Modelling the fundamental category of a directed space (*)

LECTURE NOTES

for three lectures in: "**Algebraic Topology, Concurrency and Rewriting**"

Geocal06, CIRM, Marseille-Luminy, Jan 30 - Feb 3, 2006.

Marco Grandis

Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, 16146-Genova, Italy.

e-mail: `grandis@dima.unige.it`

home page: `http://www.dima.unige.it/~grandis/`

MSC: 55Pxx, 18A40, 18D05, 68Q85.

Keywords: homotopy theory, adjunctions, reflective subcategories, lax categories, directed algebraic topology, fundamental category, concurrent processes.

Abstract. Directed Algebraic Topology is a recent field, deeply linked with ordinary and higher dimensional Category Theory. A 'directed space', e.g. an ordered topological space, has directed homotopies (which are generally non reversible) and fundamental *n-categories* (replacing the fundamental n-groupoids of the classical case). Finding a simple model of the latter is a non-trivial problem, whose solution gives relevant information on the given 'space'; a problem which is of interest for applications as well as in general Category Theory.

Here we give a presentation of the work "The shape of a category up to directed homotopy" [G3], with a new extension on 'surjective models' and a brief introduction to a 2-dimensional analysis, via the fundamental lax 2-category introduced in [G5]. Both extension are motivated by studying the singularities of 3-dimensional ordered spaces, for which the previous analysis is often insufficient.

Introduction

Directed Algebraic Topology studies 'directed spaces' in some sense, where paths and homotopies cannot generally be reversed; for instance: ordered topological spaces, 'spaces with distinguished paths', 'inequilogical spaces', simplicial and cubical sets, etc. Its present applications deal mostly with the analysis of concurrent processes (see [FGR, FRGH, Ga, GG, Go]), but its natural range covers non reversible phenomena, in any domain.

The study of invariance under directed homotopy is far richer and more complex than in the classical case, where homotopy equivalence between 'spaces' produces a plain equivalence of their fundamental groupoids, for which one can simply take - as a minimal model - the categorical skeleton. Our directed structures have a *fundamental category* $\uparrow\Pi_1(X)$; this must be studied up to appropriate notions of *directed homotopy equivalence of categories*, which are *more general* than ordinary

(*) Work supported by MIUR Research Projects.

categorical equivalence: the latter would often be of no use, since the fundamental category of an ordered topological space, for instance, is always skeletal (the same situation shows that the fundamental *monoids* $\uparrow\pi_1(X, x_0)$ can be trivial, without $\uparrow\Pi_1(X)$ being so; cf. 1.2). Such a study has been carried on in a previous work [G3]. Other references for Directed Algebraic Topology and its applications can be found there.

In [G3] we have introduced two (dual) directed notions, which take care, respectively, of variation 'in the future' or 'from the past': *future equivalence* (a symmetric version of an adjunction, with two units) and its dual, a *past equivalence* (with two counits); and studied how to combine them. *Minimal models* of a category, up to these equivalences, have been introduced to better understand the 'shape' and properties of the category we are analysing, and of the process it represents.

The paper [FRGH] has similar goals and results, based on a different categorical tool, categories of fractions. More recently, the thesis of E. Haucourt [Ha] has combined this tool with a more effective one: the quotient of a category with respect to the generalised congruence generated by a set of arrows which are to become identities; or, in the present terminology, a *normal quotient* (with respect to the ideal of discrete functors, see 2.1 - 2.2).

Now, the analysis of [G3] captures essential facts of many *planar* ordered spaces (subspaces of the ordered plane $\uparrow\mathbf{R}^2$), but may say little about objects embedded in the ordered space $\uparrow\mathbf{R}^3$, much in the same way as the fundamental group does not recognise the singularity of a 2-sphere. There seem to be two ways of exploring such higher-dimensional singularities.

The 'obvious' one would be to go for a higher dimensional study, based on the fundamental *2-category* of the directed space, in its *strict* version $\uparrow\Pi_2(X)$ - introduced and studied in [G4] - or in some *lax* version, as the ones introduced in [G5, G6]; part of this second approach, perhaps more interesting, is briefly recalled in Section 8. But one must be aware that 2-categories are complicated structures and their models, even finite, can hardly be considered to be 'simple'.

Another way, closely linked with the study of [FRGH] and [Ha], can be based on a finer analysis of $\uparrow\Pi_1(X)$ as attempted in the present Sections 5-7, with 'semi-faithful' *surjective models*. It is interesting to note that - for the hollow cube - *such a finer analysis has no counterpart outside of* directed *homotopy*: the fundamental group of the underlying topological space is trivial (see 1.6).

Notation. A homotopy φ between maps $f, g: X \to Y$ is written as $\varphi: f \to g: X \to Y$. A *preorder* relation is assumed to be reflexive and transitive; it is a (partial) *order* if it is also anti-symmetric. As usual, a preordered set is *identified* with a (small) category having at most one arrow between any two given objects. The ordered topological space $\uparrow\mathbf{R}$ is the euclidean line with the natural order. The classical properties of adjunctions and equivalences of categories are used without reference (see [M2]). **Cat** denotes the category of small categories; if C is a small category, $x \in C$ means that x is an object of C (also called a *point* of C).

1. An analysis of directed spaces

We begin with a review of the basic ideas and results of [G3], mostly taken from [G4].

1.1. Homotopy for preordered spaces. The simplest topological setting where one can study directed paths and directed homotopies is likely the category p**Top** of *preordered topological spaces* and *preorder-preserving continuous mappings*; the latter will be simply called *morphisms* or *maps*, when it is understood we are in this category. (Richer settings will be recalled in 1.7.)

In this setting, a (directed) *path* in the preordered space X is a map $a\colon \uparrow[0, 1] \to X$, defined on the standard directed interval $\uparrow\mathbf{I} = \uparrow[0, 1]$ (with euclidean topology and natural order). A (directed) *homotopy* $\varphi\colon f \to g\colon X \to Y$, *from* f *to* g, is a map $\varphi\colon X{\times}\uparrow\mathbf{I} \to Y$ coinciding with f on the lower basis of the *cylinder* $X{\times}\uparrow\mathbf{I}$, with g on the upper one. Of course, this (directed) cylinder is a product in p**Top**: it is equipped with the product topology *and* with the product preorder, where $(x, t) \prec (x', t')$ if $x \prec x'$ in X and $t \leq t'$ in $\uparrow\mathbf{I}$.

