Homotopy Structures for Algebras over a Monad

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Abstract. Let T be a monad over a category $\mathbf{A}$. Then a homotopy structure for $\mathbf{A}$, defined by a cocylinder $P: \mathbf{A} \to \mathbf{A}$, or path-endofunctor, can be lifted to the category $\mathbf{A}^T$ of Eilenberg-Moore algebras over T, provided that P is consistent with T in a natural sense, i.e. equipped with a natural transformation $\lambda: TP \to PT$ satisfying some obvious axioms.

In this way, homotopy can be lifted from well-known, basic situations to various categories of "algebras"; for instance, from topological spaces to topological semigroups, or spaces over a fixed space (*fibrewise homotopy*), or actions of a fixed topological group (*equivariant homotopy*); from categories to strict monoidal categories; from chain complexes to associative chain algebras.

The interest is given by the possibility of lifting the "homotopy operations" (as faces, degeneracy, connections, reversion, interchange, vertical composition, etc.) and their axioms from $\mathbf{A}$ to $\mathbf{A}^T$, just by verifying the consistency between these operations and $\lambda: TP \to PT$. When this holds, the structure we obtain on our category of algebras is sufficiently powerful to ensure the main general properties of homotopy.

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Introduction

The homotopy structure of topological spaces can be defined through the cylinder functor $I: \mathbf{Top} \to \mathbf{Top}$, $IX = [0, 1]\times X$, or equivalently through its right adjoint $P: \mathbf{Top} \to \mathbf{Top}$, $PY = Y^{[0, 1]}$, the cocylinder or path functor. Each of these endofunctors comes equipped with various *operations*, i.e. natural transformations between its powers (or other derived endofunctors), which produce the usual operations between homotopies of any order; for instance: faces, degeneracy, connections, symmetries and vertical composition.

An abstract setting of this type was developed in [9]; it yields a simple, yet powerful, approach to diagrammatic homotopy lemmas, the Puppe sequence of a map, homotopical stability and relations with triangulated categories. It is an enrichment of the basic situation introduced by Kan [16] and consisting of a cylinder $I: \mathbf{A} \to \mathbf{A}$ over a category, equipped with two faces $\partial^-, \partial^+: 1 \to I$ and one degeneracy $e: I \to 1$.

Here we work mostly with the dual case of a cocylinder $P: \mathbf{A} \to \mathbf{A}$, which is adequate for studying algebras over a monad (while a cylinder is more suitable for studying coalgebras over a comonad). Given a monad T over $\mathbf{A}$, and its category $\mathbf{A}^T$ of Eilenberg-Moore algebras, the cocylinder P can be lifted to $\mathbf{A}^T$, provided that P is consistent with T in a natural sense, i.e. equipped with a natural transformation $\lambda: TP \to PT$ satisfying some obvious axioms. In this way, the homotopy structure of various categories of "algebras" over **A** can be reduced to the homotopy structure of the basis.

Other approaches to abstract homotopy theory based on a cylinder functor can be found in Baues [1] and various papers by Kamps, e.g. [15]. Within Quillen's approach, Crans [5] shows how a closed model structure on the category **A** can be lifted along a right adjoint functor **A**' → **A** (not required to be monadic).

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Outline. The first section is a review of our "algebraic" setting for homotopical algebra, based on a category **A** equipped with a cocylinder endofunctor P: **A** → **A** and various operations. In Section 2, the category **A** is assumed to have a monad T = (T, η, μ) and a homotopy structure defined by the endofunctor P, consistent with T in the above sense, so that P can be lifted to the category $\mathbf{A}^T$ of Eilenberg-Moore algebras over T; if moreover the path endofunctor P has a left-adjoint I and $\mathbf{A}^T$ has coequalisers, also the lifting P^T has a left adjoint I^T (2.6-7).

This theory is applied to derive the homotopy structure of some types of algebras over spaces (as topological semigroups, monoids and groups) and categories (strict monoidal categories), in Section 3; *equivariant homotopy*, in the category G-**Top** of actions of a topological group G over spaces, is also derived from the general pattern. In Section 4, we recall the homotopy structure of chain complexes and show it is *partially* consistent with the "free-semigroup monad", producing the usual, defective homotopy structure for associative d-algebras, which lacks reversion and sum of homotopies. Finally, in Section 5, slice categories **A**\A (resp. **A**/B) are viewed as categories of algebras (resp. coalgebras) over **A**; this has applications to the topological case, namely, homotopy under A and, respectively, *fibrewise homotopy* over B.

Conventions. ⊤ and ⊥ denote the terminal and initial object of a category; if they coincide, producing the zero-object 0, the category is said to be *pointed*. The morphism $X \to \prod Y_i$ of components (f_i) is written $\langle f_i \rangle$. In a 2-graph, a 2-morphism α: f → g: A → B is generally called a homotopy, or cell, and written as α: f → g when we just want to specify its *vertical* domain and codomain. The "vertical structure" of homotopies is always written additively. Thus, the trivial endohomotopy of a morphism f is denoted by 0_f: f → f; the vertical reversion of α: f → g is written – α: g → f; and the vertical composition of α with β: g → h is written α+β: f → h. **Top** and $\mathbf{Top}^{\top}$ are respectively the categories of topological spaces and pointed spaces, while $C_*\mathbf{D}$ is the category of unbounded chain complexes over an additive category **D**.

1. Categories with a cocylinder

We give here an outline of an "algebraic" setting for homotopical algebra, based on a category **A** equipped with a cocylinder endofunctor P: **A** → **A** and various *operations*, i.e. natural transformations between powers of P, or other derived endofunctors, meant to produce the homotopies of **A**, of any order, and their operations (see [9]). All notions are exemplified in the category **Top** of topological spaces; other basic examples are given in Sections 3, 4: pointed spaces, small categories, groupoids, chain complexes; and others will be deduced, as categories of algebras over the basic ones.

1.1. The path functor. Our basic setting is dual to an abstract cylinder functor, in the sense of Kan [16]. A cocylinder is adequate to studying algebras for a monad, while a cylinder would be suitable for coalgebras over a comonad.

A *category with path* (or cocylinder) *functor*, also called P0-category, is thus a category **A** equipped with an endofunctor $P\colon \mathbf{A} \to \mathbf{A}$ and three natural transformations, called *lower face* (∂^-, or also ∂^0), *upper face* (∂^+, or also ∂^1), *degeneracy* (e)

(1) $\partial^\varepsilon\colon P \to 1, \qquad e\colon 1 \to P, \qquad \partial^\varepsilon.e = 1 \qquad (\varepsilon = -, +).$

A *cubical comonad* [7, 8], over the category **A** has also two *connections* (g^-, g^+), satisfying

(2) $g^\varepsilon\colon P \to P^2 \qquad (\varepsilon = -, +)$

(3) $g^\varepsilon.e = Pe.e = eP.e, \qquad P\partial^\varepsilon.g^\varepsilon = 1 = \partial^\varepsilon P.g^\varepsilon$

$P\partial^\varepsilon.g^\eta = e.\partial^\varepsilon = \partial^\varepsilon P.g^\eta \qquad (\varepsilon \neq \eta).$

A *symmetric* cubical comonad also has an *interchange* $s\colon P^2 \to P^2$, satisfying

(4) $s.s = 1, \qquad s.Pe = eP, \qquad P\partial^\varepsilon.s = \partial^\varepsilon P, \qquad s.g^\varepsilon = g^\varepsilon.$

In a P0-category, a *homotopy* $\alpha\colon f \to g$ between parallel maps $f, g\colon A \to B$ is defined in the obvious way, as determined by a map

(5) $\hat{\alpha}\colon A \to PB, \qquad \partial^- \hat{\alpha} = f, \qquad \partial^+ \hat{\alpha} = g;$

every map has a trivial homotopy (or vertical identity), and there is a *reduced horizontal composition* of morphisms and homotopies (for $a\colon A' \to A$, $b\colon B \to B'$)

(6) $0_f\colon f \to f, \qquad (0_f)\hat{} = eB.f = Pf.eA\colon A \to PB$

(7) $(b \circ \alpha \circ a)\colon bfa \to bga, \qquad (b \circ \alpha \circ a)\hat{} = Pb.\hat{\alpha}.a\colon A' \to PB'.$

We write $\delta = \delta A\colon \partial^- \to \partial^+\colon PA \to A$ the *structural* homotopy, represented by 1_{PA}; for every homotopy α, the representing map $\hat{\alpha}\colon A \to PB$ (often written α) is thus the only one such that $\alpha = \delta B \circ \hat{\alpha}$. The functor P can be extended to homotopies, *by the interchange* s: given $\alpha\colon f_0 \to f_1\colon A \to B$, take (and here α should not be confused with its representative map $\hat{\alpha}$)

(8) $P(\alpha) = s.P\hat{\alpha} \qquad (\partial^\varepsilon P.P(\alpha) = P\partial^\varepsilon.P\hat{\alpha} = Pf_\varepsilon)$

If, moreover, the path functor has a *reversion* $r\colon P \to P$

(9) $r.r = 1, \qquad r.e = e, \qquad \partial^-.r = \partial^+$

$g^-.r = Pr.rP.g^+, \qquad Pr.s = s.rP$

then **A** acquires an involutive vertical reversion of homotopies, preserved by P

(10) $(-\alpha)\hat{} = r.\hat{\alpha}, \qquad P(-\alpha) = s.Pr.P\hat{\alpha} = rP.s.P\hat{\alpha} = -P(\alpha).$

In **Top**, the standard path endofunctor is $P(X) = X^{[0, 1]}$, endowed with the compact-open topology (and right adjoint to the standard cylinder endofunctor, $I(X) = [0, 1] \times X$). It is a cubical comonad with interchange and reversion; faces and degeneracy are obvious; connections and symmetries are produced by the natural structure of the unit interval [0, 1] as a complemented lattice

(11) $g_0^-(t, t') = t \vee t'$, $g_0^+(t, t') = t \wedge t'$, $s_0(t, t') = (t', t)$, $r_0(t) = 1-t$.

(More precisely, [0, 1] should be viewed as a commutative, involutive *dioid* [7]: the latter term denotes a set equipped with two monoid structures, the unit of any operation being an absorbant element for the other; plainly, any lattice with 0 and 1 is a commutative dioid. A monoid is to a dioid what an augmented simplicial set is to a "cubical set with connections"; the latter were introduced and studied by Brown and Higgins [3].)

In every P0-category, the faces of P^n are indexed as $\partial_i^\varepsilon = P^{n-i}\partial^\varepsilon P^{i-1}: P^n \to P^{n-1}$, consistently with the usual face-maps of the standard cubes in **Top**

(12) $\delta_i^\varepsilon: [0, 1]^{n-1} \to [0, 1]^n$
$(t_1, ... t_{n-1}) \mapsto (t_1, ... , t_{i-1}, \varepsilon, ... t_{n-1})$ $(\varepsilon = 0, 1)$.

The faces of P^2, $\partial_1^\varepsilon = P\partial^\varepsilon$ and $\partial_2^\varepsilon = \partial^\varepsilon P$, are respectively called *vertical* and *horizontal*. In **Top**, $\partial^\varepsilon P$ takes a parametrised square $\Gamma: [0, 1]^2 \to X$ to its horizontal edge $\Gamma(-, \varepsilon)$.

1.2. Sum. Let **A** be a P0-category, with $(P, \partial^\varepsilon, e)$. A *sum* (or vertical composition) for P is a natural transformation k which allows one to define the sum of (vertically consecutive) homotopies.

First, for every object A, we assume the existence of the *composition pullback* $QA = PA \times_A PA$, or *object of composable pairs of paths*, or Q-*pullback* of A

$$(1) \qquad \begin{array}{ccc} QA & \xrightarrow{k^-} & PA \\ {\scriptstyle k^+}\downarrow & & \downarrow{\scriptstyle \partial^+} \\ PA & \xrightarrow{\partial^-} & A \end{array}$$

with projections $k^\varepsilon: QA \to PA$ (k^- yields the first path of the pair) and three faces $QA \to A$, namely $\partial^{--} = \partial^- k^-$, $\partial^{\pm} = \partial^+ k^- = \partial^- k^+$, $\partial^{++} = \partial^+ k^+$.

Second, we assume there is a given natural transformation $k: Q \to P$, *sum* or *vertical composition*, satisfying the following axioms (the last two are assumed in the presence of an interchange $s: P^2 \to P^2$, or of a reversion $r: P \to P$, respectively)

(2) $\partial^- k = \partial^- k^-$, $\partial^+ k = \partial^+ k^+$, $ke_Q = e$ $(kP.s' = s.Pk,\quad kr_Q = rk)$

where $e_Q: A \to QA$ is induced by e, while $s': PQA \to QPA$ and $r_Q: QA \to QA$ are (possibly) induced by s and r; note that s' is iso if and only if P preserves the composition pullbacks.

