ABSOLUTE LAX 2-CATEGORIES

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Abstract.

We have introduced, in a previous paper, the fundamental lax 2-category of a 'directed space' X. Here we show that, *when* X *has a* T_1*-topology*, this structure can be embedded into a larger one, with the same objects (the points of X), the same arrows (the directed paths) and the same cells (based on directed homotopies of paths), but a larger system of comparison cells.

The new comparison cells are *absolute*, in the sense that they only depend on the arrows themselves rather than on their syntactic expression, as in the usual settings of lax or weak structures. It follows that, in the original structure, *all* the diagrams of comparison cells commute, even if not constructed in a natural way and even if the composed cells need not stay within the old system.

Introduction

The purpose of this paper is solving a problem which appeared in a previous one [13]: can one define, for a directed space (see below), a fundamental *absolute* lax 2-category, with *absolute comparisons*, not depending on the syntactic expression of the arrows to be compared? The paper [13] will be cited as Part I; the reference I.2, or I.2.3, or I.2.3.4 applies to its Section 2, or its Subsection 2.3, or item (4) in the latter, respectively.

The problem can be better explained recalling the main structures in two dimensional category theory, as they have appeared in the literature:
(a) the structure of *monoidal category*, whose axioms are motivated by Mac Lane's coherence theorem for comparison cells ([16], 1963) and Kelly's reduction to two conditions ([14], 1964);
(b) the structure of (strict) *2-category* (Bénabou [2], C. Ehresmann [7], 1965);
(c) its *weak* version, a *bicategory* (Bénabou [3], 1967), where the unit and associativity laws of arrow-composition are replaced with invertible comparison cells, like $f \cong f\circ 1_x$, $(h\circ g)\circ f \cong h\circ(g\circ f)$ (and monoidal categories are included as bicategories with one object);
(d) a first *lax* version, Burroni's *pseudocategory* ([6],1971), which has non-invertible comparison cells, with the following *choice* of directions

$$f \to f\circ 1_x, \qquad f \to 1_y\circ f, \qquad (h\circ g)\circ f \to h\circ(g\circ f). \tag{1}$$

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After this paper, the study of *lax* two dimensional categories did not progress until, quite recently, Leinster introduced an *unbiased lax bicategory* ([15] Section 3.4; 2004), where all n-fold compositions $f_n \circ ... \circ f_1$ are assigned and there are comparison cells *from* each iterated composition *to* the corresponding multiple composition, as in the following examples

$$(k \circ h \circ g) \circ f \to k \circ h \circ g \circ f, \qquad (1 \circ (g \circ 1)) \circ f \to g \circ f. \tag{2}$$

Note that, here, there is no single comparison cell between $(h \circ g) \circ f$ and $h \circ (g \circ f)$, but there is one from each of them to $h \circ g \circ f$. On a more general ground, we follow (in Part I and here) his terminology, distinguishing *biased* and *unbiased* settings: a biased one is founded on binary (and nullary) compositions, while an unbiased setting works with all n-ary compositions (at least where strict laws are not assumed).

Our approach, in Part I, is based on a 'geometric guideline' derived from Directed Algebraic Topology. This recent domain studies structures having *privileged directions*, like 'directed spaces' in some sense: ordered topological spaces, *d-spaces* ('spaces with distinguished paths', cf. 1.1), simplicial and cubical sets, etc. Such objects have *directed* paths and homotopies, which cannot be reversed, generally. They can thus model non-reversible phenomena, in various domains; the existing applications deal mostly with the analysis of concurrent processes, in Computer Science (references for these applications can be found in [8, 9, 10, 12]).

Given a d-space X, the fundamental category $\uparrow\Pi_1(X)$ has been defined and studied in [10]; the (obvious) definition is recalled below (1.1). Then, in Part I, we have introduced a fundamental *biased d-lax 2-category* $\uparrow \mathrm{b}\Pi_2(X)$, where an arrow $a\colon x \to y$ is a (directed) path and a cell $[\alpha]\colon a \to b\colon x \to y$ is a 'homotopy class of (directed) homotopies', with fixed boundary (see I.2). Writing $a \otimes b$ for the concatenation of consecutive paths, our structure has comparison cells

$$1_x \otimes a \to a \to a \otimes 1_y, \qquad a \otimes (b \otimes c) \to (a \otimes b) \otimes c, \tag{3}$$

whose direction comes from the fact that, in a directed space, a comparison homotopy has to move from a concatenation to another which, at each instant $t \in [0, 1]$, *has made a longer way than the initial one*. The term 'd-lax' is meant to recall this fact: lax structures come around in different forms, the present ones being essentially different from Burroni's and Leinster's.

Similarly we have defined, in I.3, a fundamental *unbiased d-lax 2-category* $\uparrow \mathrm{u}\Pi_2(X)$, where we have multiple concatenations $a_1 \otimes ... \otimes a_n$ and new comparison cells, like

$$a \otimes (b \otimes c) \to a \otimes b \otimes c \to (a \otimes b) \otimes c. \tag{4}$$

In both cases, of course, the general notion of a *d-lax 2-category*, biased or unbiased, is explicitly given. We ended with formulating the problem (I.3.8) investigated here: is it possible to define an 'absolute' d-lax fundamental 2-category, with *absolute comparisons* $\varphi(a, b)\colon a \to b$, only depending on the actual values of the paths a, b, yet *containing* the previous *syntactic* comparisons? Note that, as a consequence, *all* diagrams of comparison cells of $\uparrow\mathrm{b}\Pi_2(X)$ and $\uparrow\mathrm{u}\Pi_2(X)$ will commute, even if not naturally constructed and even if their composed cells might not stay within the old system (cf. 2.4).

This question was suggested by the fact that a possible solution (now dealt with in 2.1), was clear from the beginning: given two paths $a, b\colon x \to y$ in the d-space X, we say that $a \prec b$ if there exist two *reparametrisation functions* $f, g\colon [0,1] \to [0,1]$ (increasing, surjective and piecewise affine) such that

$$af = bg, \qquad\qquad f \geq \mathrm{id} \geq g. \tag{5}$$

Then, we have an obvious comparison cell $\varphi(a,b) : a \to b$ constructed with two affine interpolations $\varphi_0(\mathrm{id}, f)\colon \mathrm{id} \to f$ and $\varphi_0(g, \mathrm{id})\colon g \to \mathrm{id}$, which are *directed* homotopies precisely because of the inequalities $\mathrm{id} \leq f$ and $g \leq \mathrm{id}$. (Details can be found in 2.1.)

The difficulty was proving that this cell $a \to b$ is indeed *absolute*, i.e. that it only depends on a, b and not on the choice of f, g.

This is proved here (Section 4), on the basis of a series of results on reparametrisation functions (Section 3), *but under a condition of separation*: we assume that our d-spaces *have a T_1-topology* (i.e., all points are closed). It would be interesting either to drop this restriction or to show that one cannot. In fact, assuming some separation axiom is not new in establishing two dimensional homotopy categories: the papers [4, 5] work with Hausdorff spaces.

Acknowledgements. This work was begun while I was a guest of Martin Hyland at D.P.M.M.S., Cambridge, UK, discussing with him the initial setting. Then, I got into problems with reparametrisation functions, missing the final steps (3.6-3.8) up to the end. I express my gratitude to two colleagues at my Department, Ada Aruffo and Gianfranco Bottaro, for advice and comfort while I felt stranded in an unfamiliar land, wondering whether I would ever find the way, or just a way out.

1. Review of the (unbiased) syntactic approach

We extract here, from Part I, the main points establishing the fundamental d-lax 2-category $\uparrow\mathrm{u}\Pi_2(X)$ of a d-space X, in the *unbiased* version (i.e., with multiple compositions of arrows).

1.1. Spaces with distinguished paths. As in Part I, we shall work in the setting introduced in [10].

A *d-space* is a topological space X equipped with a set dX of (continuous) maps $a\colon \mathbf{I} \to X$, defined on the standard interval $\mathbf{I} = [0,1]$; these maps, called *distinguished paths* or *d-paths*, must contain all constant paths and be closed under concatenation and 'increasing change of parametrisation' on $\mathbf{I}$: if $a\colon \mathbf{I} \to X$ is in dX and $h\colon \mathbf{I} \to \mathbf{I}$ is a continuous order-preserving function (possibly not surjective), then ah is also distinguished.

A *d-map* $f\colon X \to Y$ (or *map* of d-spaces) is a continuous mapping between d-spaces which preserves the distinguished paths: if $a \in dX$, then $fa \in dY$.

The category of d-spaces is written as d**Top**. It has all limits and colimits, constructed as in **Top** (the category of topological spaces) and equipped with the initial or final d-structure for the structural maps; for instance a path $\mathbf{I} \to \prod X_j$ is distinguished if and only if all its components $\mathbf{I} \to X_j$ are so. The forgetful functor $U\colon \mathrm{d}\mathbf{Top} \to \mathbf{Top}$ preserves thus all limits and colimits; a topological space is generally viewed as a d-space by its *natural* structure, where all (continuous) paths are directed (via the right adjoint to U).

Reversing d-paths, by the involution $r(t) = 1 - t$, yields the *reflected*, or *opposite*, d-space $RX = X^{\mathrm{op}}$, where $a \in d(X^{\mathrm{op}})$ if and only if $a^{\mathrm{op}} = ar$ is in dX.

The *standard d-interval* $\uparrow\mathbf{I} = \uparrow[0,1]$ has distinguished paths given by the (weakly) increasing maps $\mathbf{I} \to \mathbf{I}$. The *standard directed circle* $\uparrow\mathbf{S}^1 = \uparrow\mathbf{I}/\partial\mathbf{I}$ has the (obvious) quotient d-structure, where distinguished paths have to follow a precise orientation. (But note that the directed structure $\uparrow\mathbf{S}^1 \times \uparrow\mathbf{S}^1$ on the torus is not related with an orientation of this surface.)

A (directed) *path* of a d-space X is a map $\uparrow\mathbf{I} \to X$, which simply means a distinguished path in the d-structure of X itself. A (directed) *homotopy* $\varphi\colon f \to g\colon X \to Y$ is a map $\varphi\colon X \times \uparrow\mathbf{I} \to Y$ coinciding with f (resp. g) on the lower (resp. upper) basis of the *cylinder* $X \times \uparrow\mathbf{I}$. In particular, a 2-*homotopy* $\varphi\colon a \to b\colon \uparrow\mathbf{I} \to X$ is a homotopy with fixed endpoints, which means that the mapping $\varphi\colon \uparrow\mathbf{I} \times \uparrow\mathbf{I} \to X$ induces two constant paths, $\varphi(0,-)\colon a(0) \to b(0)$ and $\varphi(1,-)\colon a(1) \to b(1)$.