The fundamental category $C = \uparrow\Pi_1(X)$ has, for arrows, the classes of directed paths up to the equivalence relation *generated* by directed homotopy with fixed endpoints; composition is given by the concatenation of consecutive paths.

Note that, generally, the fundamental category of a preordered space X is *not* a preorder, i.e. can have different arrows $x \to x'$ between two given points (cf. 1.2); but any loop in X lives in a zone of equivalent points and is reversible, so that all endomorphisms of $\uparrow\Pi_1(X)$ are invertible. Moreover, if X is *ordered*, the fundamental category has no endomorphisms and no isomorphisms, except the identities, and is *skeletal*; therefore, *ordinary equivalence of categories cannot yield any simpler model*. Note also that, in this case, all the fundamental monoids $\uparrow\pi_1(X, x_0) = \uparrow\Pi_1(X)(x_0, x_0)$ are trivial. All these are crucial differences with the classical fundamental groupoid $\Pi_1(X)$ of a space, for which a model up to homotopy invariance is given by the skeleton: a family of fundamental groups $\pi_1(X, x_i)$, obtained by choosing one point in each path-connected component of X.

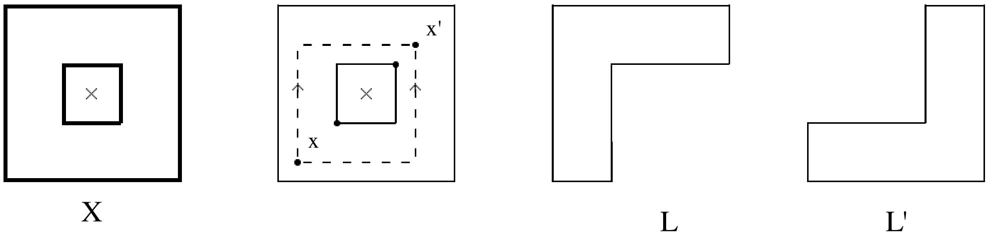
The fundamental category of a preordered space can be computed by a van Kampen-type theorem, as proved in [G2], Thm. 3.6, in a much more general setting (spaces with distinguished paths). A map of preordered spaces $f\colon X \to Y$ induces a functor $f_*\colon \uparrow\Pi_1(X) \to \uparrow\Pi_1(Y)$, and a homotopy $\varphi\colon f \to g$ induces a natural transformation $\varphi_*\colon f_* \to g_*$, generally *non* invertible. One can consider *various* notions of *directed* homotopy equivalence, for directed spaces [G2] and categories (1.3).

The forgetful functor $U\colon$ p**Top** $\to$ **Top** with values in the category of topological spaces has both a left and a right adjoint, $D \dashv U \dashv C$, where DX (resp. CX) is the space X with the *discrete* order (resp. the *coarse* preorder). Therefore, U preserves limits and colimits. *The standard embedding of* **Top** *in* p**Top** *will be the coarse one*, so that all (ordinary) paths in X are directed in CX. Note that the category of *ordered* spaces does not allow for such an embedding, and has different colimits.

1.2. The fundamental category of a square annulus. An elementary example gives some idea of the analysis developed below. Let us start from the standard *ordered* square $\uparrow[0, 1]^2$, with the euclidean topology and the product order

$$(x, y) \leq (x', y') \quad \text{if:} \quad x \leq x', \ y \leq y',$$

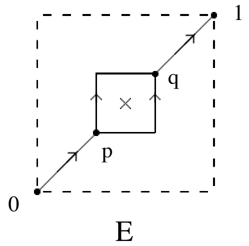
and consider the (compact) ordered subspace X obtained by taking out the *open* square $]1/3, 2/3[^2$ (marked with a cross), a sort of 'square annulus'

(1) 

Its directed paths are, by definition, the continuous *order-preserving* maps $\uparrow[0, 1] \to X$ defined on the standard ordered interval, and move 'rightward and upward' (in the weak sense). Directed homotopies of such paths are continuous order-preserving maps $\uparrow[0, 1]^2 \to X$. The fundamental category $C = \uparrow\Pi_1(X)$ has, for arrows, the classes of directed paths up to the equivalence relation *generated* by directed homotopy (with fixed endpoints, of course).

In our example, the fundamental category C has *some* arrow $x \to x'$ provided that $x \leq x'$ and both points are in L or L' (the closed subspaces represented above). Precisely, there are *two* arrows when $x \leq p = (1/3, 1/3)$ and $x' \geq q = (2/3, 2/3)$ (as in the second figure above), and *one* otherwise. This evident fact can be easily proved with the 'van Kampen' theorem recalled above, using the subspaces L, L' (whose fundamental category is the induced order).

Thus, the whole category C is easy to visualise and 'essentially represented' by the full subcategory E on four vertices 0, p, q, 1 (the central cell does not commute)

(2) 

$$0 \longrightarrow p \rightrightarrows q \longrightarrow 1$$

E

But E is far from being equivalent to C, as a category, since C *is already a skeleton*, in the ordinary sense. The situation can be analysed as follows, in E:

- the action begins at 0, from where we move to the point p,
- p is an (effective) future branching point, where we have to choose between two paths,
- which join at q, an (effective) past branching point,
- from where we can only move to 1, where the process ends.

(Definitions and properties of *regular* and *branching* points can be found below in 2.6-2.9).

In order to make precise how E can 'model' the category C, we proved in [G3] (and will recall below) that E is both *future equivalent* and *past equivalent* to C, and actually it is the 'join' of a minimal 'future model' with a minimal 'past model' of the latter.

1.3. Directed equivalences of categories. Various such notions will be developed in the next sections; here we begin with a partial sketch.

A *future equivalence* $(f, g; \varphi, \psi)$ [G3, 2.1] between the categories C, D is a symmetric version of an adjunction, with two units. It consists of a pair of functors and a pair of natural transformations (i.e., directed homotopies in **Cat**), the *units*, satisfying two coherence conditions:

(1) $f\colon C \rightleftarrows D :g$ $\qquad\qquad$ $\varphi\colon 1_C \to gf,\ \ \psi\colon 1_D \to fg,$

(2) $f\varphi = \psi f\colon f \to fgf,$ $\qquad\qquad$ $\varphi g = g\psi\colon g \to gfg$ $\qquad$ (*coherence*).

