Categorically algebraic foundations for homotopical algebra (*)

MARCO GRANDIS

Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, I - 16146 Genova, Italy

(E-mail: grandis@dima.unige.it)

Abstract. We investigate a structure for an abstract cylinder endofunctor I which produces a good basis for homotopical algebra. It essentially consists of the usual operations (faces, degeneracy, connections, symmetries, vertical composition) together with a transformation $w: I^2 \to I^2$, which we call *lens collapse* after its realisation in the standard topological case.

This structure, if somewhat heavy, has the interest of being "categorically algebraic", i.e. based on operations on functors. Consequently, it can be naturally lifted from a category **A** to its categories of diagrams $\mathbf{A}^{\mathbf{S}}$ and its slice categories **A**\X, **A**/X. Further, the dual structure, based on a cocylinder (or path) endofunctor P can be lifted to the category of **A**-valued sheaves on a site, whenever the path functor P preserves limits, and to the category Mon**A** of internal monoids, with respect to any monoidal structure of **A** consistent with P.

Mathematics Subject Classification (1991): 55U35, 18D05, 18G55, 55P, 55R05.

Key words: Homotopical algebra, abstract homotopy theory, 2-categories, homotopy, homotopy (co)-limits, (co)cylinder, cubical objects, connections, presheaves, sheaves, slice categories, internal monoids.

0. Introduction

A basic homotopy structure on a category **A** can be given through a cylinder endofunctor $I: \mathbf{A} \to \mathbf{A}$, with two faces ($\partial^-, \partial^+: 1 \to I$) and a degeneracy ($e: I \to 1$), as in Kan [23]. But homotopies defined on this basis can not even be reversed or composed, and some further structure over I has to be added, in order to get a good setting.

Among such "operations", the most commonly used (see Brown - Higgins [3, 4, 5], Spencer - Wong [30]) are the *reversion* ($r: I \to I$), the *composition* ($k: I \to J$), the *interchange* ($s: I^2 \to I^2$) and the *connections* ($g^-, g^+: I^2 \to I$), where – for the composition – $J(A) = IA +_A IA$ denotes the pasting of two cylinders, one on top of the other. The vertical reversion and composition of homotopies, respectively produced by r and k, will be written in additive notation.

Further, we add here a natural transformation $w: I^2 \to I^2$, called *lens collapse*, having the role of converting a generic *deformation*, or *double homotopy*, $\Phi: I^2A \to B$ of four homotopies $\alpha, \beta, \gamma, \delta$ (as in (1)) into a cell-homotopy $\Phi w: \alpha+\delta \to \beta+\gamma$ (as in (2)), in a bijective way

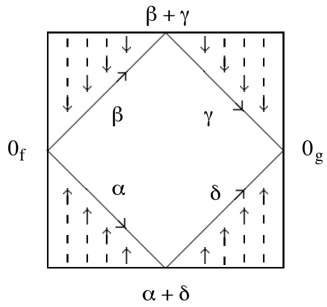
$$(1) \qquad \begin{array}{ccc} k & \xrightarrow{\gamma} & g \\ \beta\uparrow & \Phi & \uparrow\delta \\ f & \xrightarrow[\alpha]{} & h \end{array} \qquad\qquad (2) \qquad \begin{array}{ccc} f & \xrightarrow{\beta+\gamma} & g \\ 0\uparrow & \Phi w & \uparrow 0 \\ f & \xrightarrow[\alpha+\delta]{} & g \end{array}$$

(*) Work supported by MURST Research Projects.

This structure on a cylinder provides a good setting for homotopical algebra and is "categorically natural", as discussed below.

For chain complexes, w is described in 6.8. For topological spaces, it can be realised by means of the endomap w of the unit square $[0, 1]^2$ suggested by the drawing (3), which motivates our name, "lens collapse"

(3)



w collapses the vertical edges of the outer square producing two vertices of the inner one, and is bijective outside such edges; the other two vertices come from the middle points of the old horizontal edges; the inner square is then rotated and normalised.

§ 1 is a brief review of the abstract homotopical algebra developed by the author in [8, 9, 12], founded on the following two notions (weaker than the cubical analogues studied here). An h4-*category* is a sort of "relaxed" 2-category whose cells

(4) $\alpha\colon f \to g\colon A \to B$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\downarrow"] & B \end{tikzcd}$$

are thought of as homotopies, with compositions which behave well up to an assigned second-order homotopy relation $\alpha \sim \beta$. A *right homotopical* category is an h4-category having terminal object $\top$ and h-cokernels (standard homotopy cokernels) of maps, which satisfy a regularity property with respect to $\sim$ (h4-*cokernels)*. This yields a simple approach to diagrammatical lemmas, the Puppe sequence of a map, homotopical stability and relations with triangulated categories.

§ 2 introduces the notion of I4-*category*, consisting of a cylinder functor equipped with the transformations described above, including the lens collapse and a *zero collapse* $z\colon I^2 \to I$, having the effect that $\alpha{+}0 \sim \alpha$ (2.7). Such a category has a canonical h4-structure (thm. 2.9), with cell-homotopy produced by I^2 in the usual way (cf. (2)). For an I4-*homotopical* category $\mathbf{A}$ (§ 3) we further require that all the *lens conversions* $\Phi \mapsto \Phi w$ are bijective and that all the pushouts of a certain type (*cylindrical* pushouts, necessary and sufficient to form the h-pushouts, 3.2) exist and are preserved by I. Then $\mathbf{A}$ is a right homotopical h4-category, provided it has terminal object (thm. 3.5). The dual P4 case is based on a path functor P; the selfdual case IP4, on an adjunction $I \dashv P$.

These notions are "categorically natural" (§ 4-5). Thus, if $\mathbf{A}$ is I4- or P4-homotopical, the same holds for all its categories of diagrams $\mathbf{A}^{\mathbf{S}}$ (4.1-2), including the category of morphisms ($\mathbf{A}^{\mathbf{2}}$), of actions of a group ($\mathbf{A}^{G}$), of $\mathbf{A}$-valued presheaves over a fixed space, etc. Under natural hypotheses, this is also true of the slice categories $\mathbf{A}\backslash X$, $\mathbf{A}/Y$, $\mathbf{A}(X \to Y)$ (4.4-6). Further, if $\mathbf{A}$ is P4-homotopical the same holds for the category of $\mathbf{A}$-valued sheaves over an arbitrary site, provided that P preserves the existing limits (4.3), and for the category $\mathrm{Mon}\mathbf{A}$ of monoids in $\mathbf{A}$, with respect to any monoidal structure of **A** consistent with P (5.6). All these categories are therefore, in the appropriate hypotheses, right or left homotopical and the theory developed in [8, 9, 12] applies to them. This naturality property depends on the "algebraic" character of the present setting, essentially founded on operations over endofunctors of **A**.

Finally § 6 gives some basic examples: topological spaces, chain complexes, small categories, groupoids and 2-groupoids, from which various others are deduced through the categorical constructions just mentioned: diagrams, sheaves, slices and monoids.

Subsequent works will study homotopy laxifications of such constructions. For instance, in the P4-homotopical category $\mathbf{CAT}_i$ (of categories, functors and functorial isomorphisms), *h-diagrams* (indexed categories), *h-sheaves* (stacks) and *h-monoids* (monoidal categories) are more important than the strict analogues. Further, the path structure of chain complexes of modules is only in part consistent with the tensor product (6.9), which provokes the well-known deficiencies of chain algebras (the strict monoids) in adding homotopies; but, up to homotopy, this consistence subsists.

Comparing now our two approaches, the previous notions based on h4-categories are simple and seemingly adapted to develop homotopical algebra *within a given situation*, while the present cubical setting has the advantages of categorical naturality exposed above; being stronger, it inherits the previous results. Finally, even in the prime examples (as **Top**), it is generally easier to verify the lens conversion property than – directly – the axiom of regularity for h-pushouts or h-pullbacks.

To further clarify the difference, consider a category of diagrams $\mathbf{A}^\mathbf{S}$. We can not formally lift a homotopical h4-structure from **A** to $\mathbf{A}^\mathbf{S}$, essentially because of the cell-homotopy relation $\alpha \sim \beta$. On the one hand, to lift this relation pointwise to diagrams would not respect naturality (equivariance, for **S** a group), producing an (unnatural!) h4-structure, generally *not* homotopical. On the other hand, to make it "concrete" (or "algebraic") through second-order homotopies $\Phi: \alpha \to \beta$ and their operations requires a heavy structure; unless we use, as here, a cylinder or path functor to reduce homotopies of any order to maps, and their higher order operations to the basic, low order ones.

In the same way, a setting which assigns particular classes of maps (weak equivalences, (co)fibrations), as in the well known approaches of Quillen [29], Heller [18], K.S. Brown [2] and Baues [1], would not supply by itself such "naturality" results. For instance, one can define a general structure of *cofibration category* on the "bilateral" slice category $\mathbf{Top}(X \to Y)$ of *spaces under* X *and over* Y, as shown in [1, I.4]; but this is not achieved defining the cofibrations of $\mathbf{Top}(X \to Y)$ from the ones of **Top**, but deriving first the cylinder functor of the new category from the one of **Top**, through the obvious categorical procedure (as in 4.5, here), and then defining fibrations and cofibrations on this basis.

The author acknowledges with pleasure helpful discussions with Timothy Porter.

Conventions. $\top$ and $\bot$ denote the terminal and initial object of a category; if they coincide, producing the zero-object 0, the category is said to be *pointed*. In a 2-graph, a cell $\alpha: f \to g: A \to B$ is written as $\alpha: f \to g$ or $\alpha: A \Rightarrow B$ when we just want to specify the *vertical* domain and codomain (f and g) or the *horizontal* ones (A and B). **Top** and $\mathbf{Top}^\top$ are respectively the categories of topological spaces and pointed spaces, while $C_*\mathbf{D}$ is the category of unbounded chain complexes over an additive category **D**; actually, these categories are always provided with suitable additional structure concerning homotopies, as specified below. In **Top**, the cylinder is always realised as $I(X) = [0, 1]\times X$, so that $I\partial^\varepsilon: (t, x) \mapsto (t, \varepsilon, x)$ and $\partial^\varepsilon I: (t, x) \mapsto (\varepsilon, t, x)$ respectively give the horizontal and vertical faces of I^2X $(\varepsilon = 0, 1)$.

1. Basic properties of h-categories

This is an outline of a setting for homotopical algebra studied in the previous papers [8, 9]. The vertical composition of homotopies is written here as a sum.

1.1. h-Categories. An *h-category* **A** [9] is a sort of two-dimensional context, abstracting some of the nearly 2-categorical properties of topological spaces, continuous maps and homotopies.

Precisely, **A** is equipped with *cells* or *homotopies* $\alpha: f \to f': A \to B$, *vertical identities* $0_f: f \to f$ and a *reduced horizontal composition* $\circ$ of cells and maps (also written by juxtaposition)

(1) $k \circ \alpha \circ h:\ kfh \to kf'h:\ A' \to B'$ $\qquad\qquad$ $(h: A' \to A,\quad k: B \to B')$

satisfying the axiom

(hc) $1_B \circ \alpha \circ 1_A = \alpha,\qquad k \circ 0_f \circ h = 0_{kfh},\qquad k' \circ (k \circ \alpha \circ h) \circ h' = (k'k) \circ \alpha \circ (hh')$.

Formally, an h-category can be described as a category enriched over the monoidal closed category of reflexive graphs ([9], 1.3]). An equivalent notion is given in Kamps [21], def. 2.1.

In an h-category the *homotopy relation* $f \simeq f'$ (meaning that there exists a homotopy $f \to f'$ *or* $f' \to f$) need not be transitive, but it is weakly compositive: $f \simeq f'$ implies $kfh \simeq kf'h$. The morphism $f: A \to B$ is a homotopy equivalence if it has a homotopy inverse g ($gf \simeq 1$, $fg \simeq 1$); f is a fibration if it satisfies the usual lifting property of homotopies, for *each* orientation of the cells: for every $x: X \to A$, every homotopy $\eta: fx \to y$ (resp. $\eta: y \to fx$) lifts to some homotopy $\xi: x \to x'$ (resp. $\xi: x' \to x$), with $f\xi = \eta$ (and $fx' = y$).

An *h-functor* $F: \mathbf{A} \to \mathbf{B}$ preserves the whole structure, between h-categories; more generally, F is *homotopy invariant* if the existence of a cell $f \to g$ in **A** implies the existence of a cell $Ff \to Fg$ in **B**; then F preserves homotopy equivalences and (co)fibrations; every h-functor is so. If **B** is a category, equipped with the trivial h-structure given by formal vertical identities, both properties mean that F is an ordinary functor turning homotopical arrows of **A** into equal arrows of **B**. A morphism $u: F \to G: \mathbf{A} \to \mathbf{B}$ of h-functors is required to be 2-natural ($uB \circ F\alpha = G\alpha \circ uA$).

In an h-category, the *terminal* object $\top$ will always be defined by a 2-dimensional universal property, implying the usual 1-dimensional one: for every object A there is precisely one cell $A \Rightarrow \top$ (i.e. one map $\top_A: A \to \top$ and one endocell of the latter, its vertical identity). Similarly, product and sum will always mean *2-product* and *2-sum*. Instead, pullback and pushout will refer to the usual 1-dimensional property, unless we specify 2-pullback, 2-pushout.

1.2. h4-Categories. An h4-*category* is a sort of relaxed 2-category, abstracting deeper "2-categorical" properties of **Top**. Precisely, it is an h-category **A** equipped with:

a) a *vertical involution*, turning a cell $\alpha: f \to f'$ into the *reversed* (or opposite) cell $-\alpha: f' \to f$,

b) a *vertical composition*, turning two cells $\alpha: f \to f'$, $\beta: f' \to f''$ into the *sum* $\alpha+\beta: f \to f''$,

c) an equivalence relation $\sim$ (*cell-homotopy*) for cells with the same vertical domain and the same vertical codomain,

so that these axioms are satisfied:

(hc.1) $-0_f = 0_f,\qquad -(-\alpha) = \alpha,\qquad -(k\alpha h) = k(-\alpha)h,$

(hc.2) $0_f + 0_f \;=\; 0_f$, $\qquad$ $(k\alpha h) + (k\beta h) \;=\; k(\alpha+\beta)h$,

(hc.3) $-(\alpha+\beta) \;=\; (-\beta) + (-\alpha)$,

(hc.4a) $k\alpha h \sim k\alpha' h$, $\qquad -\alpha \sim -\alpha'$, $\qquad \alpha+\beta \sim \alpha'+\beta'$ $\qquad (\alpha \sim \alpha',\;\; \beta \sim \beta')$,

(hc.4b) $0_f+\alpha \sim \alpha \sim \alpha+0_{f'}$, $\;\; \alpha+(-\alpha) \sim 0_f$, $\qquad -\alpha+\alpha \sim 0_{f'}$, $\qquad (\alpha+\beta)+\gamma \sim \alpha+(\beta+\gamma)$,

(hc.4c) $k\alpha + \kappa f' \sim \kappa f + k'\alpha$ $\qquad (\alpha\colon f \to f'\colon A \to B,\;\; \kappa\colon k \to k'\colon B \to C)$.

The last property, (hc.4c), will be called *weak reduced exchange*. Intermediate notions will also be used: an h1-category has a vertical involution, satisfying (hc.1); an h2-category has a sum, satisfying (hc.2); an h3-category has both and satisfies (hc.1-3); the homotopy relation $f \simeq f'$ is then a congruence, yielding the associated category $\mathbf{A}/\!\simeq$. The corresponding notions of h1-, ... h4-*functors* are obvious.

A *strict* h4-category is an h4-category whose cell-homotopy $\sim$ is the equality; it is not difficult to see that this is equivalent to a 2-category whose cells are invertible, i.e. a *groupoid-enriched category* ([9], 1.4). The 2-category $\mathbf{Cat}_i$ of small categories, functors and natural isomorphisms is thus strict h4, as well as **Gpd** (small groupoids), while **Top**, $\mathbf{Top}^{\top}$ and $C_*\mathbf{D}$, with the usual homotopies and compositions, are non-strict h4-categories (§ 6). Every h4-category has an associated strict one, $\mathbf{A}/\!\sim$, consisting of the same objects, same maps and *tracks* (homotopies modulo $\sim$).

A *regular* h3-category, or *sesquigroupoid*, is assumed to have a *regular sum*, satisfying (hc.4b) for equality; every set of homotopies $\mathbf{A}_1(A, B)$ is then a groupoid, coherently with left and right composition *with maps*. Plainly, $C_*\mathbf{D}$ is so. (In [9], this notion is called "strict h3"; it can be formulated as an unusual enriched structure.)

1.3. h-Pushouts and h-cokernels. Let **A** be an h-category. The *h-pushout* (or standard homotopy pushout, or double mapping cylinder) of two arrows $f\colon A \to B$, $g\colon A \to C$ is an object $X = I(f, g)$ with two maps x', x'' and a homotopy $\xi\colon x'f \to x''g$ as in (1), satisfying the obvious universal property (of co-comma squares)

$$
(1)\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "x'"] \arrow[dl, phantom, "\swarrow\xi"] \\
C \arrow[r, "x''"'] & X
\end{tikzcd}
\qquad\qquad\qquad
(2)\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\top_A"'] & B \arrow[d, "x"] \arrow[dl, phantom, "\swarrow\xi"] \\
\top \arrow[r, "x''"'] & Cf
\end{tikzcd}
$$

- for every homotopy $\eta\colon y'f \to y''g\colon A \to Y$, there is exactly one morphism $a\colon I(f, g) \to Y$ such that $y' = ax'$, $y'' = ax''$, $\eta = a\xi$.

Of course, the triple $(x', x''; \xi)$ is jointly epi. In particular, if C is the terminal object $\top$ (1.1) as in diagram (2) above, this h-pushout is termed the *h-cokernel* of f, $\mathrm{hck}\, f = (x, x''; \xi)$. Then $Cf = C^-f = I(f, \top_A)$ is the (lower) *mapping cone* of f, the morphism $x''\colon \top \to Cf$ is its *vertex* and $x = c(f)$ the (main) h-cokernel map of f. $CA = C^-(1_A) = I(1_A, \top_A)$ is the (lower) *cone* of A.

The following *strict pasting* property of h-pushouts and *ordinary pushouts* will be frequently used ([9], 2.2; but the proof is standard)

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[r, "h"] \arrow[d, "x'"] \arrow[dl, "\xi"'] & B' \arrow[d, "y'"] \\
C \arrow[r, "x''"'] & X \arrow[r, "y''"'] & Y
\end{tikzcd} \tag{3}
$$

- if the triple $(x', x''; \xi)$ is the h-pushout of f and g, then the "pasted" triple $(y', y''x''; y''\xi)$ is the h-pushout of hf and g *if and only if* the right-hand square is an ordinary pushout.

If **A** is h4, the h-pushout $(x', x''; \xi)$ is said to be *h4-regular* (or an h4-pushout) if, given two maps $a_0, a_1: X \to Y$ and two cells $\vartheta: a_0x' \to a_1x': B \to Y$, $\kappa: a_0x'' \to a_1x'': C \to Y$ *coherent* with ξ, there is *some* cell $\alpha: a_0 \to a_1$ which extends ϑ and κ ($\vartheta = \alpha x'$, $\kappa = \alpha x''$). The coherence hypothesis means that we have a $\sim$-commutative diagram of homotopies, under vertical composition

$$
\begin{tikzcd}
a_0x''g \arrow[r, "\kappa g"] & a_1x''g \\
a_0x'f \arrow[u, "a_0\xi"] \arrow[r, "\vartheta f"'] \arrow[ur, phantom, "\sim"] & a_1x'f \arrow[u, "a_1\xi"']
\end{tikzcd} \tag{4}
$$

$$\vartheta f + a_1\xi \ \sim\ a_0\xi + \kappa g. \tag{5}$$

Dually one defines the h-pullback $P(f, g)$ of two converging arrows. The basic properties of regular h-pushouts and h-pullbacks, as homotopy invariance, (non-strict) pasting and reflection of homotopy equivalences, are studied in [9], § 2-3.