The fundamental category $\uparrow\Pi_1(X)$ has objects in X, and for arrows, the classes $[a]\colon x \to x'$ of directed paths, up to the equivalence relation *generated* by 2-homotopies; composition is given by the concatenation of consecutive paths, written as $[a]\otimes[b] = [a\otimes b]$ for $[a]\colon x \to x'$, $[b]\colon x' \to x''$. The category $\uparrow\Pi_1(X)$ can be computed by a van Kampen-type theorem, as proved in [10], Thm. 3.6.

An alternative setting, *inequilogical spaces*, introduced in [11] as a directed version of Dana Scott's equilogical spaces [17, 1], could also be used - but would require a more complicated procedure to concatenate paths and homotopies (cf. [11]).

1.2. Definition. An *unbiased d-lax 2-category* $\mathbf{A}$, as defined in I.3.3, consists of the following data (and properties).
(udl.0) A set of objects, Ob$\mathbf{A}$.
(udl.1) For any pair of objects x, y, a category $\mathbf{A}(x,y)$ of *maps* $a\colon x \to y$ and *cells* $\alpha\colon a \to b$, with *main*, or *upper-level*, composition $\alpha \otimes_2 \beta\colon a \to b \to c$ and units $1_a\colon a \to a$.
(udl.2) For any sequence of objects $x_0, ..., x_n$, a functor of *lower n-ary composition*

$$\mathbf{A}(x_0,x_1)\times ... \times\mathbf{A}(x_{n-1},x_n) \to \mathbf{A}(x_0,x_n), \qquad (a_1,...,a_n) \mapsto a_1\otimes ... \otimes a_n, \tag{6}$$

which, for $n = 0$, amounts to assigning an *identity* $1_{x_0}\colon x_0 \to x_0$. This defines also the *tree-composition* $\langle a_1, ..., a_n; \tau\rangle$ of a sequence of n consecutive maps, *along* a tree (cf. 1.3) with n leaves ($n > 0$); for a tree with no leaves, we have a tree-composition of identities of one object, $\langle x; \tau\rangle$.
(udl.3) For every pair of trees τ, τ' with n leaves and reparametrisation functions $r(\tau) \leq r(\tau')$ (cf. 1.4), a natural transformation (*syntactic comparison*) of ordinary functors in n

variables

$$\begin{aligned}&\varphi(\tau,\tau')\colon \langle -;\tau,\rangle \to \langle -;\tau'\rangle\colon \mathbf{A}(x_0,x_1)\times ...\times \mathbf{A}(x_{n-1},x_n) \to \mathbf{A}(x_0,x_n),\\ &\varphi(a_1,...,a_n;\tau,\tau')\colon \langle a_1,...,a_n;\tau\rangle \to \langle a_1,...,a_n;\tau'\rangle\colon a_1(x_0) \to a_n(x_n),\end{aligned} \tag{7}$$

whose general component is a cell between two tree-compositions of the same sequence of maps.
(udl.4) (*coherence*) Every diagram (universally) constructed with comparison cells, via $\otimes$- and $\otimes_2$-compositions, commutes.

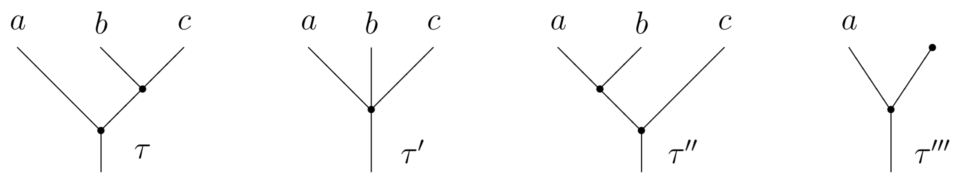
More explicitly, the last axiom means that:

$$\begin{aligned}&r(\tau) \leq r(\tau') \leq r(\tau'') \qquad \Rightarrow \qquad \varphi(\tau',\tau'')\circ\varphi(\tau,\tau') = \varphi(\tau,\tau''),\\ &r(\sigma_1) \leq r(\tau_1), ..., r(\sigma_n) \leq r(\tau_n) \qquad \Rightarrow\\ &\qquad \varphi(\sigma_1,\tau_1)\otimes ...\otimes\varphi(\sigma_n,\tau_n) = \varphi((\sigma_1,...,\sigma_n),(\tau_1,...,\tau_n)).\end{aligned} \tag{8}$$

(Note that, in the second case, $r((\sigma_1,...,\sigma_n)) \leq r((\tau_1,...,\tau_n))$.

1.3. Trees. We are using the notion of tree defined in [15], 2.3.3, which we call a *composition tree*. A tree-composition, written $\langle a_1,a_2,...,a_n;\tau\rangle$, consists of a finite tree τ whose n leaves are labelled by a sequence of n consecutive arrows $a_1,...,a_n$, as in the following examples

$$a\otimes(b\otimes c), \qquad a\otimes b\otimes c, \qquad (a\otimes b)\otimes c, \qquad a\otimes 1, \tag{9}$$



The last tree has two *shoots*, namely one leaf and one *bare shoot* ⫯

We only label leaves: there is no need of labelling bare shoots, since the corresponding identities are determined by the adjacent arrows; *unless all shoots are bare*, in which case one labelling *object* suffices: thus $\langle x;\tau\rangle$ will denote $1_x\otimes(1_x\otimes 1_x)$, if τ is the 'pruned version' of the first tree above, with all shoots bare. (In the biased case, one would only use *dichotomic trees*, with twofold bifurcations; all the examples above are of this type, except the second, which is related with ternary composition.)

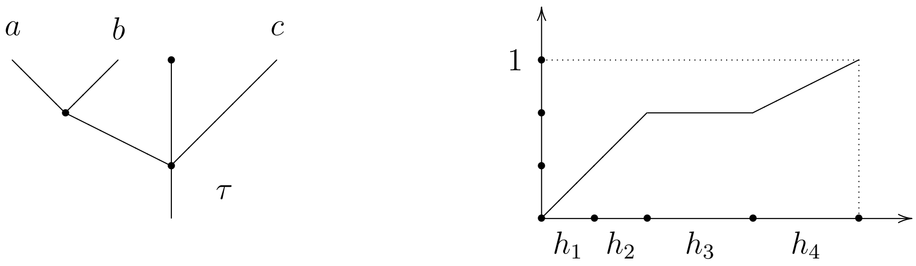
1.4. Reparametrisation functions. A *reparametrisation function* $r\colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}$ will be a (continuous) increasing, surjective, piecewise affine endofunction of the standard interval.

A composition tree has a *reparametrisation function* $r(\tau)$ defined abstractly, as recalled below; but the construction is more easily understood noting that, in a d-space X, $r(\tau)$

determines a tree-composition of paths $\langle a_1, ..., a_n; \tau\rangle$ by *reparametrising* the corresponding standard n-ary concatenation $a_1 \otimes ... \otimes a_n$

$$\langle a_1, ..., a_n; \tau\rangle = (a_1 \otimes ... \otimes a_n) \circ r(\tau) \colon \uparrow\mathbf{I} \to X. \qquad (10)$$

For instance, for the tree-composition $(a \otimes b) \otimes 1 \otimes c = \langle a, b, c; \tau\rangle$, the piecewise affine function $r = r(\tau) \colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}$ is



(11)

$$h_1 = h_2 = 1/6, \qquad h_3 = h_4 = 1/3; \qquad\qquad r(1/6) = 1/3, \qquad r(2/6) = r(4/6) = 2/3.$$

The definition of $r(\tau)$, for a composition tree with n leaves and m shoots ($m \geq n$), proceeds as follows. First, the *duration sequence* $h(\tau) = (h_1, ..., h_m)$ is defined letting h_i^{-1} be the product of the multiplicities of the nodes which precede the i-th shoot. (In the 'geometric case', h_i is the length of the time-interval on which the concatenated path goes along the i-th component.) Secondly, the *cumulative sequence* $k(\tau) = (k_1, ..., k_m)$ has $k_i = \sum_{j \leq i} h_j$, with $k_m = \sum h_i = 1$.

Finally, the reparametrisation function is affine on each interval $[k_{i-1}, k_i]$ ($i = 1, ..., m$), and increases on the latter of $1/n$ or 0, when the i-th shoot is, respectively, a leaf or bare. (An inductive definition can be found in I.3.4a.) For a pruned tree (i.e., $n = 0$), the function is the identity.

Reparametrisation functions have a rich structure, after being a monoid for composition and a lattice for the pointwise order $r \leq r'$ (the notation r' will never refer to a derivative). First, they have an n-ary concatenation, consistent with the preorder

$$(r_1 \otimes ... \otimes r_n)(t) = (i-1)/n + r_i(nt - i + 1)/n, \qquad \text{for } (i-1)/n \leq t \leq i/n. \qquad (12)$$

Furthermore, if $r \leq r'$, there is an *interpolating* directed 2-homotopy, by affine interpolation along the (directed!) segment from $r(\tau)(s)$ to $r(\tau')(s)$ (in $\uparrow\mathbf{I}$)

$$\varphi_0(r, r') \colon r \to r' \colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}, \qquad \varphi_0(r, r')(s, t) = (1-t).r(s) + t.r'(s). \qquad (13)$$

As a crucial point, its class $[\varphi_0(r, r')]$ up to 3-homotopy (with fixed boundary) is *uniquely determined* within all 2-cells $r \to r'$. In fact, if $\alpha, \beta \colon r \to r'$ are 2-homotopies, also $(\alpha \vee \beta)(s, t) = \max(\alpha(s, t), \beta(s, t))$ is so, and - plainly - there are 3-homotopies $\alpha \to \alpha \vee \beta \leftarrow \beta$.

(Reparametrisation functions, with the pointwise order and the tensor product described above, form an *ordered d-lax monoidal category*, which is a very special case of an absolute one: cf. I.3.7. Composition trees form a *pre*ordered one, cf. I.3.4. Note also that, in Part I, the 'piecewise affine' condition was only suggested as a possibility, in the general definition of reparametrisation function; but, of course, all the functions $r(\tau)$ *produced by trees* are so, by construction.)