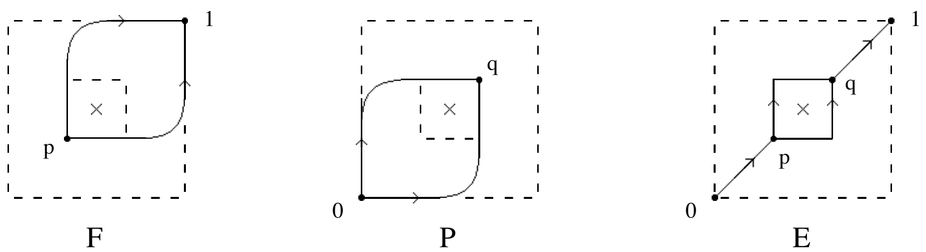
Note that the *directed homotopies* φ, ψ proceed *from* the identities *to* the composites gf, fg ('in the future'). Future equivalences compose (much in the same way as adjunctions), and yield an equivalence relation of categories.

Dually, *past equivalences* have *co*units, in the opposite direction. These two basic notions can be combined in various ways, to give various self-dual equivalence relations. First, their conjunction is called *past and future equivalence*, while *coarse equivalence* is the equivalence relation generated by them (see examples of both, in 2.5). More complex combinations will be considered in Section 3: *injective equivalence* (3.2b) and *projective equivalence* (3.2c).

An adjunction $f \dashv g$ with *invertible* counit $\varepsilon\colon fg \cong 1$ amounts to a future equivalence with invertible $\psi = \varepsilon^{-1}$. In this case, a 'split' future equivalence, D can be identified with a full reflective subcategory of C (also called a *future retract*). But, in a general future equivalence, f need not determine g. Theorem 2.3 shows that *two categories are future equivalent if and only if they can be embedded into a common one, as full reflective subcategories.*

1.4. Minimal one-dimensional models. In our example (1.2), the category $C = {\uparrow}\Pi_1(X)$ has a least *full reflective* subcategory F, which is future equivalent to C and minimal as such; its objects are a *future branching* point p (where we must choose between different ways out of it) and a *maximal* point 1 (where one cannot further proceed); they form the *future spectrum* $sp^+(C)$ (defined in 3.5;

(1)

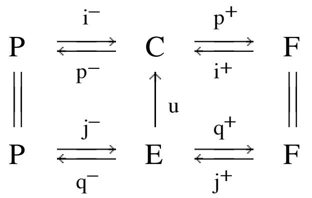


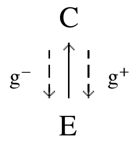
Also the full subcategory $F = Sp^+(C)$ on these objects is called a *future spectrum* of C. Dually, we have the least *full coreflective* subcategory $P = Sp^-(C)$, on the *past spectrum* $sp^-(C) = \{0, q\}$.

Together, they form the *spectral pf-presentation* of C

(2) $$P \underset{p^-}{\overset{i^-}{\rightleftarrows}} C \underset{i^+}{\overset{p^+}{\rightleftarrows}} F \qquad \begin{array}{ll} \varepsilon\colon i^-p^- \to 1_C & (p^-i^- = 1,\ p^-\varepsilon = 1,\ \varepsilon i^- = 1), \\ \eta\colon 1_C \to i^+p^+ & (p^+i^+ = 1,\ p^+\eta = 1,\ \eta i^+ = 1). \end{array}$$

Putting together the information coming from a past and a future spectrum, the *pf-spectrum* $E = Sp(C)$ is the full subcategory of C on the set of objects $sp(C) = sp^-(C) \cup sp^+(C)$ (3.6). It is linked to C by a diagram formed of four commutative squares

(3) 

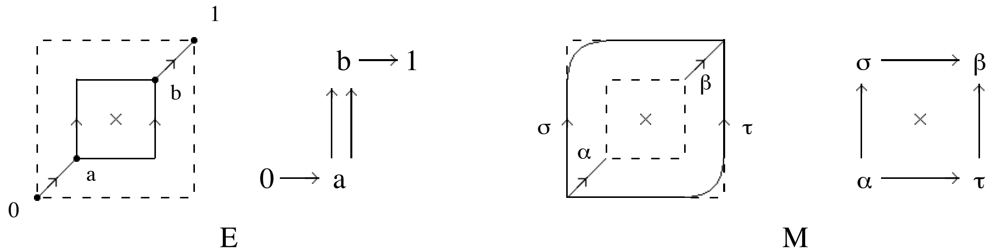


$(g^\alpha = j^\alpha p^\alpha)$.

Adding the two functors $g^\alpha = j^\alpha p^\alpha$: X → E (where $\alpha = \pm$), E becomes a *minimal injective model* of the category C, in a precise sense, which will be recalled in Section 3.

1.5. Projective models. An alternative description can be obtained with the associated *projective model* f: C → M, where M is the full subcategory of the category $\mathbf{C^2}$ containing the morphisms of C of type $f(x) = \eta x.\varepsilon x$: $i^-p^-x \to i^+p^+x$ (obtained from the adjunctions $i^- \dashv p^-$, $p^+ \dashv i^+$).

In the present case, we get the full subcategory of $\mathbf{C^2}$ on the four maps $\alpha, \beta, \sigma, \tau$

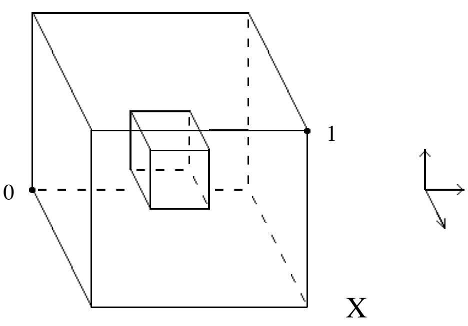
(1) 

Projective models are defined in 3.2c and 3.4b. Note that our functor f: C → M is surjective on objects but *not* full (it identifies points of X which are not comparable in the path order).

Interestingly, *this model is isomorphic to the 'category of components' of* C constructed in [FRGH]. We will see, in Section 3, that a pf-spectrum (when it exists) is an effective way of constructing both an injective model and a projective model.

1.6. The hollow cube. The analysis recalled above, based on the fundamental category, gives relevant information for *planar* ordered spaces (subspaces of $\uparrow\mathbf{R}^2$), also in much more complicated examples (see Section 4). But it may be insufficient for higher dimensional singularities.

The simplest case is a 3-dimensional analogue of our previous example, the 'hollow cube' $X \subset \uparrow[0, 1]^3$ represented below (again an ordered compact space):

(1) 

$X = \uparrow[0, 1]^3 \setminus A$

$A =]1/3, 2/3[^3$.

