\uparrow\mathcal{R}X$ (1.9): for instance as spaces *with directed paths*, a structure studied in two previous works as a setting for directed homotopy [G4, G5]. Finally, we show that the algebraic part of our results on noncommutative tori can also be obtained using Scott's equilogical spaces [Sc], instead of cubical sets.

6.1. Spaces with directed paths. In [G4, G5] we used the following setting, to develop a theory of directed homotopy. A *space with directed paths*, or *d_1-space* T, is a topological space equipped with a set d_1T of (continuous) maps $a\colon \mathbf{I} \to T$, called *directed paths* (or *distinguished paths*, or *d-paths*), satisfying three axioms:

(i) (*constant paths*) every constant map $\mathbf{I} \to T$ is distinguished,

(ii) (*reparametrisation*) d_1T is closed under composition with (weakly) increasing maps $\mathbf{I} \to \mathbf{I}$,

(iii) (*concatenation*) d_1T is closed under path-concatenation: if the d-paths a, b are consecutive in T ($a(1) = b(0)$), then their ordinary concatenation $a+b$ is also a d-path

(1) $(a+b)(t) = a(2t),$ if $0 \leq t \leq 1/2, \qquad (a+b)(t) = b(2t-1),$ if $1/2 \leq t \leq 1.$

A *directed map*, or *d_1-map* $f\colon T \to T'$, is a continuous mapping between d_1-spaces which preserves the directed paths: if $a \in d_1T$, then $fa \in d_1T'$. This category will be denoted as d_1**Top** (it was written d**Top** in [G4, G5]; here, we want to stress the *one*-dimensional character of the structure, just consisting of *paths* instead of general *cubes*). Directed homotopy has been developed on the basis of the *standard directed interval* $\uparrow\mathbf{I}$, i.e. the euclidean interval equipped with all (weakly) increasing maps $\mathbf{I} \to \mathbf{I}$.

There is now a directed realisation of cubical sets as d_1-spaces, produced by an adjunction

(2) $\mathbf{Cub} \underset{\text{cub}^1}{\overset{\uparrow\mathcal{R}_1}{\rightleftarrows}} d_1\mathbf{Top}, \qquad \uparrow\mathcal{R}_1 \dashv \text{cub}^1,$

and intermediate between the ordinary realisation $\mathcal{R}X$ (1.8) and the directed realisation $\uparrow\mathcal{R}X$ (1.9), as we show below (6.3).

Here, cub^1T is the cubical set of all mappings $x\colon \mathbf{I}^n \to T$ such that, whenever we precompose with a continuous order-preserving mapping $a\colon \mathbf{I} \to \mathbf{I}^n$, we get a distinguished path. On the other hand, the d_1-space $\uparrow\mathcal{R}_1X$ is the geometric realisation $\mathcal{R}X$ (with its topology), equipped with the finite concatenations of paths $\hat{x}a\colon \mathbf{I} \to \mathbf{I}^n \to \mathcal{R}X$, where $a\colon \mathbf{I} \to \mathbf{I}^n$ is an order-preserving map and $\hat{x}$ corresponds to some cube $x\in X_n$, as in 1.8.3.

We only need to show that the classical bijection $\mathbf{Top}(\mathcal{R}X, T) = \mathbf{Cub}(X, \square T)$ (1.8.2), which sends $f: \mathcal{R}X \to T$ to the family $g_n: X_n \to \square_n T$, $g_n(x) = f\hat{x}$, restricts to a bijection $(\uparrow\mathcal{R}_1 X, T) = (X, \mathrm{cub}^1 T)$; in fact, the map f is a morphism of the specified type if and only if, for every $x \in X_n$ and every order-preserving mapping $a: \mathbf{I} \to \mathbf{I}^n$, $f(\hat{x}a)$ is a distinguished path in T (then, finite concatenations of paths of type $\hat{x}a$ also work, because T is a d_1-space); but $f(\hat{x}a) = (f\hat{x})a = g_n(x).a$, and our condition is equivalent to saying that $g_n(x)$ is an n-cube of $\mathrm{cub}^1 T$ (for all $x \in X_n$).