The *sum*, or *vertical composition*, $\alpha+\beta: f \to h$ (also written $\alpha +_v \beta$), of a pair of *vertically consecutive* homotopies $\alpha: f \to g$ and $\beta: g \to h$, is then represented by the map

(3) $\alpha + \beta = kB.(\alpha \wedge \beta): A \to PB$ $(k^-.(\alpha \wedge \beta) = \alpha,\quad k^+.(\alpha \wedge \beta) = \beta)$

where $\alpha \wedge \beta$ denotes the obvious morphism with values in the pullback QB. This operation will acquire a "lax-regular" behaviour under additional structure (1.4). Note for now that a vertical identity $0_f: f \to f$ need not be a strict identity for the sum (even in **Top**); but we do have $0_f + 0_f = 0_f$ (and $-0_f = 0_f$), because of (2).

In particular, k^- and k^+ are (represent) consecutive homotopies, with vertical composition k

(4) $k^-: \partial^{--} \to \partial^{\pm}: QA \to A, \qquad k^+: \partial^{\pm} \to \partial^{++}: QA \to A$

$k^- + k^+ = k: \partial^{--} \to \partial^{++}.$

Q and P are not isomorphic, in general (see Section 4, for chain complexes). But in **Top** they are, and the (usual) vertical composition can be realised by taking $Q = P$ and $k = 1$, with k^- and k^+ produced by the embeddings of $[0, 1]$ into itself, as its first or second half

(5) $k_0^\varepsilon: [0, 1] \to [0, 1], \qquad k_0^-(t) = t/2, \qquad k_0^+(t) = (t+1)/2.$

(More "literally", but equivalently, QA can also be realised as the space of pairs of consecutive paths $\{(\gamma, \gamma') \in PA{\times}PA \mid \gamma(1) = \gamma'(0)\}$, with $k(\gamma, \gamma') = \gamma{*}\gamma'$, the usual path-concatenation.)

1.3. Double homotopies and 2-homotopies. Let **A** be equipped with a symmetric cubical comonad $P = (P, \partial^\varepsilon, e, g^\varepsilon, s)$. A map $\hat{\Phi}: A \to P^2B$ is considered as representing a *double homotopy* (or deformation) Φ of four homotopies, its *horizontal* and *vertical* faces (1.1)

(1) $\hat{\alpha} = \partial^-P.\hat{\Phi}, \qquad \hat{\beta} = \partial^+P.\hat{\Phi}, \qquad \hat{\rho} = P\partial^-.\hat{\Phi}, \qquad \hat{\sigma} = P\partial^+.\hat{\Phi}$

(2)
$$\begin{array}{ccc} k & \xrightarrow{\beta} & g \\ \rho\uparrow & \Phi & \uparrow\sigma \\ f & \xrightarrow[\alpha]{} & h \end{array} \qquad\qquad \begin{array}{ccc} f & \xrightarrow{\beta} & g \\ 0\uparrow & \Phi & \uparrow 0 \\ f & \xrightarrow[\alpha]{} & g \end{array}$$

arranged as in the left-hand square above. Again, we tend to use the same name for Φ and $\hat{\Phi}$.

Φ is a 2-*homotopy* (or cell-homotopy) $\Phi: \alpha \to \beta: f \to g$ if its vertical faces are trivial, as in the right-hand square above; then we write $\alpha \sim \beta$ (or $\alpha \simeq_2 \beta$).

Assume now that P has a sum $k: Q \to P$. And note that a double homotopy $\Phi: A \to P(PB)$ also represents a *vertical* homotopy $\Phi_v: \alpha \to \beta$ between its horizontal faces $\partial^\varepsilon P.\Phi$, as *maps* with values in PB. Given two double homotopies Φ_0 and Φ_1, vertically consecutive as in (3), one can thus define their vertical sum (or vertical pasting) $\Psi = \Phi_0 +_v \Phi_1$ along the common horizontal face α, through the sum of the associated vertical homotopies

(3)
$$\begin{array}{ccc} \bullet & \xrightarrow{\alpha_1} & \bullet \\ \sigma_0\uparrow & \Phi_1 & \uparrow\sigma_1 \\ \bullet & \xrightarrow{\alpha} & \bullet \\ \rho_0\uparrow & \Phi_0 & \uparrow\rho_1 \\ \bullet & \xrightarrow[\alpha_0]{} & \bullet \end{array} \qquad\qquad \begin{array}{ccc} QPB & \xrightarrow{k^-P} & P^2B \\ k^+P\downarrow & & \downarrow\partial^+P \\ P^2B & \xrightarrow[\partial^-P]{} & PB \end{array}$$

The vertical sum is realised by the Q-pullback of PB, represented above, via $kP: QPB \to P^2B$

(4) $\Psi = \Phi_0 +_v \Phi_1 = kP.(\Phi_0 \wedge_v \Phi_1): A \to P^2B, \qquad \Psi_v = (\Phi_0)_v + (\Phi_1)_v: \alpha_0 \to \alpha_1$

and the vertical faces preserve the vertical sum: $P\partial^\varepsilon.\Psi = \rho_\varepsilon + \sigma_\varepsilon$. Symmetrically, one defines the horizontal sum along an intermediate, common vertical face

(5) $s.(\Phi_0 +_h \Phi_1) \;=\; s.\Phi_0 +_v s.\Phi_1$.

Clearly, 2-homotopies are closed under vertical and horizontal sum (since $0_f + 0_f = 0_f$).

Assume now that the *composition pullbacks are preserved by* P (which is necessarily true whenever P has a left adjoint, the cylinder functor). Then the horizontal sum of double homotopies can be constructed directly, by the P-image of the Q-pullback of B

(6)
$$\begin{array}{ccc} PQB & \xrightarrow{Pk^+} & P^2B \\ {\scriptstyle Pk^-}\downarrow & & \downarrow{\scriptstyle P\partial^+} \\ P^2B & \xrightarrow[]{P\partial^-} & PB \end{array} \qquad\qquad \Phi_0 +_h \Phi_1 \;=\; Pk.(\Phi_0 \wedge_h \Phi_1)$$

because $s.Pk.(\Phi_0 \wedge_h \Phi_1) = kP.(s\Phi_0 \wedge_v s\Phi_1) = s\Phi_0 +_v s\Phi_1$. Moreover, the exchange property for horizontal and vertical sums of double homotopies holds [9].

1.4. P4-categories. We introduce now some additional structure to ensure that **A**, equipped with the homotopies ($\alpha\colon f \to g$), their operations ($b\circ\alpha\circ a$, 0_f, $-\alpha$, $\alpha+\beta$), and the second-order homotopy relation ($\alpha \sim \beta$) is a sort of "laxified 2-category". In other words, $\sim$ is an equivalence relation, compatible with the operations; the sum is $\sim$-associative, has $\sim$-identities and $\sim$-inverses and satisfies up to cell-homotopy an obvious "reduced exchange law" with the reduced horizontal composition $\circ$ (precisely, **A** satisfies the axioms of an "h4-category", as proved in [9], thm. 2.9). Consequently **A**/$\sim$ (same objects, same morphisms and *tracks* $[\alpha]\colon f \to g$ as 2-cells) is an ordinary 2-category, with invertible 2-cells. Note, however, *that we want to work in* **A**: a comma square there just yields a *weak* comma square in **A**/$\sim$.

Precisely, we want a mapping $\alpha \mapsto \Phi$ providing a homotopy α with a 2-homotopy $\Phi\colon \alpha \sim \alpha + 0$, and a mapping $\Phi \mapsto \Psi$ turning a double homotopy Φ into a 2-homotopy $\Psi\colon \alpha+\sigma \to \rho+\beta$

(1)
$$\begin{array}{ccc} k & \xrightarrow{\beta} & g \\ {\scriptstyle\rho}\uparrow & \Phi & \uparrow{\scriptstyle\sigma} \\ f & \xrightarrow[\alpha]{} & h \end{array} \qquad\qquad \begin{array}{ccc} f & \xrightarrow{\rho+\beta} & g \\ {\scriptstyle 0}\uparrow & \Psi & \uparrow{\scriptstyle 0} \\ f & \xrightarrow[\alpha+\sigma]{} & g \end{array}$$

and again we want to realise such conversions "algebraically", through operations of P.

By definition, a P4-*category* **A** has a symmetric cubical comonad P with reversion and sum, combined with a *zero collapse* $z\colon P \to P^2$ and a *lens collapse* $w\colon P^2 \to P^2$ with the following boundary and degeneracy conditions

(2)
$$\begin{array}{ccc} \partial^- & \xrightarrow{\delta+0} & \partial^+ \\ {\scriptstyle 0}\uparrow & z & \uparrow{\scriptstyle 0} \\ \partial^- & \xrightarrow[\delta]{} & \partial^+ \end{array} \qquad\qquad \begin{array}{ccc} \partial^-.\partial^-P & \xrightarrow{P\partial^-+\partial^+P} & \partial^+.\partial^+P \\ {\scriptstyle 0}\uparrow & w & \uparrow{\scriptstyle 0} \\ \partial^-.\partial^-P & \xrightarrow[\partial^-P+P\partial^+]{} & \partial^+.\partial^+P \end{array}$$

(3) $ze \;=\; e_2, \qquad we_2 \;=\; e_2$

where $\delta: \partial^- \to \partial^+$ is the structural homotopy of PA ($\hat{\delta} = 1_{PA}$) and $e_2 = eP.e = Pe.e: 1 \to P^2$ is the degeneracy of P^2. These maps z and w produce the conversions we want, $\alpha \mapsto z.\alpha$ and $\Phi \mapsto \Psi = w.\Phi$.

We say that the lens collapse is *strong*, and that **A** is *strongly* P4, if moreover all these *lens conversions* $\Phi \mapsto w\Phi$ are bijective. In other words, for every 4-tuple of homotopies $\alpha: f \to h$, $\rho: f \to k$, $\beta: k \to g$, $\sigma: h \to g$ (between parallel maps) and every 2-homotopy $\Psi: \alpha+\sigma \to \rho+\beta$, there is precisely one double homotopy Φ with boundary as in (1) such that $w\Phi$ coincides with Ψ.

Top is strongly P4, by means of the zero collapse z defined by the map z_0

(4) $z: PA \to P^2A,$ $\qquad\qquad z(\gamma) = \gamma.z_0: [0, 1]^2 \to A$

$\quad z_0: [0, 1]^2 \to [0, 1],$ $\qquad\qquad z_0(t, t') = (2t / (2 - t')) \wedge 1$

and of the lens collapse w defined by the endomap w_0 of the unit square

(5) $w: P^2A \to P^2A,$ $\qquad\qquad w(\Gamma) = \Gamma.w_0: [0, 1]^2 \to A$

$\quad w_0: [0, 1]^2 \to [0, 1]^2,$ $\qquad\qquad w_0(t, t') = (t - t{*}t', t + t{*}t')$

$\quad t{*}t' = (1 - |2t - 1|)(2t' - 1) / 2.$

The term "lens collapse" is motivated by the graphic representation of w_0 (cf. [9], Introduction). It may also be useful to see the realisation of w for chain complexes, which is simpler (4.3.3).

1.5. Regular P4-categories. Actually, one can derive a "similar" conversion $\Phi \mapsto \Psi$ from the connections, pasting Φ with two double homotopies based on its boundary and g^-, g^+

$$(1) \qquad \begin{array}{ccccccc} f & \xrightarrow{\rho} & k & \xrightarrow{\beta} & g & \xrightarrow{0} & g \\ {\scriptstyle 0}\uparrow & g^+\rho & \uparrow{\scriptstyle \rho} & \Phi & \uparrow{\scriptstyle \sigma} & g^-\sigma & \uparrow{\scriptstyle 0} \\ f & \xrightarrow[0]{} & f & \xrightarrow[\alpha]{} & h & \xrightarrow[\sigma]{} & g \end{array}$$

Often *this conversion fails to be bijective* and we have to construct a different one (as above, for **Top**). But in the "regular" case exposed below (containing chain complexes) the solution (1) works well, and we can simplify the structure omitting w and z.

A *regular* P4-category, or P4-sesquigroupoid $(P, \partial^\varepsilon, e, g^\varepsilon, s, r, k)$ is a symmetric cubical comonad over **A** with *regular* sum and reversion: this means that each set of homotopies $\mathbf{A}_1(A, B)$ (between maps from A to B) is a groupoid under the sum, with inverses supplied by reversion. Then, also the vertical and horizontal sums of double homotopies (which can be reduced to sums of homotopies, by 1.3.4-5) are groupoid laws. Moreover, **A** is *strongly P4*, with zero and lens collapses derived from the other transformations

(2) $z = eP: P \to P^2$

(3) $w = g^+.P\partial^- +_h 1 +_h g^-.P\partial^+ : P^2 \to P^2.$

In fact, direct calculations show that these maps do satisfy our boundary and coherence requirements (1.4.2-3); further, the conversion defined by w works precisely as in diagram (1)

(4) $\Phi \mapsto \Psi = w\Phi = g^+\rho +_h \Phi +_h g^-\sigma$ $\qquad\qquad (\rho = P\partial^-.\Phi, \quad \sigma = P\partial^+.\Phi)$

and is bijective (in the sense specified above, 1.4): given Ψ, take $\Phi = -_h g^+\rho +_h \Psi -_h g^-\sigma$.