1.5. Definition. Starting from a d-space X, the fundamental *unbiased d-lax 2-category* $\uparrow\mathrm{u}\Pi_2(X)$, defined in I.3.1, has the following objects, arrows, cells, compositions and comparisons (common with the biased case, up to point (d), cf. I.2.2).
(a) An *object* is a point of X.
(b) An *arrow* $a\colon x \to y$ is a (directed) path $a\colon \uparrow\mathbf{I} \to X$ with $a(0) = x$, $a(1) = y$; the *unit-arrow* $1_x\colon x \to x$ is the constant path at x.
(c) A *cell* $[\alpha]\colon a \to a'\colon x \to y$ is a homotopy class of homotopies of paths; more precisely, α is a (directed) 2-*homotopy* $a \to a'$ (with fixed endpoints), which means that the map $\alpha\colon \uparrow\mathbf{I}^2 \to X$ has the boundary represented below (the double lines represent constant paths)

$$\begin{array}{ccc} x & \xrightarrow{\;a\;} & y \\ \| & \alpha & \| \\ x & \xrightarrow[\;a'\;]{} & y \end{array} \tag{14}$$

and its homotopy class $[\alpha]$ is up to the equivalence relation generated by 3-*homotopies* $\alpha' \to \alpha''$ (with fixed boundary); the *unit-cell* $1_a\colon a \to a$ is the class of the trivial 2-homotopy $c_a(s,t) = a(s)$.
(d) The *main composition*, or *upper-level composition*, of $[\alpha]$ with $[\alpha']\colon a' \to a''\colon x \to y$ is defined by the pasting $\alpha \otimes_2 \alpha'$ of any two representatives, with respect to the second variable

$$\begin{gathered} [\alpha] \otimes_2 [\alpha']\colon a \to a''\colon x \to y, \qquad [\alpha] \otimes_2 [\alpha'] = [\alpha \otimes_2 \alpha'];\\ (\alpha \otimes_2 \alpha')(s,t) = \begin{cases} \alpha(s,2t), & 0 \leq t \leq 1/2,\\ \alpha'(s,2t-1), & 1/2 \leq t \leq 1. \end{cases} \end{gathered} \tag{15}$$

(e) The *n-ary lower composition* of (consecutive) maps and cells

$$a = a_1 \otimes a_2 \otimes ... \otimes a_n, \qquad \alpha = \alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_n, \tag{16}$$

is realised in the obvious way:

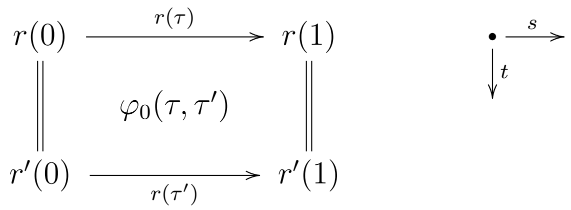
$$\begin{aligned} a(t) &= a_i(nt - i + 1), && \text{for } (i-1)/n \leq t \leq i/n,\\ \alpha(t,t') &= \alpha_i(nt - i + 1, t'), && \text{for } (i-1)/n \leq t \leq i/n. \end{aligned} \tag{17}$$

(f) The *comparison cell*

$$[\varphi(a_1, ..., a_n; \tau, \tau')]\colon \langle a_1, ..., a_n; \tau\rangle \to \langle a_1, ..., a_n; \tau'\rangle, \qquad r(\tau) \leq r(\tau'), \tag{18}$$

where $a_1, ..., a_n$ are consecutive paths and τ, τ' have n leaves, is given by the homotopy

$$\begin{aligned} &\varphi(a_1, ..., a_n; \tau, \tau') = (a_1 \otimes ... \otimes a_n) \circ \varphi_0(\tau, \tau') \colon \uparrow\mathbf{I} \times \uparrow\mathbf{I} \to X, \\ &\varphi_0(\tau, \tau') \colon \uparrow\mathbf{I} \times \uparrow\mathbf{I} \to \uparrow\mathbf{I}, \qquad \varphi_0(\tau, \tau')(s, t) = (1 - t).r(\tau)(s) + t.r(\tau')(s), \end{aligned} \tag{19}$$



which derives from the interpolating 2-homotopy $\varphi_0(\tau, \tau') = \varphi_0(r(\tau), r(\tau')) \colon r(\tau) \to r(\tau')$ (cf. (13)).

2. The absolute approach

We define absolute d-lax 2-categories (2.3), and construct the fundamental such structure $\uparrow\mathrm{L}\Pi_2(X)$ for a d-space X *having a T_1-topology.* We start from this construct, as a motivation for the abstract notion.

2.1. Absolute comparisons. Let X be always a d-space *with T_1-topology.* We proceed to construct the fundamental *absolute* (unbiased) d-lax 2-category $\uparrow\mathrm{L}\Pi_2(X)$, with the same objects, arrows and cells as $\uparrow\mathrm{u}\Pi_2(X)$, in Section 1, but extending the *syntactic* comparisons $\varphi(a_1, ..., a_n; \tau, \tau')$ of 1.5 with *absolute comparisons* $\varphi(a, b) \colon a \to b$, only depending on the actual values of the paths a, b (rather than on their being produced by tree-concatenations which are 'directly comparable').

The construction is based on the set of reparametrisation functions $\uparrow\mathbf{I} \to \uparrow\mathbf{I}$ (1.4), i.e. (continuous) increasing, surjective, piecewise affine endofunctions of the standard interval. The main results stated here will be proved in Section 4 (on the basis of various lemmas on these functions, stated and proved in Section 3).

Take two paths $a, b \colon x \to x'$ in X and say that $a \prec b$ if there exist two reparametrisation functions $f, g \colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}$ such that

$$af = bg, \qquad f \geq \mathrm{id} \geq g. \tag{20}$$

(Thus, instead of requiring, as in (18), that a, b be comparable reparametrisations of some *given* normal form, we ask that a, b have a common reparametrisation $af = bg$, with suitable inequalities.)

Then, there is a comparison, constructed with interpolating 2-homotopies (cf. (13))

$$\begin{aligned} &\varphi(a, b) \colon a \to b \qquad\qquad (f \geq \mathrm{id} \geq g), \\ &\varphi(a, b) = [a\varphi_0(\mathrm{id}, f)] \otimes_2 [b\varphi_0(g, \mathrm{id})] \colon a \to af = bg \to b, \\ &\varphi_0(\mathrm{id}, f) \colon \mathrm{id} \to f \colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}, \qquad \varphi_0(g, \mathrm{id}) \colon g \to \mathrm{id} \colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}. \end{aligned} \tag{21}$$

This *does not depend on the choice of* f, g - as we shall prove in Thm. 4.1 - and will be called the *absolute comparison* from a to b.

The relation $a \prec b$ is obviously reflexive, and $\varphi(a,a) = 1_a$. We shall prove in Thm. 4.2 that *it is also transitive, and*:

$$\varphi(a,b) \otimes_2 \varphi(b,c) = \varphi(a,c) \qquad\qquad (\text{for } \ a \prec b \prec c). \tag{22}$$

Our preorder and comparisons are also consistent with n-ary concatenation: given a finite sequence of pairs of comparable paths $a_i \prec b_i \colon x_{i-1} \to x_i (i = 1, ..., n)$, the tensor product of reparametrisation functions (defined in (12)) shows that:

$$\begin{aligned} & a_1 \otimes ... \otimes a_n \ \prec \ b_1 \otimes ... \otimes b_n, \\ & \varphi(a_1, b_1) \otimes ... \otimes \varphi(a_n, b_n) = \varphi(a_1 \otimes ... \otimes a_n, b_1 \otimes ... \otimes b_n). \end{aligned} \tag{23}$$

2.2. Reviewing syntactic comparisons. Now, coming back to the unbiased d-lax 2-category $\uparrow\mathrm{u}\Pi_2(X)$, *every syntactic comparison between multiple concatenations of paths is also an absolute one.* (Comments on this fact can be found in 2.4.)

In fact, a syntactic comparison φ can be expressed as follows (cf. (18)), writing $c = a_1 \otimes ... \otimes a_n$

$$\varphi = \varphi(a_1, ..., a_n; \tau, \tau') = [c\varphi_0(f,g)] \colon cf \to cg \qquad\qquad (f = r(\tau) \leq r(\tau') = g). \tag{24}$$

Since $f \leq g$, the Balance Lemma (3.3) will prove that there exist reparametrisation functions h, k such that $fh = gk$ and $h \geq \mathrm{id} \geq k$. But then

$$cfh = cgk \qquad\qquad (cf \prec cg), \tag{25}$$

and φ coincides with the absolute comparison $\varphi(cf, cg)$, as defined above, in (21)

$$\begin{aligned} \varphi \ &= \ [c\varphi_0(f,g)] = [c(\varphi_0(f, fh) \otimes_2 \varphi_0(gk, g))] \\ &= \ [cf\varphi_0(\mathrm{id}, h)] \otimes_2 [cg\varphi_0(k, \mathrm{id})] = \varphi(cf, cg). \end{aligned} \tag{26}$$

In particular, the relations $1 \otimes a \prec a \prec a \otimes 1$ and $a \otimes (b \otimes c) \prec (a \otimes b) \otimes c$ can be expressed by the following reparametrisation functions g, f, h, k

$$\begin{aligned} & 1 \otimes a = ag, \qquad && g(s) = \max(0, 2s - 1) \qquad && (g \leq \mathrm{id}), \\ & af = a \otimes 1, \qquad && f(s) = \min(2s, 1) \qquad && (f \geq \mathrm{id}), \end{aligned} \tag{27}$$

$$(a \otimes (b \otimes c))h = a \otimes b \otimes c = ((a \otimes b) \otimes c)k \qquad\qquad (h \geq \mathrm{id} \geq k), \tag{28}$$

$$h(s) = \begin{cases} 3s/2, & 0 \leq s \leq 1/3, \\ (3s+1)/4, & 1/3 \leq s \leq 1, \end{cases}$$

$$k(s) = \begin{cases} 3s/4, & 0 \leq s \leq 2/3, \\ (3s-1)/2, & 2/3 \leq s \leq 1. \end{cases}$$