6.2. Comments. This category $d_1\mathbf{Top}$ is insufficient for our purposes here, because it only breaks reversion, *and not the interchange symmetry*. Thus, the directed 2-sphere $\uparrow\mathcal{R}_1(\uparrow\mathbf{s}^2)$ *living there* (called $\uparrow\mathbf{S}^2$ in [G4, G5]) would get the chaotic preorder on $\uparrow H_2$.

Yet, it seems difficult to develop a reasonable homotopy theory without the interchange symmetry, which is required - for instance - to prove the homotopy invariance of the cylinder, cone and suspension functors, and deduce important properties of the (co)fibration sequence [G5]. One could use sets with distinguished cubes *closed under connections and interchange*; but then, it is perhaps simpler and more effective to use objects with distinguished paths, whose cubes automatically have connections and interchange.

6.3. Comparison of directed realisations. Finally, we link the three realisations we have considered, $\mathcal{R}X$ (1.8), $\uparrow\mathcal{R}X$ (1.9) and $\uparrow\mathcal{R}_1 X$ (6.1), showing that $\mathcal{R}$ factors through $\uparrow\mathcal{R}_1$ and the latter through $\uparrow\mathcal{R}$ (including their adjunctions).

This will be better seen introducing a variant of $d_1\mathbf{Top}$: the category $d_1\mathbf{Set}$ of *d_1-sets*, constructed as $d_1\mathbf{Top}$ using sets and mappings instead of spaces and maps, with the exception that reparametrisation is still required for increasing *maps* $\mathbf{I} \to \mathbf{I}$. (It works similarly to $d_1\mathbf{Top}$ and has the advantage of being cartesian closed.) Then we can construct the following chain of adjunctions (left adjoints as dashed arrows)

$$(1) \qquad \mathbf{Cub} \underset{c_*}{\overset{\uparrow\mathcal{R}}{\rightleftarrows}} c\mathbf{Set} \underset{v_1}{\overset{d_1}{\rightleftarrows}} d_1\mathbf{Set} \underset{u_1}{\overset{t_1}{\rightleftarrows}} d_1\mathbf{Top} \underset{C_0}{\overset{U}{\rightleftarrows}} \mathbf{Top} \qquad (\uparrow\mathcal{R}_1: \mathbf{Cub} \dashrightarrow d_1\mathbf{Top}, \quad \mathcal{R}: \mathbf{Cub} \dashrightarrow \mathbf{Top})$$

$$\uparrow\mathcal{R}_1 = t_1 d_1.\uparrow\mathcal{R}, \qquad \qquad \mathcal{R} = U.\uparrow\mathcal{R}_1.$$

At the right hand, we have the forgetful functor $U: d_1\mathbf{Top} \to \mathbf{Top}$ (forgetting distinguished paths); its right adjoint C_0 equips a topological space with the *natural* d_1-structure, where the distinguished paths are the continuous ones [G4, 1.1]. Then, the forgetful functor $u_1: d_1\mathbf{Top} \to d_1\mathbf{Set}$ has a left adjoint t_1 which equips a d_1-set with the finest topology making all its distinguished paths continuous. Finally, the functor $d_1: c\mathbf{Set} \to d_1\mathbf{Set}$ completes the distinguished paths of $c_1 K$ under the closure conditions (i)-(iii), while its right adjoint $v_1: d_1\mathbf{Set} \to c\mathbf{Set}$ produces a c-structure on a d_1-set X, saying that a mapping $\mathbf{I}^n \to X$ is distinguished if and only if, whenever we precompose with a continuous order-preserving mapping $\mathbf{I} \to \mathbf{I}^n$, we get a distinguished path.