Chain complexes, **Cat** and **Gpd** have a regular P4-structure (Sections 4, 3). On the other hand, **Top** is strongly P4, but not regular.

Our definition of a *regular* P4-category is still redundant, as the reversion is here determined by the sum, and each connection determines the other via the reversion (1.1.9). However, this definition leads to a simple presentation of the axioms, and is also suited for a "lax extension" to the homotopy structure of Stasheff's strongly homotopy associative algebras [21], as presented in [10].

1.6. Homotopy pullbacks. Let **A** be a P0-category (more generally, it would be sufficient to have a "category with homotopies", or h-category [6]).

The *h-pullback* (or standard homotopy pullback, or double mapping cocylinder) of two converging arrows $f: A \to C$, $g: B \to C$ is an object $P(f, g)$ with two maps x', x'' and a homotopy $\xi: fx' \to gx''$ as in (1), satisfying the obvious universal property (of comma squares)

(1)
$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle x'}\uparrow & \searrow{\scriptstyle \xi} & \uparrow{\scriptstyle g} \\ P(f, g) & \xrightarrow[x'']{} & B \end{array}$$

- for every $y': Y \to A$, $y'': Y \to B$ and every homotopy $\eta: fy' \to gy'': Y \to C$, there is exactly one morphism $a: Y \to P(f, g)$ such that $y' = x'a$, $y'' = x''a$, $\eta = \xi\circ a$.

In an h-pullback, the triple $(x', x''; \xi)$ is jointly monic. In **Top**, $P(f, g)$ can be realised as a subspace of the product $A{\times}PC{\times}B$

(2) $P(f, g) = \{(a, \gamma, b) \in A{\times}PC{\times}B \mid \gamma(0) = f(a),\ \gamma(1) = g(b)\}$.

By definition of homotopies, $PA = P(1_A, 1_A)$. Conversely, we show below that h-pullbacks can be constructed through path objects and ordinary pullbacks (as clearly suggested by the topological realisation (2)); the tool is the following *strict pasting* property of h-pullbacks and *ordinary pullbacks*, which is easy to verify

(3)
$$\begin{array}{ccccc} A' & \xrightarrow{h} & A & \xrightarrow{f} & C \\ {\scriptstyle y'}\uparrow & & {\scriptstyle x'}\uparrow & \searrow{\scriptstyle \xi} & \uparrow{\scriptstyle g} \\ Y & \xrightarrow[y'']{} & X & \xrightarrow[x'']{} & B \end{array}$$

- if the triple $(x', x''; \xi)$ is the h-pullback of f and g, then the "pasted" triple $(y', x''y''; \xi\circ y'')$ is the h-pullback of fh and g *if and only if* the left-hand square is an ordinary pullback.

1.7. P4-homotopical categories. A *P0-homotopical* category is a P0-category **A** satisfying the following equivalent conditions (for each pair of arrows (f, g) with the same codomain)

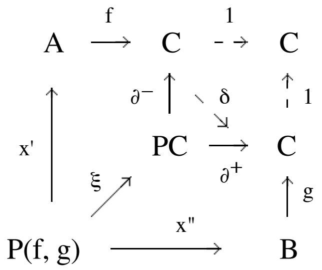
(i) the h-pullback $P(f, g)$ exists (**A** has h-pullbacks),

(ii) the (strict) limit $P(f, g)$ of the following diagram exists (**A** has *cocylindrical limits*)

$$(1) \qquad A \xrightarrow{f} C \xleftarrow{\partial^-} PC \xrightarrow{\partial^+} C \xleftarrow{g} B$$

(iii) the pair of (strict) pullbacks in the left-hand diagram (2) exists (**A** has *cocylindrical pullbacks*)

$$(2) \qquad \begin{array}{ccccc} A & \xrightarrow{f} & C & & \\ \uparrow * & & \uparrow \partial^- & & \\ P^-f & \longrightarrow & PC & \xrightarrow{\partial^+} & C \\ \uparrow * & & & & \uparrow g \\ P(f, g) & & \longrightarrow & & B \end{array}$$



The relations between the presentation of P(f, g) as an h-pullback or a cocylindrical limit are shown in the right-hand diagram (2). The equivalence of (ii) and (iii) is trivial; that of (i) and (iii) follows from the previous pasting property (1.6.3).

Note that the upper pullback in the left-hand diagram (2) yields a particular h-pullback $P^-f = P(f, 1)$, the *lower mapping cocylinder* of f. Also the composition pullback QA is a *particular cylindrical pullback*, since it can be obtained as $P^-(\partial^+) = P(\partial^+, 1)$, or symmetrically as $P(1, \partial^-)$.

Similarly, a *P4-homotopical* category **A** is a *strong* P4-category satisfying the following equivalent conditions, applying to arbitrary pairs of converging arrows (f, g):

(i') h-pullbacks P(f, g) exist and are preserved by P (P(P(f, g)) = P(Pf, Pg)),

(ii') cocylindrical limits exist and are preserved by P (*as limits*),

(iii') cocylindrical pullbacks exist and are preserved by P (*as pullbacks*).

The first characterisation proves (directly) that every power of P has then the same preservation properties. The second and third show that such preservation properties are automatic whenever P is a right adjoint. Since the composition pullbacks QA are a particular instance of the cocylindrical ones, (iii') implies that they are preserved by P; in particular, the exchange property for horizontal and vertical pastings of double homotopies holds (1.3).

1.8. I4- and IP4-categories. Dualising the previous part, an I4-*category* **A** is equipped with a cylinder functor I: **A** → **A** with operations $(\partial^\varepsilon, e, g^\varepsilon, s, r, k, z, w)$, which form a symmetric cubical monad with reversion r, vertical composition k, zero collapse z and lens collapse w

$$(1) \quad \begin{array}{lll} \partial^\varepsilon: 1 \to I, & e: I \to 1, & g^\varepsilon: I^2 \to I \\ s: I^2 \to I^2, & r: I \to I, & k: IA \to JA = IA \underset{A}{+} IA \\ z: I^2 \to I, & w: I^2 \to I^2. & \end{array}$$

Here, k takes values in the *composition pushout* $JA = IA \underset{A}{+} IA$, or J-*pushout* of A, i.e. the pushout of $\partial^+, \partial^-: A \to IA$ (obtained in **Top** by pasting two cylinders, one on top of the other). A *strong* I4-category has a bijective lens conversion $\check{\Phi} \mapsto \check{\Phi}\, w^A$ ($\check{\Phi}: I^2A \to B$).

An *I4-homotopical* category is a strong I4-category having I-preserved h-pushouts I(f, g) of diverging arrows, or equivalently having all cylindrical colimits, preserved by I (*as colimits*)

(2) $$A \xleftarrow{f} C \xrightarrow{\partial^-} IC \xleftarrow{\partial^+} C \xrightarrow{g} B;$$

the latter also include J-pushouts.

Now, an IP0- (resp IP4-) *category* is a category equipped with adjoint endofunctors $I \dashv P$ and with *consistent*, or adjoint, I0- and P0- (resp. I4- and P4-) structures, determining each other by the adjunction as we make precise now.

To begin with, let $(P, \partial^\varepsilon, e)$ be a P0-structure over $\mathbf{A}$ and $I \dashv P$; then one gets for free the unique I0-structure $(I, \bar\partial^\varepsilon, \bar e)$ *consistent* with the original data (and the unique IP0-structure completing them), transforming the original operations of P via the unit and counit of the adjunction $(u: 1 \to PI,\ v: IP \to 1)$ into the *adjoint operations* of I

(3) $\bar\partial^\varepsilon = \partial^\varepsilon I.u = (1 \to PI \to I)$ $\qquad\qquad$ $\bar e = v.Ie = (I \to IP \to 1)$.

The faces of I^n are indexed as $\bar\partial^\varepsilon_i = I^{i-1}\bar\partial^\varepsilon I^{n-i}$, consistently with the adjoint ones for P^n (1.1) and with the usual face-mappings of cubes δ^ε_i in **Top** (1.1.12). The horizontal faces $\partial^\varepsilon_2 = \partial^\varepsilon P$ of P^2 correspond thus to the horizontal faces $\bar\partial^\varepsilon_2 = I\bar\partial^\varepsilon$ of I^2. Diacritic bars will be generally omitted; but we may distinguish the components of the operations of P and I by writing the object as a sub- or superscript, respectively; for instance, $e_A: A \to PA$ and $e^A: IA \to A$.

One proceeds in the same way for connections, reversion, and interchange; for instance

(4) $\bar g^\varepsilon = vI.IvPI.I^2g^\varepsilon I.I^2u = (I^2 \to I^2PI \to I^2P^2I \to IPI \to I)$.

Let now $\mathbf{A}$ be a P4-category with $I \dashv P$, and assume that $\mathbf{A}$ has also all composition-pushouts JA. Then $\mathbf{A}$ is IP4, through the derived operations for I; these are obtained as before, after remarking that $J \dashv Q$ (in order to define the new sum $k: I \to J$). Indeed, the adjunction $(u, v): I \dashv P$ produces precisely one adjunction $(u', v'): J \dashv Q$ consistent with the projections $k^\varepsilon: Q \to P$ and the injections $k^\varepsilon: I \to J$

(5) $u': 1 \to QJ$, $\qquad\qquad$ $k^\varepsilon J.u' = Pk^\varepsilon.u: 1 \to PJ$

$v': JQ \to 1$, $\qquad\qquad$ $v'.k^\varepsilon Q = v.Ik^\varepsilon: IQ \to 1$.

An IP4-category is strongly I4 iff it is strongly P4, since the lens conversion can be equivalently realised through the cylinder $(\check\Phi \mapsto \check\Phi\, w^A)$ or the path functor $(\hat\Phi \mapsto w_B\, \hat\Phi)$.

Finally, an IP0-homotopical category is obviously an IP0-category having all h-pullbacks and h-pushouts, or equivalently having all cocylindrical limits and cylindrical colimits. And an IP4-homotopical category is an IP4-category having all cocylindrical limits and cylindrical colimits (automatically preserved by P and I, respectively); by the previous argument, this reduces to a P4-homotopical category having a left adjoint $I \dashv P$ and all cylindrical colimits.

2. Algebras for a monad

In this section, we consider a monad $T = (T, \eta, \mu)$ over a category $\mathbf{A}$. Then a homotopy structure for $\mathbf{A}$, defined by a path endofunctor P can be lifted to the category $\mathbf{A}^T$ of Eilenberg-Moore algebras over T, provided that P is consistent with T in a natural sense. We refer to Mac Lane's text [18] for the standard theory of monads, their algebras and their relations with adjunctions.

2.1. Lifting functors to algebras. Let $T = (T, \eta, \mu)$ be a monad over the category $\mathbf{A}$; as usual, $\mathbf{A}^T$ denotes the category of (Eilenberg-Moore) T-algebras, $U^T: \mathbf{A}^T \to \mathbf{A}$ the forgetful functor and $F^T: \mathbf{A} \to \mathbf{A}^T$ its left adjoint, the free-algebra functor $(F^T(A) = (TA, \mu A))$.

It is well known that the forgetful functor $U^T: \mathbf{A}^T \to \mathbf{A}$ creates the (existing) limits. For instance, given two morphisms of T-algebras $f_i: (A_i, t_i) \to (B, u)$ $(i = 1, 2)$, if their underlying maps $f_i: A_i \to B$ have a pullback A in $\mathbf{A}$, there is precisely one structure $t: TA \to A$ making the pullback-projections into T-morphisms (determined by $p_i.t = t_i.Tp_i$). And (A, t) is then the pullback of (f_1, f_2) in $\mathbf{A}^T$, with the same projections p_i

$$(1)\qquad \begin{array}{ccc} (A, t) & \xrightarrow{p_1} & (A_1, t_1) \\ {\scriptstyle p_2}\downarrow & * & \downarrow{\scriptstyle f_1} \\ (A_2, t_2) & \xrightarrow{f_2} & (B, u) \end{array} \qquad\qquad p_i.t \; = \; t_i.Tp_i: TA \to A_i .$$

We need now to consider an endomorphism $P: \mathbf{A} \to \mathbf{A}$, typically a path-endofunctor, "consistent" with the monad. But we prefer to deal, more generally, with two monads over two categories, as this distinction appears to clarify things.