The fundamental category $C = \uparrow\Pi_1(X)$ seems to say little about this space: C has an initial and a terminal object, 0 and 1, whence it is future contractible (to its object 1) and past contractible as well (to

0); its minimal injective model is the category $\mathbf{2} = \{0 \to 1\}$ (1.4).

Now, this injective model is *not* faithful, in the sense that, of the three functors $\mathbf{2} \leftrightarrows C$, the ones starting at C are not faithful. In fact, the category C is not a preorder, since C(x, y) contains two arrows when x, y are suitably placed 'around' the obstruction; a phenomenon which only appears within *directed* homotopy theory: the fundamental group of the underlying topological space is trivial, and the fundamental groupoid is codiscrete (one arrow between any two given points). We shall therefore try to extract a better information from C, using a *partially faithful surjective* model (in Sections 5 and 7).

Another approach, followed in [G4], is based on studying the fundamental 2-category $C_2 = \uparrow\Pi_2(X)$, trying to reproduce one dimension up the previous study of $\uparrow\Pi_1(X)$, for the 'square annulus' (1.4). This can also be done with more interesting lax versions [G5, G6]; the simplest will be outlined in Section 8.

1.7. Other directed structures. In a preordered space, every loop is reversible (as already remarked in 1.1); therefore, this setting has no 'directed circle' or 'directed torus'.

We briefly recall more complex directed structures, which allow for non-reversible loops. All of them have a directed interval $\uparrow\mathbf{I}$ with the structure considered above, so that all the previous constructions can be easily extended. Furthermore, all of them have a *reflection* $X \mapsto X^{op}$ extending the preorder-reversion of p**Top**.

A sufficiently general, well-behaved setting has been studied in [G2]. A *d-space* X = (X, dX) is a topological space equipped with a set dX of (continuous) maps $a\colon \mathbf{I} \to X$; these maps, called *distinguished paths* or *directed paths* or *d-paths*, must contain all constant paths and be closed under concatenation and (weakly) increasing reparametrisation.

A *d-map* $f\colon X \to Y$ (or *map* of d-spaces) is a continuous mapping between d-spaces which preserves the directed paths: if $a \in dX$, then $fa \in dY$.

The category of d-spaces is written as d**Top**. It has all limits and colimits, constructed as in **Top** and equipped with the initial or final d-structure for the structural maps; for instance a path $\mathbf{I} \to \Pi X_i$ is directed if and only if all its components $\mathbf{I} \to X_i$ are so. The forgetful functor U: d**Top** → **Top** preserves thus all limits and colimits; a topological space is generally viewed as a d-space by its *natural* structure, where all (continuous) paths are directed (via the right adjoint to U).

Reversing d-paths, by the involution $r(t) = 1 - t$, yields the *reflected*, or *opposite*, d-space $RX = X^{op}$, where $a \in d(X^{op})$ if and only if $a^{op} = ar \in dX$.

The *standard d-interval* $\uparrow\mathbf{I} = \uparrow[0, 1]$ has directed paths given by the (weakly) increasing maps $\mathbf{I} \to \mathbf{I}$. The *standard directed circle* $\uparrow\mathbf{S}^1 = \uparrow\mathbf{I}/\partial\mathbf{I}$ has the obvious d-structure, where paths have to follow a precise orientation. But note that the directed structures $\uparrow\mathbf{I}^2$, $\uparrow\mathbf{R}^2$ or the torus $\uparrow\mathbf{S}^1 \times \uparrow\mathbf{S}^1$ have nothing to do with orientation; furthermore, the Klein bottle has a natural d-structure (induced by a 'slanting' d-structure on $[0, 1]^2$), which is locally isomorphic to $\uparrow\mathbf{R}^2$.

An alternative setting, *inequilogical spaces*, introduced in other works as a directed version of Dana Scott's equilogical spaces [Sc, BBS], could also be used - but would require a more complicated procedure to concatenate paths and homotopies.

Recently, S. Krishnan has proposed a 'convenient category of locally preordered spaces' which, in contrast with the previous versions of this notion, has colimits and therefore allows for the usual constructions of homotopy theory, like mapping cones and suspension [Kr].

2. Future equivalence of categories and future invariant properties

We develop now, from [G3], the notions of future and past equivalence introduced in the previous sections.

2.1. Future equivalence. As already recalled above, a *future equivalence* $(f, g; \varphi, \psi)$ between the categories X, Y consists of a pair of functors and a pair of natural transformations (i.e., directed homotopies), the *units*, satisfying two coherence conditions:

(1) $f: X \rightleftarrows Y :g$ $\qquad\qquad \varphi: 1_X \to gf, \quad \psi: 1_Y \to fg,$

(2) $f\varphi = \psi f: f \to fgf,$ $\qquad\qquad \varphi g = g\psi: g \to gfg$ $\qquad\qquad$ (*coherence*).

Notice that f does not determine g, in general (cf. 2.5b). A property (making sense *in* a category, or *for* a category) is said to be *future invariant* if it is preserved by future equivalences.

Given $(f, g; \varphi, \psi)$ and a second future equivalence

(3) $h: Y \rightleftarrows Z :k$ $\qquad\qquad \vartheta: 1_Y \to kh, \quad \zeta: 1_Z \to hk,$

$h\vartheta = \zeta h: h \to hkh,$ $\qquad\qquad \vartheta k = k\zeta: k \to khk,$

their *composite* is:

(4) $hf: X \rightleftarrows Z :gk$ $\qquad\qquad g\vartheta f.\varphi: 1_X \to gk.hf, \quad h\psi k.\zeta: 1_Z \to hf.gk.$

Its coherence is easily checked; the composition is associative, with obvious identities. *Being future equivalent categories* is thus an equivalence relation.

2.2. Full reflective subcategories as future retracts. A special case of future equivalence is important for its own sake, but will also be shown to generate the general case (in 2.3).

A *split* future equivalence *of* F *into* X (or *of* X *onto* F) is a future equivalence $(i, p; 1, \eta)$ where the unit $1 \to pi$ is an identity

(1) $i: F \rightleftarrows X :p$ $\qquad\qquad \eta: 1_X \to ip$ $\qquad$ (the *main unit*),

$pi = 1_F,$ $\qquad\qquad p\eta = 1_p, \quad \eta i = 1_i$ $\qquad\qquad$ $(p \dashv i)$.

