Genova, March 17, 2023

Genova ricorda Lawvere

Lawvere metric spaces in Weighted Algebraic Topology

Marco Grandis

1. Lawvere metric spaces as directed and weighted spaces

Lawvere generalised metric space [Lw] (1973), or *δ-space*:

- a set X equipped with a *δ-metric*: $\delta\colon X \times X \to [0, \infty]$

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δ**Mtr**: the category of *δ-spaces and weak contractions*

$$f\colon X \to Y, \qquad \delta(x, x') \geqslant \delta(f(x), f(x'))$$

- symmetric monoidal closed:

$$X \otimes Y, \qquad \delta((x, y), (x', y')) = \delta(x, x') + \delta(y, y').$$

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δ_∞**Mtr** : the larger category of *δ-spaces and Lipschitz maps*:

$$L\delta(x, x') \geqslant \delta(f(x), f(x')) \text{ (some } L \geqslant 0).$$

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1. allowing ∞: $\rightarrow$ δ**Mtr** is complete and cocomplete
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3. The associated symmetric δ-metric (and topology):

$! : \delta\mathbf{Mtr} \rightarrow \mathbf{Mtr}$ (the reflector):

$$!\delta(x, x') = \inf_{\mathbf{x}} \left(\sum_j \left(\delta(x_{j-1}, x_j) \wedge \delta(x_j, x_{j-1})\right)\right)$$

$$\mathbf{x} = (x_0, ..., x_p), \quad x_0 = x, \quad x_p = x',$$

greatest symmetric δ-metric $\leqslant \delta$.

2. Some models and their directed structure

The standard δ-line $\delta\mathbb{R}$: $\delta(x, y) = y - x$ if $x \leqslant y$, OR ∞

(No going back. Reflector: euclidean metric. Corefl.: discrete).

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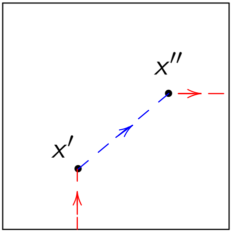

$\delta\mathbb{S}^2$

$\delta(x', x'')$, $\delta(x'', x')$

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The weighted structure of a δ-space:

a path $a\colon [0,1] \to X$ in a δ-space has a *length*, or cost:

$L(a) = \sup_{\mathbf{t}} L_{\mathbf{t}}(a) \in [0,\infty]$,

$L_{\mathbf{t}}(a) = \sum \delta(a(t_{j-1}), a(t_j))$ $(\mathbf{t}\colon 0 = t_0 < t_1 < \ldots < t_p = 1)$.

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The directed structure of a δ-space:

a path a is distinguished, or allowed, if $L(a) < \infty$
(a consequence of the weighted structure).

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DAT arose in the 1990's ([Gr], 2009; downloadable)

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directed spaces *can* have a 'direction', and non-reversible paths.

General aim: modelling (possibly) non-reversible phenomena.

Intended applications: domains where privileged directions appear:

- concurrent processes,
- traffic networks,
- space-time models,
- rewrite systems, etc.

3A. Settings for DAT

A basic setting and the most used one:

p**Top**: category of *preordered topological spaces*,

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closed under: constant paths, concatenation, partial reparam.

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Forgetful functors:

- $\mathrm{p}\mathbf{Top} \to \mathrm{d}\mathbf{Top}$ *distinguished paths*: the monotone ones,
- $\delta\mathbf{Mtr} \to \mathrm{d}\mathbf{Top}$ *distinguished paths*: $L(a) < \infty$,
- $\delta\mathbf{Mtr} \to \mathrm{p}\mathbf{Top}$ *preorder*: $x \prec x'$ if $\delta(x, x') < \infty$.

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Remarks: the p-space associated to $\delta\mathbb{S}^n (n \geqslant 1)$ has no interest
(chaotic preorder),
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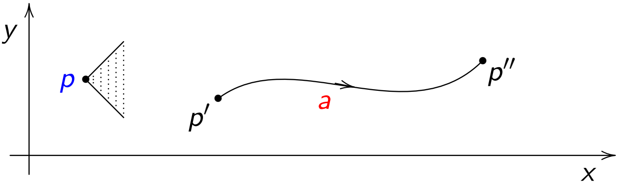
Other settings: cubical sets, inequilogical spaces, flows, etc.