2.3. Definition. An *absolute (unbiased) d-lax 2-category* $\mathbf{A}$ will consist of the following data (and properties).
(dL.0) A set of objects, Ob$\mathbf{A}$.
(dL.1) For any two objects x, y, a category $\mathbf{A}(x, y)$ of *maps* $a\colon x \to y$ and *cells* $\alpha\colon a \to b$, with *main*, or *upper-level*, composition $\alpha \otimes_2 \beta\colon a \to b \to c$ and units $1_a\colon a \to a$.
(dL.2) For any sequence of objects $x_0, ..., x_n$, a functor of *lower n-ary composition*

$$\mathbf{A}(x_0, x_1)\times...\times\mathbf{A}(x_{n-1}, x_n) \to \mathbf{A}(x_0, x_n), \qquad (a_1, ..., a_n) \mapsto a_1\otimes...\otimes a_n,$$

which, for $n = 0$, reduces to a *lower identity* 1_{x_0}.
(dL.3) A preorder $a \prec b$ on every set of maps $\mathbf{A}_1(x, y)$, together with assigned *absolute comparison cells* $\varphi(a, b)$ such that:

$$\varphi(a, b)\colon a \to b \qquad\qquad (\text{for } a \prec b), \tag{29}$$

$$\varphi(a, a) = 1_a, \qquad \varphi(a, b) \otimes_2 \varphi(b, c) = \varphi(a, c) \qquad (\text{for } a \prec b \prec c), \tag{30}$$

(φ is a functor $\mathbf{A}_1(x, y) \to \mathbf{A}(x, y)$, defined on the preorder category).
(dL.4) The preorder and its comparisons are consistent with n-ary composition: given a finite sequence $a_i \prec b_i\colon x_{i-1} \to x_i$ $(i = 1, ..., n)$ of consecutive pairs of comparable arrows

$$\begin{aligned} &a_1\otimes...\otimes a_n \ \prec\ b_1\otimes...\otimes b_n,\\ &\varphi(a_1, b_1)\otimes...\otimes\varphi(a_n, b_n) = \varphi(a_1\otimes...\otimes a_n, b_1\otimes...\otimes b_n). \end{aligned} \tag{31}$$

(dL.5) Given a pair of trees τ, τ' with n leaves and reparametrisation functions $r(\tau) \leq r(\tau')$, every sequence $(a_1, ..., a_n)$ of n consecutive arrows produces the following relation between its concatenations of types τ, τ'

$$\langle a_1, ..., a_n; \tau\rangle \prec \langle a_1, ..., a_n; \tau'\rangle; \tag{32}$$

moreover, the associated absolute comparisons (for variable $(a_1, ..., a_n)$) produce a natural transformation (*syntactic comparison*) of ordinary functors in n variables

$$\begin{aligned} &\varphi(\tau, \tau')\colon \langle -; \tau, \rangle \to \langle -; \tau'\rangle\colon \mathbf{A}(x_0, x_1)\times...\times\mathbf{A}(x_{n-1}, x_n) \to \mathbf{A}(x_0, x_n),\\ &\varphi(\tau, \tau')(a_1, ..., a_n) = \varphi(\langle a_1, ..., a_n; \tau\rangle, \langle a_1, ..., a_n; \tau'\rangle). \end{aligned} \tag{33}$$

An absolute d-lax 2-category $\mathbf{A}$ contains an *associated unbiased d-lax 2-category* $u\mathbf{A}$, just by restricting comparisons to the syntactic ones. (And the latter contains an associated *biased d-lax 2-category* $b(u\mathbf{A})$, by I.3.4c.)

2.4. Theorem. [Main Theorem] *The structure* $\uparrow\mathrm{L}\Pi_2(X)$ *constructed in 2.1 for a d-space* X *with* T_1*-topology is indeed an absolute d-lax 2-category. Its associated unbiased structure coincides with* $\uparrow\mathrm{u}\Pi_2(X)$*, giving*

$$\uparrow\mathrm{b}\Pi_2(X) \subset \uparrow\mathrm{u}\Pi_2(X) \subset \uparrow\mathrm{L}\Pi_2(X). \tag{34}$$

For the second inclusion, the only difference concerns comparison cells.

Comments. This proves that *also the syntactic comparisons are determined by the arrows they link*, and that *all* diagrams of comparison cells in $\uparrow\mathrm{b}\Pi_2(X)$ and $\uparrow\mathrm{u}\Pi_2(X)$ commute, even when not naturally constructed (but exploiting some specific coincidence). However, while the absolute comparisons of $\uparrow\mathrm{L}\Pi_2(X)$ are closed under composition (cf. (30)), the syntactic ones are only known to be closed under 'syntactic composition', in a naturally constructed diagram (cf. (8)).

Proof. See 2.1, 2.2 and the next two sections, where we will prove the arguments we have deferred. Note that, since we have proved the inclusion (34) in (26), the naturality requirement in (dL.5) only concerns the unbiased structure $\uparrow\mathrm{u}\Pi_2(X)$, and has been proved in Part I (I.3.5). ■

2.5. Functoriality. A map $f\colon X \to Y$ of d-spaces *with T_1-topology* induces a *strict* 2-functor between absolute d-lax 2-categories

$$\begin{aligned} &f_*\colon \uparrow\mathrm{L}\Pi_2(X) \to \uparrow\mathrm{L}\Pi_2(Y), \\ &f_*(x) = f(x), \qquad f_*(a\colon x \to x') = (f\circ a\colon fx \to fx'), \qquad f_*[\alpha] = [f\alpha]. \end{aligned} \tag{35}$$

This takes objects, arrows and cells of $\uparrow\mathrm{L}\Pi_2(X)$ to similar items of $\uparrow\mathrm{L}\Pi_2(Y)$, preserving the whole structure: domains, codomains, units, compositions, preorder and comparisons. Thus, if $a \prec b$ in X, we have

$$f_*(a) \prec f_*(b) \text{ in } Y, \qquad f_*(\varphi^X(a,b)) = \varphi^Y(f_*(a), f_*(b)). \tag{36}$$

Furthermore, a (directed) homotopy $\alpha\colon f \to g\colon X \to Y$, represented by a map $\alpha\colon X\times \uparrow\mathbf{I} \to Y$, induces a *lax natural transformation of 2-functors*

$$\begin{aligned} &\alpha_*\colon f_* \to g_*\colon \uparrow\mathrm{L}\Pi_2(X) \to \uparrow\mathrm{L}\Pi_2(Y), \\ &\alpha_*(x) = \alpha(x,-)\colon f(x) \to g(x), \qquad \alpha_*(a\colon x \to x') = [\hat{\alpha}_*(a)], \end{aligned} \tag{37}$$

where $\hat{\alpha}_*(a)$ is the 2-cell defined as in I.2.7.3. (The general definition of lax natural transformations can be seen in I.2.8.)

2.6. Paths up to reparametrisation. The 1-dimensional structure of $\uparrow\mathrm{L}\Pi_2(X)$ is a graph with multiple composition and a consistent preorder, $\prec$.

We shall prove, in Thm. 4.3, that the congruence (of graphs with multiple composition) generated by this preorder is characterised as:

$$a \sim b\colon \qquad \text{there exist two reparametrisation functions } f, g \text{ such that } af = bg. \tag{38}$$

The quotient of this 1-dimensional structure with respect to the congruence yields a *category*, say $\uparrow C(X)$, because of the relations $1\otimes a \prec a \prec a\otimes 1$ and $a\otimes(b\otimes c) \prec (a\otimes b)\otimes c$ already proved (2.2).

In this category, an arrow $\hat{a}\colon x \to x'$ is a class of paths *up to reparametrisation.* Plainly, the fundamental category $\uparrow\Pi_1(X)$ (1.1), where an arrow $[a]\colon x \to x'$ is a class of paths up to 2-homotopy (with fixed endpoints) is a quotient of $\uparrow C(X)$.

It would be interesting to prove that the 2-cells of $\uparrow\mathrm{L}\Pi_2(X)$ induce a (strict) 2-category structure on $\uparrow C(X)$.

3. The machinery of reparametrisation functions

The properties of reparametrisation functions investigated here will be used to prove the main results, in the next section. These functions are written $f, f', g, g'...$ (with no reference to derivatives). The notation $[u, v]$ denotes the closed interval between the real numbers u, v (also when $v \leq u$).

3.1. A PIECEWISE ANALYSIS. A reparametrisation function $f \colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}$ (order-preserving, surjective and piecewise affine) can be defined by assigning a partition (t_i) of the standard interval, together with the corresponding values $u_i = f(t_i)$

$$0 = t_0 < t_1 < ... < t_{n-1} < t_n = 1, \qquad 0 = u_0 \leq u_1 \leq ... \leq u_{n-1} \leq u_n = 1, \tag{39}$$

requiring that f be affine on each interval of the partition (t_i).

The function f is invertible (i.e., strictly increasing) if and only if the sequence (u_i) is also a partition (i.e., strictly increasing); then, its inverse is again a reparametrisation function (the roles of the sequences above being interchanged).

If, for a given f, the partition (t_i) cannot be reduced, then it is determined by f and its points t_i will be called the *characteristic* points of f. Non-minimal partitions are also useful, e.g. when one considers a finite family of functions. (A different 'piecewise analysis', working with a part of the characteristic points, will be used in the proof of 3.4.)

Finally, it will be useful to observe that any two reparametrisation functions $f, g \colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}$ satisfy a *finite-overtaking property*: one can always find a partition (t_i) such that, on each interval $[t_{i-1}, t_i]$, both f and g are affine and, in the interior: either $f < g$, or $f = g$, or $f > g$.

3.2. LEMMA. *Given a finite family $f_1, ..., f_m$ of invertible reparametrisation functions, there exists an invertible reparametrisation function h such that $f_i h \leq \mathrm{id}$, for all i. One can always choose $h \leq \mathrm{id}$. (The property also holds for invertible endomaps of $\uparrow\mathbf{I}$, with the same proof.)*

PROOF. The function $f = f_1 \vee ... \vee f_m$ is again a strictly increasing reparametrisation function, as well as its inverse, $h = f^{-1}$. For all indices i, we have $h \leq f_i^{-1}$ and $f_i h \leq \mathrm{id}$. Adding the identity to the family $f_1, ..., f_m$, we also get the last point. ■

3.3. LEMMA. [Balance Lemma, I] *Let $f, g \colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}$ be reparametrisation functions. If $f \leq g$, there exist reparametrisation functions h, k such that $fh = gk$ and $h \geq \mathrm{id} \geq k$.*

(Comments. This lemma is crucial for our main results. The name comes from the following 'interpretation': think of f, g as objects on the two dishes of weighing scales; think of h, k, as weights we are adding to make the balance, one on each dish. The next lemma will go on in this line.)