In fact, monads are the objects of a 2-category **MON** (which can be formally motivated by viewing a monad as a 2-functor $T: \mathbf{m} \to \mathbf{CAT}$, see 2.8). Given a second monad $S = (S, \eta', \mu')$ over $\mathbf{B}$ (with forgetful functor $U^S: \mathbf{B}^S \to \mathbf{B}$ and free-algebra functor F^S), a (lax) *morphism of monads* $(P, \lambda): T \to S$ is a functor $P: \mathbf{A} \to \mathbf{B}$ equipped with a natural transformation $\lambda = \lambda^P: SP \to PT$ satisfying

$$(2)\quad \lambda.\eta'P \; = \; P\eta, \qquad\qquad \lambda.\mu'P \; = \; P\mu.\lambda_2$$

$$(3)\qquad \begin{array}{ccccc} P & \xrightarrow{\eta'P} & SP & \xleftarrow{\mu'P} & S^2P \\ & {\scriptstyle P\eta}\searrow & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda_2} \\ & & PT & \xleftarrow{P\mu} & PT^2 \end{array} \qquad\qquad (\lambda_2 = \lambda T.S\lambda: S^2P \to PT^2).$$

The morphism (P, λ) is said to be *strong* if λ is an isomorphism, and *strict* if the latter is an identity; it will generally be written P, leaving λ^P understood. The composition with $P': S \to R$ is obvious

$$(4)\quad \lambda^{P'P} \; = \; (P'\circ\lambda^P).(\lambda^{P'}\circ P): \; R.P'P \to P'SP \to P'P.T.$$

A 2-cell, or *natural transformation* $\alpha: P \to Q: T \to S$, is an ordinary natural transformation $\alpha: P \to Q: \mathbf{A} \to \mathbf{B}$ making the following square commute, for any object A

$$(5)\qquad \begin{array}{ccc} SPA & \xrightarrow{S\alpha} & SQA \\ {\scriptstyle \lambda^P}\downarrow & & \downarrow{\scriptstyle \lambda^Q} \\ PTA & \xrightarrow{\alpha T} & QTA \end{array} \qquad\qquad \alpha T.\lambda^P \; = \; \lambda^Q.S\alpha.$$

Any morphism $P: T \to S$ has a canonical lifting $\overline{P}: \mathbf{A}^T \to \mathbf{B}^S$, with $U^S.\overline{P} = PU^T$

(6) $\overline{P}(A, t) = (PA, Pt.\lambda A)$, $\qquad\qquad \overline{P}(f) = P(f)$

and any natural transformation $\alpha: P \to Q$ has a unique lifting $\overline{\alpha}: \overline{P} \to \overline{Q}$, with $U^S.\overline{\alpha} = \alpha U^T$, which will also be written $\alpha: \overline{P} \to \overline{Q}$ as its components are "the same" as the ones of α

(7) $\overline{\alpha}(A, t) = \alpha A: \overline{P}(A, t) \to \overline{Q}(A, t)$.

The lifting respects the various compositions, forming a 2-functor $(-)^{-}$ from **MON** to **CAT** which takes the monad T to its category of algebras $\mathbf{A}^T$. The lifting of adjunctions will be considered later (2.7).

For T = S, an endomorphism of monads $(P, \lambda): T \to T$ will also be called a T-*functor*; it is an endofunctor $P: \mathbf{A} \to \mathbf{A}$ equipped with a natural transformation $\lambda = \lambda^P: TP \to PT$ such that

(8) $\lambda.\eta P = P\eta$, $\qquad \lambda.\mu P = P\mu.\lambda_2$ $\qquad\qquad (\lambda_2 = \lambda T.T\lambda: T^2P \to PT^2)$.

Note that P^2 is a T-functor through $\lambda^{P^2} = (P\circ\lambda^P).(\lambda^P\circ P): T.P^2 \to P^2.T$, whose lifting coincides with $(P^T)^2$.

2.2. Remarks. a) As noted in Johnstone [14] (Lemma 1, attributed to Appelgate's thesis), it is easy to see that, given two monads T and S as above and a mere functor $P: \mathbf{A} \to \mathbf{B}$, there is a bijective correspondence between natural transformations $\lambda: SP \to PT$ satisfying the conditions above (2.1.2, i.e. making P into a morphism of monads) and liftings $\overline{P}: \mathbf{A}^T \to \mathbf{B}^S$ of P. We have already given this correspondence in one direction. Conversely, let such a lifting $\overline{P}$ be given; there is a natural transformation $\hat{\lambda}: F^SP \to \overline{P}F^T$, the transpose of $P\eta: P \to PU^TF^T = U^S\overline{P}F^T$, and $\lambda = U^S\hat{\lambda}: SP \to PT$ satisfies our conditions (2.1.2). The two directions are inverse.

b) It is also relevant to note that an *op(-lax) morphism* of monads can be *extended* to the categories of Kleisli algebras [18], yielding a 2-functor $(-)_{-}$: **MON**' $\to$ **CAT** which takes the monad T to the Kleisli category $\mathbf{A}_T$.

Now, an op-morphism of monads $(K, \varphi): T \to S$ is a functor $K: \mathbf{A} \to \mathbf{B}$ equipped with a natural transformation $\varphi: KT \to SK$ making the appropriate diagram commute ("dual" to 2.1.3)

(1)
$$\begin{array}{ccccc} K & \xrightarrow{\eta' K} & SK & \xleftarrow{\mu' K} & S^2K \\ & {\scriptstyle K\eta}\searrow & \uparrow{\scriptstyle \varphi} & & \uparrow{\scriptstyle \varphi_2} \\ & & KT & \xleftarrow{K\mu} & KT^2 \end{array} \qquad (\varphi_2 = S\varphi.\varphi T: KT^2 \to S^2K).$$

The category of Kleisli algebras $\mathbf{A}_T$ has the same objects as $\mathbf{A}$, a morphism $f_T: A \to A'$ being represented by any $f \in \mathbf{A}(A, TA')$. Its canonical adjunction is

(2) $F_T: \mathbf{A} \to \mathbf{A}_T$, $\qquad F_T(A) = A$, $\qquad F_T(a: A \to A') = (\eta A'.a)_T: A \to A'$

(3) $U_T: \mathbf{A}_T \to \mathbf{A}$, $\qquad U_T(A) = TA$, $\qquad U_T(f_T: A \to A') = \mu A'.Tf: TA \to TA'$.

An op-morphism $(K, \varphi): T \to S$ has a canonical *extension* to Kleisli algebras, $\underline{K}: \mathbf{A}_T \to \mathbf{B}_S$ $(\underline{K}F_T = F_SK)$ defined as follows

(4) $\underline{K}(A) = K(A)$, $\qquad \underline{K}(f_T: A \to A') = (\varphi A'.Kf:\ KA \to KTA' \to SKA')_S$

(5) $\underline{K}F_T(a: A \to A') = \underline{K}(\eta A'.a)_T = (\varphi A'.K\eta A'.Ka)_S = (\eta' KA'.Ka)_S = F_SK(a)$.

c) If $(P, \lambda)\colon T \to T$ is an endomorphism of monads and P is part of a comonad (P, ε, δ) then λ is called a *bialgebra distributivity* in case $\varepsilon T.\lambda = T\varepsilon$ and $\delta T.\lambda = P\lambda.\lambda P.T\delta$, see MacDonald and Stone [19]; a similar case, in which P is part of a monad, is dealt with in Beck ([2], p. 120).

2.3. Monads and path-functors. By definition, a P0-structure over $\mathbf{A}$, *consistent* with our monad T, is given by an ordinary P0-structure

(1) $P\colon \mathbf{A} \to \mathbf{A}$, $\quad \partial^\varepsilon\colon P \to 1$, $\quad e\colon 1 \to P$ $\qquad (\partial^\varepsilon e = 1)$

together with a natural transformation $\lambda\colon TP \to PT$ making P into a T-functor and ∂^ε, e into T-natural transformations (2.1); in other words, the following equations hold

(2) $\lambda.\eta P = P\eta$, $\qquad \lambda.\mu P = P\mu.\lambda T.T\lambda$

$\partial^\varepsilon T.\lambda = T\partial^\varepsilon$, $\qquad eT = \lambda.Te$.

Then $(P, \partial^\varepsilon, e)$ lifts to a P0-structure for algebras, yielding a path endofunctor $P^T = \overline{P}$

(3) $P^T\colon \mathbf{A}^T \to \mathbf{A}^T$, $\qquad P^T(A, t) = (PA, Pt.\lambda)$

whose faces and degeneracy will still be written $\partial^\varepsilon\colon P^T \to 1$, $e\colon 1 \to P^T$. The forgetful functor $U^T\colon \mathbf{A}^T \to \mathbf{A}$ extends obviously to homotopies, double homotopies and 2-homotopies, preserving faces

(4) $(U^T(\alpha))^\wedge = U^T(\hat{\alpha})$, $\qquad (U^T(\Phi))^\wedge = U^T(\hat{\Phi})$.

Recalling that the lifting of P^2 is $(P^T)^2$, the same lifting property holds for a cubical comonad $(P, \partial^\varepsilon, e, g^\varepsilon)$ *consistent* with T; this means that, moreover, the connections have to satisfy

(5) $g^\varepsilon T.\lambda = P\lambda.\lambda P.Tg^\varepsilon$.

Also an interchange $s\colon P^2 \to P^2$ or a reversion $r\colon P \to P$ can be similarly lifted to P^T, provided it satisfies the following condition, respectively

(6) $sT.P\lambda.\lambda P = P\lambda.\lambda P.Ts$ $\qquad rT.\lambda = \lambda.Tr$.

Plainly, the same terminology can be used for a category $\mathbf{C}$ *monadic* over $\mathbf{A}$. This means that $\mathbf{C}$ is equipped with a functor $U\colon \mathbf{C} \to \mathbf{A}$ which has a left adjoint $F\colon \mathbf{A} \to \mathbf{C}$ $((\eta, \vartheta)\colon F \dashv U)$; and that the monad $T = UF\colon \mathbf{A} \to \mathbf{A}$ derived from the adjunction has a comparison functor K which is an isomorphism (so that we can identify $\mathbf{C}$ and $\mathbf{A}^T$)

(7) $K\colon \mathbf{C} \to \mathbf{A}^T$, $\qquad K(C) = (UC, U\vartheta\colon UFUC \to UC)$.

2.4. The functor Q of composable paths. Let P be a cubical comonad over $\mathbf{A}$ consistent with T, and assume that $\mathbf{A}$ has all Q-pullbacks

(1)
$$\begin{array}{ccc} QA & \xrightarrow{k^-} & PA \\ {\scriptstyle k^+}\downarrow & * & \downarrow{\scriptstyle \partial^+} \\ PA & \xrightarrow{\partial^-} & A \end{array}$$

Then the functor $Q\colon \mathbf{A} \to \mathbf{A}$ has a canonical T-structure $\lambda^Q\colon TQ \to QT$, making the following diagram commute (the *-marked square is the Q-pullback of TA)

$$
(2) \qquad
\begin{array}{ccccc}
TQA & & \xrightarrow{Tk^-} & & TPA \\
& \searrow \lambda^Q & & & \downarrow \lambda \\
Tk^+ \downarrow & & QTA & \xrightarrow{k^-T} & PTA \\
& & k^+T \downarrow & * & \downarrow \partial^+T \\
TPA & \xrightarrow[\lambda]{} & PTA & \xrightarrow[\partial^-T]{} & TA
\end{array}
$$

since the conditions

(3) $\lambda^Q.\eta Q \ = \ Q\eta\colon QA \to QTA, \qquad\qquad \lambda^Q.\mu Q \ = \ Q\mu.\lambda^Q T.T\lambda^Q\colon T^2QA \to QTA$

are easily deduced from the analogues for P, by composing with the projections $k^\varepsilon T$ of QTA; for instance

(4) $k^\varepsilon T.(\lambda^Q.\eta Q) \ = \ \lambda.Tk^\varepsilon.\eta Q \ = \ \lambda.\eta P.k^\varepsilon \ = \ P\eta.k^\varepsilon \ = \ k^\varepsilon T.(Q\eta).$

Moreover, the lifted functor $Q^T(A, t) = (QA, Qt.\lambda^Q A)$ is the composition-pullback for algebras, through the lifting of projections $k^\varepsilon\colon Q^T \to P^T$

$$
(5) \qquad
\begin{array}{ccc}
Q^T(A, t) & \xrightarrow{k^-} & P^T(A, t) \\
k^+ \downarrow & & \downarrow \partial^+ \\
P^T(A, t) & \xrightarrow{\partial^-} & (A, t)
\end{array}
$$

since the structure $Qt.\lambda^Q A$ is precisely the one created by U^T over the composition-pullback in $\mathbf{A}$ (2.1.1): $k^\varepsilon.(Qt.\lambda^Q A) \ = \ Pt.k^\varepsilon T.\lambda^Q A \ = \ Pt.\lambda A.Tk^\varepsilon$.

2.5. The remaining second-order structure. In the same way, we say that a P4-category structure $(P, \partial^\varepsilon, e, g^\varepsilon, r, s, k, z, w)$ over $\mathbf{A}$ is *consistent* with the monad T, if P and all the listed natural transformations are so. Then, also by the previous results on the lifting of Q, $\mathbf{A}^T$ is a P4-category, and is strong iff $\mathbf{A}$ is so.

Homotopy pullbacks can be viewed as cocylindrical limits (1.7), and are thus *created* by the forgetful functor $U^T\colon \mathbf{A}^T \to \mathbf{A}$. Finally, we get the following results.