We also say that F is a *future retract* of X. Note that p is now left adjoint to i, which is full and faithful.

Forgetting about direction, a future retract corresponds - in Topology - to a *strong deformation retract* (with an additional coherence condition, $p\eta = 1$). Here, this structure means that F is (isomorphic to) a *full reflective subcategory* of X, i.e. that there is a full embedding $i: F \to X$ with a left adjoint $p: X \to F$ (then p is essentially determined by i, and - via the universal property of the unit - can always be constructed so that the counit $pi \to 1_F$ be an identity, as we are assuming).

Equivalently, one can assign a *strictly idempotent monad* (e, η) on X

(2) $e: X \to X,$ $\qquad \eta: 1_X \to e,$ $\qquad\qquad ee = e, \quad e\eta = 1_e = \eta e.$

Indeed, given $(i, p; \eta)$, we take $e = ip$; given (e, η), we factor $e = ip$ splitting e through the subcategory F of X formed of the objects and arrows which e leaves fixed.

Dually, a *split* past equivalence, *of* P *into* X (or *of* X *onto* P) is a past equivalence $(i, p; 1, \varepsilon)$ where the counit $pi \to 1_P$ is an identity

(3) $\quad i: P \rightleftarrows X :p \qquad\qquad \varepsilon: ip \to 1_X \qquad$ (the *counit*),

$\qquad pi = 1_P, \qquad\qquad p\varepsilon = 1_p, \quad \varepsilon i = 1_i \qquad\qquad (i \dashv p).$

This amounts to saying that $i(P)$ is a *full coreflective subcategory* of X (with a choice of the coreflection making the unit $1 \to pi$ an identity); P is also called a *past retract* of X.

2.3. Theorem (Future equivalence and reflective subcategories) [G3, 2.5]. (a) A future equivalence $(f, g; \varphi, \psi)$ between X and Y (2.1) has a canonical factorisation into two split future equivalences

$$X \underset{p}{\overset{i}{\rightleftarrows}} W \underset{j}{\overset{q}{\leftrightarrows}} Y \qquad\qquad (\eta: 1_W \to ip, \;\; \eta': 1_W \to jq), \tag{1}$$

so that X and Y are full reflective subcategories of W. (It is a *mono-epi* factorisation in the category of future equivalences, through a sort of 'graph' of $(f, g; \varphi, \psi)$).

(b) Two categories are future equivalent if and only if they are full reflective subcategories of a third.

(c) A property is future invariant if and only if it is preserved by all *embeddings* of full reflective subcategories, as well as by their *reflectors*.

The proof can be seen in [G3, 2.5]. We only recall how one constructs the category W.

An object is a four-tuple $(x, y; u, v)$ such that:

(2) $\quad u: x \to gy$ (in X), $\quad v: y \to fx$ (in Y), $\qquad gv.u = \varphi x, \quad fu.v = \psi y,$

$$\begin{array}{ccc} x & \xrightarrow{u} & gy \\ & {\scriptstyle \varphi x} \searrow & \downarrow {\scriptstyle gv} \\ & & gfx \end{array} \qquad\qquad \begin{array}{ccc} y & \xrightarrow{v} & fx \\ & {\scriptstyle \psi y} \searrow & \downarrow {\scriptstyle fu} \\ & & fgy \end{array}$$

A morphism is a pair $(a, b): (x, y; u, v) \to (x', y'; u', v')$ such that:

(3) $\quad a: x \to x'$ (in X), $\quad b: y \to y'$ (in Y), $\qquad gb.u = u'.a, \quad fa.v = v'.b,$

$$\begin{array}{ccc} x & \xrightarrow{u} & gy \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle gb} \\ x' & \xrightarrow[u']{} & gy' \end{array} \qquad\qquad \begin{array}{ccc} y & \xrightarrow{v} & fx \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle fa} \\ y' & \xrightarrow[v']{} & fx' \end{array}$$

2.4. Terminal and maximal points. The existence of a map $x \to x'$ in the category X (a *path*) produces the *path preorder* $x \prec x'$ (x *reaches* x') on the points of X; the resulting *path equivalence* relation, meaning that there are maps $x \rightleftarrows x'$, is written as $x \simeq x'$. (Of course, if the category X 'is' a preorder, the path preorder coincides with the original relation.)

The following properties of an object $x \in X$ are future invariant (as proved in [G3, Lemma 2.7]):

(a) x is the *terminal* object of X,

(b) x is a *weakly terminal* object of X, i.e. a *maximum* for the path preorder $\prec$ (i.e., it can be reached from every point of X),

(c) x is *maximal* in X, for the path preorder (i.e., it can only reach the points $\simeq x$),

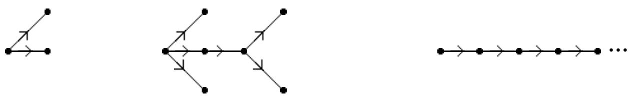
(d) x does not reach a maximal point z.

We say that a category X is *future contractible* if it is future equivalent to $\mathbf{1}$ (the singleton category $\{*\}$). It is easy to see that this happens if and only if X has a terminal object [G3, 2.6].

2.5. Other elementary examples. (a) Let us begin with a few examples, produced by finite or countable ordered sets. For preordered sets (viewed as categories), a future equivalence consists of a pair of preorder-preserving mappings $f\colon X \rightleftarrows Y :g$ such that $1_X \leq gf$ and $1_Y \leq fg$, and *is necessarily faithful*. We already know that future contractibility means having a maximum

(1) (past and future contractible),

(2)  (just future-contractible),

(3)  (just past-contractible),

(4)  (just coarse-contractible).

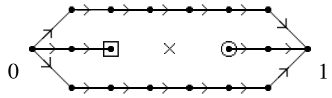
(Of all these examples, the ones is (1) are also projectively contractible, but only the first of them is also injectively contractible; see 3.2d.)

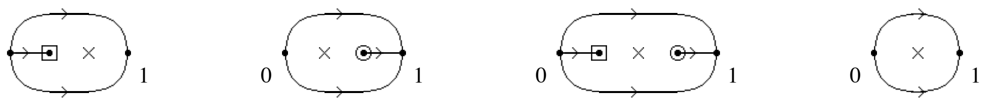
(b) Consider again (as in (3)) the ordered set $\mathbf{n}$ of natural numbers, as a category. There are future equivalences

(5) $f\colon \mathbf{n} \rightleftarrows \mathbf{n} :g, \qquad f(x) = x, \quad g(x) = \max(x, x_0), \qquad \varphi(x) = \psi(x)\colon x \leq g(x),$

where $x_0 \in \mathbf{n}$ is arbitrary (and coherence automatically holds, since our categories are preorders). Thus, f does *not* determine g.