PROOF. If f is strictly increasing, it suffices to take $h = f^{-1}g \geq \mathrm{id}$, so that $fh = g = g.\mathrm{id}$.

In the general case, the construction will be based on the reverse *relation* $f^\sharp$ of f (which, in Analysis, would be called a 'maximal monotone operator'). Note that $ff^\sharp = \mathrm{id}$, because f is surjective, and therefore

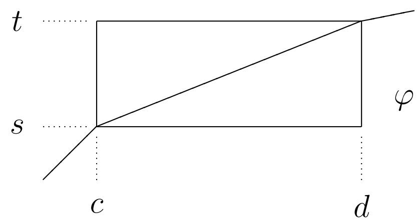
$$f\varphi = g = g.\mathrm{id}. \qquad\qquad \varphi = f^\sharp g. \tag{40}$$

We will prove that one can replace φ, the function g (in its middle occurrence) and id with reparametrisation *functions* h, u, k so that $fh = u = gk$ and all requirements above are met.

We say that $c \in \mathbf{I}$ is a *regular* or *singular* point for the relation $f^\sharp$ according to the fact that $f^\sharp(c) = f^{-1}\{c\}$ (the maximal interval on which $f(t) = c$) be a point or a non-degenerate interval. Note that, since f is piecewise affine, $f^\sharp$ has a *finite number* of singular points, corresponding to the maximal (non trivial) intervals on which f is constant.

Now, $\varphi = f^\sharp g$ is an endorelation of $\mathbf{I}$ such that
(i) φ is singular on finitely many intervals $[c_i, d_i]$ (possibly degenerate) and constant on each of them: $\varphi(x) = [s_i, t_i]$ (non degenerate) for all $x \in [c_i, d_i]$,
(ii) on the complement C of the union of these intervals, $\varphi = f^{-1}g$ is regular and $\geq$ id .

First, we can assume that all these intervals $[c_i, d_i]$ are degenerate, i.e. singular points of φ. Indeed, if $\varphi(x) = [s, t]$ on $[c, d]$, with $c < d$ and $s < t$, we can modify φ on $[c, d]$, taking the affine function whose graph is the diagonal of the rectangle, from (c, s) to (d, t) (so that the property $f\varphi = g$ stays true)



(41)

Now *the whole graph of φ is (weakly) above the diagonal*, because of (ii) (formerly, part of the 'singular rectangles' could be below the diagonal). We shall write $\varphi \geq \mathrm{id}$, for short.

Second, let us consider a point c which is singular for φ: $\varphi(c) = f^\sharp g(c) = [s, t]$, with $c \leq s < t$ (because $\varphi \geq \mathrm{id}$). We will modify locally this relation, to remove the singularity. Since $c < 1$, we will operate on a right neighbourhood $c_\varepsilon = [c, c + \varepsilon]$ (see the figure (44)).

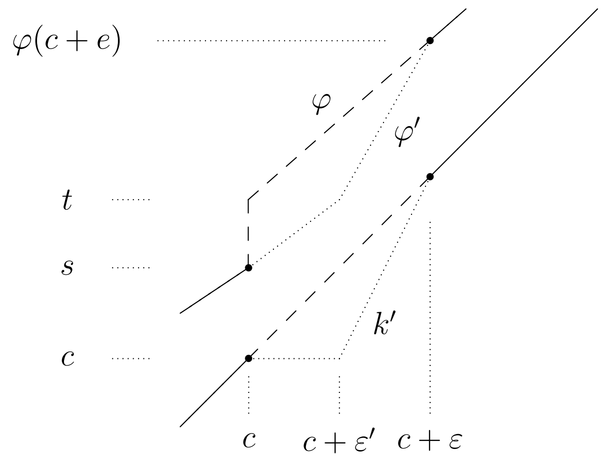
For $\varepsilon > 0$, sufficiently small, φ is regular on $]c, c + \varepsilon]$ (because there is a finite number of singular points) and *moreover* f is affine and strictly increasing on $[t, \varphi(c + \varepsilon)]$ (because $g(c)$ is a singular point of $f^\sharp$); we shall say that ε is *sufficiently small for φ at c*. Define the relation $\varphi' = \varphi_{c\varepsilon\varepsilon'}$ as follows

$$\varphi'(x) = \varphi(x), \qquad\qquad \text{for } \; x \notin c_\varepsilon, \tag{42}$$

$$\begin{aligned} &\varphi' \text{ is an affine function on } [c, c+\varepsilon'] \text{ and } [c+\varepsilon', c+\varepsilon], \text{ with:} \\ &\varphi'(c) = s, \qquad \varphi'(c+\varepsilon') = t, \qquad \varphi'(c+\varepsilon) = \varphi(c+\varepsilon), \end{aligned} \tag{43}$$

where $\varepsilon' < \varepsilon$ is *sufficiently small* so that the graph of φ' *still stays above the diagonal*: $\varepsilon' \leq t - s$. (Taking $\varepsilon' = \min(\varepsilon/2, t-s)$ satisfies these conditions, but depends on s, t, which we do not want).

In the example below, we represent the graph of φ (dashed) and φ' (dotted), with the common parts as solid lines; similarly for the identity (dashed) and its modification $k' = k_{c\varepsilon\varepsilon'}$ (defined below; dotted)



(44)

The relevant point is that the composite $u' = f\varphi'$ only depends on $g, c, \varepsilon, \varepsilon'$ (*not on* f):

$$f\varphi'(x) = f\varphi(x) = g(x), \qquad \text{for } x \notin c_\varepsilon, \tag{45}$$

$$\begin{aligned} &f\varphi' \text{ is affine on } [c, c+\varepsilon'] \text{ and } [c+\varepsilon', c+\varepsilon], \text{ with:} \\ &f\varphi'(c) = f\varphi'(c+\varepsilon') = g(c), \qquad f\varphi'(c+\varepsilon) = g(c+\varepsilon), \end{aligned} \tag{46}$$

and can also be obtained as $u' = gk'$, by the reparametrisation function $k' = k_{c\varepsilon\varepsilon'} \leq \mathrm{id}$, represented above and defined as follows (this also shows that u' is again a reparametrisation function)

$$k'(x) = x, \qquad \text{for } x \notin c_\varepsilon, \tag{47}$$

$$\begin{aligned} &k' \text{ is affine on } [c, c+\varepsilon'] \text{ and } [c+\varepsilon', c+\varepsilon], \text{ with:} \\ &k'(c) = k'(c+\varepsilon') = c, \qquad k'(c+\varepsilon) = c+\varepsilon. \end{aligned} \tag{48}$$

Now, let $\{c_1, ..., c_n\}$ be the set of all singular points of φ. Choose $\varepsilon > \varepsilon' > 0$ sufficiently small for φ, at all c_i, and such that all the intervals $[c_i - \varepsilon, c_i + \varepsilon]$ are disjoint. Then, the modifications of φ and id at the various c_i, as described above, do not interfere; operating all of them, we get two reparametrisation *functions* h, k (only depending on $f, g, \varepsilon, \varepsilon'$) such that $h \geq \mathrm{id} \geq k$ and $fh = gk = u$ (only depending on $c_1, ..., c_n, \varepsilon, \varepsilon'$). ■

3.4. Lemma. [Balance Lemma, II] *Let $f, g\colon \uparrow\mathbf{I} \to \uparrow\mathbf{I}$ be reparametrisation functions. Then, there exist reparametrisation functions h, k such that $fh = gk$. If $f, g \leq \mathrm{id}$, we can choose $h, k \leq \mathrm{id}$.*

More generally, for a finite family $f_1, ..., f_m$ of reparametrisation functions, there exist a second family $h_1, ..., h_m$ such that $f_1h_1 = ... = f_mh_m$. If all f_i are $\leq \mathrm{id}$, we can choose $h_i \leq \mathrm{id}$.

(This statement completes the previous one and can be given the same 'interpretation'.)

Proof. We shall write 'function' for *reparametrisation function*. The proof is based on the finite set $C(f)$ of *constant values* of a function f, i.e. the numbers c such that $f^{-1}\{c\}$ is a non-trivial interval. And on the fact that, precomposing with any function h, we have $C(fh) \supset C(f)$. The constant values of f coincide with the singular points of $f^\sharp$, used in the previous proof.

(One can also give here a proof similar to the previous one, but the present pattern seems clearer. Moreover, this proof also works replacing, in the definition of reparametrisation function, the condition 'piecewise affine' with the more general property of 'having a finite set of constant values'. It can likely be adapted for an infinite subset, necessarily countable.)

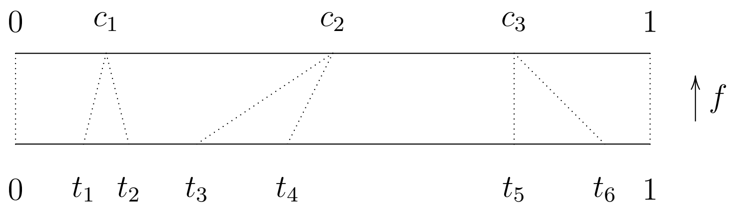
Let $C(f)$ consist of the values $c_1 < ... < c_n$ in $\uparrow\mathbf{I}$ $(n \geq 0)$. Thus, f is constant on their pre-images, which are non-trivial intervals

$$[t_{2i-1}, t_{2i}] = f^{-1}\{c_i\}, \qquad\qquad i = 1, ..., n, \tag{49}$$

while it is strictly increasing (and *piecewise* affine) on the remaining intervals $[t_{2i}, t_{2i+1}]$ $(i = 0, ..., n)$ of the partition

$$0 \leq t_1 < t_2 < ... < t_{2n-1} < t_{2n} \leq 1 \qquad\qquad (t_0 = 0,\ t_{2n+1} = 1). \tag{50}$$

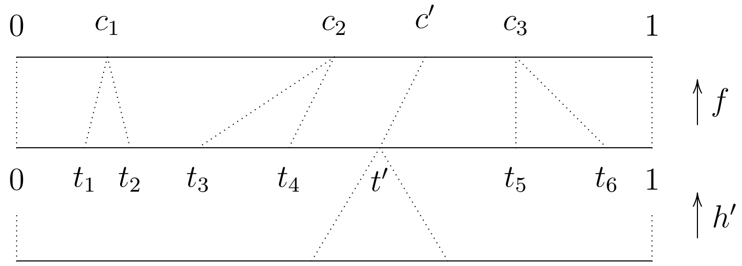
This will be called the C-*partition* of the function f (and need not contain all the characteristic points of f, cf. 3.1). It has $2n + 1$ (non-trivial) intervals *when* $c_1 > 0$ and $c_n < 1$, as in the figure below, but it has $2n$ or $2n - 1$ intervals when only one of these conditions occurs, or none (and one interval if there are no constant values: $n = 0$).