Now, it is easy to verify that every standard cube $\mathbf{I}^n$ has the final topology for the maps $\mathbf{I} \to \mathbf{I}^n$. Therefore, a mapping $\mathbf{I}^n \to T$ with values in a topological space is continuous if and only if all those precomposites are: this proves that $(-)_\square = v_1 u_1 C_0: \mathbf{Top} \to c\mathbf{Set}$ (1.9.1), whence also their left adjoints coincide, $Ut_1d_1 = t\colon c\mathbf{Set} \to \mathbf{Top}$ (1.9). Similarly, $cub^1\colon d_1\mathbf{Top} \to \mathbf{Cub}$ coincides with $c_*v_1u_1$ and $\uparrow\mathcal{R}_1 = t_1d_1.\uparrow\mathcal{R}$. Finally, $U.\uparrow\mathcal{R}_1 = Ut_1d_1.\uparrow\mathcal{R} = t.\uparrow\mathcal{R} = \mathcal{R}$.

6.4. Equilogical spaces. We end with some remarks, to be developed elsewhere, on a structure introduced by D. Scott [Sc]. An *equilogical space* (T, R) is a topological space T equipped with an equivalence relation R; a *map* f: (T, R) → (T', R') is a mapping T/R → T'/R' which admits *some* continuous lifting T → T'. The category **Eql** thus obtained contains **Top** as a full subcategory, identifying the space T with the pair $(T, =_T)$; moreover, **Eql** is cartesian closed. (We are dropping the condition that the support spaces be T_0, generally assumed but inessential; cf. [Ro].)

Singular cubes and singular homology have an obvious extension to equilogical spaces, setting $\square_n(T, R) = \mathbf{Eql}(\mathbf{I}^n, (T, R))$. And there is an embedding of equilogical spaces in c-sets (or in cubical sets)

(1) $\mathbf{Eql} \to c\mathbf{Set}, \qquad\qquad (T, R) \mapsto T_\square/R,$

consistent with singular cubes and singular homology, since a cube $\mathbf{I}^n \to (T, R)$ is the same as a mapping $\mathbf{I}^n \to T/R$ which can be continuously lifted to T, that is a distinguished cube of $T_\square/R$.

Now, it is easy to see that our result 4.2a on the group $G_\vartheta = \mathbf{Z}+\vartheta\mathbf{Z}$ acting on the real line *can also be stated in terms of the equilogical space* $(\mathbf{R}, \equiv_{G_\vartheta})$

(2) $H_*(\mathbf{R}, \equiv_{G_\vartheta}) \; = \; H_*(\mathbf{R}_\square/G_\vartheta) \; \cong \; H_*(\mathbf{T}^2).$

The deeper results on the cubical sets $C_\vartheta = \uparrow\mathbf{R}/G_\vartheta$ can be obtained with equilogical spaces equipped with an ordering. However, the directed homology of such a structure could hardly avoid the general drawbacks we have considered above, for d_1-spaces (6.2).

References

[ABS] F.A.A. Al-Agl - R. Brown - R. Steiner, *Multiple categories: the equivalence of a globular and a cubical approach*, Adv. Math. **170** (2002), 71-118.

[An] R. Antolini, *Geometric realisations of cubical sets with connections, and classifying spaces of categories*, Appl. Categ. Structures **10** (2002), 481-494.

[Bl] B. Blackadar, *K-theory for operator algebras*, Springer, Berlin 1986.

[BH1] R. Brown - P.J. Higgins, *On the algebra of cubes*, J. Pure Appl. Algebra **21** (1981), 233-260.

[BH2] R. Brown - P.J. Higgins, *Tensor products and homotopies for ω-groupoids and crossed complexes*, J. Pure Appl. Algebra **47** (1987), 1–33.

[C1] A. Connes, *C*-algèbres et géométrie différentielle*, C.R. Acad. Sci. Paris Sér. A **290** (1980), 599-604.

[C2] A. Connes, *Noncommutative geometry*, Academic Press, San Diego CA 1994.

[C3] A. Connes, *A short survey of noncommutative geometry*, J. Math. Physics **41** (2000), 3832-3866.