In **Top**, $I(f, g)$ is realised pasting the spaces B and C over the bases of the cylinder $IA = [0, 1]\times A$, along f and g (a "cylindrical colimit", 3.2.1). For Cf, the upper base of the cylinder is collapsed to a point, and $CA = IA/\{1\}\times A$. Dually, the h-pullback of two maps ending in A can be realised through the path space PA, as $P(f, g) = \{(b, \alpha, c) \in B\times PA\times C \mid \alpha(0) = f(b),\ \alpha(1) = g(c)\}$.

1.4. Cylinder and paths. Let **A** be an h-category. The *cylinder* IA of the object A is defined to be the h-pushout of the pair of identities $(1_A, 1_A)$, and comes equipped with a structural cell δ^A between its two *faces*, the lower (written ∂^- or ∂^0, according to convenience) and the upper one (∂^+, or ∂^1)

(1) $\delta^A: \partial^- \to \partial^+: A \to IA$

while its *degeneracy* $e: IA \to A$ is determined by $e\delta^A = 0: 1_A \to 1_A$. If IA exists, every homotopy $\alpha: A \Rightarrow B$ is *corepresented* by a map $\hat{\alpha}: IA \to B$ ($\hat{\alpha}.\delta^A = \alpha$).

If $\bot$ exists, $I\bot = \bot$ (because of its 2-dimensional property, 1.1). Similarly, if all cylinders exist, the cylinder functor $I: \mathbf{A} \to \mathbf{A}$ automatically preserves all the existing sums; moreover, h-pushouts can be reduced to particular ordinary colimits, as it will be shown below (3.2). It is not difficult to prove (but not needed here) that, if cylinders exist and are regular h-pushouts, the quotient $\mathbf{A}/\simeq$ provides the category of fractions of **A** with respect to homotopy equivalences ([9], 2.3, 3.3).

Dually, the *path-object* PB of the object B is the h-pullback of $(1_B, 1_B)$, with structural cell

(2) $\delta_B: \partial^- \to \partial^+: PB \to B$

where we distinguish the corresponding cylinder and path transformations by superscripts and lowerscripts. Every homotopy $\alpha: A \Rightarrow B$ is then *represented* by a map $\check{\alpha}: A \to PB$ ($\delta_B.\check{\alpha} = \alpha$). If the path functor $P: \mathbf{A} \to \mathbf{A}$ exists, it preserves all the existing products.

If all cylinder and path objects exist, we obtain a canonical adjunction $I \dashv P$ determined by the bijective correspondence $\hat{\alpha} \mapsto \check{\alpha}$, between $\mathbf{A}(IA, B)$ and $\mathbf{A}(A, PB)$.

1.5. Right homotopical categories. We end by outlining the main results of [9]; they will be at disposal of the stronger setting developed in the next sections, even if their concrete use in the present paper is marginal.

A *right semihomotopical* (resp. *right homotopical*) category **A** is an h-category (resp. h4-category) provided with terminal object $\top$ and the h-cokernel Cf of any map (resp. h4-cokernel, 1.3).

Every right semihomotopical category has a *cone* and *suspension* endofunctors $\mathbf{A} \to \mathbf{A}$

(1) $CA = C(1_A)$ (2) $\Sigma A = C(\top_A \colon A \to \top)$

where the object ΣA comes (universally) equipped with two *vertices* $a', a'' \colon \top \to A$ and a cell $ev^A \colon a'\top_A \to a''\top_A \colon A \to \Sigma A$ (*suspension evaluation of* A). Extending the standard topological situation [28], every map $f \colon A \to B$ has a natural Puppe sequence, or cofibration sequence

(3) $$A \xrightarrow{f} B \xrightarrow{x} Cf \xrightarrow{\delta} \Sigma A \xrightarrow{\Sigma f} \Sigma B \xrightarrow{\Sigma x} \Sigma Cf \longrightarrow \ldots$$

where x is the h-cokernel map of f and the differential $\delta \colon Cf \to \Sigma A$ is determined by the conditions

(4) $\delta x = a'\top_B$, $\delta x'' = a''$, $\delta \xi = ev^A$ $(\text{hck } f = (x, x''; \xi))$.

The Puppe sequence of f can be linked to the one of $x = c(f)$ through the *comparison map* $s \colon Cx \to \Sigma A$, forming a diagram which is commutative, *except for the right-hand* comparison *square*

(5) $$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "x"] \arrow[d, equal] & Cf \arrow[r, "\delta"] \arrow[d, equal] & \Sigma A \arrow[r, "\Sigma f"] \arrow[dr, phantom, "\#"] & \Sigma B \arrow[d, equal] \\ & B \arrow[r, "x"'] & Cf \arrow[r, "y"'] & Cx \arrow[r, "\delta'"'] \arrow[u, "s"] & \Sigma B \end{tikzcd}$$

(6) $s y = \delta$, $s y'' = a'$, $s \eta = 1 \colon \delta x \to a'\top_A$ $(\text{hck } x = (y, y''; \eta))$.

All this behaves well if **A** is right *homotopical*. Then, cone and suspension are homotopy invariant. For each f, the h-cokernel map $c(f)$ is a *cofibration*, the comparison map $s \colon Cx \to \Sigma A$ is a *homotopy equivalence* and the comparison square is *homotopically anti-commutative* (for the reversion of Σ, produced by the reversion of homotopies). One can thus construct a homotopically commutative *cofibration diagram*, connecting the Puppe sequence of f to the sequence of its iterated h-cokernels $x_n = c(x_{n-1})$ (for $x_0 = f$), through a sequence of homotopy equivalences (each of them being a composite of iterated suspensions of comparison maps, alternatively r-modified or not).

If moreover **A** *is pointed, with finite sums*, the suspension ΣA has a natural h-cogroup structure; the two induced structures over a double suspension $\Sigma^2 A$ have the same identity and satisfy the exchange property up to homotopy. Thus, in $\mathbf{A}/\simeq$, ΣA is an internal cogroup and $\Sigma^2 A$ an abelian one. In the *stable* case, $\mathbf{A}/\simeq$ is additive (cf. 3.7).

1.6. Homotopical categories. Dually, *left* semihomotopical and *left* homotopical categories are based on the (lower) *h-kernel* $Kf = K^-f = P(f, \perp)$ of a morphism $f \colon A \to B$

(1)
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle x}\uparrow & \searrow{\scriptstyle \xi} & \uparrow{\scriptstyle \perp} \\ Kf & \xrightarrow[x'']{} & \perp \end{array}$$

(2) hkr f = (x, x"; ξ)

the h-pullback of $\perp_B: \perp \to B$ along f. We have now the (lower) *cocone* endofunctor $KA = K^-A = K(1_A)$, the *loop* endofunctor $\Omega A = K(\perp \to A)$ equipped with two *covertices* $a', a'': A \to \perp$ and a cell $ev_A: \perp_A a' \to \perp_A a'': \Omega A \to A$ (*loop evaluation of* A), and the *fibration sequence* of a map. Actually in [8, 9] it is rather used the *upper* h-kernel $K^+f = P(\perp, f)$, together with the *lower* h-cokernel; but the existence of a reversion makes the upper and lower notions equivalent.

A (*semi-)homotopical* category is at the same time left and right (semi)homotopical; every map f has thus a *fibration-cofibration sequence*. If **A** is *pointed* semihomotopical ($\perp = \top$), the suspension and loop-endofunctor are canonically adjoint ($\Sigma \dashv \Omega$), as well as cone and cocone ($C^- \dashv K^-$).

2. Categories with a cylinder

An abstract cylinder endofunctor I makes the category **A** into an h-category; if I is provided with suitable "operations", we obtain an h4-structure. The study of h-pushouts in this frame is deferred to the next section. Categories with a cylinder functor are dealt with in [1, 10, 11,20, 22].

2.1. Cubical monads. Let **A** be a category equipped with a "homotopy system" in the sense of Kan [23], i.e. a *cylinder* endofunctor $I: \mathbf{A} \to \mathbf{A}$ with natural transformations

(1) $\quad 1 \underset{e}{\overset{\partial^\varepsilon}{\rightleftarrows}} I \qquad\qquad e\,\partial^\varepsilon = 1 \qquad\qquad (\varepsilon = -, +; \;\text{ or }\; 0, 1)$

respectively called *lower face* or *lower unit* (∂^- or ∂^0), *upper face* (∂^+ or ∂^1), *degeneracy* (e).

This produces a *semicubical* enrichment over **A**, with *morphisms* $f: A \to_p B$ *of degree* p represented by maps $f^{(p)}: I^pA \to B$, and the following composition, faces, degeneracies ($1 \leq i \leq p$)

(2) $\quad (g{\circ}f)^{(p+q)} = g^{(q)}.I^qf^{(p)}, \qquad (\partial_i^\varepsilon f)^{(p-1)} = f^{(p)}.I^{i-1}\partial^\varepsilon I^{p-i}, \qquad (e_i f)^{(p)} = f^{(p-1)}.I^{i-1}eI^{p-i}.$

The 0-morphisms are thus the ordinary maps of **A**, the 1-morphisms are the homotopies (2.2), the 2-morphisms are the deformations (2.5). The indexing of faces is consistent with the usual face-maps in **Top**, $\delta_i^\varepsilon: [0, 1]^{n-1} \to [0, 1]^n$, $(t_1,\dots t_{n-1}) \mapsto (t_1,\dots t_{i-1}, \varepsilon,\dots t_{n-1})$, for $IX = [0, 1]\times X$.

Such a structure $(I, \partial^-, \partial^+, e)$ on **A** will also be termed an *I0-category* or *cubical semimonad* or *semidiad*, while we call *cubical monad* or *diad* ([10], 1.5) on **A** a collection $(I, \partial^\varepsilon, e, g^\varepsilon)$ consisting of a semidiad and two *connections* or *main operations* (g^-, g^+), satisfying

(3) $\quad 1 \underset{e}{\overset{\partial^\varepsilon}{\rightleftarrows}} I \overset{g^\varepsilon}{\longleftarrow} I^2$

(4) $\quad e.\partial^\varepsilon = 1, \qquad\qquad e.g^\varepsilon = e_2$

(5) $g^\varepsilon.I\partial^\varepsilon = 1 = g^\varepsilon.\partial^\varepsilon I$, $\qquad g^\eta.I\partial^\varepsilon = \partial^\varepsilon.e = g^\eta.\partial^\varepsilon I \qquad (\varepsilon \neq \eta)$

where $e_2 = e.Ie = e.eI\colon I^2 \to 1$ denotes the second-order degeneracy (of I^2). The last conditions mean that ∂^ε is a unit for the corresponding operation g^ε, and absorbant for the other, g^η.

An *associative* diad, satisfying also the associativity axiom ($g^\varepsilon.Ig^\varepsilon = g^\varepsilon.g^\varepsilon I\colon I^3 \to I$), corresponds to the algebraic notion of *dioid* (a set with two monoid structures, so that the unit of any operation is absorbant for the other [10]), in the same way as the categorical notion of monad [25] corresponds to monoids. Every lattice with 0 and 1 is a dioid, commutative and idempotent. The algebraic character of cubical monads (and monads) can be made explicit through the notion of PROP, introduced by Mac Lane [24]. For the study of connections in cubical objects, see Brown-Higgins [3, 4, 5].

Two kinds of symmetry for cubical monads are relevant, reversion and interchange, which – algebraically – correspond to involutive and commutative dioids, respectively. (Of course, for an I0-category, one should discard the conditions involving connections.) A *reversion* $r\colon I \to I$ exchanges the lower structure (∂^-, g^-) with the upper one (∂^+, g^+)

(6) $r.r = 1, \qquad e.r = e, \qquad r.\partial^- = \partial^+, \qquad r.g^- = g^+.Ir.rI \ \ (= g^+.rI.Ir)$

while an *interchange* $s\colon I^2 \to I^2$ exchanges the horizontal faces $(I\partial^\varepsilon)$ with the vertical ones $(\partial^\varepsilon I)$ and is invariant under the connections

(7) $s.s = 1, \qquad eI.s = Ie, \qquad s.I\partial^\varepsilon = \partial^\varepsilon I, \qquad g^\varepsilon.s = g^\varepsilon;$

moreover, if both are present, a coherence condition is required

(8) $Ir.s = s.rI.$

In **Top**, the standard connections and symmetries are produced by the following maps of the unit interval [0, 1] and the unit square $[0, 1]^2$

(9) $g^-(t, t') = t \vee t', \qquad g^+(t, t') = t \wedge t', \qquad r(t) = 1-t, \qquad s(t, t') = (t', t).$

2.2. Homotopies. It is easy to see that an I0-category is the same as an h-category (1.1) where each object A has a cylinder IA (defined as the h-pushout of the pair $(1_A, 1_A)$, 1.4).

One implication has already been showed (1.4). Conversely, if **A** is an I0-category, the associated h-structure consists of *homotopies* $\alpha\colon f \to g\colon A \to B$ (the 1-morphisms $\alpha\colon A \to_1 B$ of the semicubical structure, 2.1), formally represented by maps $\hat{\alpha} = \alpha^{(1)}$

(1) $\hat{\alpha}\colon IA \to B, \qquad \hat{\alpha}\partial^- = f, \qquad \hat{\alpha}\partial^+ = g$

vertical identities $0_f\colon f \to f$ and reduced horizontal composition $k\alpha h$, represented by

(2) $\hat{0}_f = f.eA = eB.If\colon IA \to B$

(3) $(k\alpha h)\hat{} = k.\hat{\alpha}.Ih\colon IA' \to B' \qquad (h\colon A' \to A, \quad k\colon B \to B')$

while the cylinder IA (as a homotopy pushout for the h-structure) is supplied by the endofunctor I itself, with structural cell $\delta\colon \partial^- \to \partial^+$ represented by the map $\hat{\delta} = 1_{IA}$. Note that, in the composition (3), the homotopy αh is represented by the map $\hat{\alpha}.Ih$ (while "$\hat{\alpha}.h$" makes no sense, in general). We shall write α for $\hat{\alpha}$ when this ambiguity does not risk to produce errors.

In an I0-category, an ordinary terminal object is automatically 2-terminal (1.1), since every object A has exactly one map $IA \to T$.

An I1-*category* **A** will be an I0-category equipped with a reversion $r: I \to I$ (2.1). This produces an h1-structure, with $-\alpha: g \to f$ represented by $\hat{\alpha}.r: IA \to B$. In other words, an I1-category is the same as an h1-category with cylinders.

2.3. I3-categories. Let **A** be I1. In order to introduce the vertical composition of homotopies, we assume the existence, for every object A, of the *composition* pushout $J(A) = IA +_A IA$ (the pasting of two cylinders, one on top of the other)

(1)
$$\begin{array}{ccc} A & \xrightarrow{\partial^+} & IA \\ {\scriptstyle\partial^-}\downarrow & & \downarrow{\scriptstyle k^-} \\ IA & \xrightarrow{k^+} & JA \end{array}$$

There are thus two natural transformations $k^-, k^+: I \to J$. They supply three faces $1 \to J$ (lower, upper and middle face)

(2) $\partial^{--} = k^- \partial^-$, $\quad\partial^{++} = k^+\partial^+$, $\quad\partial^{\pm} = k^+ \partial^- = k^- \partial^+$

Moreover, the transformations e, r induce a *degeneracy* $\overline{e}$ and a *reversion* $\overline{r}$ for J

(3) $\overline{e}: JA \to A$, $\quad\overline{e}\, k^- = \overline{e}\, k^+ = e$, $\quad\overline{e}\, \partial^{--} = \overline{e}\, \partial^{++} = \overline{e}\, \partial^{\pm} = 1$

(4) $\overline{r}: JA \to JA$, $\quad\overline{r}\, k^- = k^+ r$, $\quad\overline{r} . \overline{r} = 1$, $\quad\overline{e} . \overline{r} = \overline{e}$.

An I3-*category* will be an I1-category having all the composition pushouts JA, and equipped with a *composition map* $k: I \to J$ satisfying

(5) $k\, \partial^- = k^- \partial^-$, $\quad k\, \partial^+ = k^+ \partial^+$, $\quad\overline{e}\, k = e$, $\quad\overline{r}\, k = k\, r$.

The *vertical composition*, or *sum*, $\alpha+\beta: f \to h$, of a pair of vertically consecutive homotopies $\alpha: f \to g$ and $\beta: g \to h$, is then represented by the map

(6) $(\alpha+\beta)^\wedge = (\hat{\alpha} \vee \hat{\beta}).k: IA \to B$

where $\hat{\alpha} \vee \hat{\beta}$ denotes the obvious map defined over the pushout JA. It follows easily that the vertical composition satisfies the axioms for an h3-category (1.2). In particular, k^- and k^+ are (represent) consecutive homotopies, with vertical composition k

(7) $k^-: \partial^{--} \to \partial^{\pm}$, $\quad k^+: \partial^{\pm} \to \partial^{++}$; $\quad k^- + k^+ = k: \partial^{--} \to \partial^{++}: A \to JA$.

For chain complexes, $k: IA \to JA$ is not iso (6.6). But in **Top**, one can take $JA = IA$ and $k = 1$, with k^- produced by the "first-half" embedding of the standard interval into itself, $t \mapsto t/2$. Thus the sum of homotopies is the standard one.

Omitting the reversion, one can give the notion of I2-*category*, similarly related to h2-categories.

2.4. Symmetrical I3-categories. An I0-, or I1-, or I3-category *with interchange* (or *symmetrical*) is further equipped with an interchange $s: I^2 \to I^2$ (2.1). In the last case, a new coherence condition between interchange and vertical composition is required.