2.6. Theorem. a) Let $\mathbf{A}$ be a category with a monad T. A structure of P0-, or P4-, or P0-homotopical, or P4-homotopical category over $\mathbf{A}$, made *consistent* with T by a natural transformation $\lambda\colon TP \to PT$ (as specified above, 2.3-5) can always be lifted to a structure of the same type over the category of algebras, with cocylinder

(1) $P^T\colon \mathbf{A}^T \to \mathbf{A}^T, \qquad\qquad P^T(A, t) \ = \ (PA, Pt.\lambda).$

b) If $\mathbf{A}$ is a P4-homotopical category consistent with T, P^T has a left adjoint I^T and $\mathbf{A}^T$ has all cylindrical colimits, then $\mathbf{A}^T$ is IP4-homotopical.

c) If $\mathbf{A}$ is a P0-category consistent with a monad T, $\mathbf{A}^T$ has coequalizers and P has a left adjoint I, then P^T has a left adjoint I^T, which *extends* I, in the sense that $I^T F^T \cong F^T I$ (of course, I^T is not a lifting of I, generally; see for instance the case of topological semigroups in 3.2).

Proof. a) and b) just summarise the results of this Section, together with the previous ones on IP4-homotopical categories (1.8); c) is a particular case of the following well-known theorem, which can be found in [14] (Thm. 2), together with various references to its earlier appearances.

2.7. Theorem (Lifting adjunctions to algebras). Consider a morphism of monads $P = (P, \lambda)\colon T \to S$ over the categories $\mathbf{A}$, $\mathbf{B}$ (2.1). Assume that the underlying functor $P\colon \mathbf{A} \to \mathbf{B}$ has a left adjoint $I\colon \mathbf{B} \to \mathbf{A}$ and the category of algebras $\mathbf{A}^T$ *has coequalizers*. Then, the lifted functor $\mathbf{P}\colon \mathbf{A}^T \to \mathbf{B}^S$ ($\mathbf{P}(A, t) = (PA, Pt.\lambda A)$) has a left adjoint $\mathbf{I}\colon \mathbf{B}^S \to \mathbf{A}^T$, which extends I with respect to the free-algebra functors ($\mathbf{I}F^S \cong F^T I$).

(In particular, if $P\colon \mathbf{A} \to \mathbf{A}$ is the identity, $\lambda\colon S \to T$ is a *morphism of monads* (or triples) over $\mathbf{A}$, i.e. $\lambda.\eta^S = \eta^T$ and $\lambda.\mu^S = \mu^T.\lambda_2$; this case can also be found in Beck [2], p. 119.)

Proof. The proof is outlined in [14], Thm. 2. We give a detailed argument here along the same lines.

First, let us note that, if $\mathbf{P}$ has a left adjoint $\mathbf{I}$, then $\mathbf{I}F^S$ and $F^T I$ are both left adjoints to $U^S\mathbf{P} = PU^T$, hence canonically isomorphic.

Recall that $\lambda = \lambda^P\colon SP \to PT$ satisfies the equations $\lambda.\eta'P = P\eta$, $\lambda.\mu'P = P\mu.\lambda T.S\lambda$. Write the unit and counit of $I \dashv P$ as $u\colon 1_{\mathbf{B}} \to PI$ and $v\colon IP \to 1_{\mathbf{A}}$, and the free-algebra adjunctions of $T = (T, \eta, \mu)$ and $S = (S, \eta', \mu')$ as

(1) $F^T : \mathbf{A} \rightleftarrows \mathbf{A}^T : U^T$, $\quad U^T(A, t) = A$, $\quad F^T(A) = (TA, \mu A)$

$\eta\colon 1 \to T = U^T F^T$, $\quad \varepsilon\colon F^T U^T \to 1$; $\quad U^T\varepsilon(A, t) = t\colon TA \to A$;

(2) $F^S : \mathbf{B} \rightleftarrows \mathbf{B}^S : U^S$, $\quad \eta'\colon 1 \to S = U^S F^S$, $\quad \varepsilon'\colon F^S U^S \to 1$.

To construct $\mathbf{I}$, note that the functor I, because of the adjunction $I \dashv P$, inherits a natural transformation $\lambda^*\colon IS \to TI$, often called the "mate" of $\lambda\colon SP \to PT$ (by composing unit, λ, counit)

(3) $\lambda^* = vTI.I\lambda I.ISu\colon\ IS \to ISPI \to IPTI \to TI$

and let us record the fact that

(4) $P\lambda^*.uS = PvTI.PI(\lambda I.Su).uS = PvTI.uPTI.\lambda I.Su = \lambda I.Su.$

Now, λ^* makes I into an *op*-morphism of monads (which would just allow us to extend I to free algebras, according to 2.2 b). However (since every algebra is a coequaliser of free ones, and left adjoints have to preserve the existing colimits), we get the value of $\mathbf{I}$ over the S-algebra $(B, s\colon SB \to B)$ as the coequaliser in $\mathbf{A}^T$ of the following two maps of free T-algebras

(5) $F^T I U^S\varepsilon'(B, s) = F^T Is\colon\ F^T IS(B) = (TISB, \mu ISB) \to (TIB, \mu IB) = F^T I(B)$

$\bar{\lambda} = \varepsilon F^T IU^S.F^T\lambda^* U^S\colon\ F^T IS(B) \to F^T TI(B) = F^T U^T.F^T I(B) \to F^T I(B)$

$$F^T ISU^S(B, s) \underset{\bar{\lambda}}{\overset{F^T IU^S\varepsilon'}{\rightrightarrows}} F^T IU^S(B, s) \xrightarrow{\ p\ } \mathbf{I}(B, s)$$

and its structure $\bar{s}\colon TU^T\mathbf{I}(B, s) \to U^T\mathbf{I}(B, s)$ makes p an S-morphism

$$(6)\qquad \begin{array}{ccc} TI(B) & \xrightarrow{U^Tp} & U^T\mathbf{I}(B, s) \\ {\scriptstyle \mu IU^T}\uparrow & & \uparrow{\scriptstyle \bar{s}} \\ T^2I(B) & \xrightarrow{TU^Tp} & TU^T\mathbf{I}(B, s) \end{array}$$

The functor $\mathbf{I}$ is certainly well defined since we can let its value on morphisms be the induced map on coequalizers. And the coequaliser-maps form a natural transformation $p\colon F^TIU^S \to \mathbf{I}$.

We next proceed by giving a unit and counit for the new adjunction. For the unit $\mathbf{u}\colon 1 \to \mathbf{PI}$, we want its underlying transformation $U^S\mathbf{u}\colon U^S \to U^S\mathbf{PI} = PU^T\mathbf{I}$ to be the composite

(7) $U^S\mathbf{u} = PU^Tp.P\eta IU^S.uU^S\colon\ U^S \to PIU^S \to PU^TF^TIU^S \to PU^T\mathbf{I}$;

in fact, for an S-algebra $(B, s\colon SB \to B)$, we do get a $\mathbf{B}^S$-morphism with values in $\mathbf{PI}(B, s)$ (whose structure is $P\bar{s}.\lambda U^T\mathbf{I}(B, s)$, by 2.1.6)

(8) $U^S\mathbf{u}(B, s).s = PU^Tp(B, s).(P\eta I.u)B.s = PU^Tp(B, s).PU^TF^TIs.(P\eta I.u)SB$

$= PU^Tp(B, s).P\mu IB.PU^TF^T\lambda^*B.P\eta ISB.uSB$ (by (5))

$= PU^Tp(B, s).P\mu IB.P\eta TIB.P\lambda^*B.uSB = PU^Tp(B, s).\lambda IB.SuB$ (by (4))

(9) $P\bar{s}.\lambda U^T\mathbf{I}(B, s).SU^S\mathbf{u}(B, s) = P\bar{s}.\lambda U^T\mathbf{I}(B, s).SP(U^Tp(B, s).\eta IB).SuB$

$= P\bar{s}.PT(U^Tp(B, s).\eta IB).\lambda IB.SuB$

$= PU^Tp(B, s).P\mu IB.PT\eta IB.\lambda IB.SuB = PU^Tp(B, s).\lambda IB.SuB$ (by (6)).

For the counit $\mathbf{v}\colon \mathbf{IP} \to 1$, we require that

(10) $\mathbf{v}.p\mathbf{P} = \varepsilon.F^TvU^T\colon\ F^TIU^S\mathbf{P} = F^TIPU^T \to F^TU^T \to 1.$

The solution exists (and is unique), provided we show that the following natural transformations f, g: $F^TIU^SF^SU^S\mathbf{P} \to 1$ coincide

(11) $f = \varepsilon.F^TvU^T.F^TIU^S\varepsilon'\mathbf{P}$, $\qquad g = \varepsilon.F^TvU^T.\varepsilon F^TIU^S\mathbf{P}.F^T\lambda^*U^S\mathbf{P}$.

Applying these to an object (A, t) of $\mathbf{A}^T$ we get a pair of $\mathbf{A}^T$-morphisms; it is sufficient to show that the underlying $\mathbf{A}$-morphisms, $f_0 = U^Tf(A, t)$ and $g_0 = U^Tg(A, t)$ are equal

(12) $f_0 = t.TvA.TIU^S\varepsilon'(PA, Pt.\lambda A) = t.TvA.TIPt.TI\lambda A$

$= t.Tt.TvTA.TI\lambda A = t.\mu A.TvTA.TI\lambda A$

(13) $g_0 = U^Tg(A, t) = t.TvA.U^T\varepsilon F^TIPA.U^TF^T\lambda^*PA$

$= t.TvA.\mu IPA.TvTIPA.TI\lambda IPA.TISuPA$ (by (3))

$= t.TvA.(\mu.TvT.TI\lambda)IPA.TISuPA$

$= t.(\mu.TvT.TI\lambda)A.TISPvA.TISuPA = t.\mu A.TvTA.TI\lambda A.$

Finally, to verify the triangle identities for $\mathbf{u}, \mathbf{v}$, it is sufficient to show that $U^S(\mathbf{Pv}.\mathbf{uP}) = 1$ and $(\mathbf{vI}.\mathbf{Iu}).p = p$; this follows from (7) and (10) and the triangle identities of u, v.

2.8. Formal remarks. The morphisms and 2-cells of monads we have considered above (2.1) are "natural", as soon as we consider the 2-category **m** generated by one object $*$, one arrow $t: * \to *$, and two cells $e: 1 \to t$, $m: t^2 \to t$ subject to the relations

(1) $m.et = 1 = m.te, \qquad m.mt = m.tm$

((t, e, m) can be viewed as a *monad on the object* $*$ *of the 2-category* **m**; see Kelly and Street [17] for further information regarding monads on objects of a 2-category.)

Plainly, a monad (T, η, μ) amounts to a strict 2-functor $\mathbf{T}: \mathbf{m} \to \mathbf{CAT}$, with $T = \mathbf{T}(t)$, $\eta = \mathbf{T}(e)$, $\mu = \mathbf{T}(m)$ (its domain-category being $\mathbf{A} = \mathbf{T}(*)$).

Now, given a second monad S over **B**, a *lax natural transformation of 2-functors* $\mathbf{T} \to \mathbf{S}$: $\mathbf{m} \to \mathbf{CAT}$ amounts precisely to a functor $P: \mathbf{A} \to \mathbf{B}$ (corresponding to the object $*$) and a natural transformation $\lambda: SP \to PT$ (corresponding to the generating arrow $t: * \to *$ of **m**) satisfying our conditions (2.1.2), i.e. a lax morphism $(P, \lambda): T \to S$. Similarly, a *modification* $\alpha: P \to Q: \mathbf{T} \to \mathbf{S}$ of lax natural transformations amounts to an ordinary natural transformation $\alpha: P \to Q: \mathbf{A} \to \mathbf{B}$ (corresponding to the object $*$) satisfying the appropriate condition (2.1.5).

Equivalently, one could also view a monad as a *lax functor* $\mathbf{T}: \mathbf{1} \to \mathbf{CAT}$, consisting of a category $\mathbf{A} = \mathbf{T}(*)$, an endofunctor $T = \mathbf{T}(1_*)$ and two natural transformations

(2) $\eta: 1_{\mathbf{T}(*)} \to \mathbf{T}(1_*), \qquad \mu: \mathbf{T}(1_*).\mathbf{T}(1_*) \to \mathbf{T}(1_*.1_*)$

under conditions coinciding with the axioms of monads. Lax natural transformations of such lax functors, and their modifications, would give the same notions as above.

3. Applications to topological spaces and small categories

The previous theory is applied to derive the homotopy structure of some types of algebras over spaces (topological semigroups, monoids and groups) and categories (strict monoidal categories). *Equivariant homotopy*, for spaces equipped with an action of a fixed topological group, is considered in 3.4; see Cordier and Porter [4], Moerdijk and Svensson [20].

3.1. Topological semigroups and monoids. The category **Top** of topological spaces is IP4-homotopical, with respect to the standard cylinder and path functors $I \dashv P$ (1.1), with the usual operations recalled above (connections and symmetries in 1.1; vertical composition in 1.2) plus the zero collapse and the lens collapse described in 1.4. We recall now that the category Sgr**Top** of topological semigroups is monadic over **Top** (a category of algebras), and we show below (3.2) that the IP4-homotopical structure lifts to Sgr**Top**.