[GG] P. Gaucher - E. Goubault, *Topological deformation of higher dimensional automata*, Homology, Homotopy Appl. **5** (2003), 39-82. `http://www.rmi.acnet.ge/hha/volumes/2003/n2a3/`

[Go] E. Goubault, *Geometry and concurrency: a user's guide*, in: Geometry and concurrency, Math. Structures Comput. Sci. **10** (2000), no. 4, pp. 411-425.

[G1] M. Grandis, *Cubical monads and their symmetries,* in: Proceedings of the Eleventh International Conference on Topology, Trieste 1993, Rend. Ist. Mat. Univ. Trieste, **25** (1993), 223-262.

[G2] M. Grandis, *Categorically algebraic foundations for homotopical algebra*, Appl. Categ. Structures **5** (1997), 363-413.

[G3] M. Grandis, *Higher fundamental functors for simplicial sets*, Cahiers Topologie Géom. Différentielle Catég. **42** (2001), 101-136.

[G4] M. Grandis, *Directed homotopy theory, I. The fundamental category*, Cahiers Topologie Géom. Différentielle Catég. **44** (2003), 281-316.

[G5] M. Grandis, *Directed homotopy theory, II. Homotopy constructs*, Theory Appl. Categ. **10** (2002), No. 14, 369-391 (electronic). `http://tac.mta.ca/tac/`

[GM] M. Grandis - L. Mauri, *Cubical sets and their site*, Theory Appl. Categ. **11** (2003), No. 8, 185-211 (electronic). `http://tac.mta.ca/tac/`

[HW] P.J. Hilton - S. Wylie, *Homology theory*, Cambridge Univ. Press, Cambridge 1962.

[Ja] J.F. Jardine, *Cubical homotopy theory: a beginning,* Preprint (2002), `http://www.math.uwo.ca/~jardine/`

[KP] K.H. Kamps - T. Porter, *Abstract homotopy and simple homotopy theory*, World Scientific Publishing Co., River Edge NJ 1997.

[K1] D.M. Kan, *Abstract homotopy I*, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 1092-1096.

[K2] D.M. Kan, *Abstract homotopy II*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 255-258.

[Ma] S. Mac Lane, *Homology*, Springer, Berlin 1963.

[Ms] W. Massey, *Singular homology theory,* Springer, Berlin 1980.

[MP] C.J. Mulvey - J.W. Pelletier, *On the quantisation of spaces*, J. Pure Appl. Algebra **175** (2002), 289-325.

[Mu] J.R. Munkres, *Elements of algebraic topology*, Perseus Publ., Cambridge MA, 1984.

[PV] M. Pimsner - D. Voiculescu, *Imbedding the irrational rotation C^*-algebra into an AF-algebra*, J. Operator Th. **4** (1980), 93-118.

[Ra] M. Raussen, *State spaces and dipaths up to dihomotopy*, Homotopy Homology Appl. **5** (2003), 257-280. `http://www.rmi.acnet.ge/hha/volumes/2003/n2a9/`

[R1] M.A. Rieffel, *C^*-algebras associated with irrational rotations*, Pacific J. Math. **93** (1981), 415-429.

[R2] M.A. Rieffel, *Projective modules over higher-dimensional noncommutative tori*, Canad. J. Math. **40** (1988), 257-338.

[Ro] G. Rosolini, *Equilogical spaces and filter spaces*, Categorical studies in Italy (Perugia, 1997). Rend. Circ. Mat. Palermo (2) Suppl. No. **64**, (2000), 157-175.

[Sc] D. Scott, *A new category? Domains, spaces and equivalence relations*, Unpublished manuscript (1996). `http://www.cs.cmu.edu/Groups/LTC/`

[Se] J. P. Serre, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math. **54** (1951), 425-505.

[Ta] J. Tapia, *Sur la pente du feuilletage de Kronecker et la cohomologie étale de l'espace des feuilles*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), 427-429.

[To] A.P. Tonks, *Cubical groups which are Kan*, J. Pure Appl. Algebra **81** (1992), 83-87.