(51)

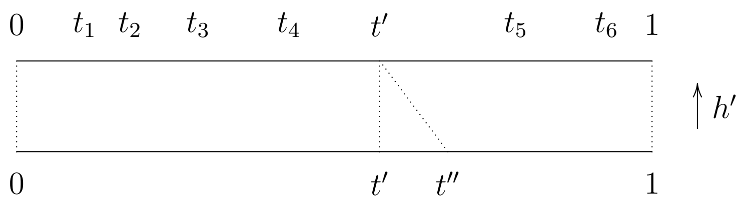
(a) First, we prove that two 'functions' f, g can be linked with an *invertible* function k, so that $f = gk$, if and only if $C(f) = C(g)$. The necessity of this condition being obvious, let us assume it holds. We want k to turn the C-partition of f (written as above, in (50)) into the corresponding partition of g; this defines k on the points $t_1, ..., t_{2n}$. Then, k is

affine on the intervals $[t_{2i-1}, t_{2i}]$ on which f is constant (and turns them into the intervals on which g is constant), while it is defined as $g^{-1}f$ on the intervals on which f is strictly increasing (and turns them into the intervals on which g is strictly increasing).
(b) Now, let us come back to the general problem. Suppose that c' is *not* a constant value for f, so that there is a unique point t' where $f(t') = c'$. Then, it is easy to construct a function h' such that $C(fh') = C(f) \cup \{c'\}$: any 'function' having precisely one constant value, at t', will do. This will introduce two new points in the C-partition of fh' (and - perhaps - move the remaining ones, which has no relevance in our argument)



(52)

Proceeding this way we get, in a finite number of steps, a composite fh with $C(fh) = C(f) \cup C(g)$. Proceeding symmetrically on g, we get a composite gk with $C(gk) = C(f) \cup C(g)$. Applying (a), we can further precompose with a suitable (invertible) function k', so that $fh = g(kk')$.
(c) Suppose now that $f, g \leq \mathrm{id}$. Then, *all the constant values of such functions are necessarily* < 1, which implies that, in point (b), *we can choose two functions* $h, k \leq \mathrm{id}$ with $C(fh) = C(f) \cup C(g) = C(gk)$, as shown in the following elementary step (adding the constant value $t' < 1$, on the closed interval from t' to some t'', with $t' < t'' < 1$)



(53)

Now, (a) tells us that there exist an *invertible* function k' such that $fh = g(kk')$, while Lemma 3.2 gives the existence of a function $k'' \leq \mathrm{id}$ such that $k'k'' \leq \mathrm{id}$. Finally, the equality $f(hk'') = g(kk'k'')$ satisfies our conditions, all the functions h, k'', k and $k'k''$ being $\leq \mathrm{id}$.
(d) For a finite family $f_1, ..., f_m$, we begin with choosing $k_1, ..., k_m$ so that all $C(f_ik_i)$ coincide with $C = C(f_1) \cup ... \cup C(f_m)$. Then we choose m invertible functions k'_i so that all $f_ik_ik'_i$ coincide with some fixed function u having $C(u) = C$ (for instance, f_1k_1). The last point is proved as above. ■

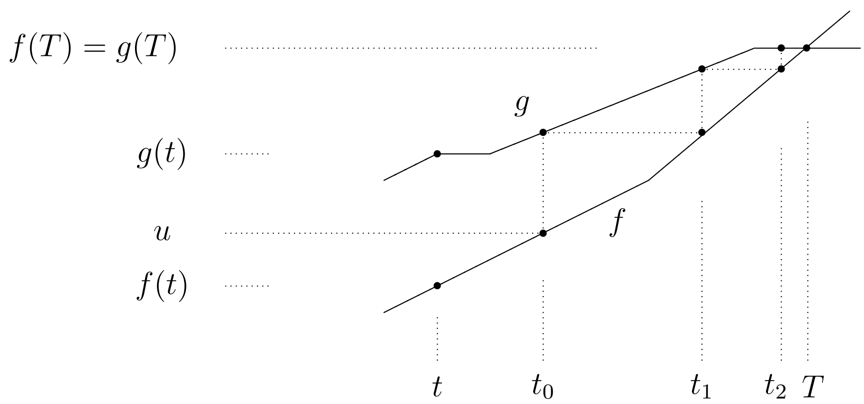
3.5. LEMMA. *Let $f, g\colon \mathbf{I} \to \mathbf{I}$ be increasing continuous functions with $f(1) = g(1)$. Let $a\colon \mathbf{I} \to X$ be a continuous mapping with values in a T_1-topological space. If $af = ag$ then, for every $t \in \mathbf{I}$, the mapping a is constant on the closed interval between $f(t)$ and $g(t)$. (Dropping the hypothesis on X, this need not be true.)*

As a consequence, the equivalence classes of the coequaliser of f, g for T_1-spaces are closed intervals.

PROOF. (This lemma is not used in the present paper, but is included as a preparation for the following two results.) Let us begin with the first statement, for some $t \in \mathbf{I}$; if $f(t) = g(t)$ there is nothing to prove, and we can suppose that $f(t) < g(t)$. Let

$$T = \min\{\tau \geq t \mid f(\tau) = g(\tau)\}, \tag{54}$$

(which exists because $f(1) = g(1)$), so that $f < g$ in the interval $[t, T[$. We want to prove that, for every $u \in [f(t), g(t)]$, $a(u) = af(T)$. The argument proceeds constructing an increasing sequence (t_n), as in the figure below



(55)

First, since $f(t) \leq u \leq g(t) \leq g(T) = f(T)$, there exists a point t_0 such that

$$\begin{aligned} &f(t_0) = u, \qquad\qquad t \leq t_0 \leq T, \\ &f(t_0) \leq g(t_0) \leq g(T) = f(T). \end{aligned} \tag{56}$$

Second, there exists a point t_1 such that

$$f(t_1) = g(t_0), \qquad\qquad t_0 \leq t_1 \leq T. \tag{57}$$

Proceeding this way, we get an increasing sequence (t_n) with $f(t_{n+1}) = g(t_n)$ (which *may* be eventually constant at T, as in the figure above), so that:

$$a(u) = af(t_0) = ag(t_0) = af(t_1) = ... = af(t_n) = ag(t_n) = af(t_{n+1}) \; ... \tag{58}$$

This sequence converges to $t' \in [t, T]$; but $f(t') = g(t')$, by continuity (and uniqueness of the limit, in $\mathbf{I}$), whence $t' = T$. Now, the continuous mapping a coincides on all $f(t_n)$, whence the constant sequence $a(u) = af(t_n)$ converges to $af(T)$ in X; thus, $af(T)$ belongs to the closure of $a(u)$ in X, and coincides with this point.

Dropping the T_1-hypothesis on X, it is easy to give a counterexample for $X = \{0, 1\}$ with the Sierpinski topology: $\{1\}$ is open, $\{0\}$ is not. Take $f(t) = t$, $g(t) = (t+1)/2$ and $a\colon \mathbf{I} \to \{0, 1\}$ the function which annihilates - precisely - on the closed subset $\{g^n(0)|n \geq 0\} \cup \{1\}$.

We end with the last statement. Let $a\colon \mathbf{I} \to \mathbf{I}/\sim$ be the T_1-coequaliser of f, g. The equivalence relation $u \sim u'$ (associated to the mapping a) is implied by the existence of points $t_0, ..., t_n \in \mathbf{I}$ such that:

$$u = f(t_0), \qquad g(t_0) = f(t_1), ..., \qquad g(t_{n-1}) = f(t_n), \qquad g(t_n) = u', \tag{59}$$

and we have proved that every interval $[f(t), g(t)]$ (recall that this notation does not require $f(t) \leq g(t)$) is contained in an equivalence class. Since, for $t = t_0, ..., t_n$, these intervals meet pairwise, the interval $[u, u']$ is also contained in an equivalence class. Finally, keeping u fixed, the equivalence class of u coincides with the union of the intervals $[u, u']$ for $u' \sim u$, which is necessarily an interval; and a closed one, since the points of $\mathbf{I}/\sim$ are closed. ■

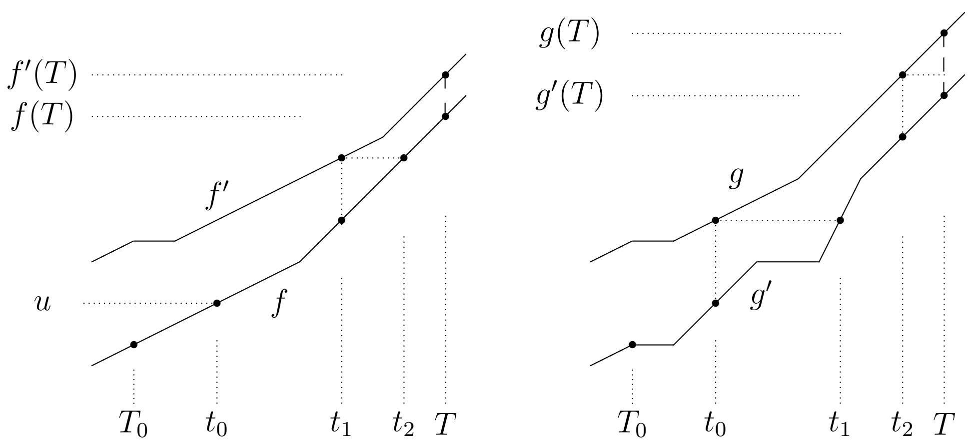
3.6. Lemma. *Let $f, g, f', g'\colon \mathbf{I} \to \mathbf{I}$ be increasing continuous functions and $a, b\colon \mathbf{I} \to X$ continuous mappings with values in a T_1-topological space, such that*

$$\begin{array}{lll} af = bg, & af' = bg', & \text{in } [T_0, T], \\ f(t) < f'(t), & g'(t) < g(t), & \text{for } T_0 < t < T, \\ a(u) = x_0, \ \text{for } u \in [f(T), f'(T)], & & b(u) = x_0, \ \text{for } u \in [g'(T), g(T)]. \end{array} \tag{60}$$

Then the mapping a is constant at x_0 on the interval $[f(T_0), f'(T)]$ while b is constant at x_0 on the interval $[g'(T_0), g(T)]$.