Consider the forgetful functor $U: \mathrm{Sgr}\mathbf{Top} \to \mathbf{Top}$ and its left adjoint F

(1) $FX = (\Sigma_{n>0} X^n, *) \qquad (x_1,\dots x_p) * (x_{p+1},\dots x_n) = (x_1,\dots x_n)$

$\eta: X \subset UFX, \qquad \vartheta: FUA \to A, \qquad \vartheta(x_1,\dots x_n) = x_1 \cdot \dots \cdot x_n$

noting that FX is the free semigroup over the underlying set $|X|$, endowed with the sum of the product topologies (the finest topology making all the embeddings $X^n \subset FX$ continuous). FX is a topological semigroup, since the juxtaposition $*: X^p \times X^q \to X^{p+q}$ is plainly continuous and every cartesian product in **Top** distributes over arbitrary sums (whence $TX \times TX = \Sigma_{p,q>0} X^p \times X^q$). It follows easily that FX is indeed the free topological semigroup over the space X.

The adjunction produces, in the standard way, a monad over **Top** (the "free-semigroup" monad for topological spaces)

(2) $T = UF$: **Top** $\to$ **Top**, $\qquad TX = \Sigma_{n>0} X^n$

$\mu = U\vartheta F: T^2 \to T, \qquad \mu((x_{11},\dots x_{1p_1}),\dots (x_{n1},\dots x_{np_n})) = (x_{11},\dots x_{np_n})$.

A T-algebra (X, t) is "the same" as a topological semigroup $(X, \cdot)$ with multiplication $\cdot = t_2: X^2 \to X$, and a map of T-algebras is a continuous homomorphism. We identify $\mathbf{Top}^T$ = Sgr**Top**.

Similarly, the category Mon**Top** of *topological monoids* is monadic over **Top**, and inherits a P4-homotopical structure. One uses now the "free-monoid" monad for **Top**, with $TX = \Sigma_{n\geq 0} X^n$.

3.2. The homotopy structure of topological semigroups. First, the cocylinder P: **Top** $\to$ **Top** preserves powers (as a right adjoint) and also sums. It is a strong T-functor, as proved by the following relations (for $\gamma, \gamma_i, \gamma_{ij} \in PX$)

(1) $\lambda: TP \to PT, \qquad \lambda X: \Sigma_{n>0} (PX)^n \cong P(\Sigma_{n>0} X^n)$

$(\gamma_1,\dots \gamma_n) \mapsto \langle\gamma_1,\dots \gamma_n\rangle: [0, 1] \to X^n$

(2) $\lambda.\eta P(\gamma) = \gamma = P\eta(\gamma)$

$\lambda.\mu P((\gamma_{11},\dots \gamma_{1p_1}), \dots (\gamma_{n1},\dots \gamma_{np_n})) = \lambda(\gamma_{11},\dots \gamma_{np_n}) = \langle\gamma_{11},\dots \gamma_{np_n}\rangle =$

$= P\mu (\langle\langle\gamma_{11},\dots \gamma_{1p_1}\rangle,\dots \langle\gamma_{n1},\dots \gamma_{np_n}\rangle\rangle) = P\mu.\lambda T.T\lambda((\gamma_{11},\dots \gamma_{1p_1}),\dots (\gamma_{n1},\dots \gamma_{np_n}))$.

P can thus be canonically lifted to topological semigroups

(3) $P^T: \mathbf{Top}^T \to \mathbf{Top}^T, \qquad P^T(X, t) = (PX, Pt.\lambda)$

which simply means that $P^T(X, \cdot)$ is the path-space $PX = X^{[0, 1]}$ with the pointwise multiplication $\gamma\cdot\gamma'(\tau) = \gamma(\tau)\cdot\gamma'(\tau)$.

Moreover, all the operations of the P4-structure of **Top** are T-transformations. Leaving apart, for the moment, $k: Q \to P$, each of the remaining transformations ∂□$^{\varepsilon}$, e, g^{ε}, r, s, z, w is defined by some continuous function between powers of the unit interval, via contravariant composition

(4) $f_0: [0, 1]^q \to [0, 1]^p$

$fX: P^pX \to P^qX, \qquad (\Gamma: [0, 1]^p \to X) \mapsto (\Gamma f_0: [0, 1]^q \to X)$

and *all such transformations are consistent with* λ, i.e. $fT.\lambda^{P^p} = \lambda^{P^q}.Tf$. In fact, after noting that the transformation making P^q a T-functor is

(5) $\lambda^{P^q} = P^{q-1}\lambda. \dots .P\lambda P^{q-2}.\lambda P^{q-1}: T.P^q \to P^q.T$

$\lambda^{P^q}X: \Sigma_{n>0} (P^qX)^n \cong P^q (\Sigma_{n>0} X^n), \qquad (\Gamma_1,\dots \Gamma_n) \mapsto \langle\Gamma_1,\dots \Gamma_n\rangle: [0, 1]^q \to X^n$

the property above is an easy consequence

(6) $fT.\lambda^{P^p} (\Gamma_1,\dots \Gamma_n) = fT \langle\Gamma_1,\dots \Gamma_n\rangle = \langle\Gamma_1 f_0,\dots \Gamma_n f_0\rangle = \lambda^{P^q} (\Gamma_1 f_0,\dots \Gamma_n f_0)$

$= \lambda^{P^q}.Tf (\Gamma_1,\dots \Gamma_n)$.

Finally, recall our choice of $Q = P$ for **Top** (1.2), with $k^\varepsilon: Q \to P$ produced by embeddings $k_0^\varepsilon: [0, 1] \to [0, 1]$ (1.2.5) and $k = 1$. Then the lifting of k^ε to topological semigroups makes $P^T A$ into the composition-pullback of A; now, $k = 1$ obviously lifts.

By 2.6, we have thus proved that $\mathbf{Top}^T = \mathrm{Sgr}\mathbf{Top}$ is IP4-homotopical, with path functor P^T. Its left adjoint cylinder functor I^T can be directly calculated as

(7) $I^T: \mathrm{Sgr}\mathbf{Top} \to \mathrm{Sgr}\mathbf{Top}, \qquad I^T(X, \cdot) = (FIX) / R$

where $IX = [0, 1]{\times}X$ is the topological cylinder and R is the congruence of semigroups over F(IX) spanned by the following relation, based on the multiplication $\cdot$ of X

(8) $(\tau, x){*}(\tau, y) \ R_0 \ (\tau, x{\cdot}y) \qquad (\tau \in [0, 1]; \ \ x, y \in X)$

so that, for every "instant" τ, the mapping $u_\tau: (X, \cdot) \to I^T(X, \cdot)$, $u_\tau(x) = [\tau, x]$ is a continuous homomorphism. The unit of the adjunction is

(9) $u: (X, \cdot) \to P^T I^T(X, \cdot) = P((FIX) / R, *), \qquad u(x): \tau \mapsto u_\tau(x) = [\tau, x].$

3.3. Topological groups. One can follow a similar procedure for topological groups. But concretely, it is simpler to lift the path functor (together with its operations)

(1) $P^T: \mathrm{Gp}\mathbf{Top} \to \mathrm{Gp}\mathbf{Top}$

by letting $P^T(X, \cdot)$ be the path-space $PX = X^{[0, 1]}$ with pointwise multiplication.

The monad procedure gives the same result (as noted in general in 2.2a). Consider the forgetful functor $U: \mathrm{Gp}\mathbf{Top} \to \mathbf{Top}$, whose left adjoint FX is the free group over the underlying set |X|, endowed with the finest group-topology making continuous the unit-embedding $\eta X: X \to UFX$ (the join of all such topologies). Letting $T = UF: \mathbf{Top} \to \mathbf{Top}$ be the associated monad (the "free-group" monad for spaces), we identify $\mathbf{Top}^T = \mathrm{Gp}\mathbf{Top}$. Now, there is a unique continuous homomorphism $\lambda X: TPX \to PTX$ such that $\lambda X.\eta PX = P(\eta X): PX \to PTX$, providing a natural transformation $\lambda: TP \to PT$, which also satisfies the "multiplicative" condition $\lambda.\mu P = P\mu.\lambda T.T\lambda$.

Thus, $\mathbf{Top}^T = \mathrm{Gp}\mathbf{Top}$ is IP4-homotopical, with path functor P^T (2.6). Its left adjoint cylinder functor I^T can be directly calculated as above (3.2.7), using of course a *group*-congruence R.

3.4. Equivariant homotopy. Let G be a topological group and G-**Top** the category of G-spaces, i.e. topological spaces Y equipped with an action $G{\times}Y \to Y$, $(g, y) \mapsto g.y$, satisfying the usual, obvious conditions.

The forgetful functor $U: G\text{-}\mathbf{Top} \to \mathbf{Top}$ has left adjoint

(1) $F(X) = G{\times}X \qquad (\eta: 1 \to UF, \ \ \vartheta: FU \to 1)$

where $G{\times}X$ is the product of topological spaces, with action $g'.(g, x) = (g'.g, x)$. This supplies a monad over **Top**

(2) $T = UF: \mathbf{Top} \to \mathbf{Top}, \qquad TX = G{\times}X$

$\eta X: X \to G{\times}X, \qquad x \mapsto (1, x)$

$\mu = U\vartheta F: T^2 \to T, \qquad \mu X: G{\times}G{\times}X \to G{\times}X, \qquad \mu X(g', g, x) = (g'.g, x)$

and a G-space Y is the same as a T-algebra $(X, t: G \times X \to X)$. The cocylinder $P: \mathbf{Top} \to \mathbf{Top}$ is a T-functor through the degenerate-path embedding $e: G \to PG$

(3) $\lambda: TP \to PT$

$\lambda X = eG \times PX: G \times PX \to P(G \times X), \qquad (g, \gamma) \mapsto \langle e(g), \gamma\rangle: [0, 1] \to G \times X$

(4) $\lambda.\eta P(\gamma) = \lambda(1, \gamma) = \langle e(1), \gamma\rangle = P\eta(\gamma)$

$\lambda.\mu P\,(g', g, \gamma) = \langle e(g'.g), \gamma\rangle = P\mu \langle e(g'), e(g), \gamma\rangle = P\mu.\lambda T.T\lambda\,(g', g, \gamma).$

P can thus be canonically lifted to G-spaces,

(5) $P^T: G\text{-}\mathbf{Top} \to G\text{-}\mathbf{Top}, \qquad P^T(X, t) = (PX, Pt.\lambda)$

which means that $P^T(X, t)$ is the path-space $PX = X^{[0, 1]}$ with the pointwise action

(6) $(g\cdot\gamma)(\tau) = g.(\gamma(\tau)).$

and a homotopy $\alpha: (X, t) \to P^T(Y, u)$ is an *equivariant homotopy*, in the usual sense.

To show the coherence of the P4-structure with λ, one can now procede as for topological semigroups, replacing 3.2.6 with

(7) $fT.\lambda^{P^p}(g, \Gamma) = fT \langle e_p(g), \Gamma\rangle = \langle e_q(g), \Gamma f_0\rangle = \lambda^{P^q}.Tf\,(g, \Gamma).$

$\mathbf{Top}^T = \mathbf{GTop}$ is thus IP4-homotopical (2.6), with path functor P^T. The cylinder functor I^T (derived from 2.7 or directly calculated as left adjoint to P^T), is given by

(8) $I^T: G\text{-}\mathbf{Top} \to G\text{-}\mathbf{Top}, \qquad I^T(X) = (FIX)/R$

where $IX = [0, 1] \times X$ is the topological cylinder and R is the congruence of G-spaces over F(IX) spanned by the following relation, based on the G-action over X

(9) $(g; \tau, x)\ R_0\ (1; \tau, g.x) \qquad (g \in G;\ \tau \in [0, 1];\ x \in X).$

3.5. Algebras for pointed topological spaces. The (pointed) category $\mathbf{Top}^\top$ of pointed topological spaces is IP4-homotopical. As well known, the P-structure comes directly from the one of **Top**, adding to the original path-space PX the constant loop at the base-point

(1) $\mathbf{P}(X, x) = (PX, x^P), \qquad x^P = e_X.x = Px.e_\top: \top \to PX$

while the cylinder **I** is formed by collapsing the subspace I{x} in the non-pointed cylinder IX

(2) $\mathbf{I}(X, x) = (IX/I\{x\}, x^I) \qquad x^I(*) = [t, x].$

($\mathbf{Top}^\top$ itself can be seen as a category of algebras over **Top**, for a monad consistent with P, see Section 5; but we are not interested in this fact here.)

