Higher fundamental groupoids for spaces (*)

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Abstract. Fundamental n-groupoids for a topological space are introduced, by techniques based on Moore paths, similar to those used in [G3] for symmetric simplicial sets. Also the 'directed case' is treated, based on a structure introduced in [G6]: a *directed topological space*, where privileged directions are assigned and paths need not be reversible; such objects are provided with fundamental n-categories, as it was done for ordinary simplicial sets in [G3].

We end by comparing the present structures with the previous ones, via a geometric realisation of symmetric and ordinary simplicial sets, as spaces and directed spaces, respectively. All this essentially agrees also with the classical treatment of Kan complexes as non-directed structures.

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Introduction

Recently, a homotopy 2-groupoid for Hausdorff spaces [HKK1] has been defined; various notions of weak n-groupoids have been considered, in connections with homotopy: cf. [Ba, HKK2, T1, T2] and their references.

Here, to define the higher fundamental groupoids $\Pi_n X$ of a topological space, we start from the space $\mathbb{P}X$ of *Moore paths*, a (strict) involutive category (2.1). The usual structure (faces, degeneracy, etc.) produces a cubical object $\mathbb{P}^*X = (\mathbb{P}^n X)_{n\geq 0}$ with associative concatenations, in fact a cubical ω-category. A quotient $\mathcal{P}X$ of the underlying set $|\mathbb{P}X|$, modulo *delays* and *regressions*, has the effect of annihilating constant paths and converting reverse paths into strict inverses: $\mathcal{P}X$ is now a groupoid (2.5), called the groupoid of *strongly reduced paths*.

From this we derive a cubical ω-groupoid $\mathcal{P}^*X$ (with connections); and then we extract from the latter an ordinary (i.e., globular) ω-groupoid $\Pi_\omega X = \mathcal{P}_* X$, the *fundamental* ω-*groupoid* of X (in the same way as one extracts a 2-category from a double one). Finally, the *fundamental* n-*groupoid* $\Pi_n X = \rho_n(\Pi_\omega X)$ is simply obtained by applying the reflector of n-groupoids, i.e. truncating $\Pi_\omega X$ at degree n and replacing the last component $\mathcal{P}_n X$ with the coequaliser of the faces $\mathcal{P}_{n+1}X \rightrightarrows \mathcal{P}_n X$.

In Section 3, we give a 'directed' version of all this. Directed Algebraic Topology is a recent subject, whose domain should be distinguished from classical Algebraic Topology by the principle that *directed spaces have privileged directions and directed paths therein need not be reversible*. Thus,

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ordinary homotopies and fundamental n-groupoids are replaced with *directed homotopies* and *fundamental n-categories*. Its applications deal with domains where privileged directions appear, like concurrent processes, traffic networks, space-time models, etc. [FGR, Ga, GG, Go, G3].

Following our setting, in [G6], a *directed topological space*, or *d-space*, is a topological space X equipped with a set dX of 'directed paths' $[0, 1] \to X$, containing all constant paths and closed under increasing reparametrisation and concatenation. Such objects, called *directed spaces* or *d-spaces*, form a category d**Top** which has general properties similar to **Top** (topological spaces): limits and colimits exist and are easily computed, and all directed intervals $\uparrow[i, j] \subset \uparrow\mathbf{R}$ are *exponentiable* (3.2.1).

One can give, now, a treatment parallel to the previous one, omitting all points depending on reversion. The space $\uparrow\mathbb{P}X$ of *directed Moore paths* produces a cubical object $\uparrow\mathbb{P}^*X = (\uparrow\mathbb{P}^nX)_{n\geq0}$. Annihilating constant paths, by the quotient of the set $|\uparrow\mathbb{P}X|$ modulo delays, we obtain the category $\uparrow\mathbf{P}X$ of *reduced paths*. The induced structure on $\uparrow\mathbf{P}^*X$ is a cubical ω-category (with connections), from which we extract the *fundamental* ω*-category* of X, $\uparrow\Pi_\omega X = \uparrow\mathbf{P}_*X$, and derive the *fundamental* n*-category* $\uparrow\Pi_nX = \rho_n(\uparrow\Pi_\omega X)$ as above.

The last section compares these constructs with the fundamental categories (groupoids) of (symmetric) simplicial sets, introduced in previous works [G3, G5]. We view a simplicial set K as a *directed* notion: it has fundamental n-categories, defined in [G3], and a directed realisation $\uparrow\mathcal{R}K$ *as a directed space*, constructed here (4.3-5), from which a comparison with the present fundamental n-categories is derived (4.6). Its symmetric analogue, a *symmetric simplicial set*, is a presheaf on the category $!\mathbb{A}$ of positive finite cardinals (characterised by generators and relations in [G4]); it can be realised as a space and has fundamental n-groupoids [G3], in comparison with the present ones (4.7).

Finally, let us note that the present approach 'essentially' agrees with the classical use of simplicial sets as a non-directed structure. First, our directed realisation $\uparrow\mathcal{R}K$ is the ordinary geometric realisation $\mathcal{R}K$ with an additional d-structure. Second, the classical homotopy groups $\pi_n(K, x)$ of a simplicial set are defined if K is a Kan complex (and coincide with the ones of $\mathcal{R}K$); but in this case, our n-homotopy monoid $\uparrow\pi_n(K, x)$ is in fact a group and coincides with the former (4.2.2).

Our study of simplicial sets aims to freeing their basic homotopy theory from the classical restriction to Kan complexes. Theoretically, it is thus possible to obtain general results as the adjunction $\uparrow\Pi_n$: **Smp** $\rightleftarrows$ n-**Cat** :N_n [G3] (or its symmetric analogue); practically, one need not replace elementary simplicial models with infinite versions satisfying the Kan condition.

1. The standard path functor

We begin by considering the lattice-like structure of the standard interval [0, 1] and the derived structure of the standard path functor in **Top**, the category of topological spaces. The index α takes values –, +.

1.1. The standard interval. The real interval $\mathbf{I} = [0, 1]$ will have the usual structure of involutive lattice, with involution $r(t) = 1 - t$. Forgetting some properties which are not relevant for homotopy and uselessly restraining, e.g. the idempotence of the lattice operations, the structure we are interested in is a *commutative, involutive cubical monoid* [G1]: a set equipped with two structures of commutative monoid (the *connections* g^α), where the unit (or *face* ∂^α) of each operation is an absorbent element for the other, and the involution (or *reversion* r) turns each structure into the other; adding the *degeneracy* e and the *interchange* s (which are determined by the cartesian structure)

$$(1) \qquad \{*\} \underset{e}{\overset{\partial^\alpha}{\rightleftarrows}} \mathrm{I} \overset{g^\alpha}{\longleftarrow} \mathrm{I}^2 \qquad\qquad r: \mathbf{I} \to \mathbf{I}, \qquad s: \mathbf{I}^2 \to \mathbf{I}^2,$$

$$\partial^-(*) = 0, \qquad \partial^+(*) = 1, \qquad e(t) = *, \qquad g^-(t, t') = t \vee t', \qquad g^+(t, t') = t \wedge t',$$

$$r(t) = 1 - t, \qquad s(t, t') = (t', t),$$

our seven maps satisfy rather obvious axioms (whose dual can be found below, 1.2.2).

The standard cylinder $I(X) = X \times \mathbf{I}$ inherits seven natural transformations, which will be denoted with the same letters: $\partial^\alpha: 1 \to I$, and so on; they satisfy similar axioms.

1.2. The cubical comonad of paths. The *path endofunctor* $P: \mathbf{Top} \to \mathbf{Top}$, $PX = X^{[0, 1]}$ (right adjoint to I) also inherits seven natural transformations (produced by the maps 1.1.1, contravariantly), which collect the basic properties of homotopy: two faces ∂^α, a degeneracy e, two connections g^α and two symmetries, the reversion r and the interchange s

$$(1) \qquad 1 \underset{e}{\overset{\partial^\alpha}{\leftrightarrows}} P \overset{g^\alpha}{\rightrightarrows} P^2 \qquad\qquad r: P \to P, \qquad s: P^2 \to P^2,$$

$$\partial^-(a) = a(0), \qquad e(x)(t) = x, \qquad g^-(a)(t, t') = a(t \vee t'), \ \ldots$$

They satisfy the axioms of a *cubical comonad with symmetries* [G1, G2]):

(2) $\partial^\alpha.e = 1$, $\qquad g^\alpha.e = Pe.e \ (= eP.e)$ $\qquad$ (*degeneracy*),

$Pg^\alpha.g^\alpha = g^\alpha P.g^\alpha$ $\qquad P\partial^\alpha.g^\alpha = 1 = \partial^\alpha P.g^\alpha$ $\qquad$ (*associativity, unit*),

$P\partial^\alpha.g^\beta = e.\partial^\alpha = \partial^\alpha P.g^\beta$ $\qquad$ (*absorbency*; $\alpha \neq \beta$),

$r.r = 1$, $\qquad r.e = e$, $\qquad \partial^-.r = \partial^+$, $\qquad g^-.r = Pr.rP.g^+$,

$s.s = 1$, $\qquad s.Pe = eP$, $\qquad P\partial^\alpha.s = \partial^\alpha P$, $\qquad s.g^\alpha = g^\alpha$,

$Pr.s = s.rP$ $\qquad$ (*symmetries*).