4. DAT: elementary examples in $\mathrm{p}\mathbf{Top}$

Euclidean plane $\mathbb{R}^2$ with order relation:



$$(x, y) \leqslant (x', y') \iff |y' - y| \leqslant x' - x.$$

- cone of the future at the point p (the set of points $\geqslant p$),
- directed path a from p' to p'' (a monotone map $[0, 1] \to \mathbb{R}^2$),
- there is no (directed) path from p'' to p',
- every loop is constant.

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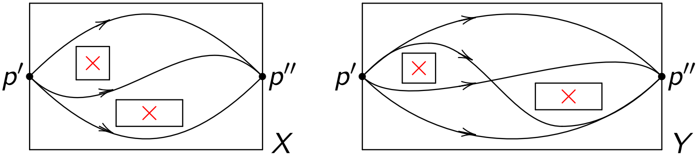
Two subspaces X, Y, with directed paths from p' to p''



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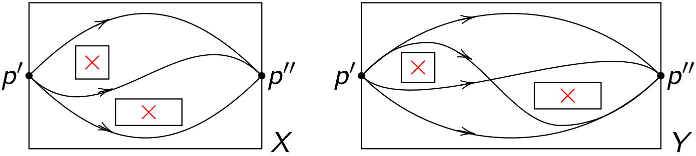

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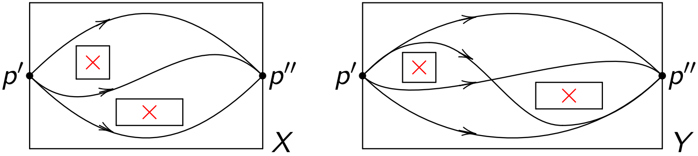

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Interpretations:

1. a stream with two islands; order: upper bound for relative v,
2. time and one-dim. space, with linear obstacles; order: $v \leqslant 1$,

(3. execution paths of two concurrent automata with conflict of resources; order: time progression [FGR].)

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3. DAT studies spaces by directed homotopy and homology (enriched versions):
- $\uparrow\Pi_1 : \mathrm{d}\mathbf{Top} \to \mathbf{Cat}$, fundamental category of a d-space,
- $\uparrow H_n : \mathrm{d}\mathbf{Top} \to \mathrm{p}\mathbf{Ab}$, n-th homology preordered group.

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4. Examples:

$\uparrow\Pi_1(\uparrow\mathbb{S}^1)$ = a subcategory of the groupoid $\Pi_1(\mathbb{S}^1)$,

$\uparrow H_1(\uparrow\mathbb{S}^1) = (\mathbb{Z}, \leqslant)$, ordered abelian group.

5. Something on Weighted Algebraic Topology

Weighted Algebraic Topology: Reference: [Gr], Chapter 6.

An enriched version of DAT:

paths have a *length* (or cost, duration,...) in $[0, \infty]$,
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w**Top**: category of *w-spaces and cost-decreasing maps*

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C*-algebra: involutive Banach $\mathbb{C}$-algebra, with

$$(xy)^* = y^* x^*, \qquad ||xx^*|| = ||x||^2.$$

Gelfand duality: $C : \mathbf{HCmp} \to \mathrm{UComC}^*\mathbf{Alg}^{\mathrm{op}}$, equivalence of cat.

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An example: ϑ is an irrational number.

$G_\vartheta = \mathbb{Z} + \vartheta\mathbb{Z}$: ordered subgroup of $\mathbb{R}$; acts on $\mathbb{R}$ by translations

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This 'object' *should* have H_1 generated by two cycles, of length 1 and ϑ! (Non-trivial and independent in DAT and WAT).

6A. Interpretation in Non-commutative Geometry

This trivial space is replaced by a non-commutative C*-algebra:

A_ϑ: the *irrational rotation algebra* associated with ϑ

generated by two unitary elements u, v: $vu = \exp(2\pi\vartheta).uv$.

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Classifications (up to isomorphism or strong Morita equivalence):

$A_\vartheta \cong A_{\vartheta'} \iff \vartheta' \in \pm\vartheta + \mathbb{Z}$,

$A_\vartheta \sim_M A_{\vartheta'}$: also characterised

by fractional action of 2×2 matrices with integral entries.

6B. Interpretation in Weighted Algebraic Topology

This trivial space is replaced by a w-space:

$W_\vartheta = \mathrm{w}\mathbb{R}/G_\vartheta$: the *irrational rotation w-space* associated with ϑ

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$D_\vartheta = \uparrow\mathbb{R}/G_\vartheta$: a quotient of the d-line (increasing paths)

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- DAT gives a 'Non-commutative Topology', without weights.
- The quotient $\delta\mathbb{R}/G_\vartheta$ has the chaotic δ-metric and topology.

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