Proof. Let us take $u \in [f(T_0), f'(T)]$ and prove that $a(u) = x_0$. First, we assume that the inequalities $f(t) < f'(t)$ and $g'(t) < g(t)$ also hold at T_0.

The argument is an extension of the one used in 3.5: we construct a (finite or infinite) sequence working, alternatively, on the pair f, f' or g, g', as in the following figure



(61)

Looking at the left figure, we have $f(T_0) \leq u \leq f'(T)$. If $u \geq f(T)$, then $a(u) = x_0$ and we are done; otherwise $u < f(T)$ and there exists a point t_0 such that

$$f(t_0) = u, \qquad T_0 \leq t_0 \leq T \qquad (a(u) = af(t_0) = bg(t_0)). \tag{62}$$

Now, looking at the right figure, $g'(t_0) \leq g(t_0) \leq g(T)$. If $g(t_0) \geq g'(T)$, then $a(u) = bg(t_0) = x_0$; otherwise, $g(t_0) < g'(T)$ and there exists a point t_1 such that

$$g'(t_1) = g(t_0), \qquad T_0 \leq t_0 \leq t_1 \leq T \qquad (a(u) = bg(t_0) = bg'(t_1) = af'(t_1)). \tag{63}$$

Coming back to the left diagram, since $f(t_1) \leq f'(t_1) \leq f'(T)$, either $f'(t_1) \geq f(T)$ (and we are finished) or there exists a point t_2 such that

$$f(t_2) = f'(t_1), \qquad T_0 \leq t_0 < t_1 \leq t_2 \leq T \qquad (a(u) = af'(t_1) = af(t_2) = bg(t_2)). \tag{64}$$

Proceeding this way, either we get $a(u) = x_0$ in a finite number of steps (as in the figure above) or we construct an increasing sequence (t_n) with

$$\begin{aligned} &g'(t_{2n+1}) = g(t_{2n}), \qquad f(t_{2n+2}) = f'(t_{2n+1}) \qquad (n \geq 0), \\ &a(u) = af(t_0) = bg(t_0) = bg'(t_1) = af'(t_1) = af(t_2) = ... \end{aligned} \tag{65}$$

The sequence (t_n) converges to some $t' \in [T_0, T]$; but $f(t') = f'(t')$, by continuity (and uniqueness of limit in $\mathbf{I}$), whence our hypotheses (including the initial assumption $f(T_0) < f'(T_0)$) imply that $t' = T$ (and the interval $[f(T), f'(T)]$ is degenerate). Now, the continuous mapping a coincides on all $f(t_{2n})$, whence the constant sequence $a(u) = af(t_{2n})$ converges to $af(T)$ in X; since $a(u)$ is closed in X, $a(u) = af(T) = x_0$.

Finally, let us drop the initial assumption at T_0. Applying the previous argument to all intervals $[T_1, T]$, with $T_0 < T_1 < T$, we have that $a(u) = x_0$, for $u \in]f(T_0), f'(T)]$ (which is not empty, because $f(T_0) \leq f(T_1) < f'(T_1) \leq f'(T)$). Thus, the point $af(T_0)$ belongs to the closure of x_0, which is reduced to this point. ■

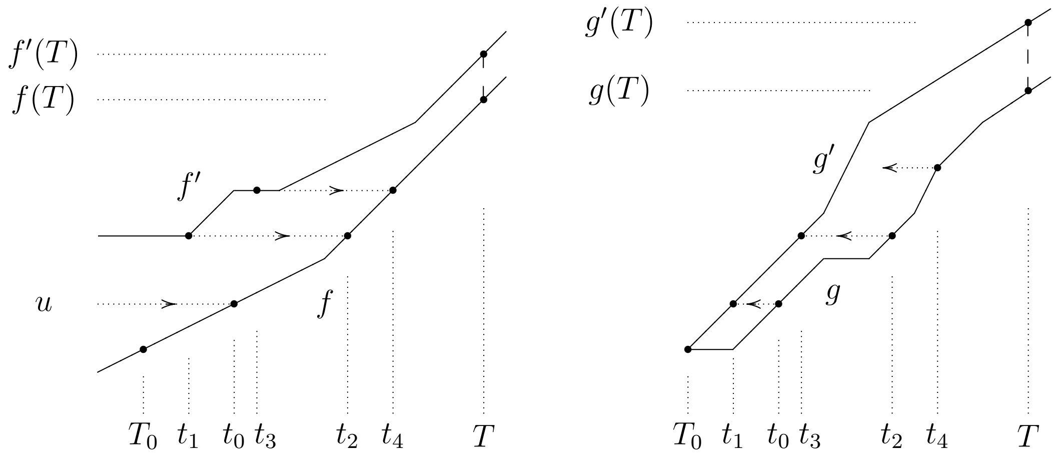
3.7. Lemma. *Let $f, g, f', g' \colon \mathbf{I} \to \mathbf{I}$ be increasing continuous functions and $a, b \colon \mathbf{I} \to X$ continuous mappings with values in a T_1-topological space, such that*

$$\begin{array}{lll} af = bg, & af' = bg', & \textit{in } [T_0, T], \\ f(t) < f'(t), & g'(t) < g(t), & \textit{for } T_0 < t < T, \\ a(u) = x_0, \ \textit{for } u \in [f(T), f'(T)], & & b(u) = x_0, \ \textit{for } u \in [g(T), g'(T)], \\ (f(T_0) = f'(T_0)) \quad \text{or} \quad (g(T_0) = g'(T_0)). & & \end{array} \tag{66}$$

Then the mapping a is constant at x_0 on the interval $[f(T_0), f'(T_0)]$ while b is constant at x_0 on the interval $[g(T_0), g'(T_0)]$. (The statement is similar to the previous one, but the inequality between g, g' is reversed.).

Proof. We choose the case $g(T_0) = g'(T_0)$. Taking $u \in [f(T_0), f'(T_0)]$, we shall prove that $a(u) = x_0$.

First, we assume that $f(T_0) < f'(T_0)$. Working again, alternatively, on the pair f, f' or g, g', as in the figure below, we construct now a (finite or infinite) sequence (t_n) such that both *subsequences* (t_{2n}) and (t_{2n+1}) are increasing



(67)

Looking at the left figure, we have $f(T_0) \le u \le f'(T_0) \le f'(T)$. If $u \ge f(T)$, then $a(u) = x_0$ and we are done; otherwise $u < f(T)$ and there exists a point t_0 such that

$$f(t_0) = u, \qquad T_0 \le t_0 \le T \qquad (a(u) = af(t_0) = bg(t_0)). \tag{68}$$

Now, looking at the right figure, $g'(T_0) = g(T_0) \le g(t_0) \le g'(t_0)$, and there exists a point t_1 such that

$$g'(t_1) = g(t_0), \qquad T_0 \le t_1 \le t_0 \le T \qquad (a(u) = bg(t_0) = bg'(t_1) = af'(t_1)). \tag{69}$$

Coming back to the left diagram, we have $f(t_0) = u \le f'(T_0) \le f'(t_1) \le f'(T)$. If $f'(t_1) \ge f(T)$, we are finished: $a(u) = af'(t_1) = x_0$; otherwise $f'(t_1) < f(T)$ and there exists a point t_2 such that

$$f(t_2) = f'(t_1), \qquad T_0 \le t_0 \le t_2 \le T \qquad (a(u) = af'(t_1) = af(t_2) = bg(t_2)). \tag{70}$$

At the right, we have now $g'(t_1) = g(t_0) \le g(t_2) \le g(T) \le g'(T)$, and there exists a point t_3 such that

$$g'(t_3) = g(t_2), \qquad t_1 \le t_3 \le T \qquad (a(u) = bg(t_2) = bg'(t_3) = af'(t_3)). \tag{71}$$

Proceeding this way, either we get $a(u) = x_0$ in a finite number of steps or we construct a sequence (t_n) where (t_{2n}) and (t_{2n+1}) are increasing, and:

$$\begin{aligned} &g'(t_{2n+1}) = g(t_{2n}), \qquad f(t_{2n+2}) = f'(t_{2n+1}) \qquad (n \ge 0), \\ &a(u) = af(t_0) = bg(t_0) = bg'(t_1) = af'(t_1) = af(t_2) = ... \end{aligned} \tag{72}$$

The sequence (t_{2n}) converges to some $t' \in [T_0, T]$; and $f(t') = f'(t')$, by continuity (and uniqueness of limit in $\mathbf{I}$), whence our hypotheses imply that $t' = T$ (and the interval $[f(T), f'(T)]$ is degenerate). Now, the continuous mapping a coincides on all $f(t_{2n})$, whence the constant sequence $a(u) = af(t_{2n})$ converges to $af(T)$ in X; since $a(u)$ is closed in X, $a(u) = af(T) = x_0$. Dropping the initial assumption at T_0, we proceed as in the last point of the proof of 3.6. ∎

3.8. LEMMA. *Let $f, g, f', g' \colon \mathbf{I} \to \mathbf{I}$ be reparametrisation functions and $a, b \colon \mathbf{I} \to X$ continuous mappings with values in a T_1-topological space, such that*

$$af = bg, \qquad\qquad af' = bg'. \tag{73}$$

Then

$$a(f \wedge f') = b(g \wedge g'), \qquad\qquad a(f \vee f') = b(g \vee g'). \tag{74}$$

PROOF. This is the second crucial point of this section. We start with noting that the thesis certainly holds at every point $t \in \mathbf{I}$ where the condition $P(t)$ is true

$$P(t)\colon \qquad (f(t) \leq f'(t) \text{ and } g(t) \leq g'(t)) \text{ or } (f(t) \geq f'(t) \text{ and } g(t) \geq g'(t)).$$