The trivial path at $x \in X$ will be written $0_x = e(x)$, while $-a = r(a)$ will denote the reverse path.

1.3. Concatenation. Moreover, we have to consider the concatenation a+b of two consecutive paths a, b,

(1) $(a+b)(t) = a(2t)$, if $0 \leq t \leq 1/2$, $\qquad (a+b)(t) = b(2t - 1)$, if $1/2 \leq t \leq 1$,

and we need to describe formally also this procedure. This is done by another natural transformation

(2) $k: QX \to PX$, $\qquad k(a, b) = a+b$,

defined on the space of pairs of consecutive paths, the *concatenation pullback* $QX = PX \times_X PX$

(3) $QX = \{(a, b) \mid \partial^+a = \partial^-b\}$

$$\begin{array}{ccc} QX & \xrightarrow{k^+} & PX \\ {\scriptstyle k^-}\downarrow & & \downarrow{\scriptstyle \partial^-} \\ PX & \xrightarrow{\partial^+} & X \end{array}$$

The concatenation k satisfies the following axioms [G2]:

(4) $\partial^-k = \partial^-k^-$, $\quad \partial^+k = \partial^+k^+$, $\quad ke_Q = e$,

$kr_Q = rk$, $\quad kP.s' = s.Pk$,

where $e_Q: X \to QX$, $r_Q: QX \to QX$, and $s': PQX \to QPX$ are the obvious maps induced by e, r, and s respectively.

2. Higher fundamental groupoids for topological spaces

The functor of Moore paths $\mathbb{P}$ has a similar structure but an associative concatenation, which goes down to its quotient, the functor of strongly reduced paths $\mathcal{P}$; the latter produces our higher homotopy groupoids (2.7).

2.1. Moore paths. A first way of obtaining an associative concatenation is to consider *Moore paths*, or *paths with duration* (here slightly modified with respect to the usual presentation), forming the following space

(1) $\mathbb{P}X = \Sigma_{rs} X^{[i, j]}$,

a sum of compact-open topologies. The sum is indexed over the set r**s** of *supports*, the real compact intervals $[i, j] \subset \mathbf{R}$, where $i \leq j$ are integers, for simplicity. $\mathbb{P}X$ consists thus of all pairs (ρ, a) where $\rho \in r\mathbf{s}$ is a support and $a: \rho \to X$ is a continuous mapping. The advantage on concatenation is paid with the existence of infinitely many constant paths at the same point $x \in X$, one for each support $[i, j]$.

$\mathbb{P}$ has obvious faces, degeneracy, connections and symmetries, similar to the ones of P (1.2) and described below by simply specifying the resulting support (except for faces)

(2) $\partial^-(a: [i, j] \to X) = a(i)$, $\quad \partial^+(a: [i, j] \to X) = a(j)$,

$\mathrm{supp}(e(x)) = [0, 0]$, $\quad \mathrm{supp}(g^\alpha(\rho, a)) = \rho{\times}\rho$,

$\mathrm{supp}(r(\rho, a)) = -\rho$, $\quad \mathrm{supp}(s(\rho{\times}\sigma, a)) = \sigma{\times}\rho$.

Such transformations *nearly* form a cubical comonad with symmetries: the only axiom which fails (with respect to the list 1.2.2) is absorbency of connections, 'because' of constant paths. In fact, $\mathbb{P}\partial^+.g^- = \partial^+\mathbb{P}.g^-: \mathbb{P} \to \mathbb{P}$ does not coincide with $e.\partial^+$; the former map sends a pair (ρ, a) to a constant path having the same duration, while $e.\partial^+(\rho, a)$ has a fixed duration, $[0, 0]$.

Concatenation makes $\mathbb{P}X$ into a strict involutive category (in **Top**). Formally, it is defined on the pullback $\mathbb{Q}X = \mathbb{P}X \times_X \mathbb{P}X$ of consecutive pairs of Moore paths

(3) $k: \mathbb{Q} \to \mathbb{P}$: **Top** $\to$ **Top**, $\quad k(\rho, a; \sigma, b) = (\rho+\sigma, a+b)$,

and $a+b = c$ has support $\rho+\sigma = [\rho^- + \sigma^-, \rho^+ + \sigma^+]$ and pasting point at $\rho^+ + \sigma^-$

(4) $c(t) = a(t - \sigma^-)$, for $\rho^- + \sigma^- \leq t \leq \rho^+ + \sigma^-$,

$c(t) = b(t - \rho^+)$, for $\rho^+ + \sigma^- \leq t \leq \rho^+ + \sigma^+$.

Then, k does satisfy the axioms of concatenation (1.3.4). Implicitly, we have used on r**s** a structure of commutative involutive monoid, in additive notation:

(5) $[i, j] + [i', j'] = [i+i', j+j']$, $0 = [0, 0]$, $-[i, j] = [-j, -i]$.

Now, we have to identify all constant paths at the same point, and to force reverse paths to become inverses; a few technical tools are needed.

2.2. Tolerance sets. As in [G3], a very simple combinatorial structure will be of much use. A *tolerance set* X is a set equipped with a tolerance relation x!x' (reflexive and symmetric); it can be viewed as an elementary simplicial complex, where a finite subset is distinguished if and only if all its pairs belong to the tolerance relation. A *tolerance map*, or *combinatorial mapping*, between such objects preserves the tolerance relation. The resulting category **Tol** has all limits and colimits and is cartesian closed ([G3], 1.4).

The *combinatorial (integral) line* **Z** is the set of integers with the structure of *contiguity*. Precisely, **Z** is a tolerance set, with tolerance relation i!j if $|i - j| \leq 1$. An integral interval $[i, j]_{\mathbf{Z}} \subset \mathbf{Z}$ has the induced structure $(i \leq j)$.

The geometric realisation of **Z** (as a simplicial complex) is the real line **R**. A combinatorial mapping $f: \mathbf{Z} \to \mathbf{Z}$ has a *real extension* (denoted by the same letter), its geometric realisation

(1) $f: \mathbf{R} \to \mathbf{R}$, $f(i+t) = (1-t).f(i) + t.f(i+1)$ $(0 \leq t \leq 1)$,

which is piecewise affine (affine on each real interval $[i, i+1]$, $i \in \mathbf{Z}$). Similarly, a combinatorial mapping $f: [i, j]_{\mathbf{Z}} \to [i', j']_{\mathbf{Z}}$ between integral intervals has a *real extension* to the corresponding real intervals. And a product of combinatorial mappings $f_1 \times ... \times f_n \in \mathbf{Tol}(\mathbf{Z}^n, \mathbf{Z}^n)$ has a real extension in $\mathbf{Top}(\mathbf{R}^n, \mathbf{R}^n)$. (This extension, multi-affine on every 'elementary cube', can be viewed as a cubical geometric realisation; but can also be derived from the classical geometric realisation of simplicial complexes, cf. 4.7.)

2.3. Delays and regressions. In particular, we are interested in the *elementary delays* $\delta_i: \mathbf{Z} \to \mathbf{Z}$ ([G3], 2.3) and the *elementary regressions* $\delta_i: \mathbf{Z} \to \mathbf{Z}$ ([G3], 3.5)

(1) $\delta_i(t) = t$ if $t \leq i$, $\delta_i(t) = t - 1$ otherwise,

(2) $\gamma_i(t) = t$ if $t \leq i$, $\gamma_i(t) = t - 2$ otherwise.