In fact, the composition pushout J(IA) in (1) determines a map $\overline{s}: JIA \to IJA$, and the square (2) is asked to commute (note that $\overline{s}$ is iso whenever I preserves the composition pushouts)

$$(1)\qquad \begin{tikzcd}[column sep=small] & \mathrm{IA} \arrow[rr, "\mathrm{I}\partial^+"] \arrow[dd, dashed, "\mathrm{I}\partial^-" near start] & & \mathrm{I}^2\mathrm{A} \arrow[dd, "\mathrm{Ik}^-"] \\ \mathrm{IA} \arrow[ur, "1"] \arrow[rr, "\partial^+\mathrm{I}" near start] \arrow[dd, "\partial^-\mathrm{I}"'] & & \mathrm{I}^2\mathrm{A} \arrow[ur, "s"] \arrow[dd, "k^-\mathrm{I}" near start] & \\ & \mathrm{I}^2\mathrm{A} \arrow[rr, dashed, "\mathrm{Ik}^+" near start] & & \mathrm{IJA} \\ \mathrm{I}^2\mathrm{A} \arrow[ur, dashed, "s"'] \arrow[rr, "k^+\mathrm{I}"] & & \mathrm{JIA} \arrow[ur, "\bar{s}"'] & \end{tikzcd} \qquad\qquad (2)\qquad \begin{tikzcd} \mathrm{I}^2\mathrm{A} \arrow[r, "k\mathrm{I}"] \arrow[d, "s"'] & \mathrm{JIA} \arrow[d, "\bar{s}"] \\ \mathrm{I}^2\mathrm{A} \arrow[r, "\mathrm{Ik}"'] & \mathrm{IJA} \end{tikzcd}$$

In every I0-category with interchange, the functor I *is homotopy invariant*. Indeed, given a homotopy $\hat{\alpha}$: IA $\to$ B, with $\hat{\alpha}.\partial^\varepsilon = f_\varepsilon$, $\mathrm{I}\hat{\alpha}.s$ is a homotopy from If_0 to If_1 ($\mathrm{I}\hat{\alpha}.s.\partial^\varepsilon\mathrm{I} = \mathrm{I}\hat{\alpha}.\mathrm{I}\partial^\varepsilon = \mathrm{If}_\varepsilon$). More precisely, and here one should not confuse α with $\hat{\alpha}$, we set

(3) $\quad \mathrm{I}(\alpha)\hat{}\ = \mathrm{I}\hat{\alpha}.s$

making I into an h-functor, which is also h1 (preserves reversion) in the symmetrical I1-case

(4) $\quad \mathrm{I}(k\circ\alpha\circ h) = \mathrm{I}k\circ\mathrm{I}(\alpha)\circ\mathrm{I}h, \qquad\qquad \mathrm{I}(0_f) = 0_{\mathrm{I}f}, \qquad\qquad \mathrm{I}(-\alpha) = -\,\mathrm{I}\alpha.$

Finally, if **A** is symmetrical I3, I is an h3-functor, i.e. preserves also the sum of homotopies

(5) $\quad \mathrm{I}(\alpha+\beta) = \mathrm{I}\alpha + \mathrm{I}\beta \qquad\qquad (\alpha\colon f \to g\colon A \to B, \quad \beta\colon g \to h\colon A \to B)$

since the pushout JIA in (1) shows that $\mathrm{I}(\hat{\alpha} \vee \hat{\beta}).\bar{s} = (\mathrm{I}\hat{\alpha}.s) \vee (\mathrm{I}\hat{\beta}.s)$, and (2) implies that

(6) $\quad \mathrm{I}(\alpha+\beta) = \mathrm{I}(\hat{\alpha} \vee \hat{\beta}).\mathrm{Ik}.s = \mathrm{I}(\hat{\alpha} \vee \hat{\beta}).\bar{s}.k\mathrm{I} = (\mathrm{I}\hat{\alpha}.s) \vee (\mathrm{I}\hat{\beta}.s).k\mathrm{I} = \mathrm{I}\alpha + \mathrm{I}\beta.$

2.5. Deformations and their pasting. Let **A** be I0. A 2-morphism $\Phi\colon A \to_2 B$ of the semicubical structure (2.1), represented by a map $\hat{\Phi} = \Phi^{(2)}\colon \mathrm{I}^2A \to B$ defined over a second-order cylinder will be viewed as a *deformation*, or *double homotopy*, of four homotopies, its *horizontal* and *vertical* faces, forming the *boundary* $\partial\Phi$

$$(1)\qquad \begin{tikzcd} h \arrow[r, "\alpha_1"] & k \\ f \arrow[u, "\rho_0"] \arrow[r, "\alpha_0"'] \arrow[ur, phantom, "\Phi"] & g \arrow[u, "\rho_1"'] \end{tikzcd}$$

(2) $\quad \hat{\alpha}_\varepsilon = (\partial_1^\varepsilon\,\Phi)\hat{}\ = \hat{\Phi}.\mathrm{I}\partial^\varepsilon$

(3) $\quad \hat{\rho}_\varepsilon = (\partial_2^\varepsilon\,\Phi)\hat{}\ = \hat{\Phi}.\partial^\varepsilon\mathrm{I}.$

Also here, we tend to use the same name for Φ and $\hat{\Phi}$. An I0-category will be said to be *flat* if two deformations having the same boundary necessarily coincide (see. 6.3-4). The boundaries of $\beta\circ\alpha$ (for $\alpha\colon a_0 \to a_1\colon A \to B$ and $\beta\colon b_0 \to b_1\colon B \to C$, see 2.1.2) and g^-, $g^+\colon \mathrm{I}^2A \to \mathrm{IA}$ (if such connections exist in **A**) are

$$(4)\ \begin{tikzcd} b_0 a_1 \arrow[r, "\beta a_1"] & b_1 a_1 \\ b_0 a_0 \arrow[u, "b_0\alpha"] \arrow[r, "\beta a_0"'] \arrow[ur, phantom, "\beta\circ\alpha"] & b_1 a_0 \arrow[u, "b_1\alpha"'] \end{tikzcd} \qquad (5)\ \begin{tikzcd} \partial^+ \arrow[r, "0"] & \partial^+ \\ \partial^- \arrow[u, "\delta"] \arrow[r, "\delta"'] \arrow[ur, phantom, "g^-"] & \partial^+ \arrow[u, "0"'] \end{tikzcd} \qquad (6)\ \begin{tikzcd} \partial^- \arrow[r, "\delta"] & \partial^+ \\ \partial^- \arrow[u, "0"] \arrow[r, "0"'] \arrow[ur, phantom, "g^+"] & \partial^- \arrow[u, "\delta"'] \end{tikzcd}$$

Assume now that **A** is I3. Note (and this remark should be used with care, to avoid confusion with the stronger notion of cell-homotopy studied below) that the map $\hat{\Phi}\colon \mathrm{I(IA)} \to B$ can also be thought to represent the associated *horizontal* homotopy $\Phi_h\colon \hat{\rho}_0 \to \hat{\rho}_1$ between its vertical faces, as *maps* defined over IA. Given two deformations Φ_0 and Φ_1 disposed as in (7), one can thus define their horizontal pasting, or sum $\Psi = \Phi_0 +_h \Phi_1$ along the common vertical face ρ, corresponding to the sum of the associated horizontal homotopies

(7)
$$\begin{tikzcd} \bullet \arrow[r, "\alpha_1"] & \bullet \arrow[r, "\beta_1"] & \bullet \\ \bullet \arrow[u, "\rho_0"] \arrow[r, "\alpha_0"'] \arrow[ur, phantom, "\Phi_0"] & \bullet \arrow[u, "\rho"'] \arrow[r, "\beta_0"'] \arrow[ur, phantom, "\Phi_1"] & \bullet \arrow[u, "\rho_1"'] \end{tikzcd}$$

(8)
$$\begin{tikzcd} \bullet \arrow[r, "\alpha_1 + \beta_1"] & \bullet \\ \bullet \arrow[u, "\rho_0"] \arrow[r, "\alpha_0 + \beta_0"'] \arrow[ur, phantom, "\Psi"] & \bullet \arrow[u, "\rho_1"'] \end{tikzcd}$$

(9) $\Psi = \Phi_0 +_h \Phi_1 = (\Phi_0 \vee \Phi_1).\mathrm{kI}\colon \mathrm{I}^2\mathrm{A} \to \mathrm{B}, \qquad \Psi_h = (\Phi_0)_h + (\Phi_1)_h\colon \hat{\rho}_0 \to \hat{\rho}_1 .$

The pasting is thus realised through the J-pushout (10) of IA and the map $\mathrm{kI}\colon \mathrm{I}^2\mathrm{A} \to \mathrm{J(IA)}$

(10)
$$\begin{tikzcd} \mathrm{IA} \arrow[r, "\partial^+\mathrm{I}"] \arrow[d, "\partial^-\mathrm{I}"'] & \mathrm{I}^2\mathrm{A} \arrow[d, "\mathrm{k}^-\mathrm{I}"] \\ \mathrm{I}^2\mathrm{A} \arrow[r, "\mathrm{k}^+\mathrm{I}"] & \mathrm{JIA} \end{tikzcd}$$

(11)
$$\begin{tikzcd} & \mathrm{IA} \arrow[rr, "\partial^+\mathrm{I}"] \arrow[dd, dashed, "\partial^-\mathrm{I}"] & & \mathrm{I}^2\mathrm{A} \arrow[dd, "\mathrm{k}^-\mathrm{I}"] \\ \mathrm{A} \arrow[ur, "\partial^\varepsilon"] \arrow[rr, "\partial^+"] \arrow[dd, "\partial^-"'] & & \mathrm{IA} \arrow[ur, "\mathrm{I}\partial^\varepsilon"] \arrow[dd, "\mathrm{k}^-"] & \\ & \mathrm{I}^2\mathrm{A} \arrow[rr, dashed, "\mathrm{k}^+\mathrm{I}"] & & \mathrm{JIA} \\ \mathrm{IA} \arrow[ur, "\mathrm{I}\partial^\varepsilon"] \arrow[rr, "\mathrm{k}^+"'] & & \mathrm{JA} \arrow[ur, "\mathrm{J}\partial^\varepsilon"'] & \end{tikzcd}$$

and the horizontal faces of $\Phi_0 +_h \Phi_1$ are indeed the sum of the horizontal faces of Φ_ε, as claimed in (8), because of the commutative cube (11) whose front pushout defines $\alpha_\varepsilon \vee \beta_\varepsilon$

(12) $\Psi.\mathrm{I}\partial^\varepsilon = (\Phi_0 \vee \Phi_1).\mathrm{kI}.\mathrm{I}\partial^\varepsilon = (\Phi_0 \vee \Phi_1).\mathrm{J}\partial^\varepsilon.\mathrm{k} = (\alpha_\varepsilon \vee \beta_\varepsilon).\mathrm{k} = \alpha_\varepsilon + \beta_\varepsilon.$

If the I3-category **A** is *regular*, or *has a regular sum* (i.e., is regular h3, 1.2), then also the horizontal sum of deformations is *regular*, and yields a groupoid structure on each set $\mathbf{A}(\mathrm{I}^2\mathrm{A}, \mathrm{B})$, with identities and inverses given by

(13) $0^h_\rho = \mathrm{e}.\mathrm{I}\hat{\rho} = 0_1 \circ \rho, \qquad -_h \Phi = \Phi.\mathrm{rI}.$

In a *symmetrical* I3-category one can define the "vertical pasting" along an intermediate, common horizontal face

(14) $(\Phi_0 +_v \Phi_1).\mathrm{s} = \Phi_0.\mathrm{s} +_h \Phi_1.\mathrm{s} \qquad (0^v_\alpha = \hat{\alpha}.\mathrm{Ie} = \alpha \circ 0_1, \quad -_v \Phi = \Phi.\mathrm{Ir}).$

The vertical pasting can also be defined in an I3-category *whose composition pushouts are preserved by* I; and in this case one can prove the exchange property for horizontal and vertical pastings of deformations. If there is an interchange, the two definitions agree (2.4.1-2).

2.6. Cell homotopies. As usual, a *cell-homotopy*, or 2-*homotopy*, $\Phi\colon \alpha_0 \sim \alpha_1$, between parallel cells $\alpha_\varepsilon\colon f \to g\colon A \to B$, is a deformation Φ with horizontal faces α_ε and vertical faces degenerate

$$(1) \qquad \begin{array}{ccc} f & \xrightarrow{\alpha_1} & g \\ 0\uparrow & \Phi & \uparrow 0 \\ f & \xrightarrow[\alpha_0]{} & g \end{array} \qquad\qquad (2) \quad \hat{\Phi}.I\partial^\varepsilon = \hat{\alpha}_\varepsilon, \qquad \hat{\Phi}.\partial^\varepsilon I = 0.$$

If **A** is I3, the previous pasting operations restrict to cell-homotopies

a) given two cell-homotopies $\Phi_0: \alpha_0 \sim \alpha_1: f_0 \to g$ and $\Phi_1: \beta_0 \sim \beta_1: g \to f_1$, the horizontal pasting along g is defined through the J-pushout of IA (2.5.10)

(3) $\Phi_0 +_h \Phi_1 = (\Phi_0 \vee \Phi_1).kI: \alpha_0{+}\beta_0 \sim \alpha_1{+}\beta_1: f_0 \to f_1$

b) given two cell-homotopies $\Phi_0: \alpha_0 \sim \beta: f \to g$ and $\Phi_1: \beta \sim \alpha_1: f \to g$, the vertical pasting along β

(4) $\Phi_0 +_v \Phi_1 = (\Phi_0.s \;+_h\; \Phi_1.s).s : \alpha_0 \sim \alpha_1: f \to g$

exists, provided **A** has interchange (recall that $0_f + 0_f = 0_f$, 2.3).

2.7. I4-categories. We need now a mapping $\alpha \mapsto \Phi$ turning a homotopy α into a cell-homotopy Φ: $\alpha \sim \alpha + 0$, and a mapping $\Phi \mapsto \Psi$ turning a deformation Φ of four homotopies, as in (1), into a cell-homotopy $\Psi: \alpha{+}\delta \to \beta{+}\gamma$, as in (2)

$$(1) \qquad \begin{array}{ccc} k & \xrightarrow{\gamma} & g \\ \beta\uparrow & \Phi & \uparrow \delta \\ f & \xrightarrow[\alpha]{} & h \end{array} \qquad\qquad (2) \qquad \begin{array}{ccc} f & \xrightarrow{\beta+\gamma} & g \\ 0\uparrow & \Psi & \uparrow 0 \\ f & \xrightarrow[\alpha+\delta]{} & g \end{array}$$

and we want to realise such conversions "algebraically", through transformations of the cylinder.

An I4-*category* **A** has a structure $(I, \partial^\varepsilon, e, g^\varepsilon, r, s, J, k, z, w)$ combining a cubical monad with reversion and interchange (2.1) with a symmetrical I3-structure (2.3-4); further, it has a *zero collapse* $z: I^2 \to I$ and a *lens collapse* $w: I^2 \to I^2$ with the following boundary and degeneracy conditions

$$(3) \qquad \begin{array}{ccc} \partial^- & \xrightarrow{\delta+0} & \partial^+ \\ 0\uparrow & z & \uparrow 0 \\ \partial^- & \xrightarrow[\delta]{} & \partial^+ \end{array} \qquad\qquad (4) \qquad \begin{array}{ccc} I\partial^-.\partial^- & \xrightarrow{\partial^- I + I\partial^+} & I\partial^+.\partial^+ \\ 0\uparrow & w & \uparrow 0 \\ I\partial^-.\partial^- & \xrightarrow[I\partial^- + \partial^+ I]{} & I\partial^+.\partial^+ \end{array}$$

(5) $e.z = e_2$ (6) $e_2.w = e_2$

where $\delta: \partial^- \to \partial^+: A \to IA$ is the structural cell of the cylinder IA ($\hat{\delta} = 1_{IA}$) and $e_2 = e.Ie = e.eI: I^2 \to 1$ is the degeneracy of I^2.

These maps z and w produce the conversions we want, $\hat{\alpha} \mapsto \hat{\alpha}.z$ and $\hat{\Phi} \mapsto \hat{\Psi} = \hat{\Phi}.w$

(7) $\alpha.z.I\partial^- = \alpha.1_{IA} = \alpha$, $\alpha.z.I\partial^+ = \alpha.(\delta{+}0) = \alpha{+}0$, $\alpha.z.\partial^\varepsilon I = 0$

(8) $\Phi w.I\partial^- = \Phi.(I\partial^-{+}\partial^+ I) = \alpha{+}\delta$, $\Phi w.I\partial^+ = \beta{+}\gamma$, $\Phi w.\partial^\varepsilon I = \Phi.0 = 0$.

We say that the lens collapse is *strong*, and that **A** is *strong* I4, if moreover all these *lens conversions* $\Phi \mapsto \Phi w$ are bijective. In other words, for every fourtuple of homotopies $\alpha: f \to h$, $\beta: f$

$\to$ k, γ: k $\to$ g, δ: h $\to$ g (between parallel maps) and every cell-homotopy Ψ: $\alpha+\delta \to \beta+\gamma$, there exists precisely one deformation Φ with boundary (1) such that Φw coincides with Ψ. It can be noted that some results (e.g. thm. 3.5) just need the surjectivity of the conversion, while other (e.g. the naturality properties of § 4) do need the bijectivity.

Plainly, a *flat* I4-category (2.5) is strong. **Top** is strong I4, non flat, through the zero collapse

(9) $z: [0, 1]^2 \to [0, 1], \qquad z(t, t') = (2t/(2-t')) \wedge 1$

and the endomap w of the unit square $[0, 1]^2$ described in the Introduction (0.3), which motivates the name of lens collapse.

2.8. Regular I4-categories. Actually, one can derive a "similar" conversion $\Phi \mapsto \Psi$ from the connections, pasting Φ with two deformations based on g^-, g^+

$$(1)\qquad \begin{array}{ccccccc} f & \xrightarrow{\beta} & k & \xrightarrow{\gamma} & g & \xrightarrow{0} & g \\ 0\uparrow & \beta g^+ & \uparrow\beta & \Phi & \uparrow\delta & \delta g^- & \uparrow 0 \\ f & \xrightarrow[0]{} & f & \xrightarrow[\alpha]{} & h & \xrightarrow[\delta]{} & g \end{array}$$

In general, e.g. in **Top**, this conversion fails to be bijective and is not sufficient for our purposes. But in the regular case this works well, and we can simplify the structure omitting w (and z as well, since here $\alpha+0 = \alpha$).

A *regular* I4-category, or I4-sesquigroupoid (I, ∂^ε, e, g^ε, r, s, J, k) will be thus a symmetrical I3-structure (over **A**) with consistent connections and *regular sum* (1.2). Then **A** is *strong I4*, with zero and lens collapses derived from the other transformations

(2) $z = \mathrm{I}e: \mathrm{I}^2 \to \mathrm{I}$ (3) $w = (\partial^-\mathrm{I}.g^+) +_h 1 +_h (\partial^+\mathrm{I}.g^-): \mathrm{I}^2 \to \mathrm{I}^2$.

Direct calculations show that these maps do satisfy our boundary and coherence requirements (2.7.3-6); further, the conversion defined by w works precisely as in diagram (1)

(4) $\Phi \mapsto \Psi = \Phi w = \beta g^+ +_h \Phi +_h \delta g^- \qquad (\beta = \partial^-\mathrm{I}.\Phi, \quad \delta = \partial^+\mathrm{I}.\Phi)$

and is bijective (in the sense specified above, 2.7) because of the regular behaviour of the sum of deformations (2.5): given Ψ, take $\Phi = -_h \beta g^+ +_h \Phi -_h \delta g^-$.

Top is strong I4, neither flat nor regular. Chain complexes form a regular, non flat I4-category. $\mathbf{Cat}_i$ and **Gpd** are examples of the flat, regular case. (One could further reduce the redundancy of the regular structure deducing the reversion, $r = -\delta$, and one connection.)

2.9. Theorem. Every I4-category **A** is h4, with respect to the h3-structure previously considered (2.3) and the cell-homotopy relation $\sim$ defined by I^2 (2.6). The functor I is h4.