(c) Now we consider some finite categories, generated by the directed graphs drawn below; the outer cells, marked with a cross, do not commute and these categories are not preorders. The category represented in (6) is future equivalent to the first in (7), past equivalent to the second, past and future equivalent to the third and coarse-equivalent to the last

(6) 

(7) 

This shows a situation of interest in concurrency. There is a given *starting point* 0, which is minimal (2.4), but *not* initial *nor* the unique minimal point (generally); and a given *ending point* 1, which is maximal. Moreover:

- 0 is also a *future branching* point, where one has to choose among different ways of going *forward*; being such is a future invariant property (cf. 2.9);

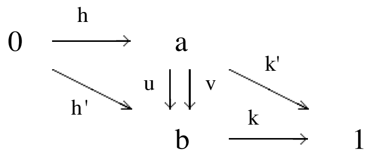
- a *square-marked* point is a *deadlock*, i.e. a maximal *unsafe vertex* (from where one cannot reach 1); this is again a future invariant property (2.4);

- a *circle-marked* point is a minimal *unreachable vertex* (which cannot be reached from 0); being such is a past invariant property (according to the dual of 2.4);

- 1 is a *past branching* point, preserved by past equivalences (2.9).

The 'past and future model' above (the third category in (7)) *preserves all these properties*, while the coarse one only recognises that there are two paths from 0 to 1.

(d) Finally, the following category (described by generators and relations)

(8) 

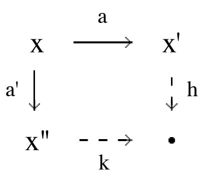
$uh = vh = h', \quad ku = kv = k',$

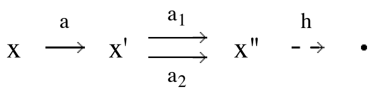
has an initial object (0) and a terminal one (1): it is past and future contractible, but *not* faithfully so. Note also that, in the future contraction, all the components of the unit $(x \to 1)$ are epi.

2.6. Future regularity [G3, 6.1]. A morphism $a: x \to x'$ in X is said to be V^+-*regular* if it satisfies condition (i), O^+-*regular* if it satisfies (ii), and *future regular* if it satisfies both:

(i) given $a': x \to x''$, there is a commutative square $ha = ka'$ (V^+-*regularity*),

(ii) given $a_i: x' \to x''$ such that $a_1a = a_2a$, there is some h such that $ha_1 = ha_2$ (O^+-*regularity*),

(1) 



Future regular morphisms are closed under composition (2.7), but they are not invertible, in general. The equivalence relation $\sim^+$ in ObX generated by the existence of a future regular morphism between two objects is called *future regularity equivalence*. The future regularity class of an object x is written $[x]^+$.

In a category with finite colimits or with terminal object, all morphisms are future regular. In a preordered set, all arrows are O^+-regular, and future regularity coincides with V^+-regularity.

On the other hand, we say that a is V^+-*branching* if it is not V^+-regular; that it is O^+-*branching* if it is not O^+-regular; that it is a *future branching morphism* if it falls in (at least) one of the previous cases, i.e. if it is not future regular. In the category represented below, at the left, the morphism a is V^+-branching and O^+-regular, while at the right a is O^+-branching and V^+-regular

(2)
$$\begin{array}{ccc} x & \xrightarrow{a} & x' \\ {\scriptstyle a'}\downarrow & & \\ x'' & & \end{array} \qquad\qquad \begin{array}{ccc} x & \xrightarrow{a} & x' \\ & \searrow^{b} & {\scriptstyle a_1}\downarrow\downarrow{\scriptstyle a_2} \\ & & x'' \end{array} \qquad (b = a_1a = a_2a).$$

Dually, we have V^--*regular*, O^--*regular*, *past regular* morphisms and the corresponding *branching* morphisms; the *past regularity equivalence* $\sim^-$ and its *past regularity classes* $[x]^-$.

2.7. Lemma [G3, 6.2]. (a) V^+-regular, O^+-regular and future regular morphisms form (wide) subcategories, containing all the isomorphisms.

(b) If a composite ba is V^+-regular (resp. O^+-regular), then the first map a (resp. the second map b) is also.

2.8. Theorem (Future equivalence and regular morphisms) [G3, 6.3-6.4]. Given a future equivalence $f: X \rightleftarrows Y :g$, with natural transformations $\varphi: 1 \to gf$, $\psi: 1 \to fg$, we have:

(a) all the components φx and ψy are future regular morphisms,

(b) the functors f and g *preserve* V^+-regular, O^+-regular and future regular morphisms,

(c) the functors f and g *preserve* V^+-branching, O^+-branching and future branching morphisms (i.e. *reflect* V^+-regular, O^+-regular and future regular morphisms),

It follows that a future equivalence $f: X \rightleftarrows Y :g$ *induces a bijection*

(1) $(\mathrm{Ob}X)/\sim^+ \rightleftarrows (\mathrm{Ob}Y)/\sim^+$,

between the quotients of objects up to future regularity equivalence; f and g preserve and reflect the future regularity equivalence relations $\sim^+$.

2.9. Branching points. We consider now future invariant properties *of points* of a category X. We have already seen some of them, concerning maximal points (2.4). A point x is said to be V^+-*regular* if it satisfies (i), O^+-*regular* if it satisfies (ii), *future regular* if it satisfies both:

(i) every arrow starting from x is V^+-regular (equivalently, two arrows starting from x can always be completed to a commutative square),

(ii) every arrow starting from x is O^+-regular (equivalently, given an arrow $a: x \to x'$ and two arrows $a_i: x' \to x''$ such that $a_1a = a_2a$, there exists an arrow h such that $ha_1 = ha_2$).

We say that x is a V^+-*branching point* in X if it is not V^+-regular (i.e., if there is some arrow starting from x which is V^+-branching); that x is an O^+-*branching point* if it is not O^+-regular; that x is a *future branching point* if it falls in at least one of the previous cases, i.e. if it is not future regular.

Note now that, in the fundamental category C considered in 1.2, the starting point 0 is V^+-branching, but the choice between the different paths starting from it can be deferred, while at the point p the choice must be made. To distinguish these situations, we say that a future branching point is *effective* when every future regular map starting from it is a split mono. (In the fundamental category of a preordered or ordered space, this amounts to an isomorphism or an identity, respectively.)