A monoid or a group in $\mathbf{Top}^\top$ is the same as in **Top**, which we have already considered. But a semigroup in $\mathbf{Top}^\top$ is a *topological semigroup* X *with an assigned idempotent element* x. $\mathrm{Sgr}\mathbf{Top}^\top$ is the category of algebras of the "free-semigroup monad" over $\mathbf{Top}^\top$

(3) $T: \mathbf{Top}^\top \to \mathbf{Top}^\top, \qquad T(X, x) = \Sigma_{n>0}\,(X, x)^n$

where the powers and the sum belong now to the category $\mathbf{Top}^{\top}$ (recall that the sum of pointed spaces has the base-points identified). $\mathbf{P}$ (which does not preserve sums) is a (non-strong) T-functor through the transformation

(4) $\lambda\colon \mathrm{T}\mathbf{P} \to \mathbf{P}\mathrm{T}, \qquad\qquad \lambda X\colon \Sigma_{n>0} (\mathbf{P}X)^n \to \mathbf{P}\, (\Sigma_{n>0} X^n)$

$(\gamma_1,\dots \gamma_n) \mapsto \langle\gamma_1,\dots \gamma_n\rangle\colon [0, 1] \to X^n$

which is plainly consistent with the identifications in our sums of pointed spaces. One shows now, as above, that $\mathrm{Sgr}\mathbf{Top}^{\top}$ is IP4-homotopical, with

(5) $\mathbf{P}^{\mathrm{T}}(X, x, \cdot) = (\mathrm{P}X, x^{\mathrm{P}}$, pointwise multiplication of paths).

3.6. Categories and natural equivalences. We show here that the category **Cat** of small categories and functors is regular IP4-homotopical, *with homotopies given by natural isomorphisms*. (This produces a 2-category structure $\mathbf{Cat}_{\mathrm{i}}$ with invertible cells; in other words, a groupoid-enriched category, or also a *strict* h4-category, see 1.4.)

Recall that **Cat** is cartesian closed, with $[X, Y] = Y^X$ the category of functors $X \to Y$ and their natural transformations. The role of the standard interval in **Top** is now played by the *undiscrete* groupoid $\mathbf{i}$ on two objects, say 0, 1 (a groupoid is undiscrete if each hom-set has one element), which consists of "the free isomorphism". It supplies a cylinder and a cocylinder functor, $I \dashv P$, for **Cat**

(1) $IX = \mathbf{i}{\times}X, \qquad PY = Y^{\mathbf{i}}$

where $Y^{\mathbf{i}}$ "is" the full subcategory of $Y^{\mathbf{2}}$ whose objects are the isomorphisms of Y. Cylinders and cocylinders respectively corepresent and represent homotopies in the above sense, since a natural iso $\alpha\colon f \to g\colon X \to Y$ is the same as a functor $\check{\alpha}\colon IX \to Y$, or a functor $\hat{\alpha}\colon X \to PY$.

As in **Top**, we generate the operations of I and P through operations on the "interval". First, $\mathbf{i}$ is a commutative, involutive *dioid*-object in **Cat** (1.1). Indeed, $\mathbf{i}{\times}\mathbf{i}$ is the undiscrete groupoid on four objects, displayed in (2), and its operations $(\partial^\varepsilon, g^\varepsilon, s, r)$ are functors determined by their action on the objects, as follows, for $\varepsilon, i, j = 0, 1$ (while $e\colon \mathbf{i} \to \mathbf{1} = \top$ need not be defined)

(2)
$$\begin{array}{ccc} (0,1) & \longleftrightarrow & (1,1) \\ \updownarrow & \nwarrow\!\!\nearrow\!\!\swarrow\!\!\searrow & \updownarrow \\ (0,0) & \longleftrightarrow & (1,0) \end{array}$$

(3) $\partial^\varepsilon\colon \mathbf{1} \to \mathbf{i}, \qquad \partial^\varepsilon(0) = \varepsilon$

$g^\varepsilon\colon \mathbf{i}{\times}\mathbf{i} \to \mathbf{i}, \qquad g^-(i, j) = i \vee j, \qquad g^+(i, j) = i \wedge j$

$s\colon \mathbf{i}{\times}\mathbf{i} \to \mathbf{i}{\times}\mathbf{i}, \qquad s(i, j) = (j, i)$

$r\colon \mathbf{i} \to \mathbf{i}, \qquad r(i) = 1 - i.$

Further, $\mathbf{i}$ has a (regular) vertical composition k. Note that $\mathbf{i} +_{\mathbf{1}} \mathbf{i}$ is the undiscrete groupoid on three objects; writing them as "0, 1/2, 1", the functor k is the inclusion

(4) $k\colon \mathbf{i} \to \mathbf{i} +_{\mathbf{1}} \mathbf{i}, \qquad k(0) = 0, \qquad k(1) = 1.$

This structure, transferred to the cylinder and path functors, makes **Cat** into a regular IP4-category. Since **Cat** is complete and cocomplete, it is also IP4-homotopical.

The lens collapse w is determined as in 1.5; it collapses "vertically" $\mathbf{i}\times\mathbf{i}$ on its main diagonal

(5) $w: \mathbf{i}\times\mathbf{i} \to \mathbf{i}\times\mathbf{i}, \qquad w(i, j) = (i, i).$

All this plainly restricts to the category **Gpd** of groupoids. Note that **Cat** has also a "larger" homotopy structure, with homotopies consisting of arbitrary natural transformations. The groupoid **i** should then be replaced with the ordinal category $\mathbf{2} = \{0 < 1\}$, and the groupoid $\mathbf{i}\times\mathbf{i}$ with the order category $\mathbf{2}\times\mathbf{2}$. The operations considered above can be extended, *except* of course for the reversion, which can only be partially surrogated by a *generalised reversion* $r: \mathbf{2} \to \mathbf{2}^{op}$. On groupoids, both structures coincide.

3.7. Strict monoidal categories. The category Mon**Cat** of (small) *strict* monoidal categories and strict monoidal functors can be studied along the same lines as topological monoids, in 3.2: the P4-homotopical structure of **Cat** is consistent with the monad, and lifts to an IP4-homotopical structure for Mon**Cat**.

First, the latter is monadic over **Cat**, through the forgetful functor U and its left adjoint F

(1) $F: \mathbf{Cat} \rightleftarrows \mathrm{Mon}\mathbf{Cat}: U, \qquad \eta: 1 \to UF, \qquad \vartheta: FU \to 1$

$FX = (\Sigma_{n\geq 0}\, X^n, \otimes) \qquad (x_1,\dots x_p) \otimes (x_{p+1},\dots x_n) = (x_1,\dots x_n)$

$\eta: X \subset UFX, \qquad \vartheta: FUA \to A, \qquad \vartheta(x_1,\dots x_n) = x_1\otimes \dots \otimes x_n$

where FX, the free strict-monoidal category over X, is the sum of the power categories X^n.

Also here, the cocylinder $P: \mathbf{Cat} \to \mathbf{Cat}$, $PX = X^{\mathbf{i}} \subset X^{\mathbf{2}}$ considered above (3.6.1) preserves powers (as a right adjoint) and also sums; it is a strong T-functor (same calculations as in 3.2), by identifying an n-tuple of isomorphisms in X with an isomorphism of X^n

(2) $\lambda: TP \to PT, \qquad \lambda X: \Sigma_{n>0}\, (PX)^n \cong P\, (\Sigma_{n>0}\, X^n)$

$(\gamma_1,\dots \gamma_n) \mapsto \langle\gamma_1,\dots \gamma_n\rangle: \mathbf{i} \to X^n .$

Similarly one deals with Mon**Gpd**, the 2-category of (small) strict-monoidal groupoids.

Of course, in order to get *monoidal categories* in the usual relaxed sense one should consider *h-algebras* in **Cat**, satisfying the axioms of T-algebra up to specified, coherent homotopies (natural isomorphisms).

4. Applications to chain complexes

We recall the IP4-homotopical structure of chain complexes. Of course, homotopies are the usual ones and the basic structure is classical; but the connections are less known, while the lens collapse and its lifting property were introduced in [9]. This structure is shown to be partially consistent with the "free-semigroup monad", producing a homotopy structure for associative d-algebras which lacks reversion and sum of homotopies. In a graded module, $\bar{x} = (-1)^{\deg x}.x$.

4.1. Chain complexes. Let us begin considering the category $C_*\mathbf{D}$ of (unbounded) chain complexes $A = ((A_n), (\partial_n))$ over an additive category **D**. A **D**-map between finite biproducts $f: \oplus A_j \to \oplus B_i$ of components f_{ij} will be written "on variables", as $f(x_1, ... x_n) = (\Sigma f_{1j} x_j, ... \Sigma f_{mj} x_j)$; this allows one to calculate as in a category of modules, and can be formally justified by setting $x_j = pr_j: \oplus A_j \to A_j$.

To fix notation, a homotopy in $C_*\mathbf{D}$ is written as in (1), and satisfies (2)

(1) $\alpha: f \to g: A \to B,$ $\qquad \alpha = (f, \alpha_\bullet, g)$

(2) $-f + g = \partial.\alpha_\bullet + \alpha_\bullet.\partial$ $\qquad (-f_n + g_n = \partial_{n+1} \alpha_n + \alpha_{n-1} \partial_n)$

where $\alpha_\bullet = (\alpha_n): |A| \to_1 |B|$ is a map of graded objects, of degree 1, the *centre* of α.

Homotopies are represented (or defined) by a path endofunctor P

(3) $(PA)_n = A_n \oplus A_{n+1} \oplus A_n,$ $\qquad \partial(a, x, b) = (\partial a, -a - \partial x + b, \partial b)$

equipped with faces, degeneracy, vertical reversion and vertical composition

(4) $\partial^\varepsilon: P \to 1,$ $\qquad \partial^-(a, x, b) = a, \qquad \partial^+(a, x, b) = b$

(5) $e: 1 \to P,$ $\qquad e(a) = (a, 0, a)$

(6) $r: PA \to PA,$ $\qquad r(a, x, b) = (b, -x, a)$

(7) $k: PA \times_A PA \to PA,$ $\qquad k(a, x, c, y, d) = (a, x+y, d)$

which produce the usual, regular sum of homotopies.

They are also corepresented by a cylinder $I \dashv P$

(8) $(IA)_n = A_n \oplus A_{n-1} \oplus A_n,$ $\qquad \partial(a, x, b) = (\partial a - x, -\partial x, \partial b + x).$

P and I respectively preserve the existing limits and colimits; both preserve finite biproducts. Note that kernels and cokernels need not exist in $C_*\mathbf{D}$, since $\mathbf{D}$ is only assumed to be additive.

4.2. The second-order structure. $C_*\mathbf{D}$ is a regular IP4-category, studied more in detail in [9], 6.5-8. The second-order path functor P^2 and its operations are (write $\xi = (a, x, b;\ u, z, v;\ c, y, d)$ and $z' = -x + u + \partial z - v + y$)

(1) $(P^2A)_n = (A_n \oplus A_{n+1} \oplus A_n) \oplus (A_{n+1} \oplus A_{n+2} \oplus A_{n+1}) \oplus (A_n \oplus A_{n+1} \oplus A_n)$

(2) $\partial(\xi) = (\partial a,\ -a - \partial x + b,\ \partial b;\ -a - \partial u + c,\ z',\ -b - \partial v + d;\ \partial c,\ -c - \partial y + d,\ \partial d)$

(3) $\partial^- P(\xi) = (a, x, b), \qquad \partial^+ P(\xi) = (c, y, d)$

$P\partial^-(\xi) = (a, u, c), \qquad P\partial^+(\xi) = (b, v, d)$

(4) $g^-(a, x, b) = (a, x, b;\ x, 0, 0;\ b, 0, b),$ $\qquad g^+(a, x, b) = (a, 0, a;\ 0, 0, x;\ a, x, b)$

(5) $s(\xi) = (a, u, c;\ x, -z, y;\ b, v, d).$

The homotopy pullback of $f: A \to C$ and $g: B \to C$ is

(6) $(P(f, g))_n = A_n \oplus C_{n+1} \oplus B_n,$ $\qquad \partial(a, x, b) = (\partial a, -fa - \partial x + gb, \partial b).$

4.3. Lens collapse. Consider four homotopies $\alpha, \beta, \rho, \sigma: X \to PA$, connecting four maps $a, b, c, d: X \to A$, as below. By 4.2.1-2, a double homotopy $\Phi: X \to P^2A$ with boundary

(1) $\begin{array}{ccc} c & \xrightarrow{\beta} & d \\ \rho \uparrow & \Phi & \uparrow \sigma \\ a & \xrightarrow[\alpha]{} & b \end{array}$ $\qquad \alpha = (a, x, b)\colon a \to b, \qquad \rho = (a, u, c)\colon a \to c$

$\qquad \beta = (c, y, d)\colon c \to d, \qquad \sigma = (b, v, d)\colon b \to d$

amounts to a map $z = \Phi_\bullet\colon |X| \to_2 |A|$ of graded objects, of degree 2, the *centre* of Φ, satisfying

(2) $\partial z - z\partial = x + v - y - u\colon |X| \to_1 |A|$

where $x + v - y - u = (\alpha + \sigma - \beta - \rho)_\bullet$ is the centre of the anti-clockwise endohomotopy of $a\colon X \to A$ determined by the boundary, the endohomotopy *associated* to Φ.