Proof. a) The cell-homotopy relation $\sim$ is an equivalence. The transitive property follows from the vertical pasting of cell-homotopies (2.6 b), which is possible because of the interchange. Moreover

- for $\alpha: f_0 \to f_1: A \to B$, take $\Phi = \alpha.\mathrm{I}e: \mathrm{I}^2A \to B$, so that $\Phi: \alpha \sim \alpha$

(1) $\Phi.\mathrm{I}\partial^\varepsilon = \alpha.\mathrm{I}e.\mathrm{I}\partial^\varepsilon = \alpha, \qquad \Phi.\partial^\varepsilon\mathrm{I} = \alpha.\mathrm{I}e.\partial^\varepsilon\mathrm{I} = \alpha.\partial^\varepsilon.e = f_\varepsilon.e = 0.$

- for $\Phi: \alpha \sim \beta: f_0 \to f_1: A \to B$, take $\Psi = \Phi.\mathrm{Ir}: I^2A \to B$, so that $\Psi: \beta \sim \alpha$

(2) $\Psi.\mathrm{I}\partial^\varepsilon = \Phi.\mathrm{I}(r\partial^\varepsilon) = \Phi.\mathrm{I}\partial^\eta \quad (\varepsilon \neq \eta), \qquad \Psi.\partial^\varepsilon\mathrm{I} = \Phi.\mathrm{Ir}.\partial^\varepsilon\mathrm{I} = \Phi.\partial^\varepsilon\mathrm{I}.r = 0.$

b) $\alpha_0 \sim \alpha_1$ implies $k\alpha_0 h \sim k\alpha_1 h$ (for $\alpha_\varepsilon: f_0 \to f_1: A \to B$, $h: A' \to A$, $k: B \to B'$). Choose $\Phi: \alpha_0 \sim \alpha_1$ and consider the map $\Psi = k.\Phi.I^2h: I^2A' \to B'$

(3) $\Psi.\mathrm{I}\partial^\varepsilon = k.\Phi.\mathrm{I}^2h.\mathrm{I}\partial^\varepsilon = k.\Phi.\mathrm{I}\partial^\varepsilon.\mathrm{Ih} = k.\hat{\alpha}_\varepsilon.\mathrm{Ih} = k\,\alpha_\varepsilon\,h$

$\Psi.\partial^\varepsilon\mathrm{I} = k.\Phi.\mathrm{I}^2h.\partial^\varepsilon\mathrm{I} = k.\Phi.\partial^\varepsilon\mathrm{I}.\mathrm{Ih} = k.0.\mathrm{Ih} = 0.$

c) $\alpha_0 \sim \alpha_1$ implies $-\alpha_0 \sim -\alpha_1$. Taking $\Psi = \Phi.\mathrm{rI}$, we have

(4) $\Psi.\mathrm{I}\partial^\varepsilon = \Phi.\mathrm{rI}.\mathrm{I}\partial^\varepsilon = \Phi.\mathrm{I}\partial^\varepsilon.r = -\alpha_\varepsilon, \qquad \Psi.\partial^\varepsilon\mathrm{I} = \Phi.\mathrm{rI}.\partial^\varepsilon\mathrm{I} = \Phi.\partial^\varepsilon\mathrm{I}.r = 0.$

d) Given $\Phi_0: \alpha_0 \sim \alpha_1: f_0 \to g$ and $\Phi_1: \beta_0 \sim \beta_1: g \to f_1$, define $\Psi: \alpha_0+\beta_0 \sim \alpha_1+\beta_1: f_0 \to f_1$ as the horizontal pasting of the cell-homotopies Φ_0 and Φ_1 (2.6 a).

e) We already know that $\alpha + 0_g \sim \alpha$, because of the zero collapse (2.7); the symmetrical property follows by reversion: $0_f + \alpha = -(-\alpha + 0_f) \sim -(-\alpha) = \alpha$. A cell-homotopy $-\alpha + \alpha \sim 0_f$ is provided by the connection g^-, through the composition $\Phi = \alpha.g^-.\mathrm{rI}.w: I^2A \to B$ whose boundary is shown here below (check from right to left)

$$(5) \qquad \begin{tikzcd} \bullet \arrow[r, "0"] & \bullet \\ \bullet \arrow[u, "0"] \arrow[r, "-\alpha+\alpha"'] \arrow[ur, phantom, "\Phi"] & \bullet \arrow[u, "0"'] \end{tikzcd} \xrightarrow{\ w\ } \begin{tikzcd} \bullet \arrow[r, "0"] & \bullet \\ \bullet \arrow[u, "0"] \arrow[r, "-\alpha"'] & \bullet \arrow[u, "\alpha"'] \end{tikzcd} \xrightarrow{\ \mathrm{rI}\ } \begin{tikzcd} \bullet \arrow[r, "0"] & \bullet \\ \bullet \arrow[u, "\alpha"] \arrow[r, "\alpha"'] & \bullet \arrow[u, "0"'] \end{tikzcd} \xrightarrow{\ g^-\ } \bullet \xrightarrow{\alpha} \bullet$$

f) To prove the $\sim$-associativity of the sum, $(\alpha+\beta) + \gamma \sim \alpha + (\beta+\gamma)$, note that the deformations $\Phi_0 = \alpha.g^-: I^2A \to B$ and $\Phi_1 = \beta.g^+: I^2A \to B$ have a horizontal pasting Φ in (7)

$$(6) \qquad \begin{tikzcd} \bullet \arrow[r, "0"] & \bullet \arrow[r, "\beta"] & \bullet \\ \bullet \arrow[u, "\alpha"] \arrow[r, "\alpha"'] \arrow[ur, phantom, "\Phi_0"] & \bullet \arrow[u, "0"] \arrow[r, "0"'] \arrow[ur, phantom, "\Phi_1"] & \bullet \arrow[u, "\beta"'] \end{tikzcd} \qquad\qquad (7) \qquad \begin{tikzcd} \bullet \arrow[r, "0+\beta"] & \bullet \arrow[r, "0+\gamma"] & \bullet \\ \bullet \arrow[u, "\alpha"] \arrow[r, "\alpha+0"'] \arrow[ur, phantom, "\Phi"] & \bullet \arrow[u, "\beta"] \arrow[r, "\beta+0"'] \arrow[ur, phantom, "\Psi"] & \bullet \arrow[u, "\gamma"'] \end{tikzcd}$$

which can be further pasted along β with the deformation Ψ similarly constructed from β and γ, producing a deformation Θ. By lens conversion, $\Theta w: I^2A \to B$ is a cell-homotopy which implies the thesis, through the previous results

(8) $\Theta w: \quad ((\alpha+0) + (\beta+0)) + \gamma \sim \alpha + ((0+\beta) + (0+\gamma)).$

g) As to the weak reduced exchange, for $\alpha: f \to f': A \to B$ and $\kappa: k \to k': B \to C$

(9) $\kappa f + k'\alpha \sim k\alpha + \kappa f': \; kf \to k'f': \; A \to C$

it suffices to compose α and κ (in the cubical enriched structure supplied by I) and apply lens conversion to get a cell-homotopy $\Phi = \kappa.\mathrm{I}\alpha.w: I^2A \to C$ as required

$$(10)\qquad \begin{tikzcd} \bullet \arrow[r, "k\alpha+\kappa f'"] & \bullet \\ \bullet \arrow[u, "0"] \arrow[r, "\kappa f+k'\alpha"'] \arrow[ur, phantom, "\Phi"] & \bullet \arrow[u, "0"'] \end{tikzcd} \quad \xrightarrow{\ w\ } \quad \begin{tikzcd} \bullet \arrow[r, "\kappa f'"] & \bullet \\ \bullet \arrow[u, "k\alpha"] \arrow[r, "\kappa f"'] & \bullet \arrow[u, "k'\alpha"'] \end{tikzcd} \quad \xrightarrow{\ I\alpha\ } \quad \bullet \xrightarrow{\ \kappa\ } \bullet$$

h) Finally, we already know that I is an h3-functor, with $\mathrm{I}(\alpha)\hat{}\ = \mathrm{I}\hat{\alpha}.\mathrm{s}$ (2.4.3), and we must prove it preserves the cell-homotopy relation $\sim$. Given a cell-homotopy $\Phi\colon \alpha_0 \sim \alpha_1$, we turn it into the cell-homotopy $\mathrm{I}\Phi\colon \mathrm{I}(\alpha_0) \sim \mathrm{I}(\alpha_1)$, through the composed symmetry $\mathrm{s}' = \mathrm{sI.Is}\colon \mathrm{I}^3 \to \mathrm{I}^3$

(11) $\Phi\colon \alpha_0 \sim \alpha_1\colon f_0 \to f_1\colon A \to B$ $\qquad (\hat{\Phi}.\mathrm{I}\partial^\varepsilon = \hat{\alpha}_\varepsilon, \quad \hat{\Phi}.\partial^\varepsilon\mathrm{I} = 0_{f_\varepsilon})$

(12) $(\mathrm{I}\Phi)\hat{}\ = \mathrm{I}\hat{\Phi}.\mathrm{sI.Is}\colon \mathrm{I}^2(\mathrm{IA}) \to \mathrm{IB}$

(13) $\mathrm{I}\hat{\Phi}.\mathrm{sI.Is.I}\partial^\varepsilon\mathrm{I} = \mathrm{I}\hat{\Phi}.\mathrm{sI.II}\partial^\varepsilon = \mathrm{I}\hat{\Phi}.\mathrm{II}\partial^\varepsilon.\mathrm{s} = \mathrm{I}(\hat{\Phi}.\mathrm{I}\partial^\varepsilon).\mathrm{s} = \mathrm{I}\hat{\alpha}_\varepsilon.\mathrm{s} = \mathrm{I}(\alpha_\varepsilon)$

(14) $\mathrm{I}\hat{\Phi}.\mathrm{sI.Is.}\partial^\varepsilon\mathrm{II} = \mathrm{I}\hat{\Phi}.\mathrm{sI.}\partial^\varepsilon\mathrm{II.s} = \mathrm{I}\hat{\Phi}.\mathrm{I}\partial^\varepsilon\mathrm{I.s} = \mathrm{I}(\hat{\Phi}.\partial^\varepsilon\mathrm{I}).\mathrm{s} = \mathrm{I}(\hat{0}_{f_\varepsilon}).\mathrm{s} = \mathrm{I}(0_{f_\varepsilon}) = 0_{\mathrm{I}f_\varepsilon}.$

2.10. P4- and IP4-categories. Dualising 2.7, a P4-*category* **A** is equipped with a cubical comonad with reversion and interchange $(\mathrm{P}, \partial^\varepsilon, \mathrm{e}, \mathrm{g}^\varepsilon, \mathrm{r}, \mathrm{s})$ [10], together with vertical composition k, zero collapse z and lens collapse w

(1) $\mathrm{k}\colon \mathrm{QA} = \mathrm{PA} \times_{\mathrm{A}} \mathrm{PA} \to \mathrm{PA}, \qquad\qquad \mathrm{z}\colon \mathrm{P} \to \mathrm{P}^2, \qquad\qquad \mathrm{w}\colon \mathrm{P}^2 \to \mathrm{P}^2$

where k acts on the *composition pullback* $\mathrm{QA} = \mathrm{PA} \times_{\mathrm{A}} \mathrm{PA}$, or *object of composable paths* (the pullback of $\partial^+, \partial^-\colon \mathrm{PA} \to \mathrm{A}$).

An IP0- (resp IP4-) *category* is a category equipped with adjoint endofunctors $\mathrm{I} \dashv \mathrm{P}$ and with *consistent* I0- and P0- (resp. I4- and P4-) structures, determining each other through the adjunction. For instance, if $(\mathrm{I}, \partial^\varepsilon, \mathrm{e}, \mathrm{g}^\varepsilon)$ is a cubical monad and $\mathrm{I} \dashv \mathrm{P}$, one gets a cubical comonad $(\mathrm{P}, \partial^\varepsilon_\bullet, \mathrm{e}_\bullet, \mathrm{g}^\varepsilon_\bullet)$ for P, through the unit and counit of the adjunction $(\eta\colon 1 \to \mathrm{PI},\ \vartheta\colon \mathrm{IP} \to 1)$

(2) $\partial^\varepsilon_\bullet = \vartheta.\partial^\varepsilon\mathrm{P} = (\mathrm{P} \to \mathrm{IP} \to 1)$ $\qquad$ (3) $\mathrm{e}_\bullet = \mathrm{Pe}.\eta = (1 \to \mathrm{PI} \to \mathrm{P})$

(4) $\mathrm{g}^\varepsilon_\bullet = \mathrm{P}^2\vartheta.\mathrm{P}^2\mathrm{g}^\varepsilon\mathrm{P}.\mathrm{P}\eta\mathrm{IP}.\eta\mathrm{P} = (\mathrm{P} \to \mathrm{PIP} \to \mathrm{P}^2\mathrm{I}^2\mathrm{P} \to \mathrm{P}^2\mathrm{IP} \to \mathrm{P}^2).$

In this way, *the horizontal faces* $\mathrm{I}\partial^\varepsilon$ *of* I^2 *correspond to the faces* $\partial^\varepsilon_\bullet\mathrm{P}$ of P^2, which are thus considered as *horizontal*.

In a P0-category, one has thus the following situation, parallel to the one of I0-categories (2.5-6). A deformation $\check{\Phi}\colon \mathrm{A} \to \mathrm{P(PB)}$ also represents the associated *vertical* homotopy $\Phi_{\mathrm{v}}\colon \check{\alpha}_0 \to \check{\alpha}_1$ between its horizontal faces $\partial^\varepsilon\mathrm{P}.\check{\Phi}$, as *maps* with values in PB. A cell-homotopy $\Phi\colon \alpha_0 \sim \alpha_1$ is a deformation with vertical faces degenerate, as in 2.6 $(\partial^\varepsilon\mathrm{P}.\check{\Phi} = \check{\alpha}_\varepsilon,\ \mathrm{P}\partial^\varepsilon.\check{\Phi} = 0)$, and the boundary of the deformation $\beta\circ\alpha$ is the same as in the cylinder case (2.5.4), with vertical faces $\mathrm{P}\partial^\varepsilon.(\beta\circ\alpha) = (\partial^\varepsilon\beta).\alpha$. The faces of P^n are indexed as $\partial^\varepsilon_i = \mathrm{P}^{n-i}\partial^\varepsilon\mathrm{P}^{i-1}$, consistently with the ones of I^n (2.1.2; and with the usual face-maps δ^ε_i in **Top**).

An IP4-category is strong I4 iff it is strong P4, since the lens conversion can be equivalently realised through the cylinder or the path functor

(5) $\hat{\Phi} \mapsto \hat{\Phi}\mathrm{w}^{\mathrm{A}} \qquad (\hat{\Phi}\colon \mathrm{I}^2\mathrm{A} \to \mathrm{B})$ $\qquad$ (6) $\check{\Phi} \mapsto \mathrm{w}_{\mathrm{B}}\,\check{\Phi} \qquad (\check{\Phi}\colon \mathrm{A} \to \mathrm{P}^2\mathrm{B}).$

3. I4-homotopical categories

For I0-categories, the existence of h-pushouts reduces to the existence of particular pushouts, the "cylindrical" ones (3.2); similarly, in the I4 case, the regularity of h-pushouts can be reduced to the preservation of such colimits by I (3.3-4). These properties yield the definition of I0- and I4-homotopical category. As noted in 3.2, it is not convenient to assume the existence of all pushouts.

3.1. Some remarks on pushouts. a) First, we say that a map $t\colon X \to Y$ in the category $\mathbf{A}$ *has all pushouts* (or that $\mathbf{A}$ has all t-pushouts)

(1)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "*"] & \bullet \arrow[d, "t'"] \\ Y \arrow[r] & \bullet \end{tikzcd}$$

(2)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "t"'] \arrow[dr, phantom, "*"] & \bullet \arrow[r, "g"] \arrow[d, "t'"] \arrow[dr, phantom, "*"] & \bullet \arrow[d, "t''"] \\ Y \arrow[r] \arrow[rr, bend right] & \bullet \arrow[r, dashed] & \bullet \end{tikzcd}$$

if the pushout of t along an arbitary map f exists; this yields a map t' which again has all pushouts, as proved in (2) – use the pushout of t along a composite gf, and "factorise" it through (1).

b) Consequently, if t has all pushouts and a functor $F\colon \mathbf{A} \to \mathbf{B}$ preserves them, then F preserves also the pushouts of t'.

c) It will be useful to note once and for all that *pushouts preserve pushouts*. Precisely, given a commutative diagram (3) formed by the pasting of two "divergent" cubes (hidden vertices and edges are not drawn), if the faces "A", "B", "C" are pushouts and all four pairs (f_i, g_i) of horizontal maps have a pushout, then the resulting new face "D" in (4) is a pushout (as it is easy to check)

(3)
$$\begin{tikzcd}[column sep=small] & B_2 \arrow[dr] & & A_2 \arrow[ll, "f_2"'] \arrow[rr, "g_2"] \arrow[dr] & & C_2 \arrow[dr] & \\ B_1 \arrow[ur] \arrow[dr] & & B_4 & & A_4 \arrow[ll, "f_4"'] \arrow[rr, "g_4"] & & C_4 \\ & B_3 \arrow[ur] & & A_3 \arrow[ll, "f_3"] \arrow[rr, "g_3"'] \arrow[ur] & & C_3 \arrow[ur] & \end{tikzcd}$$

(4)
$$\begin{tikzcd}[column sep=small] & B_2 \arrow[rr, "k_2"] \arrow[dr] & & D_2 \arrow[dr] & & C_2 \arrow[ll, "h_2"'] \arrow[dr] & \\ B_1 \arrow[ur] \arrow[dr] & & B_4 \arrow[rr, "k_4"] & & D_4 & & C_4 \arrow[ll, "h_4"'] \\ & B_3 \arrow[rr, "k_3"'] \arrow[ur] & & D_3 \arrow[ur] & & C_3 \arrow[ll, "h_3"] \arrow[ur] & \end{tikzcd}$$

3.2. I0-homotopical categories. We say that the I0-category $\mathbf{A}$ is *I0-homotopical* if it satisfies the following equivalent conditions (applying to any pair of arrows f, g with the same domain)

(i) the h-pushout I(f, g) exists,

(ii) the (ordinary) colimit I(f, g) of the following diagram exists ($\mathbf{A}$ has *cylindrical colimits*)

$$(1) \qquad \mathrm{B} \xleftarrow{f} \mathrm{A} \xrightarrow{\partial^-} \mathrm{IA} \xleftarrow{\partial^+} \mathrm{A} \xrightarrow{g} \mathrm{C}$$

(iii) the three ordinary pushouts in (2) exist (**A** has *cylindrical pushouts*)

$$(2) \qquad \begin{tikzcd} & \mathrm{A} \arrow[r, "f"] \arrow[d, "\partial^-"] \arrow[dr, phantom, "*"] & \mathrm{B} \arrow[d] \\ \mathrm{A} \arrow[r, "\partial^+"] \arrow[d, "g"'] \arrow[dr, phantom, "*"] & \mathrm{IA} \arrow[r] \arrow[d] \arrow[dr, phantom, "*"] & \mathrm{I}^-f \arrow[d] \\ \mathrm{C} \arrow[r] & \mathrm{I}^+g \arrow[r] & \mathrm{I}(f, g) \end{tikzcd} \qquad\qquad (3) \qquad \begin{tikzcd} \mathrm{A} \arrow[r, dashed, "1"] \arrow[d, dashed, "1"'] & \mathrm{A} \arrow[r, "f"] \arrow[d, "\partial^-"] \arrow[dl, dashed, "\delta"'] & \mathrm{B} \arrow[dd, "x'"] \\ \mathrm{A} \arrow[r, "\partial^+"'] \arrow[d, "g"'] & \mathrm{IA} \arrow[dr, "\xi"] & \\ \mathrm{C} \arrow[rr, "x''"] & & \mathrm{I}(f, g) \end{tikzcd}$$

The equivalence of (i) and (iii) follows from 1.3.3; that of (ii) and (iii) is trivial. The presentation of I(f, g) as a cylindrical colimit is shown in (3).

The upper pushout in (2) yields a particular double mapping cylinder $\mathrm{I}^-f = \mathrm{I}(f, 1)$, the *lower mapping cylinder* of f (which pastes B on the lower base of IA, along f). Also the composition pushout JA (2.3.1) *is a particular cylindrical pushout*, since it can be obtained as the mapping cylinder $\mathrm{I}^-(\partial^+) = \mathrm{I}(\partial^+, 1)$, or symmetrically as $\mathrm{I}^+(\partial^-) = \mathrm{I}(1, \partial^-)$.

If **A** is I0-homotopical and has a terminal object, it is right semihomotopical as an h-category (1.5). The h-cokernel $\mathrm{C}f = \mathrm{C}^-f = \mathrm{I}(f, \top_{\mathrm{A}})$ can equivalently be obtained through the h-pushout 1.3.2, or the *conical* colimit of the diagram (4), or the two *conical* pushouts exhibited in (5)

$$(4) \qquad \mathrm{B} \xleftarrow{f} \mathrm{A} \xrightarrow{\partial^-} \mathrm{IA} \xleftarrow{\partial^+} \mathrm{A} \longrightarrow \top$$

$$(5) \qquad \begin{tikzcd} & \mathrm{A} \arrow[r, "f"] \arrow[d, "\partial^-"'] \arrow[dd, bend left, "\partial"] & \mathrm{B} \arrow[dd, "x'"] \\ \mathrm{A} \arrow[r, "\partial^+"] \arrow[d, "\top"'] \arrow[dr, phantom, "*"] & \mathrm{IA} \arrow[d] \arrow[r, phantom, "*"] & {} \\ \top \arrow[r] & \mathrm{C}^-\mathrm{A} \arrow[r] & \mathrm{C}^-f \end{tikzcd}$$

where $\partial\colon \mathrm{A} \to \mathrm{C}^-\mathrm{A}$ is the embedding of the base of the cone.

Finally, some remarks on the assumption of *cylindrical* colimits. Requiring the existence of all pushouts would be of scarce utility here, and even mask the real points of interest. For instance, the h-structure of the category C_***D** of chain complexes is founded on the pre-additive structure of **D**; h-pushouts and cylindrical pushouts, as well as h-pullbacks, exist as soon as **D** is additive, which is the real situation of interest for the homotopy of chain complexes; but arbitrary pushouts exist iff **D** has them; e.g., it is not the case for free abelian groups. On the other hand, it would be possible to restrict our assumption to the existence of h-cokernels, or *conical* colimits. But this condition is somewhat unnatural in a setting based on cylinders; and there are cases where the terminal object is missing, but one can nevertheless consider h-cokernels with respect to some generalisation of the latter.