Dually, we have the notions of V^--, O^-- and *past regular* (resp. *branching*) *point* in X, and *effective past branching points*.

Theorem (Future equivalence and branching points) [G3, Thm. 6.6]. The following properties of a point are *future invariant* (i.e., invariant up to future equivalence):

(a) being a V^+-regular, or an O^+-regular, or a future regular point,

(b) being a V^+-branching, or an O^+-branching, or a future branching point, or an effective one.

3. Injective and projective models associated with spectra

Injective and surjective models, defined in 3.2, will be our main tool. A pf-presentation of a category, formed of a past and a future retract (3.3), produces an injective model (3.4a) and a projective one (3.4b). Spectral presentations (3.5-3.6) give minimal projective models.

3.1. Pf-equivalences. We are interested in considering categories which are *at the same time* future and past equivalent. But an *unrelated* pair formed of a past equivalence and a future equivalence between the same categories is not an effective tool.

A *pf-equivalence from* X *to* Y [G3, 3.1] is a pair formed of a past equivalence $(f, g^-; \varepsilon_X, \varepsilon_Y)$ and a future equivalence $(f, g^+; \eta_X, \eta_Y)$ sharing the same functor $f: X \to Y$, and also satisfying a further *pf-coherence* condition (2) linking the two pairs:

(1) $f : X \rightleftarrows Y : g^-, g^+,$

$$\varepsilon_X: g^-f \to 1_X, \quad \varepsilon_Y: fg^- \to 1_Y, \qquad f\varepsilon_X = \varepsilon_Y f: fg^-f \to f, \quad \varepsilon_X g^- = g^-\varepsilon_Y: g^-fg^- \to g^-,$$

$$\eta_X: 1_X \to g^+f, \quad \eta_Y: 1_Y \to fg^+, \qquad f\eta_X = \eta_Y f: f \to fg^+f, \quad \eta_X g^+ = g^+\eta_Y: g \to g^+fg^+,$$

(2) $$\begin{array}{ccc} g^- & \xrightarrow{g^-\eta_Y} & g^-fg^+ \\ {\scriptstyle \eta_X g^-}\downarrow & = & \downarrow{\scriptstyle \varepsilon_X g^+} \\ g^+fg^- & \xrightarrow[g^+\varepsilon_Y]{} & g^+ \end{array}$$ (*pf-coherence*).

A pf-equivalence yields a natural transformation, the *comparison* from past to future

(3) $g: g^- \to g^+: Y \to X, \qquad g = \varepsilon_X g^+.g^-\eta_Y = g^+\varepsilon_Y.\eta_X g^-.$

which - when convenient - is seen as a functor $g: Y \to X^{\mathbf{2}}$

(4) $g: Y \to X^{\mathbf{2}}, \qquad gy: g^-y \to g^+y, \qquad g(b) = (g^-b, g^+b).$

A pf-equivalence is often written as $f: X \rightleftarrows Y$ or $f: X \rightleftarrows Y : g^\alpha$, leaving the rest understood.

The coherence condition (2) is automatically satisfied in two important cases, considered below: when f is faithful or surjective on objects [G3, Lemma 3.3].

3.2. Injective and surjective models. (a) A pf-equivalence $f: X \rightleftarrows Y : g^\alpha$ is called a *pf-injection*, or *pf-embedding*, if the functor f is a *full embedding* (i.e., full, faithful and injective on objects). Pf-embeddings compose, with the composition of pf-equivalences (2.1). Then, we say that Y is an *injective* model of X.

It is easy to see that a pf-embedding $f: X \rightleftarrows Y : g^\alpha$ amounts to these three functors together with the *two* natural transformations *at* Y, satisfying the conditions below

(1) $\varepsilon_Y: fg^- \to 1_Y$ (the *main counit*), $\eta_Y: 1_Y \to fg^+$ (the *main unit*),

$fg^-\varepsilon_Y = \varepsilon_Y fg^-$, $fg^+\eta_Y = \eta_Y fg^+$.

(b) We say that E is a *minimal injective model* of X [G3, 5.2] if:

(i) E is an injective model of every injective model E' of X,

(ii) every injective model E' of E is isomorphic to E.

We also say that E is a *strongly minimal injective model* if it satisfies the stronger condition (i'), together with (ii):

(i') E is an injective model of every category injectively equivalent to X,

where two categories are said to be *injectively equivalent* if they can be linked by a finite chain of pf-embeddings, forward or backward [G3, 4.1].

We will see in 3.6 that a pf-spectrum of a category produces a *strongly minimal injective model* of the latter.

(c) A pf-equivalence $f: X \rightleftarrows Y : g^\alpha$ is called a *pf-surjection* if the functor f is surjective on objects, and a *pf-projection* if, moreover, the associated functor $g: Y \to X^{\mathbf{2}}$ (3.1.4) is a full embedding. Then, we say that Y is a *surjective* or *projective* model of X, respectively. Also in this case, coherence is automatic (as already observed at the end of 3.1). We have already observed that the functor f need not be full (1.5). *Projective equivalence* of categories is defined by a finite chain of pf-projections, forward or backward.

(d) It is easy to see that a category X is *injectively contractible* (i.e., injectively equivalent to **1**) if and only if it is pointed (i.e., it has a zero object) [G3, 5.4]. A category X with non-isomorphic initial and terminal object is injectively modelled by the ordinal $\mathbf{2} = \{0 \to 1\}$, with an obvious pf-embedding $i: \mathbf{2} \rightleftarrows X : g^\alpha$ (*not* split) which is actually *the strongly minimal injective model* of X. (In particular, the category **2**, which is the *standard interval* of **Cat**, is *not* contractible in this sense.)

On the other hand, on the projective side, the existence of the initial and terminal objects is sufficient (and necessary) to make a category X projectively equivalent to **1**, via the split pf-projection $p: X \rightleftarrows \mathbf{1} : i^\alpha$, with $i^-(*) = 0$ and $i^+(*) = 1$.