The submonoid of *integral delays* $D_1 \subset \mathbf{Tol}(\mathbf{Z}, \mathbf{Z})$, generated by the elementary delays, consists of all surjective increasing mappings (necessarily combinatorial), which are the identity for t sufficiently small in **Z**, and strictly increasing for t sufficiently large. We write $\bar{D}_1$ the larger submonoid of *generalised integral delays,* generated by elementary delays and elementary translations; it consists of all surjective increasing mappings $\mathbf{Z} \to \mathbf{Z}$ which are strictly increasing for t sufficiently small or large; finally, we write Γ_1 the submonoid generated by $\bar{D}_1$ and elementary regressions, which is formed of all surjective combinatorial mappings $\mathbf{Z} \to \mathbf{Z}$ which are strictly increasing for t sufficiently small or large (but not necessarily increasing).

Their real extensions are written as $rD_1 \subset r\bar{D}_1 \subset r\Gamma_1 \subset \mathbf{Top}(\mathbf{R}, \mathbf{R})$.

Elementary delays and regressions satisfy the following identities

$$\text{(3)} \quad \delta_i.\delta_{j+1} = \delta_j.\delta_i \quad (i \leq j),$$

$$\gamma_i.\gamma_{j+2} = \gamma_j.\gamma_i \quad (i \leq j), \qquad \gamma_i.\gamma_{i+2} = \gamma_i.\gamma_{i+1} = \gamma_i.\gamma_i,$$

$$\delta_i.\gamma_{j+1} = \gamma_j.\delta_i \quad (i \leq j), \qquad \delta_i.\gamma_j = \gamma_j.\delta_{i+2} \quad (i \geq j).$$

Therefore, the submonoids D_1, $\bar{D}_1$ and Γ_1, as well as their real extensions, satisfy a *cofiltering property*

(4) for any two maps f_1, f_2, there are maps e_1, e_2 such that $f_1e_1 = f_2e_2$,

which derives immediately from (3) for the generators, and can easily be extended to general arrows.

We form now three subcategories of **Tol**, $\mathbf{s} \subset \mathbf{d}^{op} \subset \mathbf{c}^{op}$ (as in [G3], Section 3). Their objects are the finite integral intervals $[i, j]_\mathbf{Z}$. The category **s** is discrete; its object-set is equipped with a structure of commutative involutive monoid, as in 2.1.5. Then $\mathbf{d}^{op}$ is the category of *generalised delays*, or surjective increasing maps $d: [i', j']_\mathbf{Z} \to [i, j]_\mathbf{Z}$; it is generated by translations and elementary delays $\delta_i: [h, k]_\mathbf{Z} \to [\delta_i(h), \delta_i(k)]_\mathbf{Z}$. Finally, $\mathbf{c}^{op}$ is the category of all combinatorial surjective mappings $c: [i', j']_\mathbf{Z} \to [i, j]_\mathbf{Z}$, and is generated by translations, elementary delays and elementary regressions.

These categories have (isomorphic) *real analogues* $r\mathbf{s} \subset r\mathbf{d}^{op} \subset r\mathbf{c}^{op} \subset \mathbf{Top}$: the objects are the compact real intervals with integral extrema, the mappings are the real extensions of the previous maps. The first category has already been used to define $\mathbb{P}$, the others will be used to introduce two new 'path functors' $\mathbf{P}$, $\mathcal{P}$.

2.4. Reduced paths. As a second way of getting an associative concatenation, one can identify the paths of the *set* $|\mathbb{P}X|$ 'up to delays', and therefore the various results given by concatenation on different supports; this produces an involutive category $\mathbf{P}X$ where all constant paths become identities.

The functor $\mathbf{P}: \mathbf{Top} \to \mathbf{Set}$ can be obtained as a colimit based on the category $r\mathbf{d} \subset \mathbf{Top}^{op}$ (2.3)

$$\text{(1)} \quad \mathbf{P}X = \mathrm{Colim}_{r\mathbf{d}}\, \mathbf{Top}([i, j], X), \qquad \mathbb{P}X \twoheadrightarrow \mathbf{P}X = |\mathbb{P}X|/\!\equiv,$$

and also as a quotient of the *set* $|\mathbb{P}X|$ modulo the congruence *up to delays*: $(\rho, a) \equiv (\sigma, b)$ means that there exist two real delays $d, d' \in rD_1 \subset \mathbf{Top}(\mathbf{R}, \mathbf{R})$ such that $ad = bd'$ (2.3). Congruence is an equivalence relation, by the cofiltering property (2.3.4).

Note that *any path is congruent to its* (integral) *translations*, since $a\delta_i(t)$ is the path a 'delayed of one unit', when $i \leq \rho^-(a)$. Therefore we can equivalently use, in the definition of the previous congruence $\equiv$, the larger submonoid $r\bar{D}_1 \subset \mathbf{Top}(\mathbf{R}, \mathbf{R})$ of *generalised real delays*. A congruence class of paths $a^\bullet$ will be called a *reduced path* of X.

It is easy to show that *all admissible concatenations of two given paths* a, b *are congruent*: let c be one of them, derived from supports ρ, σ; then, varying ρ^- or σ^+ has no effect on c, while increasing ρ^+ (resp. σ^-) of one unit yields a concatenation $c' = c\delta_i$ delayed of one unit at a suitable instant. Thus, the concatenation $k: \mathbb{Q} \to \mathbb{P}$ induces a concatenation $k: \mathbf{Q} \to \mathbf{P}$ defined over the pullback $\mathbf{Q}X$ *of pairs of consecutive reduced paths*

(2) $k: \mathbf{Q} \to \mathbf{P}: \mathbf{Top} \to \mathbf{Set}, \qquad \mathbf{Q}X = \mathbf{P}X \times_{|X|} \mathbf{P}X = (|\mathbb{P}X| \times_{|X|} |\mathbb{P}X|)/(\equiv \times \equiv),$

and each $\mathbf{P}X$ is again a strict involutive category on X.

2.5. Strongly reduced paths. One can transform the involution of $\mathbf{P}X$ in a strict inversion, by a further identification of paths 'up to regressions', so that any concatenation $a - a$ is identified with the trivial path at the initial point $\partial^- a$.

The endofunctor $\mathcal{P}$ of *strongly reduced paths* is defined as a colimit based on the category $\mathbf{rc} \subset \mathbf{Top}^{op}$ (2.3), or equivalently as a quotient modulo the action of the monoid $r\Gamma_1 \subset \mathbf{Top}(\mathbf{R}, \mathbf{R})$ (2.3)

(1) $\mathcal{P}X = \mathrm{Colim}_{\mathbf{rc}} \mathbf{Top}([i, j], X) = |\mathbb{P}X| / r\Gamma_1,$

again, $(\rho, a), (\sigma, b) \in |\mathbb{P}X|$ are identified if there exist two maps $\gamma, \gamma' \in r\Gamma_1$ such that $a\gamma = b\gamma'$. $\mathcal{P}X$ is thus a groupoid.

2.6. The fundamental ω-groupoid. In dimension n, we shall replace D_1 with the submonoid $D_n \subset \mathbf{Tol}(\mathbf{Z}^n, \mathbf{Z}^n)$ of n-*dimensional delays*, generated by the elementary delays $\delta_i \times \mathbf{Z}^{n-1}$, $\mathbf{Z} \times \delta_i \times \mathbf{Z}^{n-2}, \ldots, \mathbf{Z}^{n-1} \times \delta_i$. Similarly, we have the submonoids $D_n \subset \bar{D}_n \subset \Gamma_n \subset \mathbf{Tol}(\mathbf{Z}^n, \mathbf{Z}^n)$ and their real extensions $rD_n \subset r\bar{D}_n \subset r\Gamma_n \subset \mathbf{Top}(\mathbf{R}^n, \mathbf{R}^n)$.

The set-valued functor

(1) $\mathbf{P}^{(n)}X = \mathrm{Colim}_{\mathbf{rd}^n} \mathbf{Top}([i_1, j_1] \times \ldots \times [i_n, j_n], X) = |\mathbb{P}^n X| / \equiv_n ,$

is produced by the congruence $\equiv_n$ defined by the real extension $rD_n \subset \mathbf{Top}(\mathbf{R}^n, \mathbf{R}^n)$, formed of *real* n-*dimensional delays*. In particular, $\mathbf{P}^{(0)}(X) = |X|$ and $\mathbf{P}^{(1)}(X) = \mathbf{P}X$.