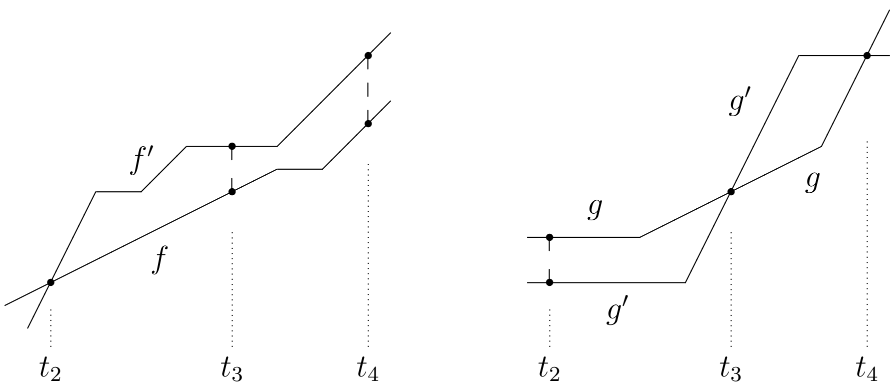
Furthermore, it also holds at every point t making $Q(t)$ true

$$Q(t)\colon \qquad (a \text{ is constant on } [f(t), f'(t)]) \text{ and } (b \text{ is constant on } [g(t), g'(t)]).$$

In fact, one can (easily) give examples where the property P holds everywhere (but not Q), and (trivially) others where $a = b$ is constant and Q holds everywhere (for arbitrary f, g, f', g'). The general situation is a 'mixture' of these cases, as in the figure below. *We shall prove, in fact, that the disjunction $P \vee Q$ is always true in* $\mathbf{I}$. It will be useful to note that the negation of $P(t)$ can be written as:

$$P'(t)\colon \qquad (f(t) < f'(t) \text{ and } g(t) > g'(t)) \text{ or } (f(t) > f'(t) \text{ and } g(t) < g'(t)).$$

By the finite-overtaking property (3.1), there exists a partition $0 = t_0 < t_1 < ... < t_n = 1$ of $\mathbf{I}$ such that, on the interior of each of its intervals, we always have $P(t)$ or always $P'(t)$



(75)

Assume that this partition is minimal, so that at all points t_i we have

$$f(t_i) = f'(t_i) \qquad \text{or} \qquad g(t_i) = g'(t_i), \tag{76}$$

and possibly both. (One can note that P'-*intervals* can be *contiguous*, while P-*intervals* cannot, because of minimality and of being defined by weak inequalities; but these facts have no relevance below.)

We prove now that, on each of these intervals:
(i) either P is always true or Q is always true,
(ii) Q always holds at the endpoints.

Proceeding leftwards, consider the interval $[t_{i-1}, t_i]$ and assume we have already proved the thesis for all intervals at its right. In particular, we have $Q(t_i)$ (which holds trivially at the beginning, for $i = n$). Now, if P holds on $[t_{i-1}, t_i]$, (as in the interval $[t_3, t_4]$ of the figure above), then (i) holds trivially there; applying Lemma 3.7, the properties $Q(t_i)$ and (76) give $Q(t_{i-1})$. Otherwise, P' holds in $]t_{i-1}, t_i[$ (as in $]t_2, t_3[$, above) and, applying Lemma 3.6, the property $Q(t_i)$ implies Q on the whole interval $[t_{i-1}, t_i]$. ■

4. Proof of the main results

We prove three theorems which contain the results we have used to construct $\uparrow\mathrm{L}\Pi_2(X)$ (2.1) and to determine the congruence of paths up to reparametrisation (2.5).

4.1. THEOREM. *Let X be a d-space with T_1-topology, and $a, b\colon \uparrow\mathbf{I} \to X$ two paths with the same endpoints. If $a \prec b$ (cf. (20)), the definition of $\varphi(a, b)\colon a \to b$ in (21) does not depend on the choice of f, g.*

PROOF. Let $af = bg$, $af' = bg'$, with $f \geq \mathrm{id} \geq g$ and $f' \geq \mathrm{id} \geq g'$. Applying Lemma 3.8 we can assume $f \leq f'$, $g \leq g'$ (since we can replace f, g with $f \wedge f'$ and $g \wedge g'$, respectively).

Now, we apply the Balance Lemma (3.3) to $f \leq f'$: there exist reparametrisation functions h, k such that $h \geq \mathrm{id} \geq k$ and $fh = f'k$. This allows us to form the following diagram, with a new path c

$$\begin{array}{ccccc} & & af = bg & & \\ & \overset{\varphi_1}{\nearrow} & \downarrow \lambda & \overset{\psi_1}{\searrow} & \\ a & \xrightarrow{\ \varphi_3\ } & c & \xrightarrow{\ \psi_3\ } & b \\ & \underset{\varphi_2}{\searrow} & \downarrow \lambda' & \underset{\psi_2}{\nearrow} & \\ & & af' = bg' & & \end{array} \tag{77}$$

$$\mathrm{id} \leq f \leq f', \qquad g \leq g' \leq \mathrm{id}, \qquad h \geq \mathrm{id} \geq k, \qquad c = afh = bgh = af'k = bg'k.$$

At the left, we have three arrows φ_i of type $[a\varphi_0(\mathrm{id}, -)]$. At the right, we have three arrows ψ_i of type $[b\varphi_0(-, \mathrm{id})]$. In the middle, the arrows λ, λ' are simultaneously of both types

$$\begin{aligned} \lambda &= [a\varphi_0(f, fh)] = [af\varphi_0(\mathrm{id}, h)] = [bg\varphi_0(\mathrm{id}, h)] = [b\varphi_0(g, gh)], \\ \lambda' &= [a\varphi_0(f'k, f')] = [b\varphi_0(g'k, g')]. \end{aligned} \tag{78}$$

Therefore, the four triangles above commute and the outer diamond as well. For instance:

$$\varphi_1 \otimes_2 \lambda = a[\varphi_0(\mathrm{id}, f)] \otimes_2 a[\varphi_0(f, fh)]) = a[\varphi_0(\mathrm{id}, fh)] = \varphi_3. \tag{79}$$

■

4.2. THEOREM. *Let X be a d-space with T_1-topology. The relation $a \prec b$ (defined in (20)) is transitive, and:*

$$\varphi(a, b) \otimes_2 \varphi(b, c) = \varphi(a, c) \qquad\qquad (\textit{for } a \prec b \prec c). \tag{80}$$

PROOF. Assume that

$$af = bg, \qquad f \geq \mathrm{id} \geq g; \qquad\qquad bf' = cg', \qquad f' \geq \mathrm{id} \geq g'; \tag{81}$$

and apply the Balance Lemma (3.3) to the functions 'working' on the middle path b, that is $g \leq \mathrm{id} \leq f'$. There exist reparametrisation functions h, k with $gh = f'k$ and $h \geq \mathrm{id} \geq k$. Therefore $a \prec c$

$$a(fh) = bgh = bf'k = c(g'k), \qquad\qquad fh \geq \mathrm{id} \geq g'k, \tag{82}$$

and we have a diagram whose upper row is $\varphi(a, b) \otimes_2 \varphi(b, c)$, while the lower one is $\varphi(a, c)$

$$\begin{array}{ccccccccc}
a & \xrightarrow{\varphi_1} & af = bg & \xrightarrow{\chi_1} & b & \xrightarrow{\chi_2} & bf' = cg' & \xrightarrow{\psi_1} & c \\
& \searrow^{\varphi_2} & \downarrow \lambda & & & & \uparrow \lambda' & \nearrow_{\psi_2} & \\
& & afh = bgh & = & = & = & bf'k = cg'k & &
\end{array} \tag{83}$$

This diagram is shown to commute, much as in 4.1. In fact:
- the arrows φ_i are of type $[a\varphi_0(\mathrm{id}, -)]$,
- the arrows χ_i are of type $[b\varphi_0(-, -)]$,
- the arrows ψ_i are of type $[c\varphi_0(-, \mathrm{id})]$,
- the arrow λ is of type $[a\varphi_0(-, -)]$ *and* of type $[b\varphi_0(-, -)]$:

$$\lambda = [a\varphi_0(f, fh)] = [af\varphi_0(\mathrm{id}, h)] = [bg\varphi_0(\mathrm{id}, h)] = [b\varphi_0(g, gh)], \tag{84}$$

and symmetrically λ' is of type $[b\varphi_0(-, -)]$ *and* $[c\varphi_0(-, -)]$. It is now immediate to see that (83) commutes. ■

4.3. THEOREM. *In* $\uparrow\mathrm{L}\Pi_2(X)$*, the congruence* $a \sim b$ (of graphs with multiple composition) *generated by the preorder $\prec$ is characterised as:*
(a) $a \sim b$*:* there exist two reparametrisation functions f, g such that $af = bg$.

It also coincides with the equivalence relation generated by the following two equivalence relations:
(b) $a \sim_1 b$*:* there exist two reparametrisation functions $f, g \leq \mathrm{id}$ such that $af = bg$,
(c) $a \sim_2 b$*:* there exist two reparametrisation functions $f, g \geq \mathrm{id}$ such that $af = bg$.

Proof. Again, 'function' will mean *reparametrisation function.*

First, the relation (a) is transitive: if $af = bg$ and $bf' = cg'$, apply the II Balance Lemma (3.4) to the functions g, f': there exist two functions h, k such that $gh = f'k$, whence $a(fh) = bgh = bf'k = c(g'k)$. It is also consistent with multiple composition, because of the tensor product of our functions (cf. (12)). The same properties hold for the relations (b) and (c), with the same proof (based on 3.4, again).

Second, let us prove that (a) implies the transitive relation generated by (b) and (c), the converse being obvious. If $af = bg$, the inequality $f \wedge \mathrm{id} \leq f$ implies (by the Balance Lemma, 3.3) the existence of functions h, k such that $h \geq \mathrm{id} \geq k$ and $(f \wedge \mathrm{id})h = fk$, so that

$$a \sim_1 a(f \wedge \mathrm{id}) \sim_2 a(f \wedge \mathrm{id})h = afk \sim_1 af; \qquad (85)$$

similarly, we can link b with $bg = af$.

Finally, it is obvious that $a \sim b$ is implied by $a \prec b$, as the latter condition says that $af = bg$ with $f \geq \mathrm{id} \geq g$. Conversely, to show that our equivalence relation is generated by the preorder, it suffices to consider the relations (b) and (c); let $af = bg$; if $f, g \leq \mathrm{id}$, then $af \prec a$ and $bg \prec b$; on the other hand, if $f, g \geq \mathrm{id}$, then $a \prec af$ and $b \prec bg$. ∎

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