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A FEW POINTS ON DIRECTED ALGEBRAIC TOPOLOGY

by Marco Grandis (*)

Dedicated to Charles Ehresmann, on the centennial of his birth

Abstract

Directed Algebraic Topology studies 'directed spaces' in some sense, where paths and homotopies cannot generally be reversed; for instance: simplicial and cubical sets, ordered topological spaces, 'spaces with distinguished paths', 'inequilogical spaces', etc. Its present applications deal mostly with the analysis of concurrent processes (see [2, 3] and references there), but its natural range should cover non reversible phenomena, in any domain.

Here, after a review of a series of papers devoted to this subject ([4] to [8]), we shall give some hints at future developments. A wider literature can be found in the papers mentioned above.

Directed spaces can be studied with directed versions of the classical tools of Algebraic Topology. Thus, the *directed homology groups* $\uparrow H_n(X)$ introduced in [5, 6] are *preordered* abelian groups. Similarly, the *fundamental category* $\uparrow\Pi_1(X)$ [4, 7] replaces the classical fundamental groupoid and allows one to study situations where all directed loops are trivial. The study of higher fundamental categories has begun [8].

Directed Algebraic Topology has thus a deep interaction with ordinary and higher dimensional category theory. Unexpected links with Noncommutative Geometry have appeared, brought about by orbit spaces or spaces of leaves which would be trivial in ordinary topology but can be realised in both domains. For instance, the well-known irrational-rotation C*-algebras A_ϑ represent 'noncommutative spaces' which can also be modelled by cubical sets and inequilogical spaces, so that the classification of the former by *ordered* K-theory [10, 11] corresponds to the classification of the latter by *directed* homology [5]. Other links have appeared between the notion of *root* of a category developed by A.C. Ehresmann [1], for modelling biological systems, and our study of the fundamental category [7].

Finally, we shall examine some links with Differential Geometry and describe the beginning of a formal setting for Directed Algebraic Topology, based on Kan ideas (formalising the (co)cylinder functor, cf. [9]) rather than on Quillen model structures,

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which do not seem to be adapted to abstract privileged directions and directed homotopies.

I am grateful for the opportunity of presenting these results in this Conference and on this Journal. The position of Directed Algebraic Topology, at the confluence of Topology, Geometry and Category Theory, can presumably be viewed as coherent with the research lines pursued by Charles and Andrée C. Ehresmann, in their work and with this Journal.

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