For a space X, these functors produce a *cubical set* $\mathbf{P}^*X$ *with connections* (g_i) [BH1, BS] *and symmetries* (the interchanges s_i and the involutions r_i)

(2) $|X| = \mathbf{P}^{(0)}(X) \Leftarrow \mathbf{P}^{(1)}(X) \Lleftarrow \mathbf{P}^{(2)}(X) \leftleftarrows \ldots \mathbf{P}^{(n)}(X) \leftleftarrows \ldots$

(only faces are displayed); the structural maps (for $1 \leq i \leq n$; $\alpha = \pm$)

(3) $\partial_i^\alpha: \mathbf{P}^{(n)} \to \mathbf{P}^{(n-1)}, \qquad e_i: \mathbf{P}^{(n-1)} \to \mathbf{P}^{(n)},$
$g_i^\alpha: \mathbf{P}^{(n)} \to \mathbf{P}^{(n+1)},$
$s_i: \mathbf{P}^{(n+1)} \to \mathbf{P}^{(n+1)}, \qquad r_i: \mathbf{P}^{(n)} \to \mathbf{P}^{(n)},$

are induced by the following natural transformations of the powers of the Moore path functor $\mathbb{P}$

(4) $\partial_i^\alpha = \mathbb{P}^{i-1}\partial^\alpha\mathbb{P}^{n-i}: \mathbb{P}^n \to \mathbb{P}^{n-1}, \qquad e_i = \mathbb{P}^{i-1}e\mathbb{P}^{n-i}: \mathbb{P}^{n-1} \to \mathbb{P}^n,$
$g_i^\alpha = \mathbb{P}^{i-1}g^\alpha\mathbb{P}^{n-i}: \mathbb{P}^n \to \mathbb{P}^{n+1},$
$s_i = \mathbb{P}^{i-1}s\mathbb{P}^{n-i}: \mathbb{P}^{n+1} \to \mathbb{P}^{n+1}, \qquad r_i = \mathbb{P}^{i-1}r\mathbb{P}^{n-i}: \mathbb{P}^n \to \mathbb{P}^n.$

Moreover, there is an i-*composition* $k_i: \mathbf{P}^{(n)} \times_{\mathbf{P}^{(n-1)}} \mathbf{P}^{(n)} \to \mathbf{P}^{(n)}$ (defined on the pullback with respect to the i-faces $\partial_i^\alpha: \mathbf{P}^{(n)} \to \mathbf{P}^{(n-1)}$), induced by the concatenation in direction i of the functor $\mathbb{P}$ (as for $n = 1$, in 2.4)

(5) $k_i = \mathbb{P}^{i-1}k\mathbb{P}^{n-i}: \mathbb{P}^{i-1}\mathbb{Q}\mathbb{P}^{n-i} \to \mathbb{P}^n$.

Finally, $\mathbf{P}^*X$ is a (small) cubical ω-category with connections [ABS] and symmetries.

$\mathbf{P}^*X$ contains an *involutive ω-category* $\mathbf{P}_*X$ (with involutions r_i for all composition laws)

(6) $|X| = \mathbf{P}_0(X) \leftrightarrows \mathbf{P}_1(X) \leftrightarrows \ldots \mathbf{P}_n(X) \leftrightarrows \ldots \qquad (\mathbf{P}_*X)$,

obtained in the obvious way. First $\mathbf{P}_i(X) = \mathbf{P}^{(i)}(X)$ for $i = 0, 1$; then $\mathbf{P}_2(X) \subset \mathbf{P}^{(2)}(X)$ is the subset of those double reduced paths whose first faces are degenerate; $\mathbf{P}_3(X) \subset \mathbf{P}^{(3)}(X)$ contains those triple cells whose last faces end in $\mathbf{P}_2(X)$, while the first and second are degenerate in their last direction (i.e., belong to the image of $e_2: \mathbf{P}_1(X) \to \mathbf{P}_2(X)$); and so on:

(7) $a \in \mathbf{P}_n(X)$ if:
- $a \in \mathbf{P}^{(n)}X$,
- $\partial^\alpha(a) = \partial^\alpha_n(a) \in \mathbf{P}_{n-1}X$,
- $\partial^\alpha_i(a) \in \mathrm{Im}(e^{n-1}_{n-1}: \mathbf{P}^{(n-2)}X \to \mathbf{P}^{(n-1)}X)$, for $0 \leq i < n$,

always keeping in the structure the last faces.

Each component has involutions $r_i: \mathbf{P}^{(n)} \to \mathbf{P}^{(n)}$, for $1 \leq i \leq n$ (which can be extended, letting $r_i = 1$ for $i > n$). Note that $\mathbf{P}^*X$ and $\mathbf{P}_*X$ 'contain the same information': it has been recently proved in [ABS] that (edge-symmetric) ω-*cubical categories with connections* are equivalent to ω-*categories* (by the procedure described in (7)).

Replacing $\mathbf{P}$ with $\mathcal{P}$ (and $\bar{D}_n$ with Γ_n), in these constructs, we obtain a (strict) *cubical* ω-*groupoid with connections and interchanges* $\mathcal{P}^*X$

(8) $\mathcal{P}^{(n)}X = \mathrm{colim}_{\mathbf{rc}^n} \mathbf{Top}([i_1, j_1] \times \ldots \times [i_n, j_n], X) = |\mathbb{P}^nX|/r\Gamma_n$,

and, within the latter, the *fundamental* ω-*groupoid*, defined as in (7)

(9) $\Pi_\omega(X) = \mathcal{P}_*X$,

Again, $\mathcal{P}^*X$ and $\mathcal{P}_*X$ 'contain the same information', since cubical ω-groupoids with connections are equivalent to (globular) ω-groupoids, a fact already known from [BH1, BH2].

A congruence class of an n-cube $a: [i_1, j_1] \times \ldots \times [i_n, j_n] \to X$ in $\mathbf{P}_nX$ (resp. in $\mathcal{P}_nX$) will be written as $a^\bullet$ (resp. $a^{\bullet\bullet}$); one can always use a representative defined on a cube $[i, j]^n$.

2.7. Fundamental n-groupoids. Finally, the *fundamental* n-*groupoid* of a space X

(1) $\Pi_n(X) = \rho_n\mathcal{P}_*(X)$,

is produced by the reflector ρ_n: ω-**Gpd** → n-**Gpd** of the (skeletal) embedding n-**Gpd** ⊂ ω-**Gpd**. Explicitly, $\Pi_n(X)$ only differs in degree n from the n-globular set produced by truncation

(2) $|X| = \mathcal{P}_0(X) \leftrightarrows \mathcal{P}_1(X) \leftrightarrows \ldots \mathcal{P}_{n-1}(X) \leftrightarrows \Pi_n(X)$

and the *set* $\Pi_n(X)$ (containing all lower cells as degenerate elements) is the coequaliser of the two faces ending in $\mathcal{P}_n(X)$, i.e. the quotient $\mathcal{P}_n(X)/\simeq_{n+1}$ modulo the equivalence relation $a \simeq_{n+1} b$: there exists $A \in \mathcal{P}_{n+1}(X)$ with $\partial^-A = a$, $\partial^+A = b$.

For a *pointed* space (X, x), the group of endocells of $\Pi_n(X)$ at the degenerate n-tuple strongly reduced path at the base point coincides with the usual n-homotopy group $(n \geq 1)$

(3) $\pi_n(X, x) = \Pi_n(X)(x, x) = \Omega^{(n)}(X, x)/\simeq_{n+1} = [\mathbf{S}^n, (X, x)]$,

since it can be viewed as the quotient of the involutive n-category $\Omega^{(n)}(X, x)$ of n-loops (those strongly reduced n-tuple paths $a \in \mathcal{P}^{(n)}(X)$ whose faces are constant at the base point), modulo the (n+1)-homotopy relation $\simeq_{n+1}$; and Γ_n-congruence implies the latter ([G3], 3.5).

3. Higher fundamental categories for directed topological spaces

The fundamental category $\uparrow\Pi_1 X$ of a *directed topological space*, introduced in [G6], is extended to any dimension $\leq \omega$, by techniques similar to the previous ones and to those used in [G3] for the higher fundamental categories of simplicial sets.

3.1. Directed topological spaces. Let us begin recalling some basic notions of [G6].

A *directed topological space* $X = (X, dX)$, or *d-space*, is a topological space equipped with a set dX of (continuous) maps $a: \mathbf{I} \to X$, defined on the standard interval $\mathbf{I} = [0, 1]$; these maps, called *directed paths* or *d-paths*, must contain all constant paths and be closed under (weakly) increasing reparametrisation and concatenation.