3.3. I4-homotopical categories. An *I4-homotopical* category **A** will be a *strong* I4-category satisfying the following conditions (which are easily seen to be equivalent, by 3.2 and interchange)

(i) h-pushouts I(f, g) exist and are preserved by the functor I (I(I(f, g)) = I(If, Ig))

(ii) cylindrical colimits (3.2.1) exist and are preserved by I (*as colimits*)

(iii) cylindrical pushouts (3.2.2) exist and are preserved by I (*as pushouts*).

The first property proves (directly) that every power of I has then the same preservation properties. The second and third show that such preservation properties are automatic whenever I is a right adjoint. Since composition pushouts are a particular instance of the cylindrical ones (3.2), (iii) implies that they are preserved by I, so that the exchange property for horizontal and vertical pastings of deformations holds (final remark in 2.5). It is also useful to recall that the zero and lens collapses are superfluous in the regular case (2.8).

If **A** is I4-homotopical with terminal object, also the cone functor C: **A** → **A** preserves the cylindrical pushouts; indeed, C is defined by a pushout, and "pushouts preserve themselves" (3.1 c).

The notion of I4-homotopical *subcategory* **A**' is obvious – the whole structure, including the cylindrical colimits and their structural maps, can be restricted to **A**', where the colimit properties still hold. If **A**' is a full subcategory of **A**, all this simply means that the functor I restricts to **A**', which is closed in **A** under cylindrical colimits.

3.4. Cubical h-pushouts. Note first that the ordinary pushout (X, x, y) of the maps f: A → B, g: A → C, is preserved by I iff it is a *2-pushout*, with respect to I-homotopies: given two homotopies β: B ⇒ Y, γ: C ⇒ Y such that $\beta \circ f = \gamma \circ g$: A ⇒ Y, there exists precisely one homotopy α: X ⇒ Y such that $\alpha \circ x = \beta$, $\alpha \circ y = \gamma$.

A similar characterisation for I-preserved h-pushouts can be given through the semicubical structure over **A** defined by I (2.1). The diagram 1.3.1 (ξ: x'f → x"g: A → X) is an I-preserved h-pushout iff, given two homotopies β: B ⇒ Y, γ: C ⇒ Y and a deformation (2.5) Φ with horizontal faces $\beta \circ f$, $\gamma \circ g$: A ⇒ Y, there is exactly one homotopy α: X ⇒ Y such that $\alpha \circ x' = \beta$, $\alpha \circ x'' = \gamma$, $\alpha \circ \xi = \Phi$. Moreover, if this is the case, the boundary $a_\varepsilon = \hat{\alpha}.\partial^\varepsilon$ of α is determined by the boundaries of β and γ, together with the vertical faces $\hat{\Phi}.\partial^\varepsilon \mathrm{I}$ of the deformation

(1) $a_\varepsilon\, x' = \hat{\beta}.\partial^\varepsilon$, $\quad a_\varepsilon\, x'' = \hat{\gamma}.\partial^\varepsilon$, $\quad \hat{\Phi}.\partial^\varepsilon \mathrm{I} = a_\varepsilon \circ \xi$.

Indeed, take $\hat{\Phi}.s$: $\hat{\beta}$.If → $\hat{\gamma}$.Ig: IA → Y. If ξ: x'f → x"g is an I-preserved h-pushout, there is one map $\hat{\alpha}$: IX → Y such that $\hat{\beta} = \hat{\alpha}.\mathrm{I}x'$, $\hat{\gamma} = \hat{\alpha}.\mathrm{I}x''$ and $\hat{\Phi}.s = \hat{\alpha} \circ \mathrm{I}(\xi) = \hat{\alpha}.\mathrm{I}(\hat{\xi}).s = (\alpha \circ \xi)\hat{}\,.s$; and conversely. One can also prove that, in an I4-homotopical category, h-pushouts (being preserved by all powers of I) satisfy a stronger universal property, for maps of every degree (*cubical h-pushouts*).

3.5. Theorem. Every I4-homotopical category **A** has h4-pushouts; the cylinder is homotopically invariant. If **A** has a terminal object, then it is right-homotopical with respect to the h4-structure produced by the cylinder; also the cone and suspension functors are homotopically invariant.

Proof. We already know that **A** is h4 and has h-pushouts, while I is invariant (thm. 2.9; 2.4). We prove now the h4-regularity of h-pushouts, essentially from the preservation of cylindrical colimits by I and the *strong* lens-conversion property. If ⊤ exists, the invariance of C and Σ follows from 1.5.

Consider the h-pushout $X = I(f, g)$ in (1). Given two maps $a_0, a_1: X \to Y$ and two cells $\vartheta: a_0x' \to a_1x': B \to Y$, $\kappa: a_0x'' \to a_1x'': C \to Y$ forming a $\sim$-commutative diagram (2) of cells, we have to prove that there is some homotopy $\alpha: a_0 \to a_1: X \to Y$ such that $\vartheta = \alpha x'$ and $\kappa = \alpha x''$

$$\text{(1)}\qquad \begin{array}{ccccc} A & \xrightarrow{f} & B & & \\ {\scriptstyle g}\downarrow & \xi\swarrow & \downarrow{\scriptstyle x'} & & \\ C & \xrightarrow[x'']{} & X & \underset{a_1}{\overset{a_0}{\rightrightarrows}} & Y \end{array} \qquad\qquad \text{(2)}\qquad \begin{array}{ccc} a_0x''g & \xrightarrow{\kappa g} & a_1x''g \\ {\scriptstyle a_0\xi}\uparrow & \sim & \uparrow{\scriptstyle a_1\xi} \\ a_0x'f & \xrightarrow[\vartheta f]{} & a_1x'f \end{array}$$

Choose a cell-homotopy $\Psi: I^2A \to Y$ representing (2). By hypothesis (2.7), it can be lifted to a deformation $\Phi: I^2A \to Y$ as in (4), whose lens conversion is $\Phi w = \Psi$

$$\text{(3)}\qquad \begin{array}{ccc} \bullet & \xrightarrow{a_0\xi + \kappa g} & \bullet \\ {\scriptstyle 0}\uparrow & \Psi & \uparrow{\scriptstyle 0} \\ \bullet & \xrightarrow[\vartheta f + a_1\xi]{} & \bullet \end{array} \qquad\qquad \text{(4)}\qquad \begin{array}{ccc} \bullet & \xrightarrow{\kappa g} & \bullet \\ {\scriptstyle a_0\xi}\uparrow & \Phi & \uparrow{\scriptstyle a_1\xi} \\ \bullet & \xrightarrow[\vartheta f]{} & \bullet \end{array}$$

Now, the last remark above (3.4) would allow us to conclude, factoring Φ through the homotopy ξ. More elementarily, without using the cubical enriched structure, consider the h-pushout $IX = I(I(f, g)) = I(If, Ig)$, with $I(\xi) = I(\hat{\xi}).s$

$$\text{(5)}\qquad \begin{array}{ccccc} IA & \xrightarrow{If} & IB & & \\ {\scriptstyle Ig}\downarrow & I\xi\swarrow & \downarrow{\scriptstyle Ix'} & & \\ IC & \xrightarrow[Ix'']{} & IX & \overset{}{\underset{\alpha}{\dashrightarrow}} & Y \end{array}$$

Since $\Phi s.\partial^- I = (\vartheta\circ f)\hat{} = \hat{\vartheta}.If$, and $\Phi s.\partial^+ I = \hat{\kappa}.Ig$, the triple $(\hat{\vartheta}, \hat{\kappa}, \Phi s)$ determines a homotopy $\hat{\alpha}: IX \to Y$ which lifts $\hat{\vartheta}$, $\hat{\kappa}$

(6) $\hat{\alpha}.Ix' = \hat{\vartheta}$, $\qquad\qquad \hat{\alpha}.Ix'' = \hat{\kappa}$, $\qquad\qquad \hat{\alpha}.I\xi = \Phi.s$

and links a_0 and a_1, as detected by the jointly epi triple $(x', x''; \xi)$

(7) $(\hat{\alpha}.\partial^\varepsilon).x' = \hat{\alpha}.Ix'.\partial^\varepsilon = \hat{\vartheta}.\partial^\varepsilon = a_\varepsilon.x'$ $\qquad\qquad (\hat{\alpha}.\partial^\varepsilon).x'' = a_\varepsilon.x''$

$(\hat{\alpha}.\partial^\varepsilon).\hat{\xi} = \hat{\alpha}.I\hat{\xi}.\partial^\varepsilon I = \hat{\alpha}.I\xi.s.\partial^\varepsilon I = \Phi.\partial^\varepsilon I = (a_\varepsilon.\xi)\hat{} = a_\varepsilon.\hat{\xi}$.

3.6. P4-homotopical and IP4-homotopical categories. Dually, a P4-*homotopical* category **A** is a strong P4-category with P-preserved h-pullbacks (or *path limits*, or *path pullbacks*).

Clearly, for every pair of arrows f, g with the same codomain, the path limit $P(f, g)$ is the h-pullback of f and g, *presented* as the ordinary limit of the following diagram

$$\text{(1)}\qquad B \xrightarrow{f} A \xleftarrow{\partial^-} PA \xrightarrow{\partial^+} A \xleftarrow{g} C.$$

The composition pullback QA (2.10.1) is a particular path pullback. If **A** has initial object $\bot$, we also have the (lower) h-kernel, or homotopy fibre, $Kf = K^-f = P(f, \bot_A)$.

An *IP0-* (resp *IP4-*) *homotopical category* is an IP0- (resp. IP4-) category with cylindrical colimits and path limits. Such colimits and limits are automatically preserved by I and P, respectively, because of the adjunction. The notion of IP4-subcategory is obvious (cf. 3.3).

3.7. Triangulated categories. The relations between *stable* right homotopical and triangulated categories studied in [12] can be easily converted into the present setting; the interested reader is referred to this paper for the terminology.

Let **A** be a *pointed* I4-homotopical category. Then **A** satisfies always the "3×3 condition for h-cokernels" ([12], 3.2-3), linked to the octahedral axiom of Verdier. Consequently, if **A** is also *strictly stable homotopical* with finite sums, the category $\mathbf{A}/\simeq$ is triangulated with respect to its suspension and the *triangles* defined by the beginning of the Puppe sequences of maps.

Indeed, we already know that **A** is (pointed) right homotopical (3.5); applying the theorem 3.4 of [12], it suffices to verify two conditions. First, the fact that C preserves the pushouts of the structural maps $\partial\colon A \to CA$; since these are particular conical pushouts, this property has already been remarked (3.2-3). Second, the existence of an interchange for C, i.e. an involution $s\colon C^2 \to C^2$ exchanging its faces $(s.C\partial = \partial C)$; but it is known, and easy to verify, that in the pointed case the interchange $s\colon I^2 \to I^2$ of the cylinder induces one for the cone ([10], 3.7 b).

4. Categories of diagrams, sheaves and slices

We consider, here and in the next section, various situations in which our cubical notions can be naturally lifted from a ground category **A** to a second category **E** provided with a forgetful functor U: $\mathbf{E} \to \mathbf{A}$ (or with a family $U_i\colon \mathbf{E} \to \mathbf{A}$).

Here we prove that this holds for every category of diagrams $\mathbf{A}^\mathbf{S}$ (4.1-2) and, under natural conditions, for the slice categories **A**\X, **A**/X, **A**(u) (4.4-6). For the category of sheaves over a site, the lifting is fairly easy for the path functor (4.3), while it would require stronger hypotheses – and a sheafification procedure to start with – to construct the cylinder (we do not work out this part here).

For homotopy in categories of diagrams and equivariant homotopy, see Dror Farjoun [7], Brown - Loday [6], Moerdijk - Svensson [27] and their references. For sheaves in general categories, over a space, see J. Gray [13]; for set-valued sheaves over a site, see Mac Lane - Moerdijk [26]. For the homotopy theory of (strict or relaxed) slice categories of spaces see James [19], Baues [1], Hardie - Kamps [14, 15, 16], Hardie - Kamps - Porter [17] and their references.

4.1. Categories of diagrams. Consider the category of diagrams $\mathbf{A}^\mathbf{S}$, i.e. functors $\mathbf{S} \to \mathbf{A}$ defined over a small category **S** with their natural transformations. An object $A = ((A_i), (A_\iota))$ is thus a collection indexed over the objects i and the arrows $\iota\colon i \to j$ of **S**, satisfying the functorial properties. This includes, for instance:

- the power $\mathbf{A}^S$, for any set S (as a discrete category),

- the category $\mathbf{A}^\mathbf{2}$ of morphisms of **A** (**2** is the ordinal category $\{0 \to 1\}$),

- the category $\mathbf{A}^\mathbb{Z}$ of unbounded towers of **A** ($\mathbb{Z}$ is the order category of integers),

- the category $\mathbf{A}^G$ of actions in **A** of a fixed group, or monoid, G (as a one-object category),

- the category Psh (X, **A**) = $\mathbf{A}^\mathbf{S}$ of presheaves of **A** over a fixed topological space X ($\mathbf{S}^{op}$ is the category of open subsets of X, with their inclusion mappings).

We are interested in lifting the structure of $\mathbf{A}$ to $\mathbf{A}^{\mathbf{S}}$, along the (jointly faithful) family of evaluation functors $U_i: \mathbf{A}^{\mathbf{S}} \to \mathbf{A}$, $A \mapsto A_i$. Equivalently, one can embed $\mathbf{A}^{\mathbf{S}}$ into $\mathbf{A}^{\mathrm{Ob}\,\mathbf{S}}$.

As a first step, if $\mathbf{A}$ is an h-category, $\mathbf{A}^{\mathbf{S}}$ has a canonical h-structure. A (*natural* or *equivariant*) homotopy $\alpha: f \to g: A \to B$ in $\mathbf{A}^{\mathbf{S}}$ is defined to be a family of $\mathbf{A}$-homotopies $\alpha_i: f_i \to g_i: A_i \to B_i$ ($i \in \mathrm{Ob}\,\mathbf{S}$) which is natural in the obvious sense provided by the reduced horizontal composition of $\mathbf{A}$

(1) $\quad B_\iota \circ \alpha_i \;=\; \alpha_j \circ A_\iota \qquad\qquad (\iota: i \to j \;$ in $\mathbf{S}$).

4.2. Proposition. If $\mathbf{A}$ is I0 (resp. I4, strong I4, I0-homotopical, I4-homotopical), the category of diagrams $\mathbf{A}^{\mathbf{S}}$ has a canonical structure of the same kind, by a pointwise lifting $\mathbf{I}$ of the cylinder. Consequently, if $\mathbf{A}$ is I4-homotopical and has $\top$, $\mathbf{A}^{\mathbf{S}}$ is *right homotopical*. Similar results hold for the P- and IP-analogues.

(As already stressed in the Introduction, the cell-homotopy relation $\alpha \sim \beta$ in $\mathbf{A}^{\mathbf{S}}$ is produced by the second-order cylinder $\mathbf{I}^2$ (2.6), and is stronger than the *pointwise lifting* of the corresponding relation of $\mathbf{A}$ ($\alpha_i \sim \beta_i$, for every i in $\mathbf{S}$), which is not of interest.)

Proof. For an I0-category $\mathbf{A}$, the I0-structure $(\mathbf{I}, \partial^\varepsilon, e)$ of $\mathbf{A}^{\mathbf{S}}$ is plainly

(1) $\quad (\mathbf{I}A)_i = I(A_i), \qquad (\mathbf{I}A)_\iota = I(A_\iota); \qquad\qquad (\partial^\varepsilon A)_i = \partial^\varepsilon A_i, \qquad (eA)_i = eA_i.$

Similarly, one lifts pointwise an I4-structure $(I, \partial^\varepsilon, e, g^\varepsilon, r, k, s, z, w)$; note that the composition pushouts $\mathbf{J}(A) = \mathbf{I}A +_A \mathbf{I}A$ exist and are pointwise calculated in $\mathbf{A}^{\mathbf{S}}$, which allows one to lift the vertical composition $k: I \to J$.

If the I4-structure of $\mathbf{A}$ is strong, take in $\mathbf{A}^{\mathbf{S}}$ a fourtuple of homotopies $\alpha, \beta, \gamma, \delta$ disposed as in (2), together with a cell-homotopy $\Psi: \alpha{+}\delta \to \beta{+}\gamma$. For every index i there exists precisely one deformation Φ_i in $\mathbf{A}$ with boundary (3) whose lens conversion $\Phi_i w$ coincides with Ψ_i

(2) $\quad \begin{tikzcd} k \arrow[r, "\gamma"] & g \\ f \arrow[u, "\beta"] \arrow[r, "\alpha"'] \arrow[ur, phantom, "?"] & h \arrow[u, "\delta"'] \end{tikzcd} \qquad\qquad$ (3) $\quad \begin{tikzcd} k_i \arrow[r, "\gamma_i"] & g_i \\ f_i \arrow[u, "\beta_i"] \arrow[r, "\alpha_i"'] \arrow[ur, phantom, "\Phi_i"] & h_i \arrow[u, "\delta_i"'] \end{tikzcd}$

and the family $\Phi = (\Phi_i)$ is indeed a morphism $I^2A \to B$, i.e. makes each diagram (4) commute, because the (injective) lens conversion turns (4) into (5), which commutes by hypothesis

(4) $\quad \begin{tikzcd} I^2A_i \arrow[r, "\Phi_i"] \arrow[d, "I^2A_\iota"'] & B_i \arrow[d, "B_\iota"] \\ I^2A_j \arrow[r, "\Phi_j"'] & B_j \end{tikzcd} \qquad\qquad$ (5) $\quad \begin{tikzcd} I^2A_i \arrow[r, "\Psi_i"] \arrow[d, "I^2A_\iota"'] & B_i \arrow[d, "B_\iota"] \\ I^2A_j \arrow[r, "\Psi_j"'] & B_j \end{tikzcd}$

In the I0-homotopical case, h-pushouts (or, equivalently, cylindrical limits) exist in $\mathbf{A}^{\mathbf{S}}$ and are pointwise calculated in $\mathbf{A}$. In the I4-homotopical case, the preservation of cylindrical limits by I automatically lifts to $\mathbf{A}^{\mathbf{S}}$. The P-case follows now by duality, and the IP-case from the previous ones.

4.3. Theorem. Let $\mathbf{A}$ be P0, or P4, or strong P4, or P0-homotopical, or P4-homotopical, and assume that the path functor P preserves all the existing limits (as it certainly happens if it is a left adjoint).

Then the category Shv(**S**, **A**) of sheaves of **A** over a small site **S** is a subcategory of the same kind (P0, or P4, etc.) in the category of presheaves Psh(**S**, **A**) = $\mathbf{A}^{\mathbf{S}^{op}}$.

Proof. Also to fix notation, recall that a *sieve* s of the object i in **S** is a right ideal of maps having codomain i (if $\iota \in s$, also $\iota\kappa$ does, whenever the composition is defined). By hypothesis, **S** is a small category equipped with a Grothendieck topology J; this assigns to every object i a set J(i) of sieves of i (those which "cover" i), under three well known axioms abstracting the behaviour of the (downwards closed) open coverings of open subsets in a space ([26], III.2, Def. 1).

Shv(**S**, **A**) is the full subcategory of Psh(**S**, **A**) consisting of those presheaves $A = ((A_i), (\iota^*))$ which are sheaves, i.e. satisfy the following limit condition. For each object i and each sieve $s \in J(i)$, consider the (small) diagram $A|s$ in **A** having the following vertices and arrows

(1) $A_\iota = A_{\mathrm{dom}\,\iota}$ $(\iota \in s)$

(2) $(x_{\iota,\kappa}: A_\iota \to A_{\iota\kappa}) = (\kappa^*: A_{\mathrm{dom}\,\iota} \to A_{\mathrm{dom}\,\kappa})$ $(\iota \in s,\ \ \mathrm{cod}\kappa = \mathrm{dom}\iota)$

then A_i is required to be the limit in **A** of this diagram, through the projections

(3) $\iota^*: A_i \to A_\iota = A_{\mathrm{dom}\,\iota}$ $(\iota \in s)$.