The lens collapse w for the path functor is determined by the previous structure (1.5.3), as

(3) $w\colon P^2A \to P^2A, \qquad w(a, x, b;\ u, z, v;\ c, y, d) = (a, x{+}v, d;\ 0, z, 0;\ a, u{+}y, d).$

The lens conversion turns thus a double homotopy $\Phi\colon X \to P^2A$ with boundary $\alpha, \beta, \rho, \sigma$ into the cell-homotopy $w\Phi\colon \alpha{+}\sigma \to \rho{+}\beta$, *preserving the centre* (note that Φ and $w\Phi$ have the same associated endohomotopy $\alpha + \sigma - \beta - \rho$), which directly shows it is bijective.

4.4. Chain algebras. Let us restrict now our attention to the category **Dm** $= C_*(R\text{-}\mathbf{Mod})$ of d-modules, i.e. unbounded chain complexes of R-modules, for a fixed commutative unitary ring R. Now, the path and cylinder functor (as well as their operations) can be obtained as $PA = \mathrm{Hom}(\mathbf{i}, A)$ and $IA = \mathbf{i}{\otimes}A$, via the closed monoidal structure of **Dm** and the "standard interval" **i**, a complex concentrated in degrees 0 and 1

(1) $\mathbf{i} = (\ \ldots 0 \to R \to R{\oplus}R \to 0 \ldots\ ), \qquad \partial_1(x) = (-x, x).$

Consider also the category **Da** of *associative d-algebras*, over R; this will mean a d-module A equipped with an associative product consistent with the differential ($\partial(x.y) = \partial x.y + \bar{x}.\partial y$). This category is equivalent to the category **DA** of *augmented unitary* associative d-algebras, which is more commonly used but often presents more complicated constructions.

Da is monadic over **Dm**, and again we identify $\mathbf{Dm}^T = \mathbf{Da}$

(2) $F : \mathbf{Dm} \rightleftarrows \mathbf{Da} : U, \qquad \eta\colon 1 \to UF, \qquad \vartheta\colon FU \to 1$

$FX = (\Sigma_{n>0} X^{\otimes n}, \otimes) \qquad (x_1{\otimes} \ldots {\otimes}x_p) \otimes (x_{p+1}{\otimes} \ldots {\otimes}x_n) = x_1{\otimes} \ldots {\otimes}x_n$

$\eta\colon X \subset UFX, \qquad \vartheta\colon FUA \to A, \qquad \vartheta(x_1{\otimes} \ldots {\otimes}x_n) = x_1{\cdot} \ldots {\cdot}x_n$

(3) $T = UF\colon \mathbf{Dm} \to \mathbf{Dm}, \qquad TX = \Sigma_{n>0} X^{\otimes n}$

$\mu = U\vartheta F\colon T^2 \to T$

$\mu((x_{11}{\otimes}\ldots{\otimes}x_{1p_1}) \otimes\ldots\otimes (x_{n1}{\otimes} \ldots {\otimes}x_{np_n})) = x_{11}{\otimes}\ldots{\otimes}x_{np_n}.$

The P0-structure of **Dm** is consistent with the monad, through the natural transformation

(4) $\lambda\colon TP \to PT, \qquad \lambda X\colon \Sigma_{n>0} (PX)^{\otimes n} \to P(\Sigma_{n>0} X^{\otimes n})$

$(x_1, z_1, y_1) \otimes\ldots\otimes (x_n, z_n, y_n) \mapsto$

$(x_1{\otimes}\ldots{\otimes}x_n,\ \Sigma_i (\bar{x}_1{\otimes}\ldots{\otimes}\bar{x}_{i-1}{\otimes}z_i{\otimes}y_{i+1}{\otimes}\ldots{\otimes}y_n),\ y_1{\otimes}\ldots{\otimes}y_n)$

so that $P^T(X, t) = (PX, Pt.\lambda)$ is the path-module PX with the usual multiplication

(5) $(x_1, z_1, y_1).(x_2, z_2, y_2) = (x_1.x_2, \ \bar{x}_1.z_2 + z_1.y_2, \ y_1.y_2)$.

Da = **Dm**^{T} is thus IP0-homotopical, with path functor **P** = P^T. The symmetric cubical comonad $(P, \partial^\varepsilon, e, g^\varepsilon, s)$ over **Dm** is consistent with the monad and lifts to algebras. This structure, studied in [8], is already sufficient to develop the basic notions of homotopical algebra. On the other hand, *the reversion, composition and lens collapse* are just consistent with the monad *up to homotopy*; which is "why" **Da** lacks reversion and sum of homotopies (as well known) and lens collapse. Again, this makes evident the interest of studying a homotopy relaxation of algebras (see [21, 10]).

The cylinder functor **I** left adjoint to **P** can be calculated as a quotient of the free chain algebra FI|A| over the cylinder of the underlying chain complex, imposing some relations which come from the multiplicative structure of A [8].

5. Slice categories and fibrewise homotopy

Slice categories **A**\A (resp. **A**/B) can be viewed as categories of algebras (resp. coalgebras) over **A**, finding again the lifting results for the path (resp. cylinder) functor exposed in [9].

For the homotopy theory of (strict or relaxed) slice categories of spaces see Baues [1], James [13], Hardie and Kamps [11, 12] and their references; homotopy in **Top**/B is called *fibrewise homotopy*.

5.1. Objects under A **as algebras.** The classical topological example of a slice category is **Top**$^⊤$ = **Top**\⊤, the category of pointed spaces or "spaces under the point" ⊤ = {*}: an object (X, t) is a map t: ⊤ → X in **Top**; pointed maps are defined coherently.

In the same way, if **A** is a category and A an object, the slice category **A**\A of *objects under* A has objects (X, t), with t: A → X in **A**; a morphism f: (X, t) → (X', t') is given by an **A**-map f: X → X' such that f.t = t'.

If **A** has finite sums, the (obvious) forgetful functor U: **A**\A → **A** is monadic over **A**. Indeed, U has a left adjoint F: **A** → **A**\A, with unit η: 1 → UF, counit ϑ: FU → 1

(1) FX = (X+A, j: A ⊂ X+A) $\qquad\qquad$ ηX: X ⊂ X+A

ϑ(X, t): (X+A, j) → (X, t), $\qquad\qquad$ $\vartheta(x) = x, \quad \vartheta(a) = t(a)$

supplying a monad over **A**

(2) T = UF: **A** → **A**, $\qquad\qquad$ TX = X+A

ηX: X ⊂ X+A, $\qquad\qquad$ $\mu = U\vartheta F: T^2 \to T$

(μX: X+A+A → X+A, $\qquad\qquad$ $\mu.in_1 = in_1, \quad \mu.in_2 = \mu.in_3 = in_2$.)

It is easy to see that a T-algebra (X, τ: X+A → X) has just to satisfy $\tau.\eta$ = idX (the multiplication axiom being trivially satisfied) and reduces to an arbitrary object under A, (X, t: A → X), with t = τ.j. We identify thus **A**^{T} = **A**\A.

5.2. The homotopy structure of Top\A. First, the cocylinder P: **Top** → **Top** preserves sums. It is a T-functor, with λX the obvious embedding defined by the degenerate-path embedding e: A → PA

(1) $\lambda\colon TP \to PT,$ $\qquad\qquad$ $\lambda X\colon PX{+}A \subset P(X{+}A) = PX + PA$

$\lambda.\eta P = P\eta\colon PX \subset P(X{+}A)$

$\lambda.\mu P = P\mu.\lambda T.T\lambda\colon PX + A + A \to PX + PA.$

The canonical lifting to T-algebras, $P^T\colon \mathbf{Top}^T \to \mathbf{Top}^T$

(2) $P^T(X, \tau) = (PX, P\tau.\lambda),$ $\qquad\qquad$ $P\tau.\lambda = (PX{+}A \subset P(X{+}A) \to PX)$

acts equivalently on a space under A, in the obvious way

(3) $P^T(X, t\colon A \to X) = (PX, Pt.e).$

To prove that the operations of the P4-structure of **Top** are T-transformations, we proceed as for topological semigroups (3.2). Taking into account our choice of $Q = P$ for **Top** (1.2), all the operations $(\partial\square^\varepsilon, e, g^\varepsilon, r, s, k^\varepsilon, z, w)$ are defined by suitable continuous function between powers of the unit interval, through contravariant composition

(4) $f_0\colon [0, 1]^q \to [0, 1]^p$

$fX\colon P^pX \to P^qX$ $\qquad\qquad$ $(\Gamma\colon [0, 1]^p \to X) \mapsto (\Gamma f_0\colon [0, 1]^q \to X)$

and all such transformations are consistent with λ, i.e.

(5) $fT.\lambda^{P^p} = \lambda^{P^q}.Tf\colon T.P^p \to P^q.T.$

In fact, after noting that the transformation making P^q a T-functor is the embedding

(6) $\lambda^{P^q} = P^{q-1}\lambda. \ldots .P\lambda P^{q-2}.\lambda P^{q-1}\colon T.P^q \to P^q.T$

$\lambda^{P^q}X\colon P^qX + A \subset P^qX + P^qA$

the property (5) above is an easy consequence. Moreover, the lifting of k^ε to T-algebras makes $P^T(X, t)$ into the composition-pullback of (X, t).

By 2.6, we have thus proved that $\mathbf{Top}^T = \mathbf{Top}\backslash A$ is IP4-homotopical, with path functor P^T. The cylinder functor I^T (derived from 2.7 or directly calculated) is

(7) $I^T\colon \mathbf{Top}\backslash A \to \mathbf{Top}\backslash A,$ $\qquad\qquad$ $I^T(X, t\colon A \to X) = (IX/R, t')$

where $IX = [0, 1]{\times}X$ is the usual cylinder and R is the congruence spanned by the following relation (making $t'(a) = [\tau, t(a)]$ well defined)

(8) $(\tau, t(a))\ R_0\ (\tau', t(a))$ $\qquad\qquad$ $(\tau, \tau' \in [0, 1];\ \ a \in A).$

5.3. Objects over B **as coalgebras and fibrewise homotopy.** Dually, if **A** has finite products and B is any object, the forgetful functor $U\colon \mathbf{A}/B \to \mathbf{A}$ is comonadic over **A**. The comonad (T, ϑ, δ) is given by the right adjoint R

(1) $RX = (X{\times}B, pr_2\colon X{\times}B \to B),$ $\qquad$ $\eta\colon 1 \to RU,$ $\qquad$ $\vartheta\colon UR \to 1$

(2) $T = UR\colon \mathbf{A} \to \mathbf{A},$ $\qquad\qquad$ $TX = X{\times}B$

$\vartheta\colon T \to 1,$ $\qquad\qquad$ $\vartheta X = pr_1\colon X{\times}B \to X$

$\delta = R\eta U\colon T \to T^2$ $\qquad\qquad$ $\delta X = X{\times}\mathrm{diag}\colon X{\times}B \to X{\times}B{\times}B.$

Its coalgebras $(X, \tau: X \to X{\times}B)$ (satisfying $pr_1.\tau = idX$) can be identified with objects $(X, t: X \to B)$ over B, with $t = pr_2.\tau$. Thus, $\mathbf{A}^T = \mathbf{A}/B$ ($\mathbf{A}^T$ denotes now the category of T-*coalgebras*).

Take $\mathbf{A} = \mathbf{Top}$ and B any (non empty) space. A morphism $f: (X, t: X \to B) \to (Y, u: Y \to B)$ is also called a *fibre map* over B, since it takes t-fibres into u-fibres. The cylinder functor $I = [0, 1]\times - : \mathbf{Top} \to \mathbf{Top}$ is a strong T-functor, with $\lambda X: I(X{\times}B) \to (IX){\times}B$ the canonical homeomorphism. One can now lift the cylinder functor of **A**

(3) $I^T(X, \tau: X \to X{\times}B) = (IX, \lambda X.I\tau: IX \to I(X){\times}B)$

$(\ I^T(X, t: X \to B) = (IX, pr_2.It: IX \to B)\ )$

with the whole structure making $\mathbf{Top}^T = \mathbf{Top}/B$ an IP4-homotopical category (2.6).

This structure is the usual one, concerning *fibrewise homotopies*. Indeed, given two fibre maps f, g: (X, t) → (Y, u), a map $\alpha: I^T(X, t) \to (Y, u)$ whose faces are f and g amounts to an ordinary homotopy $\alpha: f \to g: X \to Y$ such that, for all $x \in X$, the path $\alpha(\tau, x)$ $(\tau \in [0, 1])$ is contained in a u-fibre of Y (necessarily, the one over $t(x) = uf(x) = ug(x)$).

Again, the cocylinder functor P^T, right adjoint to I^T can be derived from 2.7, or - more simply - from the above description of homotopies

(4) $\mathbf{Top}^T (I^T(X, t), (Y, u)) = \mathbf{Top}^T ((X, t), P^T(Y, u))$

(5) $P^T: \mathbf{Top}/B \to \mathbf{Top}/B$

$P^T(Y, u: Y \to B) = (P'Y, u'),$ $\qquad\qquad P'(Y) = \{\lambda \in PY \mid t\circ\lambda \text{ is constant}\}.$

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