A *directed map* $f: X \to Y$, or *d-map*, or *map* of d-spaces, is a continuous mapping between d-spaces which preserves the directed paths: if $a \in dX$, then $fa \in dY$. The category of d-spaces is written as d**Top**. It has all limits and colimits, constructed as in **Top** and equipped with the initial or final d-structure for the structural maps; for instance a path $\mathbf{I} \to \Pi X_k$ is directed if and only if all its components $\mathbf{I} \to X_k$ are so. The forgetful functor U: d**Top** → **Top** preserves all limits and colimits; its right adjoint C^0: **Top** → d**Top** equips a space X with the natural d-structure, where all paths are distinguished.

Reversing d-paths, by the involution $r(t) = 1 - t$, gives the *reflected*, or *opposite*, d-space $RX = X^{op}$, where $a \in d(X^{op})$ iff $a^{op} = ar \in dX$. A d-space is *symmetric* if it is invariant under reflection. More generally, it is *reflexive*, or self-dual, if it is isomorphic to its reflection.

The category p**Top** of preordered topological spaces (equipped with a reflexive, transitive relation) has an obvious embedding in d**Top**: a path $\mathbf{I} \to X$ is directed if it is (weakly) increasing. But d**Top** is more general, and has pastings (colimits) and homotopy constructs like mapping cones, which cannot be obtained with preorders, nor with 'local preorders' ([G6], 1.4).

The *directed real line*, or *d-line* $\uparrow\mathbf{R}$, is the euclidean line with d-structure derived from the usual order. Similarly, its cartesian power in d**Top**, the n*-dimensional real d-space* $\uparrow\mathbf{R}^n$, derives from the product order ($x \leq x'$ iff $x_i \leq x_i'$ for all i). A directed interval $\uparrow[i, j]$ has the subspace structure of the d-line; the *standard d-interval* is $\uparrow\mathbf{I} = \uparrow[0, 1]$; the *standard d-cube* $\uparrow\mathbf{I}^n$ is its n-th power, and a subspace of $\uparrow\mathbf{R}^n$. The directed sphere $\uparrow\mathbf{S}^n = \uparrow\mathbf{I}^n/\partial\mathbf{I}^n$ $(n > 0)$ is defined as a quotient in d**Top**, collapsing to a point the boundary of the directed n-cube; thus, $\uparrow\mathbf{S}^1$ derives from an (obvious) 'local order', while the higher directed spheres cannot be so obtained [G6]. These d-spaces are not symmetric (for $n > 0$), yet reflexive.

3.2. Directed homotopies. The directed interval $\uparrow\mathbf{I} = \uparrow[0, 1]$ has the same 'lattice-like' structure considered in 1.1, except reversion (which is not a d-map). It is *exponentiable* ([G6], thm. 1.7): the cylinder functor $\uparrow I = - \times \uparrow\mathbf{I}$ has a right adjoint, the (directed) *path functor*, or *cocylinder* $\uparrow P$

(1) $\uparrow P\colon d\mathbf{Top} \to d\mathbf{Top}, \qquad \uparrow P(Y) = Y^{\uparrow\mathbf{I}},$

where the d-space $Y^{\uparrow\mathbf{I}}$ is the set of d-paths $d\mathbf{Top}(\uparrow\mathbf{I}, Y)$ with the usual compact-open topology and the d-structure where a map $c\colon \mathbf{I} \to d\mathbf{Top}(\uparrow\mathbf{I}, Y)$ is directed if, for all increasing maps $h, k\colon \mathbf{I} \to \mathbf{I}$, the derived path $t \mapsto c(h(t))(k(t))$ is in dY.

Again, the structure of $\uparrow\mathbf{I}$ in $d\mathbf{Top}$ produces a cubical comonad on $\uparrow P$, as in 1.2.2, with interchange but without reversion; concatenation works as in 1.3, by the concatenation pullback $\uparrow QX = \uparrow PX \times_X \uparrow PX$.

A (directed) *homotopy* $\varphi\colon f \to g\colon X \to Y$ is defined as a map $X \to \uparrow PY = Y^{\uparrow\mathbf{I}}$ whose two faces, $\partial^\pm(\varphi) = \partial^\pm.\varphi\colon X \to Y$ are f and g, respectively. In particular, a path is a homotopy between two points, $a\colon x \to x'\colon \{*\} \to X$. The structure of d-homotopies essentially consists of

(a) *whisker composition* of maps and homotopies:

$v\circ\varphi\circ u\colon vfu \to vgu \qquad (v\circ\varphi\circ u = \uparrow Pv.\varphi.u\colon \uparrow X' \to \uparrow PY'),$

(b) *trivial homotopies*: $0_f\colon f \to f \qquad (0_f = ef\colon X \to \uparrow PY),$

(c) *concatenation* of homotopies: $\varphi+\psi\colon f \to h;$

the last is defined via $\uparrow Q$ (for $\varphi\colon f \to g$, $\psi\colon g \to h$).

The homotopy relation $f \simeq g$ is the equivalence relation generated by the existence of a directed homotopy between two maps; for an analysis of this relation, see [G3] 2.4, 2.7.

3.3. Step sets. We need now the directed counterpart of tolerance sets (2.2). As in [G3], a *step set* X is a set equipped with a *precedence* relation, or *step* relation $x \prec x'$ (just reflexive). A *step map*, or *combinatorial mapping* between such objects preserves the step relation. The resulting category is written **Stp**. It will be useful to note that a step set can be viewed as a simplicial set: an n-simplex of X is any word $(x_0,..., x_n)$ in X where $i \leq j$ implies $x_i \prec x_j$.

The *directed combinatorial (integral) line* $\uparrow\mathbf{Z}$ is the set of integers with the structure of *consecutivity*: $i \prec j$ if $0 \leq j - i \leq 1$. Integral intervals $\uparrow[i, j]_{\mathbf{Z}} \subset \uparrow\mathbf{Z}$ have the induced structure. The directed geometric realisation of $\uparrow\mathbf{Z}$ ([G6], 4.5) is the d-line $\uparrow\mathbf{R}$. Again, as in 2.2.1, a step map $f\colon \uparrow\mathbf{Z} \to \uparrow\mathbf{Z}$ has a *real extension* $f\colon \uparrow\mathbf{R} \to \uparrow\mathbf{R}$ in $d\mathbf{Top}$, affine on each elementary interval with integral bounds $[i, i+1]$; and a product of step maps $f_1\times...\times f_n\colon \uparrow\mathbf{Z}^n \to \uparrow\mathbf{Z}^n$ has a real extension in $d\mathbf{Top}(\uparrow\mathbf{R}^n, \uparrow\mathbf{R}^n)$. (This extension is simplicial and cubical at the same time, cf. 4.6.)

3.4. Higher fundamental categories. Higher homotopies in $d\mathbf{Top}$ can be treated as in the reversible case (Section 2), except of course that reversion, regressions (2.3) and all their consequences are missing. Let X be a directed topological space.

We have now the endofunctor $\uparrow\mathbb{P}\colon d\mathbf{Top} \to d\mathbf{Top}$ of *directed Moore paths* and the functor $\uparrow\mathbf{P}\colon d\mathbf{Top} \to \mathbf{Set}$ of *directed reduced paths*, related by a natural transformation

(1) $|\uparrow\mathbb{P}X| \twoheadrightarrow \uparrow\mathbf{P}X,$

$$\uparrow\mathbb{P}X \;=\; \Sigma_{\mathbf{rs}}\, X^{\uparrow[i,\,j]}, \qquad\qquad \uparrow\mathbf{P}X \;=\; \mathrm{Colim}_{\mathbf{rd}}\, \mathrm{d}\mathbf{Top}(\uparrow[i, j], X) \;=\; |\uparrow\mathbb{P}X|/\!\equiv,$$

since $\mathbf{rs} \subset \mathrm{r}\mathbf{d} \subset \mathrm{d}\mathbf{Top}^{\mathrm{op}}$: all generalised delays are step-maps $\uparrow\mathbf{Z} \to \uparrow\mathbf{Z}$. Similarly, $D_n \subset \mathbf{Stp}(\uparrow\mathbf{Z}^n, \uparrow\mathbf{Z}^n)$, and we have set-valued functors corresponding to 2.6.1

(2) $\uparrow\mathbf{P}^{(n)}X \;=\; \mathrm{Colim}_{\mathbf{rd}^n}\, \mathrm{d}\mathbf{Top}([i_1, j_1] \times ... \times [i_n, j_n], X) \;=\; |\uparrow\mathbb{P}^nX|/\!\equiv_n$.