Note that the diagram $A|s$ is defined over the category cat(s), with objects $\iota \in s$, arrows (ι,κ): $\iota \to \iota\kappa$ $(\iota \in s,\ \mathrm{cod}\kappa = \mathrm{dom}\iota)$ and composition $(\iota\kappa,\lambda).(\iota,\kappa) = (\iota,\kappa\lambda)$. Since **A** is not required to be complete, the sheaf condition can not be given via products and equalisers; also, the direct formulation is often more manageable.

Assume now that **A** is equipped with a (path) endofunctor P which preserves the existing limits. Then the path functor of presheaves (4.2), $\mathbf{P}A = ((PA_i), (P\iota^*))$, restricts trivially to sheaves.

Because of 3.3, we just need to show that this full subcategory is closed in Psh(**S**, **A**) under composition or path pullbacks, whenever they exist in **A** (and therefore in Psh(**S**, **A**)). This follows from the following elementary fact: if X is the pullback of $A \to B \leftarrow C$ in Psh(**S**, **A**), and A, B, C are sheaves, so is X. Indeed, the proof of the sheaf condition for X reduces to a straightforward diagram-chasing in (4), for $\iota \in s$ and $\mathrm{cod}\kappa = \mathrm{dom}\,\iota$ (again, hidden edges are not drawn)

(4)
$$\begin{tikzcd}
 & A_i \arrow[r, "\iota^*"] \arrow[dr] & A_\iota \arrow[r, "\kappa^*"] \arrow[dr] & A_{\iota\kappa} \arrow[dr] & \\
X_i \arrow[ur] \arrow[dr] & & C_i \arrow[r, "\iota^*"] & C_\iota \arrow[r, "\kappa^*"] & C_{\iota\kappa} \\
 & B_i \arrow[r, "\iota^*"] \arrow[ur] & B_\iota \arrow[r, "\kappa^*"] \arrow[ur] & B_{\iota\kappa} \arrow[ur] &
\end{tikzcd}$$

4.4. Slice categories. Let **A** be an h-category. The classical topological example of a slice category is $\mathbf{Top}^\top = \mathbf{Top}\backslash\top$, the category of pointed spaces or "spaces under the point" $\top = \{*\}$. An object (A, x) is a map $x: \top \to A$ in **Top**; pointed maps and pointed homotopies are defined coherently, producing an h-category.

In the same way, if **A** is an h-category and X an object, the slice category **A**\X of *objects under* X has a canonical h-structure making the forgetful functor U: **A**\X → **A** an h-functor. A cell α: (A, x) ⇒ (A', x') in **A**\X is given by a cell $\alpha: A \Rightarrow A'$ in **A** such that $\alpha \circ x = 0_{x'}$, as in (1)

(1) $$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "x"'] & X \arrow[d, Rightarrow, "0_{x'}"] \\ A \arrow[r, Rightarrow, "\alpha"] & A' \end{tikzcd}$$

(2) $$\begin{tikzcd} A \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "0_y"'] & A' \arrow[d, "y'"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

Dually, we have the slice h-category $\mathbf{A}/Y \cong (\mathbf{A}^{op}\backslash Y)^{op}$ of *objects under* Y, whose cells make the diagram (2) commutative. To unify the argument, one can consider a self-dual situation (as, for instance, in [19] and [1, I.4.3], for $\mathbf{A} = \mathbf{Top}$).

Given a fixed map $u: X \to Y$ in $\mathbf{A}$, one has the h-category $\mathbf{A}(u) = \mathbf{A}\backslash X/(Y, u) = \mathbf{A}/Y\backslash(X, u)$ *of objects under* X *and over* Y. An object is a triple (A,x,y) with $yx = u$, and a cell $\alpha: (A,x,y) \Rightarrow (A',x',y')$ is a homotopy α of $\mathbf{A}$ making (4) commute

(3) $$X \xrightarrow{x} A \xrightarrow{y} Y$$

(4) $$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, equal] & A \arrow[r, Rightarrow, "0_y"] \arrow[d, Rightarrow, "\alpha"] & Y \arrow[d, equal] \\ X \arrow[r, Rightarrow, "0_{x'}"] & A' \arrow[r, "y'"] & Y \end{tikzcd}$$

$\mathbf{A}(u)$ extends the previous two cases, since $\mathbf{A}\backslash X \cong \mathbf{A}(X \to \top)$ and $\mathbf{A}/Y \cong \mathbf{A}(\bot \to Y)$ as h-categories, provided $\mathbf{A}$ has initial and terminal object (no real restriction here, since such objects can always be formally and easily added). Recall that $P\top = \top$ and $I\bot = \bot$ (1.4). On the other hand, $\mathbf{A}(u)$ has the following initial and terminal object (under no assumptions on $\mathbf{A}$)

(5) $\boldsymbol{\bot} = (X, 1, u)$ (6) $\boldsymbol{\top} = (Y, u, 1)$.

One could similarly consider the larger category of objects (A,x,y) with no assumption on the composite yx. But the extension is only apparent, because this category breaks into the sum of its connected components, consisting of the various $\mathbf{A}(u)$, for $u \in \mathbf{A}(X, Y)$, each with its own initial and terminal object (5), (6).

4.5. Cylinder and path. If $\mathbf{A}$ is I0 and the X-degeneracy $e^X: IX \to X$ has all pushouts, also $\mathbf{A}(u)$ is I0. The new cylinder functor $\mathbf{I}$ is constructed through the (*-marked) e^X-pushout and the commutative diagram

(1) $$\begin{tikzcd} X \arrow[r, dashed, "\partial^\varepsilon"] \arrow[d, dashed, "x"'] & IX \arrow[r, "e"] \arrow[d, "Ix"'] & X \arrow[r, equal] \arrow[d, "x^I"] \arrow[dl, phantom, "*"] & X \arrow[d, "u"] \\ A \arrow[r, dashed, "\partial^\varepsilon"'] & IA \arrow[r] \arrow[rr, bend right, "0_y"'] & \mathbf{I}(A,x,y) \arrow[r, "y^I"] & Y \end{tikzcd}$$

with the induced faces $\partial^\varepsilon: (A,x,y) \to \mathbf{I}(A,x,y)$ and degeneracy $e: \mathbf{I}(A,x,y) \to (A,x,y)$. Note the abuse of notation in (1): formally, $\mathbf{I}(A,x,y)$ is a triple containing also the maps x^I, y^I.

Dually, if $\mathbf{A}$ is P0 and the Y-degeneracy $e_Y: Y \to PY$ has all pullbacks, also $\mathbf{A}(u)$ is P0, with path functor $\mathbf{P}$ given by an e_Y-pullback

(2)
$$\begin{tikzcd}
\mathbf{P}(A,x,y) \arrow[r] \arrow[d, "y^P"'] \arrow[dr, phantom, "*"] & PA \arrow[d, "Py"] \\
Y \arrow[r, "e"] & PY
\end{tikzcd}$$

In particular (take $Y = \top$), if **A** is P0 also **A**\X is so, with path functor

(3) $\mathbf{P}(A, x) = (PA, \hat{0}_x)$, $\qquad\qquad \hat{0}_x = e_A.x = Px.e_X \colon X \to PA.$

More complete results for the cylinder of **A**(u) are given here below (4.6). Those concerning the path functor are obtained by duality.

4.6. Theorem. Let **A** be an h-category. Consider an arbitrary ("bilateral") slice category **A**(u), for $u\colon X \to Y$ in **A**.

a) If **A** is I0 (resp. I0-homotopical), so is **A**(u), provided that the X-degeneracy $e^X\colon IX \to X$ has all pushouts in **A** (3.1); the cylinder functor **I** of **A**(u) is described above (4.5.1). (The condition on e^X is trivial for $X = \bot$ and $\mathbf{A}(u) = \mathbf{A}/Y$.)

b) If **A** is I4, or strong I4, or I4-homotopical, so is **A**(u), provided that all the above e^X-pushouts exist and are preserved by I. (Again, this holds trivially for $X = \bot$ and $\mathbf{A}(u) = \mathbf{A}/Y$.)

Proof. a) The I0 case has already been considered above (4.5). In the I0-homotopical case, the h-pushout $\mathbf{I}(f, g)$ of the maps $f\colon (A,x,y) \to (A',x',y')$, $g\colon (A,x,y) \to (A'',x'',y'')$ is the **A**-colimit

(1)
$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[drr, "\partial^-"] & & X \arrow[ll, dashed, "x"'] \arrow[rr, dashed, "x"] & & A \arrow[dll, "\partial^+"'] \arrow[dr, "g"] & \\
A' \arrow[drrr, "i'"'] & & & \mathbf{I}(A,x,y) \arrow[d, "\xi"] & & & A'' \arrow[dlll, "i''"] \\
& & & \mathbf{I}(f, g) & & &
\end{tikzcd}$$

equipped with the morphism $x^I = i'x' = i''x''\colon X \to \mathbf{I}(f, g)$ and the morphism $y^I\colon \mathbf{I}(f, g) \to Y$ whose components from A', A'', $\mathbf{I}(A,x,y)$ are respectively y', y'' and $y^I\colon \mathbf{I}(A,x,y) \to Y$.

b) The proof depends on some formal properties of the slice category **A**(u), which in part we have implicitly used in defining the cylinder **I**, its faces and degeneracies:

(i) assume that the functor $F\colon \mathbf{A} \to \mathbf{A}$ is equipped with a natural transformation $e\colon F \to 1$ (*degeneracy*) whose X-component $e_X\colon FX \to X$ has all pushouts; then F can be lifted to an endofunctor **F** of **A**(u), defined through the following pushout, as **I** in 4.5.1

(2)
$$\begin{tikzcd}
FX \arrow[r, "e"] \arrow[d, "Fx"'] \arrow[dr, phantom, "*"] & X \arrow[r, equal] \arrow[d, "x^F"] & X \arrow[d, "u"] \\
FA \arrow[r, "p"] \arrow[rr, bend right, "y.e_A"'] & \mathbf{F}(A,x,y) \arrow[r, "y^F"] & Y
\end{tikzcd}$$

(ii) every *morphism* $(F, e) \to (F', e')$ of such pairs lifts to $(\mathbf{F}, e) \to (\mathbf{F}', e')$,

(iii) given such a pair (F, e), $\mathbf{F}^2$ lifts F^2 with respect to the "derived" degeneracy $e_2 = Fe.e = eF.e$: $1 \to F^2$, provided that F preserves the pushouts along e_X: $FX \to X$

$$
(3) \qquad
\begin{tikzcd}
F^2X \arrow[r, "Fe"] \arrow[d, "F^2x"'] & FX \arrow[r, "e"] \arrow[d, "FxF"] \arrow[dl, phantom, "*"] & X \arrow[d, "\hat{x}"] \arrow[dl, phantom, "*"] \\
F^2A \arrow[r, "Fp"] & F\mathbf{F}(A,x,y) \arrow[r] & \mathbf{F}^2(A,x,y)
\end{tikzcd}
$$

Let now **A** be an I4-category. The previous arguments allow one to lift I and I^2, together with connections, reversion and interchange. The composition pushout $JA = IA \underset{A}{+} IA$ in **A** lifts to the slice category, together with its transformation $k: I \to J$, provided we prove that the degeneracy $\bar{e}$: $JX \to X$ has pushout along any map $f: JX \to Y$. First form the e^X-pushouts of the restrictions f^ε, producing two maps u^ε, each of them having all pushouts; then form their pushout Z; it is easy to check that this is the pushout of f along $\bar{e}$, through the maps $h = v^\varepsilon u^\varepsilon$: $Y \to Z$ and $h' = h f k^- \partial^+ = h f k^+ \partial^-$: $X \to Z$

$$
(4) \qquad
\begin{tikzcd}
X \arrow[rrrr, "h^-"] \arrow[dd, equal] & & & & \bullet \arrow[dd, "v^-"] \\
& IX \arrow[ul, "e"'] \arrow[rr, "f^-"] \arrow[d, "k^-"'] & & Y \arrow[ur, "u^-"] \arrow[d, equal] & \\
X & JX \arrow[l, "\bar{e}"'] \arrow[rr, "f"] & & Y \arrow[r, dashed, "h"] & Z \\
& IX \arrow[u, "k^+"] \arrow[rr, "f^+"] \arrow[dl, "e"'] & & Y \arrow[u, equal] \arrow[dr, "u^+"] & \\
X \arrow[uu, equal] \arrow[rrrr, "h^+"] & & & & \bullet \arrow[uu, "v^+"']
\end{tikzcd}
$$

The lifted functor **J** has transformations $k^-, k^+ : \mathbf{I} \to \mathbf{J}$, which still form the pushout of ∂^-, ∂^+: $1 \to \mathbf{I}$, because "pushouts preserve pushouts" (3.1 c).

For the strong I4 case, take four parallel homotopies $\alpha, \beta, \gamma, \delta$: $\mathbf{I}(A,x,y) \to (A',x',y')$ in $\mathbf{A}(u)$, disposed as in (5) and a cell-homotopy Ψ: $\alpha+\delta \to \beta+\gamma$ as in (6)

$$
(5) \qquad
\begin{tikzcd}
k \arrow[r, "\gamma"] & g \\
f \arrow[u, "\beta"] \arrow[r, "\alpha"'] \arrow[ur, phantom, "?"] & h \arrow[u, "\delta"']
\end{tikzcd}
\qquad\qquad\qquad
(6) \qquad
\begin{tikzcd}
f \arrow[r, "\beta+\gamma"] & g \\
f \arrow[u, "0"] \arrow[r, "\alpha+\delta"'] \arrow[ur, phantom, "\Psi"] & g \arrow[u, "0"']
\end{tikzcd}
$$

Lift all these morphisms in **A**, by composing with the canonical projections $IA \to \mathbf{I}(A,x,y)$ and $I^2A \to \mathbf{I}^2(A,x,y)$. We get a cell-homotopy $\hat{\Psi}$: $\hat{\alpha}+\hat{\delta} \to \hat{\beta}+\hat{\gamma}$ in **A**, as it results from the commutative diagram below, where $\partial_\bullet$ is any of the four faces of the second-order cylinder

$$\begin{array}{lccccc} & I^2A & \xrightarrow{q} & \mathbf{I}^2(A,x,y) & \xrightarrow{\Psi} & (A',x',y') \\ (7) & \partial_\bullet \uparrow & & \uparrow \partial_\bullet & & \| \\ & IA & \xrightarrow{p} & \mathbf{I}(A,x,y) & \longrightarrow & (A',x',y') \end{array}$$

Applying the strong lens conversion property in **A**, there is precisely one **A**-map $\hat{\Phi}$: $I^2A \to A'$ in **A**, whose boundary is $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$ and such that $\hat{\Phi}w = \hat{\Psi}$. It suffices to prove that $\hat{\Phi}$ induces a map $\mathbf{I}^2(A,x,y) \to (A',x',y')$ through the pushout (3), i.e. that $\hat{\Phi}.I^2x = x'e_2$: $I^2X \to A'$; again, this equality is detected by the (injective) conversion mapping in **A**

(8) $\hat{\Phi}.I^2x.w = \hat{\Phi}.w.I^2x = \hat{\Psi}.I^2x = x'e_2 = x'e_2.w$

(9) $\partial(\hat{\Phi}.I^2x) = (\hat{\alpha}.Ix, \hat{\beta}.Ix, \hat{\gamma}.Ix, \hat{\delta}.Ix) = (0_{x'}, 0_{x'}, 0_{x'}, 0_{x'}) = \partial(x'e_2)$.

The I4-homotopical case is now a trivial consequence of the previous ones.

5. Monoids in monoidal homotopical categories

Monoids in a category *agree* with limits and the path endofunctor. We show that the category Mon**A** of internal monoids in a *monoidal P4-homotopical* category is P4-homotopical, with an enriched path functor **P** which lifts the original P of **A**. The dual results concern the cylinder structure for internal *co*monoids. Instead, a cylinder functor $\mathbf{I} \dashv \mathbf{P}$ for Mon**A** may very well exist, but generally does not lift the original I (cf. 6.2, 6.9). Given a homotopy α: A ⇒ B, the map A → PB which represents it will be written here as α or $\hat{\alpha}$, instead of $\check{\alpha}$.

5.1. Monoidal categories. In this section the ground category **A** is always equipped with a monoidal structure $(\otimes, E)$, whose coherence isomorphisms are not named. As usual, the term "monoidal" is replaced by *cartesian* when the structure is defined by the categorical product.

A (lax) *monoidal (endo)functor* F = (F, e, t): **A** → **A** comes with a morphism e and a natural transformation t

(1) $e: E \to FE$, $\qquad t_{AB}: FA{\otimes}FB \to F(A{\otimes}B)$

satisfying three coherence conditions, which can be written as follows once the coherence isomorphisms of the tensor are understood

(2) $t.(e{\otimes}1_F) = 1_F = t.(1_F{\otimes}e)$, $\qquad t.(t{\otimes}1_F) = t.(1_F{\otimes}t)$.

(More precisely, the third is given by

$(t.t{\otimes}1_F\text{: } (FA{\otimes}FB){\otimes}FC \to F(A{\otimes}B){\otimes}FC \to F((A{\otimes}B){\otimes}C) \cong F(A{\otimes}(B{\otimes}C))) =$

$= (t.1_F{\otimes}t\text{: } (FA{\otimes}FB){\otimes}FC \cong FA{\otimes}(FB{\otimes}FC) \to FA{\otimes}F(B{\otimes}C) \to F(A{\otimes}(B{\otimes}C))).)$

F is *strong monoidal* if e and t are iso, and *strict monoidal* if they are identical; the identity of **A** is strict monoidal. The composite HF of two monoidal endofunctors has structure

(3) $e^{HF} = He^F.e^H\text{: } E \to HFE$ $\qquad$ (4) $t^{HF} = Ht^F.t^H\text{: } HFA \otimes HFB \to HF(A{\otimes}B)$.

A *monoidal transformation* u: F → G: **A** → **A** is a natural transformation of monoidal functors consistent with their structure (e, t), in the obvious sense

(5) $e^G = u.e^F \colon E \to GE$ (6) $t^G.u{\otimes}u = u.t^F \colon FA{\otimes}FB \to G(A{\otimes}B)$.

5.2. Definition. A *monoidal h-category* amounts to a monoidal category $\mathbf{A} = (\mathbf{A}, \otimes, E)$ provided with a *consistent* h-structure, i.e., the tensor product $-\otimes-\colon \mathbf{A}{\times}\mathbf{A} \to \mathbf{A}$ is an h-functor. Similarly one defines monoidal hi-categories, for i = 1 to 4. But we are mostly interested in such structures as produced by a path functor.

First, a *monoidal P0-structure* on the monoidal category $\mathbf{A} = (\mathbf{A}, \otimes, E)$ consists of a monoidal functor (P, e_0, t)

(1) $P\colon \mathbf{A} \to \mathbf{A}$, $e_0\colon E \to PE$, $t_{AB}\colon PA \otimes PB \to P(A{\otimes}B)$

together with a P0-structure $(P, \partial^\varepsilon, e)$ whose transformations $\partial^\varepsilon\colon P \to 1$ and $e\colon 1 \to P$ are monoidal. The map e_0 coincides with the component $e_E\colon E \to PE$ (5.1.5), and will be written thus.