The family $\uparrow\mathbf{P}^*(X) = (\uparrow\mathbf{P}^{(n)}(X))_{n\geq 0}$ of all n-dimensional directed reduced path sets is a cubical set with connections and interchange (but no reversion). The concatenation of consecutive reduced paths makes $\uparrow\mathbf{P}^*(X)$ into a cubical ω-category with interchange (only faces and degeneracies are drawn)

(3) $|X| \;=\; \uparrow\mathbf{P}^{(0)}(X) \;\leftrightarrows\; \uparrow\mathbf{P}^{(1)}(X) \;\;\overset{\leftleftarrows}{\rightrightarrows}\;\; ... \;\; \uparrow\mathbf{P}^{(n)}(X) \;\;\overset{\leftleftarrows}{\rightrightarrows}\;\; ... \qquad\qquad (\uparrow\mathbf{P}^*(X)).$

This contains the *fundamental* ω-*category* $\uparrow\Pi_\omega X = \uparrow\mathbf{P}_*(X)$, defined as in 2.6.6

(4) $|X| \;=\; \uparrow\mathbf{P}_0(X) \;\leftrightarrows\; \uparrow\mathbf{P}_1(X) \;\leftrightarrows\; ... \;\; \uparrow\mathbf{P}_n(X) \;\leftrightarrows\; ... \qquad\qquad (\uparrow\Pi_\omega X).$

The reflector ρ_n: ω-**Cat** → n-**Cat** produces the *fundamental* n-*category* $\uparrow\Pi_nX = \rho_n(\uparrow\Pi_\omega X)$

(5) $|X| \;=\; \uparrow\mathbf{P}_0(X) \;\leftrightarrows\; \uparrow\mathbf{P}_1(X) \;\leftrightarrows\; ... \;\; \uparrow\mathbf{P}_{n-1}(X) \;\leftrightarrows\; \uparrow\Pi_nX;$

again, the *set* $\uparrow\Pi_n(X)$ is the coequaliser of the faces $\uparrow\mathbf{P}_{n+1}(X) \rightrightarrows \uparrow\mathbf{P}_n(X)$, i.e. the quotient $\uparrow\mathbf{P}_n(X)/\simeq_{n+1}$ modulo the equivalence relation spanned by the relation: there exists $A \in \uparrow\mathbf{P}_{n+1}(X)$ with $\partial^-A = a$, $\partial^+A = b$.

In particular, $\uparrow\Pi_1X$ coincides with the fundamental category of a d-space defined in [G6], where an arrow is a path up to homotopy with fixed endpoints.

For a *pointed* d-space (X, x), the n-dimensional *homotopy monoid* $(n \geq 1)$ is the monoid of endomaps at the degenerate n-tuple reduced path at the base point, and is abelian for $n \geq 2$:

(6) $\uparrow\pi_n(X, x) \;=\; \uparrow\Pi_n(X)(x, x) \;=\; [\uparrow\mathbf{S}^n, (X, x)],$

the last expression is proved as in 2.7.3; the directed sphere $\uparrow\mathbf{S}^n$ is defined in 3.1.

3.5. Comparison with topological spaces. For a d-space X, there is an obvious comparison

(1) $\uparrow\Pi_n(X) \;\to\; \Pi_n(UX), \qquad\qquad a^\bullet \mapsto a^{\bullet\bullet}, \;\; [a] \mapsto [a],$

sending a reduced p-dimensional d-path to the corresponding strongly reduced path, and similarly for their homotopy classes (for $p = n$).

Now, for a *topological space* X with the natural d-structure C^0X (3.1), a p-dimensional d-path $\uparrow[i, j]^p \to C^0X$ is the same as an ordinary path $[i, j]^p \to X$, so that our monoids (and fundamental category) coincide with the ordinary homotopy groups (and fundamental groupoid)

(2) $\uparrow\pi_n(C^0X, x) \;=\; \pi_n(X, x), \qquad\qquad \uparrow\Pi_1(C^0X) \;=\; \Pi_1(X).$

In higher fundamental categories, the relation is more complicated: the reversion of p-dimensional paths in $\uparrow\Pi_n(C^0X)$ does not produce inverses if $p < n$ (because regression is not taken into account); on the other hand, for (n–1)-dimensional paths a, b

(3) $\uparrow\Pi_n(C^0X)(a^\bullet, b^\bullet) \;=\; \Pi_n(X)(a^{\bullet\bullet}, b^{\bullet\bullet}).$

4. Comparison with simplicial sets and symmetric simplicial sets

We end by a comparison with the homotopy of simplicial sets, based on previous works [G3, G4, G5].

4.1. Synopsis. A simplicial set K (even though a directed notion in itself) is classically treated as a non-directed structure. Thus, its ordinary geometric realisation $\mathcal{R}K$ is just a space; moreover, if K is a Kan complex, it has intrinsic homotopy groups $\pi_n(K, x)$ which coincide with the ones of $\mathcal{R}K$. The classical homotopy theory of simplicial sets, mostly based on Kan complexes, can be found in [Ma, Cu, GZ, GJ].

We recall below a homotopy theory developed in [G3, G5], with the purpose of freeing the bases of combinatorial homotopy from the restriction to Kan conditions.

In this approach, simplicial sets have directed homotopies, parametrised on $\uparrow\mathbf{Z}$, and homotopy monoids (as recalled in 4.2); a directed realisation $\uparrow\mathcal{R}$ with values in directed spaces (constructed below, in 4.3-5); a comparison with the present homotopy monoids of d-spaces (4.6)

$$(1)\qquad \begin{array}{ccc} \mathbf{Smp} & \xrightarrow{\ \uparrow\mathcal{R}\ } & \mathrm{d}\mathbf{Top} \\ {\scriptstyle \mathrm{Sym}}\downarrow & \searrow^{\mathcal{R}} & \downarrow{\scriptstyle U} \\ !\mathbf{Smp} & \xrightarrow[\ \mathcal{R}\ ]{} & \mathbf{Top} \end{array}$$

On the other hand, the category $!\mathbf{Smp}$ of *symmetric* simplicial sets (recalled in 4.7) has reversible homotopies parametrised on the integral line $\mathbf{Z}$ (with the tolerance structure of contiguity, 2.2); homotopy groups; a realisation $\mathcal{R}$ with values in ordinary spaces; a comparison with ordinary homotopy groups.

The diagonal of the commutative diagram (1) shows that the classical realisation $\mathcal{R}$: $\mathbf{Smp} \to \mathbf{Top}$ is coherent with the other realisations we are considering here, via the forgetful functor U and a symmetrisation functor Sym (4.7); this 'coherence' also holds for homotopy monoids and groups, as discussed in 4.2.

4.2. A review of directed homotopy of simplicial sets [G3]. We have already noted that the category **Smp** of simplicial sets contains the category **Stp** of step sets (3.3). Thus, the directed integral line $\uparrow\mathbf{Z}$, with the precedence relation of consecutivity ($i \prec j$ if $i \leq j \leq i+1$) is a simplicial set; and so are the integral intervals $\uparrow[i, j]_{\mathbf{Z}}$ and $\uparrow\mathbf{2} = \uparrow[0, 1]_{\mathbf{Z}} = \{0 < 1\}$.

The path functor $\uparrow P(K) \subset K^{\uparrow\mathbf{Z}}$ introduced in [G3] contains, as 0-simplices, those *lines* a: $\uparrow\mathbf{Z} \to K$ which are eventually constant, at the left and at the right. A *directed homotopy* φ: $f^- \to f^+$: $A \to K$ is a map φ: $A \to P(K)$ with faces $\partial^\alpha\varphi = f^\alpha$. It is *bounded* if it takes values in some bounded-path object $K^{\uparrow[i, j]}$; in particular, an *immediate homotopy* φ: $A \to K^{\uparrow\mathbf{2}}$ is the same as an ordinary simplicial homotopy, and a bounded homotopy is a finite concatenation of immediate ones. If A is *finite*, every homotopy φ: $f^- \to f^+$: $A \to K$ is bounded; whereas, if K is *Kan*, immediate homotopy is an equivalence relation and every bounded homotopy (with values in K) can be replaced with an immediate one. (Note that $\uparrow\mathbf{Z}$ is contractible by general homotopies, but not by bounded ones: cf. [G3], 4.3 and [G5], 3.2).

The fundamental n-categories $\uparrow\Pi_n(K)$ $(n \leq \omega)$ of a simplicial set have been introduced in [G3], via 'path functors' $\uparrow\mathbb{P}$, $\uparrow P$, $\uparrow\mathbf{P}$ similar to the present ones for d-spaces (Section 3). It was proved there that $\uparrow\Pi_n$ preserves all colimits, and is left adjoint to an n-nerve functor N_n: n-**Cat** $\rightarrow$ **Smp** $(n \leq \omega)$. This produces the n-homotopy monoids of a pointed simplicial set, which can also be expressed as a colimit on a system of finite directed combinatorial spheres ([G3], 4.4):

(1) $\uparrow\pi_n(K, x) \; = \; \uparrow\Pi_n(K)(x, x) \; = \; \mathrm{colim}_k \, [\uparrow\mathbf{s}^n_k, (K, x)]$.

Precisely, the *directed* k-*collapsed* n-*sphere* $\uparrow\mathbf{s}^n_k = \uparrow\mathbf{Z}^n/\sim_k$ is obtained by collapsing to a point all the points of $\uparrow\mathbf{Z}^n$ out of the cube $[1,k]^n$ (for $k \geq 2$); it can be viewed as the surface of a 'pyramid' with basis $[1,k]^n$ and vertex at the base-point [0]. The maps $q^n_k\colon \uparrow\mathbf{s}^n_{k+1} \rightarrow \uparrow\mathbf{s}^n_k$ of the system are the canonical projections. The ordinary realisation of $\uparrow\mathbf{s}^n_k$ is the sphere $\mathbf{S}^n$; the realisation of q^n_k is a homeomorphism.

Thus, *if* K *is a Kan complex*, by our preceding remarks on bounded homotopies and the simplicial approximation theorem

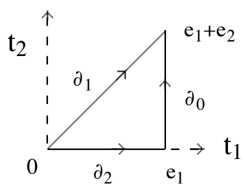
(2) $\uparrow\pi_n(K, x) \; = \; [\uparrow\mathbf{s}^n_k, (K, x)] \; = \; [\mathbf{S}^n, (\mathcal{R}K, x)] \; = \; \pi_n(\mathcal{R}K, x)$,

in other words, the present n-homotopy monoid of a pointed *Kan complex* is a group and coincides with the classical n-homotopy group.

4.3. Standard ordered simplices. But a simplicial set can be realised *as a directed space*, enriching the usual geometric realisation and pasting *ordered simplices* in d**Top**. (Similarly, one can realise cubical sets by pasting *ordered cubes* $\uparrow\mathbf{I}^n$ [G6]).

First, we realise the representable simplicial set $\Delta[n]$ as an *ordered subspace* $\uparrow\Delta^n$ of $\uparrow\mathbf{R}^n$, the *standard* n-*simplex*, so that the natural order $t \leq t'$ induce the order we want on faces

(1) $\uparrow\Delta^n \; = \; \{t \in \uparrow\mathbf{R}^n \mid 1 \geq t_1 \geq ... \geq t_n \geq 0\} \; \subset \; \uparrow\mathbf{R}^n$,



$\uparrow\Delta^2$

$\uparrow\Delta^n$ is the convex hull of the chain of its vertices, a totally ordered subset of $\uparrow\mathbf{R}^n$ (isomorphic to the positive ordinal $[n] = n+1 = \{0,... n\}$)

(2) $v_0 \; < \; v_1 \; < \; v_2 \; < \; ... \; < \; v_n, \qquad\qquad v_i \; = \; e_1 + ... + e_i \qquad\qquad (v_0 = 0)$,

where (e_i) is the canonical linear basis of $\mathbf{R}^n$. A point $t \in \uparrow\Delta^n$ can be uniquely written as a convex combination of the vertices (two fixed coordinates t_0, t_{n+1} are added, to simplify formulae)

(3) $t \; = \; (1-t_1).v_0 + (t_1-t_2).v_1 + (t_2-t_3).v_2 + ... + (t_n-0).v_n \; = \; \Sigma_i \, (t_i - t_{i+1}).v_i$,

$\qquad t_i - t_{i+1} \; \geq \; 0, \qquad\qquad \Sigma_i \, (t_i - t_{i+1}) \; = \; 1 \qquad\qquad (t_0 = 1, \; t_{n+1} = 0)$.

Note that $t \in \Delta^{\circ n}$ (the interior) iff $1 > t_1 > ... > t_n > 0$, iff all *barycentric coordinates* $u_i = t_i - t_{i+1}$ are positive, for $i = 0,..., n$. (To realise $\uparrow\Delta^n$ in the hyperplane $\Sigma\, u_i = 1$ of $\mathbf{R}^{n+1}$ one should equip the latter with a complicated order, derived from the change of coordinates $t \mapsto u$, namely: $u_n \leq u'_n$, $u_n + u_{n-1} \leq u'_n + u'_{n-1}, ...$)

Every d-map $x: [n] \to \uparrow\mathbf{R}^m$, i.e. every (weakly) increasing sequence $x_0 \leq ... \leq x_n$, has an affine extension to $\uparrow\Delta^n$

(4) $x: \uparrow\Delta^n \to \uparrow\mathbf{R}^m, \qquad x(t) = \Sigma_i (t_i - t_{i+1}) x_i = x_0 + t_1.(x_1 - x_0) + ... + t_n.(x_n - x_{n-1});$

the right-hand expression shows that x is increasing, i.e. a map $x: \uparrow\Delta^n \to \uparrow\mathbf{R}^m$. Moreover,

(5) $x(\uparrow\Delta^n) \subset \Delta(x), \qquad x(\uparrow\Delta^{\circ n}) \subset \Delta^\circ(x),$

where $\Delta(x)$ is the convex hull of the points $x_0,..., x_n$ in $\mathbf{R}^m$ and $\Delta^\circ(x)$ is the interior of $\Delta(x)$ in the affine subspace which it spans.

4.4. Directed geometric realisation. The category $\mathbb{A}$ of positive finite ordinals has thus a canonical embedding $\uparrow\Delta: \mathbb{A} \to \mathbf{pTop} \subset \mathbf{dTop}$, with cofaces and codegeneracies obtained by extending the ones of $\mathbb{A}$ $(i = 0,..., n)$

(1) $\partial_i: \uparrow\Delta^{n-1} \to \uparrow\Delta^n, \qquad \partial_i(t_0, t_1,..., t_n) = (t_0, t_1,..., t_i, t_i,..., t_n),$

$e_i: \uparrow\Delta^{n+1} \to \uparrow\Delta^n, \qquad e_i(t_0, t_1,..., t_{n+2}) = (t_0, t_1,..., \hat{t}_{i+1},..., t_{n+2}).$

The *directed singular simplicial set* of a d-space X, and its left adjoint, the *directed geometric realisation* of a simplicial set K, can now be constructed as in the ordinary case (cf. [GJ])

(2) $\uparrow\mathcal{R} : \mathbf{Smp} \rightleftarrows \mathbf{dTop} : \uparrow\mathcal{S},$

$\uparrow\mathcal{S}_n(X) = \mathbf{dTop}(\uparrow\Delta^n, X), \qquad \uparrow\mathcal{R}(K) = \int^{[n]} K_n{\bullet}\uparrow\Delta^n,$

the d-space $\uparrow\mathcal{R}(K)$ being the pasting in $\mathbf{dTop}$ of K_n copies of $\uparrow\Delta^n$ $(n \geq 0)$, along faces and degeneracies (the coend of the functor $K{\bullet}\uparrow\Delta: \mathbb{A}^{op}\times\mathbb{A} \to \mathbf{dTop}$). Every simplicial map $f: K \to L$ has a canonical realisation

(3) $\hat{f} = \uparrow\mathcal{R}(f): \uparrow\mathcal{R}(K) \to \uparrow\mathcal{R}(L).$

The adjunction $U \dashv C^0$ (3.1) between spaces and d-spaces gives back the usual realisation $\mathcal{R} = U.\uparrow\mathcal{R}: \mathbf{Smp} \to \mathbf{Top}$, left adjoint to the usual singular functor $\mathcal{S} = \uparrow\mathcal{S}.C^0: \mathbf{Top} \to \mathbf{Smp}$.