Our setting reduces thus, more explicitly, to a P0-structure $(P, \partial^\varepsilon, e)$ on **A** together with a natural transformation $t\colon P(-){\otimes}P(.) \to P(-{\otimes}.)$ satisfying the following coherence conditions

(2) $t.(P{\otimes}e_E) = 1_P = t.(e_E{\otimes}P)$, $t.(t{\otimes}1_P) = t.(1_P{\otimes}t)$

(3) $t.(e{\otimes}e) = e\colon A{\otimes}B \to P(A{\otimes}B)$ (4) $\partial^\varepsilon t = \partial^\varepsilon{\otimes}\partial^\varepsilon\colon PA{\otimes}PB \to A{\otimes}B$

showing that t can be viewed as a homotopy $t\colon \partial^-{\otimes}\partial^- \to \partial^+{\otimes}\partial^+\colon PA{\otimes}PB \to A{\otimes}B$, trivial over $A{\otimes}B$ ($t{\circ}(e{\otimes}e) = 0_1\colon A{\otimes}B \Rightarrow A{\otimes}B$). The definition of $\alpha{\otimes}\beta$ is clear, because of (4)

(5) $\alpha\colon f_0 \to f_1\colon A \to C$, $\beta\colon g_0 \to g_1\colon B \to D$

(6) $\alpha{\otimes}\beta\colon f_0{\otimes}g_0 \to f_1{\otimes}g_1\colon A{\otimes}B \to C{\otimes}D$, $\widehat{(\alpha{\otimes}\beta)} = t.(\hat{\alpha}{\otimes}\hat{\beta})\colon A{\otimes}B \to P(C{\otimes}D)$

and straightforward calculations show that **A** is now a monoidal h-category

(7) $0_f{\otimes}0_g = 0_{f{\otimes}g}$, $(k'{\otimes}k''){\circ}(\alpha{\otimes}\beta){\circ}(h'{\otimes}h'') = (k'{\circ}\alpha{\circ}h'){\otimes}(k''{\circ}\beta{\circ}h'')$.

It is also natural to define the tensor product of a homotopy with a map as follows (of course, the result need not be a trivial homotopy, notwithstanding the additive notation)

(8) $\alpha{\otimes}g = \alpha{\otimes}0_g$, $f{\otimes}\beta = 0_f{\otimes}\beta$.

Finally, given two deformations $\Phi\colon A \to P^2C$, $\Psi\colon B \to P^2D$,

(9) $\Phi{\otimes}\Psi = Pt.t.(\hat{\Phi}{\otimes}\hat{\Psi}) = (A{\otimes}B \to P^2C{\otimes}P^2D \to P(PC{\otimes}PD) \to P^2(C{\otimes}D))$

and it is straightforward to check that

(10) $\partial^\varepsilon P\,(\Phi{\otimes}\Psi) = (\partial^\varepsilon P\Phi){\otimes}(\partial^\varepsilon P\Psi)$, $P\partial^\varepsilon\,(\Phi{\otimes}\Psi) = (P\partial^\varepsilon\Phi){\otimes}(P\partial^\varepsilon\Psi)$

(11) $(\gamma{\otimes}\delta){\circ}(\alpha{\otimes}\beta) = (\gamma{\circ}\alpha){\otimes}(\delta{\circ}\beta)$

so that cell-homotopies are closed under tensor product.

5.3. Monoids. Consider now the category Mon**A** of (strict) *monoids* in the monoidal category **A**. An object $A = (|A|, i, m)$ is an object $|A|$ of **A**, equipped with a unit and a multiplication

(1) $i\colon E \to |A|$, $m\colon |A|{\otimes}|A| \to |A|$

satisfying the usual axioms $(m(i{\otimes}1) = 1 = m(1{\otimes}i);\ m(m{\otimes}1) = m(1{\otimes}m))$. A (homo)morphism $f\colon A \to B$ is given by an **A**-morphism $f\colon |A| \to |B|$ consistent with the structure ($fi_A = i_B$, $fm_A =$

$m_B(f{\otimes}f)$). The forgetful functor $|{-}|$: Mon**A** $\to$ **A** will often be omitted. There is also an intermediate forgetful functor with values in the slice category **A**\E, of interest for h-kernels (5.4)

(2) U: Mon**A** $\to$ **A**\E, $\qquad\qquad$ U(A, i, m) = (A, i).

The path-functor **P** of Mon**A** is constructed by enriching P|A| with the following multiplicative structure, which combines the multiplicative structures of P and A, (e_E, t) and (i, m) respectively

(3) $\mathbf{P}(A) = (PA, i^P, m^P)$, $\qquad\qquad$ $\mathbf{P}(f) = P|f|$

(4) $i^P = (Pi.e\colon E \to PE \to PA)$ $\qquad\qquad$ $m^P = (Pm.t\colon PA \otimes PA \to P(A{\otimes}A) \to PA)$.

Now, the P0-structure $(\partial^\varepsilon, e)$ of **A** can be easily lifted to monoids, making Mon**A** into a P0-category. For instance, $e_A\colon A \to PA$ is a homomorphism because $t.e{\otimes}e = e$ (5.2.3)

(5)
$$\begin{tikzcd}
E \arrow[r, "i"] \arrow[d, equal] & A \arrow[d, "e"] & A{\otimes}A \arrow[l, "m"'] \arrow[d, "e{\otimes}e"] \\
E \arrow[r, "Pi.e"] & PA & PA{\otimes}PA \arrow[l, "Pm.t"']
\end{tikzcd}$$

In the same way, we get general properties for lifting monoidal functors and transformations to Mon**A** (similar to the general properties considered for slice categories in 4.6 (i-iii)).

(i) Every monoidal endofunctor F: **A** $\to$ **A** lifts to a functor **F**: Mon**A** $\to$ Mon**A**. The lifting preserves the composition; in particular, $\mathbf{F}^2$ lifts F^2.

(ii) Every monoidal transformation u: F $\to$ G of such functors lifts to u: **F** $\to$ **G**.

(iii) The forgetful functor Mon**A** $\to$ **A** creates pullbacks: given two homomorphisms $f_j\colon A_j \to B$ (j = 1, 2) and their pullback $p_j\colon A \to A_j$ in **A**, there is precisely one monoid structure (i, m) over A which makes the projections p_i into homomorphisms

(6) $p_j\, i = i_j\colon E \to A_j$, $\qquad\qquad$ $p_j\, m = m_j.p_j{\otimes}p_j\colon A{\otimes}A \to A_j{\otimes}A_j \to A_j$

and under this structure, A is the pullback in Mon**A**. The associativity of m in A derives from the same property in A_j, through the following diagram (j = 1, 2)

(7)
$$\begin{tikzcd}
 & A_j{\otimes}A_j{\otimes}A_j \arrow[rr, "1{\otimes}m_j"] \arrow[dd, dashed] & & A_j{\otimes}A_j \arrow[dd, "m_j"] \\
A{\otimes}A{\otimes}A \arrow[ur] \arrow[rr, "1{\otimes}m"] \arrow[dd, "m{\otimes}1"'] & & A{\otimes}A \arrow[ur] \arrow[dd, "m"'] & \\
 & A_j{\otimes}A_j \arrow[rr, dashed] & & A_j \\
A{\otimes}A \arrow[ur] \arrow[rr, "m"] & & A \arrow[ur, "p_j"'] &
\end{tikzcd}$$

where all faces – except the front one – are known to commute, and the pair (p_1, p_2) is jointly monic; analogously one proves the unit axiom.

(iv) if the monoidal P0-category **A** has all composition pullbacks $QA = PA \times_A PA$ (with projections k^ε: QA $\to$ PA), the monoidal structure of P determines a similar structure for the functor Q of composable paths

(8) $k^{\varepsilon} e^{Q} = e^{P}: E \to PE,$ $\quad\quad$ $k^{\varepsilon} t^{Q} = t^{P}.k^{\varepsilon}\otimes k^{\varepsilon}: QA\otimes QB \to P(A\otimes B)$

and the lifting **Q**: Mon**A** $\to$ Mon**A** of Q coincides with the pullback functor provided by (iii).

5.4. h-Pullbacks. We say that **A** is *monoidal P0-homotopical* if it is monoidal P0 and P0-homotopical. We have thus the path limit P(f, g) in **A** of every diagram

(1) $$B \xrightarrow{f} A \xleftarrow{\partial^-} PA \xrightarrow{\partial^+} A \xleftarrow{g} C$$

and, as in 5.3 (iii), the forgetful functor creates a monoid structure **P**(f, g) on this limit, whenever f and g are in Mon**A**. In other words, if **A** is monoidal P0-homotopical, then Mon**A** is P0-homotopical.

Mon**A** has always an initial object, provided by the canonical monoid structure on the unit E of $\otimes$ ($E = E \cong E\otimes E$); taking $g = e: E \to A$, **P**(f, e) supplies the (lower) homotopy fibre **K**f in Mon**A**. The initial object is preserved by U: Mon**A** $\to$ **A**\E (5.3.2) (not by $|-|$: Mon**A** $\to$ **A**, in general; the initial object of **A** may have no monoid structure, as in **Top**). Thus, the h-kernels of Mon**A** are created by U: they are the h-kernels of **A**\E, equipped with the unique monoid structure consistent with the structural maps.

5.5. The P4 case. A *monoidal P4*-structure on the monoidal category **A** = (**A**, $\otimes$, E) will consist of a monoidal path functor (P, e_E, t)

(1) $P: \mathbf{A} \to \mathbf{A},$ $\quad\quad$ $e_E: E \to PE,$ $\quad\quad$ $t_{AB}: PA \otimes PB \to P(A\otimes B)$

together with a P4-structure $(\partial^{\varepsilon}, e, g^{\varepsilon}, r, k, s, z, w)$ for P consisting of *monoidal* transformations; we implicitly use the fact that the functor Q of composable paths in **A** is monoidal (5.3.iv). Assuming the structure is monoidal P4, it is *monoidal strong P4* or *monoidal regular P4* or *monoidal P4-homotopical* if it is so in the ordinary (non-monoidal) sense. Clearly, in these four cases, Mon**A** is respectively P4, or strong P4, or regular P4, or P4-homotopical. *Monoidal P1-* or *P3*-structures are similarly defined. The main results on Mon**A** here obtained are summarised below.

5.6. Theorem. If **A** is *monoidal* P0, or P0-homotopical, or P4, or strong P4, or regular P4, or P4-homotopical, the category Mon**A** of strict internal monoids has a canonical structure of the same kind (i.e. P0 or P0-homotopical, etc.), described above. Moreover, if the monoidal strong P4-category **A** has all pullbacks, preserved by P, then it is monoidal P4-homotopical.

5.7. Tensor and sum of homotopies. Let **A** be monoidal P3. Then it is monoidal h3, i.e. the tensor product is consistent with the reversion and the sum of homotopies

(1) $(-\alpha)\otimes(-\beta) = -\alpha\otimes\beta: f_1\otimes g_1 \to f_0\otimes g_0$ $\quad\quad$ $(\alpha: f_0 \to f_1,\ \beta: g_0 \to g_1)$

(2) $(\alpha+\alpha') \otimes (\beta+\beta') = (\alpha\otimes\beta) + (\alpha'\otimes\beta'): f_0\otimes g_0 \to f_2\otimes g_2$ $\quad\quad$ $(\alpha': f_1 \to f_2,\ \ \beta': g_1 \to g_2).$

The proof is based on the fact that r, k are monoidal transformations; for (2), one also uses the commutativity of the left-hand square in (3), "detected" by the jointly monic pair k^{ε} through the right-hand square (which commutes by definition of t^{Q}, 5.3.8)

$$\begin{array}{lccccc}
 & A{\otimes}B & \xrightarrow{(\hat\alpha \vee \hat\alpha') \otimes (\hat\beta \vee \hat\beta')} & QC{\otimes}QD & \xrightarrow{k^{\varepsilon}{\otimes}k^{\varepsilon}} & PC{\otimes}PD \\
(3) & \| & & \downarrow t^Q & & \downarrow t^P \\
 & A{\otimes}B & \xrightarrow[(\alpha\otimes\beta)^{\wedge} \vee (\alpha'\otimes\beta')^{\wedge}]{} & Q(C{\otimes}D) & \xrightarrow[k^{\varepsilon}]{} & P(C{\otimes}D)
\end{array}$$

It follows that *biadditivity holds for tensors of a homotopy with a map* (5.2.8), and all tensors of homotopies can be *reduced* to such particular ones

(4) $(-\alpha){\otimes}g = -\alpha{\otimes}g, \qquad f \otimes (-\beta) = -f{\otimes}\beta$

(5) $(\alpha+\alpha'){\otimes}g = \alpha{\otimes}g + \alpha'{\otimes}g, \qquad f{\otimes}(\beta+\beta') = f{\otimes}\beta + f{\otimes}\beta'$

(6) $\alpha{\otimes}\beta = \alpha{\otimes}g_0 + f_1{\otimes}\beta = f_0{\otimes}\beta + \alpha{\otimes}g_1$.

Finally, if **A** is monoidal P4, it is also monoidal h4, i.e. the tensor product preserves the cell-homotopy relation, as already proved (see the final remark of 5.2). Moreover, since the interchange is monoidal, it is easy to show that $(s\Phi){\otimes}(s\Psi) = s(\Phi{\otimes}\Psi)$.

6. Applications

Spaces, small categories and chain complexes give rise to basic IP4-homotopical structures, from which other are deduced through the categorical procedures of the two previous sections.

6.1. Spaces. a) The category **Top** of topological spaces is IP4-homotopical, with respect to the usual cylinder and path functors, the usual operations recalled above (connections and symmetries in 2.1; vertical composition in 2.3) plus the zero collapse constructed in 2.7.9 and the lens collapse described in the Introduction. Of course the whole left-homotopical structure is trivial (all h-kernels are empty), because the initial object $\bot = \emptyset$ is absolute (every map and every cell to $\bot$ is an identity).

b) The (pointed) category $\mathbf{Top}^{\top}$ of pointed topological spaces is IP4-homotopical (4.6). As well known, the P-structure comes directly from the one of **Top** (4.5.3) while the cylinder $\mathbf{I}(A, x) = (IA/I\{x\}, x^I)$ is formed by collapsing the subspace $I\{x\}$ in the non-pointed cylinder IA (4.5.1).

c) More generally, if $u: X \to Y$ is a continuous map, the category **Top**(u) of *spaces under* X *and over* Y is IP4-homotopical, with cylinder and path functor as in 4.5. This includes the categories **Top**\X of spaces under X (take $Y = \top$) and **Top**/Y of spaces over Y (take $X = \bot$).

Note that $\mathbf{Top}^{\top}(u)$ does not produce anything new. Indeed, if $u: X \to Y$ is in $\mathbf{Top}^{\top}$ and $|u|$ is its underlying map in **Top**, the forgetful functor from $\mathbf{Top}^{\top}(u)$ to $\mathbf{Top}(|u|)$ is an isomorphism.

d) The categories $\mathbf{Top}^{\mathbf{S}}$ and $(\mathbf{Top}^{\top})^{\mathbf{S}}$ of **S**-diagrams of spaces or pointed spaces are IP4-homotopical. This includes $\mathbf{Top}^{\mathbf{2}}$, $\mathbf{Top}^{\mathbb{Z}}$, G-spaces, presheaves of spaces on a fixed space and the pointed analogues. Further, for any site **S**, Shv(**S**, **Top**) and Shv(**S**, $\mathbf{Top}^{\top}$) are P4-homotopical.

6.2. Topological monoids. Clearly, **Top** is cartesian P4-homotopical (5.6). The path functor P, which preserves limits, is strong monoidal

(1) $t: PA \times PB \to P(A{\times}B), \qquad t(\alpha, \beta) = \langle\alpha, \beta\rangle: [0, 1] \to A{\times}B$

and it is easy to check that all the transformations of the P4-structure are monoidal; for instance, for the path-reversion $r(\alpha)(\tau) = \alpha(1-\tau)$, we have $t\,(r{\times}r)\,(\alpha, \beta) = \langle r\,\alpha, r\,\beta\rangle = r\,\langle \alpha, \beta\rangle = r.t\,(\alpha, \beta)$.

The category Mon**Top** of topological monoids is thus P4-homotopical. The enriched path functor **P** has pointwise multiplication $(\alpha.\beta)(\tau) = \alpha(\tau).\beta(\tau)$, and the h-kernels are created by the forgetful functor U: Mon**Top** $\to$ **Top**\E = $\mathbf{Top}^{\top}$

(2) $\mathbf{K}^{-}f = \{(a, \beta) \in A{\times}PB \mid \beta(0) = f(a),\ \beta(1) = e_B\}$.

Mon**Top** is IP4. The left adjoint $\mathbf{I} \dashv \mathbf{P}$ is easy to construct (and *does not lift* I). Take the "free object" functor F, left adjoint to the forgetful one

(3) F: **Top** $\to$ Mon**Top**, $\qquad F(X) = X^{+} = \Sigma_{n\geq 0}\, X^{n}$

which turns the space X into the free monoid of words over the underlying set, with the sum of the product topologies X^n. Now, given a topological monoid A, consider $FI|A|$, consisting of words $(\tau_1, a_1, ... \tau_n, a_n)$ over $[0, 1]{\times}A$; and divide it out modulo the monoid-congruence spanned by the relations provided by the algebraic structure of A

(4) $\mathbf{I}(A) = (FI|A|)\,/\sim,\qquad (\tau, e_A) \sim e,\qquad (\tau, a, \tau, b) \sim (\tau, a.b)$

so to make the obvious mapping $\eta\colon A \to \mathbf{PI}(A)$, $\eta(a)(\tau) = [(\tau, a)]$, into a (continuous) homomorphism, the unit of our adjunction $\mathbf{I} \dashv \mathbf{P}$.

Also $\mathbf{Top}^{\top}$ is cartesian P4-homotopical, but its monoids are again the topological monoids. Instead, under the smash product $A{\wedge}B = (A{\times}B)\,/\,(A{+}B)$, with identity S^0 and

(5) $t\colon PA{\wedge}PB \to P(A{\wedge}B),\qquad t\,[\alpha, \beta] = p.\langle\alpha, \beta\rangle\colon [0, 1] \to A{\times}B \to A{\wedge}B$

P is monoidal, non-strong, and $(\mathbf{Top}^{\top}, \wedge)$ is monoidal P4-homotopical. Its monoids are the topological monoids with *zero* (an absorbant element).

6.3. Categories and natural equivalences. We show now that the 2-category $\mathbf{Cat}_{\mathrm{i}}$ of small categories, functors and *functorial isomorphisms* (as homotopies) is IP4-homotopical. Various more complex homotopical constructs can be derived, through diagrams, sheaves and slice procedures.

Recall that the category **Cat** of small categories is cartesian closed, with $[X, Y] = Y^X$ the category of functors $X \to Y$ and their natural transformations; **Cat** is actually a 2-category (with the natural transformations). But we are interested in the sub-2-category $\mathbf{Cat}_{\mathrm{i}}$, which has invertible cells; it is thus a groupoid-enriched category, or in other words a strict h4-category.

Clearly, the homotopies are corepresentable and representable, through the *undiscrete* groupoid **i** on two objects (say 0, 1) "representing the free isomorphism" (a groupoid is undiscrete if each hom-set has one element)

(1) $IX = \mathbf{i}{\times}X \qquad\qquad$ (2) $PX = X^{\mathbf{i}}$

where $X^{\mathbf{i}}$ "is" the full subcategory of $X^{\mathbf{2}}$ whose objects are the isomorphisms of X. This **i** is a commutative, involutive dioid-object in $(\mathbf{Cat}, \times)$. $\mathbf{i}{\times}\mathbf{i}$ is the undiscrete groupoid on four objects

(3)
$$\begin{tikzcd}
(0,1) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & (1,1) \arrow[d, leftrightarrow] \arrow[dl, leftrightarrow] \\
(0,0) \arrow[r, leftrightarrow] & (1,0)
\end{tikzcd}$$

and the structural functors are determined by their action on the objects, as follows (ε, i, j = 0, 1)

(4) $\partial^\varepsilon\colon \mathbf{1} \to \mathbf{i}$, $\quad e\colon \mathbf{i} \to \mathbf{1}$, $\quad g^\varepsilon\colon \mathbf{i}{\times}\mathbf{i} \to \mathbf{i}$, $\quad r\colon \mathbf{i} \to \mathbf{i}$, $\quad s\colon \mathbf{i}{\times}\mathbf{i} \to \mathbf{i}{\times}\mathbf{i}$

$\partial^\varepsilon(0) = \varepsilon$, $\quad g^-(i, j) = i\vee j$, $\quad g^+(i, j) = i\wedge j$, $\quad r(i) = 1 - i$, $\quad s(i, j) = (j, i)$.

Further, **i** has a (regular) vertical composition k. Note that $\mathbf{i} \underset{\mathbf{1}}{+} \mathbf{i}$ is the undiscrete groupoid on three objects; writing them as "0, $^1/_2$, 1", the functor k is the inclusion $\mathbf{i} \subset \mathbf{i} \underset{\mathbf{1}}{+} \mathbf{i}$.