The directed realisation of $\uparrow\mathbf{2}$, $\uparrow\mathbf{Z}$ and $\uparrow\mathbf{s}^1 = \{* \to *\}$ are the directed interval $\uparrow\mathbf{I}$, the directed line $\uparrow\mathbf{R}$ and the directed circle $\uparrow\mathbf{S}^1$, respectively. A directed homotopy can be viewed as a simplicial map $K\times\uparrow\bar{\mathbf{Z}} \to L$ (on the extended integral line) and realised as a homotopy $\uparrow\mathcal{R}K\times\uparrow\bar{\mathbf{R}} \to \uparrow\mathcal{R}L$ (on the extended real line $\uparrow[-\infty, +\infty]$).

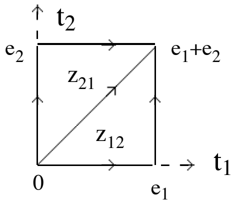
4.5. Proposition (Directed realisation of cubes). For all $n \geq 0$:

(1) $\uparrow\mathcal{R}(\uparrow\mathbf{2}^n) = \uparrow\mathbf{I}^n, \qquad \uparrow\mathcal{R}(\uparrow\mathbf{Z}^n) = \uparrow\mathbf{R}^n.$

Proof. We already know that this is true for $n \leq 1$. Forgetting about direction (which here simply appears as order), the topological aspect follows from the fact that the ordinary geometric realisation preserves finite products of countable simplicial sets [Ma, 14.4].

Taking order into account, recall that a non-degenerate n-simplex of $\uparrow\mathbf{2}^n$ is a strictly increasing sequence $(z_0,..., z_n)$ of vertices of $\uparrow\mathbf{2}^n$, corresponding to a permutation $\sigma \in S_n$

(2) $z_\sigma = (0, e_{\sigma(1)}, e_{\sigma(1)} + e_{\sigma(2)},..., e_1+...+ e_n),$



$\uparrow\Delta^2$

Thus, its extension $z_\sigma\colon \uparrow\Delta^n \to \uparrow\mathbf{R}^n$ (4.3.4) gives an isomorphism of ordered sets, between $\uparrow\Delta^n$ and the corresponding *elementary* n-*tetrahedron* of $\uparrow\mathbf{R}^n$

(3) $\hat{z}_\sigma\colon \uparrow\Delta^n \to z_\sigma(\Delta^n) \subset \uparrow\mathbf{R}^n$.

Pasting these extensions along faces, we get $\uparrow\mathbf{I}^n$. Finally, the simplicial set $\uparrow\mathbf{Z}^n$ is an obvious pasting (a colimit) of cubes $\uparrow\mathbf{2}^n$. Since realisation (a left adjoint) preserves colimits, also the second formula of the thesis is proved.

4.6. The directed comparison. It follows that the realisation of a simplicial cube $a\colon \uparrow\mathbf{2}^n \to K$ is a directed topological cube $\hat{a}\colon \uparrow[0, 1]^n \to \uparrow\mathbb{R}K$; while an n-tuple path $a\colon \uparrow\mathbf{Z}^n \to K$ has a realisation $\hat{a}\colon \uparrow\mathbf{R}^n \to \uparrow\mathbb{R}K$, affine on all elementary n-tetrahedra (4.4.3) and eventually constant, at the left and the right, in each real variable.

The geometric realisation of a delay is a real delay (2.3; 3.4); the same holds for their products $f_1\times\ldots\times f_n\colon \uparrow\mathbf{Z}^n \to \uparrow\mathbf{Z}^n$ (by 4.5). Moreover, if certain faces of a are trivial (constant at the base point), so are the corresponding faces of $\hat{a}$. This defines our two natural comparisons, a homomorphism and an ω-functor

(1) $\varphi_n\colon \uparrow\pi_n(K, x) \to \uparrow\pi_n(\uparrow\mathbb{R}K, x), \qquad \varphi_n[a] = [\hat{a}],$

$\Phi_\omega\colon \uparrow\Pi_\omega(K) \to \uparrow\Pi_\omega(\uparrow\mathbb{R}K), \qquad \Phi_\omega(a^\bullet) = \hat{a}^\bullet;$

and it would be interesting to have results on their being isomorphisms or n-equivalences, respectively.

Finally, let us note a lucky coincidence. Our extension of a product of step maps $f_1\times\ldots\times f_n\colon \uparrow\mathbf{Z}^n \to \uparrow\mathbf{Z}^n$ is a geometric realisation with respect to the structure of *simplicial* set of $\uparrow\mathbf{Z}^n$ and, at the same time, with respect to its structure of *cubical* set. Indeed, for a combinatorial mapping $g\colon \mathbf{Z}^n \to \mathbf{Z}^n$, the cubical realisation is multi-affine on every 'elementary cube', while the simplicial one is affine on every 'elementary tetrahedron'; but, if g is of the previous type, the two extensions coincide. Thus, our construction of $\uparrow\Pi_n$ and $\uparrow\pi_n$ would also give a comparison with the homotopy groups of cubical sets.

4.7. Realisation of symmetric simplicial sets. A *symmetric simplicial set* K [G3] is a presheaf on the category $!\mathbb{A}$ of positive finite *cardinals*; its structure, besides faces and degeneracies, has a coherent action of the symmetric group S_{n+1} on the component K_n. A simplicial complex A can be viewed as such a presheaf, by letting an n-simplex be any word $(x_0,\ldots, x_n)$ of elements of A whose support is a distinguished subset; thus, the category of simplicial complexes is embedded in the category **!Smp** of symmetric simplicial sets (as the cartesian closed full subcategory of *simple presheaves*, where every item is determined by the indexed family of its vertices [G3]). The forgetful functor **!Smp** → **Smp** has a left adjoint, where $(\mathrm{Sym}K)_n = S_{n+1}\times K_n$ is the set obtained by freely permuting the original n-simplices.

The fundamental n-groupoids $\Pi_n(K)$ ($n \leq \omega$) and the homotopy groups $\pi_n(K, x)$ have been introduced in [G3], by techniques similar to the present ones (in Section 2): an n-dimensional path in K is map $\mathbf{Z}^n \to K$ which is eventually constant, at the left and the right, in each variable (where the integral line $\mathbf{Z}$ has the structure of contiguity, 2.2). It was proved there that Π_n: !**Smp** $\to$ n-**Gpd** preserves all colimits, and is left adjoint to a symmetric nerve functor M_n: n-**Gpd** $\to$!**Smp**.

The realisation $\mathcal{R}$: !**Smp** $\to$ **Top** of symmetric simplicial sets can be easily defined (see [G4], Section 6): the representable presheaf $!\Delta[n]$ is realised by the usual standard simplex Δ^n (with the obvious action of the permutation group S_{n+1}, derived from permuting vertices); then the procedure is extended by colimits. This extends the classical realisation of simplicial complexes (cf. [Sp]).

Now, the realisation of $\mathbf{2}^n$, the codiscrete simplicial complex on 2^n points, is not $\mathbf{I}^n$, but Δ^{2^n-1}; thus, $\mathbf{2}^2$ becomes a solid tetrahedron and $\mathbf{Z}^2$ becomes a sort of 3-dimensional 'bubble wrap'. However, we can embed $\mathbf{I}^n$ in Δ^{2^n-1}, by the multi-affine mapping sending the vertex $\alpha = (\alpha_1,..., \alpha_n) \in \{0, 1\}^n$ to the vertex e_α of the canonical basis of $\mathbf{R}^{2^n}$

(1) $(t_1,..., t_n) \mapsto \Sigma_\alpha u_\alpha.e_\alpha$, $\qquad u_\alpha = r^{1+\alpha_i}(t_1).r^{1+\alpha_2}(t_2). ... .r^{1+\alpha_n}(t_n)$,

(where $r(t) = 1-t$). Thus, an n-tuple path $a: \mathbf{Z}^n \to X$ can be realised as a real n-dimensional path

(2) $\hat{a}: \mathbf{R}^n \to \mathcal{R}X$,

restricting the geometric realisation on Δ^{2^n-1}. Again, we have two natural comparisons, a homomorphism and an ω-functor

(3) $\varphi_n: \pi_n(K, x) \to \pi_n(\mathcal{R}K, x)$, $\qquad \Phi_\omega: \Pi_\omega(K) \to \Pi_\omega(\mathcal{R}K)$,

and it would be interesting to prove, in all generality, that φ_n is iso. In [G5], this fact has been proved for a relevant particular case: *when* K *'is' a simplicial complex.*

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