This structure, transferred to the cylinder and path functors, makes $\mathbf{Cat}_\mathrm{i}$ into a strict IP4-category. It is flat (2.5), because the diagonal maps in (3) are determined by the boundary ones. Since **Cat** is complete and cocomplete, $\mathbf{Cat}_\mathrm{i}$ is IP4-homotopical. All h-kernels, produced by the initial category $\perp = \mathbf{0}$, are empty; instead the right homotopical structure is not trivial (and h-cokernels are easily calculated). The lens collapse $w\colon \mathbf{i}{\times}\mathbf{i} \to \mathbf{i}{\times}\mathbf{i}$ is determined as in 2.8.3; it collapses "vertically" $\mathbf{i}{\times}\mathbf{i}$ on its main diagonal ($w(i, j) = (i, i)$).

All this plainly restricts to the 2-category **Gpd** of groupoids, functors and natural transformations (necessarily iso). For the larger 2-category **Cat** (with all natural transformations), the groupoid **i** should be replaced with the ordinal category $\mathbf{2} = \{0 < 1\}$, and the groupoid $\mathbf{i}{\times}\mathbf{i}$ with the order category $\mathbf{2}{\times}\mathbf{2}$. The structure considered above can thus be extended, *except* of course for the reversion, which can only be partially surrogated by a *generalised reversion* $r\colon \mathbf{2} \to \mathbf{2}^{\mathrm{op}}$ [10].

$\mathbf{Cat}_\mathrm{i}$ and **Gpd** are also cartesian P4-homotopical. Mon$\mathbf{Cat}_\mathrm{i}$ is the P4-homotopical category of (small) *strict* monoidal categories. In order to get *monoidal categories* in the usual relaxed sense, one should consider *h-monoids* in $\mathbf{CAT}_\mathrm{i}$, satisfying the monoid axioms up to specified, coherent homotopies (functorial isomorphisms); this will be studied in the sequel.

6.4. Two-dimensional groupoids. Homotopy properties of 2-groupoids are considered in [27], to which we refer for basic definitions and terminology. We show that the 2-category 2-**Gpd** of small 2-groupoids, homomorphisms (i.e., 2-functors) and natural transformations is IP4-homotopical.

The category 2-**Cat** of small 2-categories is cartesian closed, with $[X, Y] = Y^X$ the 2-category of 2-functors $X \to Y$, their natural transformations and their modifications. Similarly for 2-**Gpd**.

Take the *undiscrete* 2-groupoid $\underline{\alpha} = \mathbf{i}_2$ on two objects, 0 and 1, "representing the free cell" (a 2-groupoid X is undiscrete if each hom-groupoid X(x, y) is isomorphic to **i**). It corepresents and represents homotopies, through the following cylinder and path functors

(1) $IX = \underline{\alpha}{\times}X$, $\qquad$ (2) $PX = X^{\underline{\alpha}}$

(an object of $X^{\underline{\alpha}}$ can be identified to an arbitrary cell $\alpha\colon f \to g\colon x \to y$ in X).

This $\underline{\alpha}$ is a commutative, involutive internal dioid in 2-**Gpd**. Indeed, $\underline{\alpha}{\times}\underline{\alpha}$ is the undiscrete 2-groupoid on four objects (0, 0), ... (1, 1); the structural functors (∂^ε, g^ε, r, s) are again described by their action on the objects, as in the 1-dimensional case (6.3.4). Also $\underline{\alpha} \underset{\mathbf{1}}{+} \underline{\alpha}$ (the undiscrete 2-groupoid on three objects, 0, $^1/_2$, 1), the (regular) vertical composition k, the lens collapse w are defined as in 6.3.

Transferring these operations to the cylinder and path functors, 2-**Gpd** becomes a flat, strict IP4-category. Since 2-**Gpd** is complete and cocomplete, it is also IP4-homotopical.

6.5. Chain complexes. Finally, we show that the category $C_*\mathbf{D}$ of unbounded chain complexes $A = ((A_n), (\partial_n))$ over an additive category $\mathbf{D}$ is a regular IP4-homotopical category, and therefore homotopical. Of course, homotopies are the usual ones and the basic structure is classical; but the connections are less known, the lens collapse map and its lifting property seemingly new.

Some general remarks on duality will reduce calculations. Writing X*, f* the object and arrow corresponding to X and f in the opposite category, the anti-isomorphism

(1) $C_*\mathbf{D} \to C_*(\mathbf{D}^{op}), \qquad A = ((A_n), (\partial_n)) \mapsto A' = ((A^*_{-n}), (\partial^*_{-n+1}))$

shows that $(C_*\mathbf{D})^{op}$ is again a category of chain complexes, and allows one to derive the cylinder functor I of $C_*\mathbf{D}$ from the path functor P^{op} of $C_*(\mathbf{D}^{op})$

(2) $I(A) = (P^{op}(A'))'$

and similarly for cones, suspensions, h-pushouts. We also note that here $(C_*\mathbf{D})^{\mathbf{S}} \cong C_*(\mathbf{D}^{\mathbf{S}})$ is nothing new; again, if $\mathbf{D}$ is complete, $\mathrm{Shv}(\mathbf{S}, C_*\mathbf{D}) \cong C_*(\mathrm{Shv}(\mathbf{S}, \mathbf{D}))$.

A $\mathbf{D}$-map between finite biproducts $f\colon \oplus A_j \to \oplus B_i$ of components f_{ij} will be written "on variables", as $f(x_1, ... x_n) = (\Sigma f_{1j} x_j, ... \Sigma f_{mj} x_j)$; this allows one to calculate as in a category of modules, and can be formally justified by setting $x_j = pr_j\colon \oplus A_j \to A_j$.

6.6. The path functor. To fix notation, a homotopy in $C_*\mathbf{D}$ is written as in (1), and satisfies (2)

(1) $\alpha\colon f \to g\colon A \to B, \qquad \alpha = (f, \alpha_\bullet, g)$

(2) $-f + g = \partial\alpha_\bullet + \alpha_\bullet\partial \qquad (-f_n + g_n = \partial_{n+1}\alpha_n + \alpha_{n-1}\partial_n)$

where $\alpha_\bullet = (\alpha_n)\colon |A| \to |B|$ is a map of graded objects, of degree 1, the *centre* of α. The reduced horizontal composition and the vertical identities are

(3) $k\alpha h = (kfh, k\alpha_\bullet h, kgh), \qquad 0_f = (f, 0, f).$

Homotopies are represented by a path endofunctor P, with structural homotopy δ

(4) $(PA)_n = A_n \oplus A_{n+1} \oplus A_n, \qquad \partial(a, h, b) = (\partial a, -a - \partial h + b, \partial b)$

(5) $\delta\colon \partial^- \to \partial^+\colon PA \to A, \qquad \partial^\varepsilon(a^-, h, a^+) = a^\varepsilon, \qquad \delta_n(a, h, b) = h.$

The P3-structure is given by the above transformations $\partial^-, \partial^+\colon P \to 1$, with the obvious degeneracy $(e(a) = (a, 0, a))$, vertical reversion $(-_v)$ and composition $(+_v)$

(6) $-_v\colon PA \to PA, \qquad -_v(a, h, b) = (b, -h, a)$

(7) $+_v\colon PA \times_A PA \to PA, \qquad (a, h, c) +_v (c, k, d) = (a, h+k, d)$

which produce the usual, regular sum of homotopies. By duality (6.5), homotopies are also corepresented by a cylinder

(8) $(IA)_n = A_n \oplus A_{n-1} \oplus A_n, \qquad \partial(a, h, b) = (\partial a - h, -\partial h, \partial b + h)$

and we have an IP3-structure. P and I respectively preserve the existing limits and colimits; both preserve finite biproducts.

6.7. The cubical comonad. P is a cubical comonad with interchange. The second order path-object has the following components and differential (with $z' = -h + u + \partial z - v + k$)

(1) $(P^2A)_n = (A_n \oplus A_{n+1} \oplus A_n) \oplus (A_{n+1} \oplus A_{n+2} \oplus A_{n+1}) \oplus (A_n \oplus A_{n+1} \oplus A_n)$

(2) $\partial(a, h, b;\ u, z, v;\ c, k, d) =$

$$= (\partial a,\ -a - \partial h + b,\ \partial b;\ \ -a - \partial u + c,\ z',\ -b - \partial v + d\ ;\ \ \partial c,\ -c - \partial k + d,\ \partial d).$$

It is convenient to represent the "variable" $\xi = (a, h, b;\ u, z, v;\ c, k, d)$ of P^2A as a square diagram, so that its faces $\partial^{\varepsilon}P$ and $P\partial^{\varepsilon}$ appear as horizontal or vertical edges, respectively

(3)
$$\begin{tikzcd} c \arrow[r, "k"] & d \\ a \arrow[u, "u"] \arrow[r, "h"'] \arrow[ur, phantom, "z"] & b \arrow[u, "v"'] \end{tikzcd} \qquad \begin{array}{ll} \partial^{-}P\,(\xi) = (a, h, b), & \partial^{+}P\,(\xi) = (c, k, d) \\ P\partial^{-}\,(\xi) = (a, u, c), & P\partial^{+}\,(\xi) = (b, v, d) \end{array}$$

The connections g^{-} and g^{+} completing the P-structure appear thus in their geometrical meaning

(4)
$$\begin{tikzcd} b \arrow[r, "0"] & b \\ a \arrow[u, "h"] \arrow[r, "h"'] \arrow[ur, phantom, "0"] & b \arrow[u, "0"'] \end{tikzcd}$$

(5)
$$\begin{tikzcd} a \arrow[r, "h"] & b \\ a \arrow[u, "0"] \arrow[r, "0"'] \arrow[ur, phantom, "0"] & a \arrow[u, "h"'] \end{tikzcd}$$

(6) $g^{-}(a, h, b) = (a, h, b;\ h, 0, 0;\ b, 0, b)$ (7) $g^{+}(a, h, b) = (a, 0, a;\ 0, 0, h;\ a, h, b)$.

Similarly, the interchange $s\colon P^2A \to P^2A$ is obtained through a reflection with respect to the "main diagonal", as in **Top**, together with a sign-change in the central term

(8)
$$\begin{tikzcd} c \arrow[r, "k"] & d \\ a \arrow[u, "u"] \arrow[r, "h"'] \arrow[ur, phantom, "z"] & b \arrow[u, "v"'] \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} b \arrow[r, "v"] & d \\ a \arrow[u, "h"] \arrow[r, "u"'] \arrow[ur, phantom, "-z"] & c \arrow[u, "k"'] \end{tikzcd}$$

(9) $s\,(a, h, b;\ u, z, v;\ c, k, d) = (a, u, c;\ h, -z, k;\ b, v, d)$

and s is obviously involutive, turns the horizontal faces into the vertical ones, makes the connections g^{ε} commutative ($g^{-}.s = g^{-}$, since g^{-} in (4) is invariant under such reflection and change) and is consistent with the degeneracy e. As a consequence of s, P is homotopy invariant (2.4).

Finally, $C_*\mathbf{D}$ is a regular IP4-homotopical category (and therefore h4 and homotopical). Indeed, all path limits exist (and are preserved by P): for $f\colon A \to C$, $g\colon B \to C$

(10) $(P(f, g))_n = A_n \oplus C_{n+1} \oplus B_n$, $\partial(a, c, b) = (\partial a, -fa - \partial c + gb, \partial b)$.

6.8. Lens collapse. Let be given four homotopies $\alpha, \beta, \gamma, \delta\colon X \to PA$, connecting four maps $a, b, c, d\colon X \to A$, as below. By 6.7.1-2, a deformation $\Phi\colon X \to P^2A$ with boundary $(\alpha, \beta, \gamma, \delta)$

(1)
$$\begin{tikzcd} c \arrow[r, "\gamma"] & d \\ a \arrow[u, "\beta"] \arrow[r, "\alpha"'] \arrow[ur, phantom, "\Phi"] & b \arrow[u, "\delta"'] \end{tikzcd} \qquad \begin{array}{ll} \alpha = (a, h, b)\colon a \to b, & \beta = (a, u, c)\colon a \to c \\ \gamma = (b, k, d)\colon b \to d, & \delta = (c, v, d)\colon c \to d \end{array}$$

amounts to a map $z = \Phi_\bullet\colon |X| \to |A|$ of graded objects, of degree 2, the *centre* of Φ, satisfying a differential condition related to the (anti-clockwise) endohomotopy $\alpha + \delta - \gamma - \beta$ of a

(2) $\partial z - z\partial = h + v - k - u\colon |X| \to |A|$ $\qquad\qquad (h + v - k - u = (\alpha + \delta - \gamma - \beta)_\bullet)$.

The (strong) lens collapse $w\colon P^2A \to P^2A$ follows from the regular structure: the formula 2.8.3 ($w = \partial^-I.g^+ +_h 1 +_h \partial^+I.g^-$) supplies the (geometrically obvious) solution

(3)
$$\begin{array}{ccc} c & \xrightarrow{k} & d \\ u\uparrow & z & \uparrow v \\ a & \xrightarrow[h]{} & b \end{array} \qquad \mapsto \qquad \begin{array}{ccc} a & \xrightarrow{u+k} & d \\ 0\uparrow & z & \uparrow 0 \\ a & \xrightarrow[h+v]{} & d \end{array}$$

(4) $w\,(a, h, b;\ u, z, v;\ c, k, d) = (a, h{+}v, d;\ 0, z, 0;\ a, u{+}k, d)$.

It is easy to verify directly that w is strong. Given four homotopies $\alpha, \beta, \gamma, \delta\colon X \to PA$ disposed as above, in (1), the lens conversion turns a deformation $\Phi\colon X \to P^2A$ with this boundary into the cell-homotopy $w\Phi\colon \alpha{+}\delta \to \beta{+}\gamma$, *preserving the centre* (Φ and $w\Phi$ have the same associated endohomotopy $\alpha + \delta - \gamma - \beta$) and is therefore bijective.

6.9. Chain algebras. The category **Dm** = C_*(R-**Mod**) of chain complexes of modules over the commutative unitary ring R is monoidal (closed), with unit R (in degree zero)

(1) $(A\otimes B)_n = \bigoplus_{p+q=n} A_p\otimes B_q$, $\qquad\qquad \partial(a\otimes b) = (\partial a)\otimes b + (-1)^{\deg a}\, a\otimes(\partial b)$.

The path-functor P (6.6.4) becomes lax-monoidal when equipped with the natural transformation

(2) $t\colon (PA)\otimes(PB) \to P(A\otimes B)$

(3) $t\,((a', a, a'')\otimes(b', b, b'')) = (a'\otimes b',\ a\otimes b'' + (-1)^{\deg a'}\, a'\otimes b,\ a''\otimes b'')$.

DA = Mon**Dm**, the category of associative, unitary chain R-algebras, is thus P0-homotopical. The symmetrical cubical comonad structure $(\partial^\varepsilon, e, g^\varepsilon, s)$ consists of monoidal transformations and lifts to algebras. This structure, studied in [11], is already sufficient to develop the basic notions of homotopical algebra. Instead, the reversion, composition and lens collapse are just monoidal up to homotopy; **Dm** is *not* monoidal P4, which is "why" **DA** lacks "algebraic" reversion and sum of homotopies (as well known). Again, this makes evident the interest of studying homotopy relaxations of § 5; a work on the homotopy structure of Stasheff's A_∞-algebras is in preparation.

As in the case of topological monoids (6.2), the cylinder functor **I** left adjoint to **P** can be constructed as a quotient of the free chain algebra FI|A| over the cylinder of the underlying chain complex, imposing some relations which come from the multiplicative structure of A [11].

References

[1] H.J. Baues: *Algebraic homotopy*, Cambridge Univ. Press, 1989.

[2] K.S. Brown: Abstract homotopy theory and generalised sheaf cohomology, *Trans. Amer. Math. Soc.* **186** (1973), 419–458.

[3] R. Brown and P.J. Higgins: On the algebra of cubes, *J. Pure Appl. Algebra* **21** (1981), 233–260.

[4] R. Brown and P.J. Higgins: Colimit theorems for relative homotopy groups, *J. Pure Appl. Algebra* **22** (1981), 11–41.

[5] R. Brown and P.J. Higgins: Tensor products and homotopies for ω-groupoids and crossed complexes, *J. Pure Appl. Algebra* **47** (1987), 1–33.

[6] R. Brown and J.L. Loday: Van Kampen theorems for diagrams of spaces, *Topology* **26** (1987), 311–335.

[7] E. Dror Farjoun: Homotopy and homology of diagrams of spaces, in: *Algebraic Topology, Proceedings,* Seattle 1985, Lecture Notes in Math. **1286**, Springer-Verlag, 1987, pp. 93–134.

[8] M. Grandis: On the categorical foundations of homological and homotopical algebra, *Cahiers Top. Géom. Diff. Catég.* **33** (1992), 135–175.

[9] M. Grandis: Homotopical algebra in homotopical categories, *Appl. Categorical Structures* **2** (1994), 351–406.

[10] M. Grandis: Cubical monads and their symmetries, in: *Proceedings of the Eleventh International Conference on Topology*, Trieste 1993, *Rend. Ist. Mat. Univ. Trieste* **25** (1993), 223–268.

[11] M. Grandis: Cubical homotopical algebra and cochain algebras, *Ann. Mat. Pura Appl.* **170** (1996), 147-186.

[12] M. Grandis: Homotopical algebra and triangulated categories, *Math. Proc. Cambridge Philos. Soc.* **118** (1995), 259–295.

[13] J.W. Gray: Sheaves with values in a category, *Topology* 3 (1965), 1–18.

[14] K.A. Hardie and K.H. Kamps: Homotopy over B and under A, *Cahiers Topol. Géom. Diff. Catég.* **28** (1987), 183–195.

[15] K.A. Hardie and K.H. Kamps: Track homotopy over a fixed space, *Glasnik Mat.* **24** (1989), 161–179.

[16] K.A. Hardie and K.H. Kamps: Variations on a theme of Dold, *Can. Math. Soc. Conf. Proc.* **13** (1992), 201–209.

[17] K.A. Hardie, K.H. Kamps and T. Porter: The coherent homotopy category over a fixed space is a category of fractions, *Topology Appl.* **40** (1991), 265–274.

[18] A. Heller: Stable homotopy categories, *Bull. Amer. Math. Soc.* **74** (1968), 28–63.

[19] I.M. James: Ex homotopy theory I, *Ill. J. of Math.* **15** (1971), 329–345.

[20] K.H. Kamps: Faserungen und Cofaserungen in Kategorien mit Homotopiesystem, Dissertation, Saarbrücken 1968.

[21] K.H. Kamps: Über einige formale Eigenschaften von Faserungen und h-Faserungen, *Manuscripta Math.* **3** (1970), 237–255.

[22] K.H. Kamps: Kan-Bedingungen und abstrakte Homotopietheorie, *Math. Z.* **124** (1972), 215–236.

[23] D.M. Kan: Abstract homotopy II, *Proc. Nat. Acad. Sci. U.S.A.* **42** (1956), 255–258.

[24] S. Mac Lane: Categorical algebra, *Bull. Amer. Math. Soc.* **71** (1965), 40–106.

[25] S. Mac Lane: *Categories for the working mathematician*, Springer-Verlag, 1971.

[26] S. Mac Lane and I. Moerdijk: *Sheaves in geometry and logic*, Springer-Verlag, 1992.

[27] I. Moerdijk and J.A. Svensson: Algebraic classification of equivariant homotopy 2-types, I, *J. Pure Appl. Algebra* **89** (1993), 187–216.

[28] D. Puppe: Homotopiemengen und ihre induzierten Abbildungen, I, *Math. Z.* **69** (1958), 299–344.

[29] D.G. Quillen: *Homotopical algebra*, Lect. Notes in Math. **43**, Springer-Verlag, 1967.

[30] C.B. Spencer and Y.L. Wong: Pullback and pushout squares in a special double category with connection, *Cahiers Topol. Géom. Diff. Catég.* **24** (1983), 161–192.