3.3. Pf-presentations. A *pf-presentation* of the category X [G3, 4.2] is a diagram consisting of a *past retract* P (i.e., full coreflective subcategory) and a *future retract* F (i.e., full reflective subcategory) of X

$$(1)\qquad P \underset{p^-}{\overset{i^-}{\rightleftarrows}} X \underset{i^+}{\overset{p^+}{\rightleftarrows}} F \qquad \begin{array}{ll} \varepsilon: i^-p^- \to 1_X & (p^-i^- = 1,\ p^-\varepsilon = 1,\ \varepsilon i^- = 1), \\ \eta: 1_X \to i^+p^+ & (p^+i^+ = 1,\ p^+\eta = 1,\ \eta i^+ = 1). \end{array}$$

We have thus two adjunctions $i^- \dashv p^-$, $p^+ \dashv i^+$, and a composed one, from P to F, which is no longer split, with the following counit and unit

(2) $p^+\varepsilon i^+\colon p^+i^-.p^-i^+ \to p^+i^+ = 1_F$, $\qquad p^-\eta i^-\colon 1_P = p^-i^- \to p^-i^+p^+i^-$ $\qquad (p^+i^- \dashv p^-i^+)$.

3.4. The associated models. (a) These data produce an *associated injective model* E, the full subcategory of X on $\mathrm{Ob}P \cup \mathrm{Ob}F$, which has a pf-embedding $u\colon E \rightleftarrows X :r^\alpha$.

More precisely (as proved in [G3, 4.3]), these data can be uniquely completed to a diagram with four commutative squares

$$(1) \qquad \begin{array}{ccccc} P & \underset{p^-}{\overset{i^-}{\rightleftarrows}} & X & \underset{i^+}{\overset{p^+}{\leftrightarrows}} & F \\ \| & & \uparrow u & & \| \\ P & \underset{q^-}{\overset{j^-}{\rightleftarrows}} & E & \underset{j^+}{\overset{q^+}{\leftrightarrows}} & F \end{array} \qquad\qquad \begin{array}{c} X \\ r^- \downarrow \uparrow \downarrow r^+ \\ E \end{array}$$

The lower row is a pf-presentation of E, with the unique natural transformations $\varepsilon_E\colon j^-q^- \to 1_E$ and $\eta_E\colon 1_E \to j^+q^+$ such that

(2) $u\varepsilon_E = \varepsilon u$, $\qquad u\eta_E = \eta u$.

Moreover, letting $r^\alpha = j^\alpha p^\alpha\colon X \to E$ $(\alpha = \pm)$, we get a pf-embedding $u\colon E \rightleftarrows X :r^\alpha$, with the natural transformations $\varepsilon_E, \varepsilon, \eta_E, \eta$; E becomes thus the injective model *generated* by the given pf-presentation of X.

(b) But the given pf-presentation of X also produces an *associated projective model* $f\colon X \rightleftarrows M :g^\alpha$ [G3, Thm. 4.6]. M is the full subcategory of the category of morphisms $X^{\mathbf{2}}$, on the objects

(3) $f(x) = \eta x.\varepsilon x\colon i^-p^-x \to i^+p^+x$,

while the functors g^- and g^+ are the restrictions of domain and codomain, respectively.

3.5. Spectra [G3, 7.2]. Recall that we have defined, in the set of objects $\mathrm{Ob}X$, the equivalence relation $x \sim^+ x'$ of future regularity, with equivalence classes $[x]^+$ (2.6).

A *future spectrum* $sp^+(X)$ of the category X is a subset of objects such that:

(sp^+.1) $sp^+(X)$ contains precisely one object, written $sp^+(x)$, in every future regularity class $[x]^+$,

(sp^+.2) for every $x \in X$ there is precisely one morphism $\eta x\colon x \to sp^+(x)$ in X,

(sp^+.3) every morphism $a\colon x \to sp^+(x')$ factors as $a = h.\eta x$, for a unique $h\colon sp^+(x) \to sp^+(x')$.

The second condition can be equivalently written as:

(sp^+.2') for every $x \in X$, $sp^+(x)$ is the terminal object of the full subcategory on $[x]^+$.

As proved in [G3, 7.4], a functor $i\colon F \to X$ embeds F as a future retract of X if and only if:

(a) the category F has precisely one object in each future regularity class; the functor i is a future retract (i.e., it has a left adjoint $p\colon X \to F$ with $pi = 1_F$ as counit); moreover the unit-component $x \to ip(x)$ is the unique X-morphism with these endpoints.

Also the full subcategory $Sp^+(X)$ of X on this set of objects is called the *future spectrum*. The future spectrum (when it exists) is the least future retract of the given category [G3, 7.3]. This full subcategory, as well as its embedding in X, is determined up to a *canonical* isomorphism (and is thus more strictly determined than the ordinary skeleton; cf. [G3, 7.5]).

Dually we have the *past spectrum* $sp^{-}(X)$ and its full subcategory $Sp^{-}(X)$.

It is easy to see that a category has future spectrum **1** if and only if it is future equivalent to **1**, if and only if it has a terminal object.

3.6. Spectral presentations. The *spectral pf-presentation* of X [G3, 7.6] is a diagram where

(1) $$P \underset{p^-}{\overset{i^-}{\rightleftarrows}} X \underset{i^+}{\overset{p^+}{\rightleftarrows}} F \qquad \begin{array}{ll} \varepsilon\colon i^-p^- \to 1_X & (p^-i^- = 1,\ p^-\varepsilon = 1,\ \varepsilon i^- = 1), \\ \eta\colon 1_X \to i^+p^+ & (p^+i^+ = 1,\ p^+\eta = 1,\ \eta i^+ = 1). \end{array}$$

(i) P is *the* past spectrum and F *the* future spectrum of X,

(ii) given $x \in \mathrm{Ob}P$ and $x' \in \mathrm{Ob}F$, if $x \cong x'$ in X then $x = x'$ (*linked choice*).

Such a presentation exists if and only if X has a past spectrum and a future one, since the linked-choice condition can always be realised, replacing each object of P with its isomorphic copy in F, if any. The set of objects produced by this linked choice is called the *pf-spectrum* of X, or *spectral model*

(2) $sp(X) = \mathrm{Ob}P \cup \mathrm{Ob}F = sp^{-}(X) \cup sp^{+}(X)$.

The full subcategory Sp(X) on these objects is also called the *pf-spectrum* of X. As a crucial fact, Sp(X) is a *strongly minimal injective model* of X [G3, Thm. 8.4], determined up to isomorphism (in fact, a unique *coherent* isomorphism, in a sense made precise in [G3, Thm. 8.6]).

The projective model $X \to M$ associated to the spectral pf-presentation (as in 3.4) is called the *spectral projective model* of X.

4. Examples

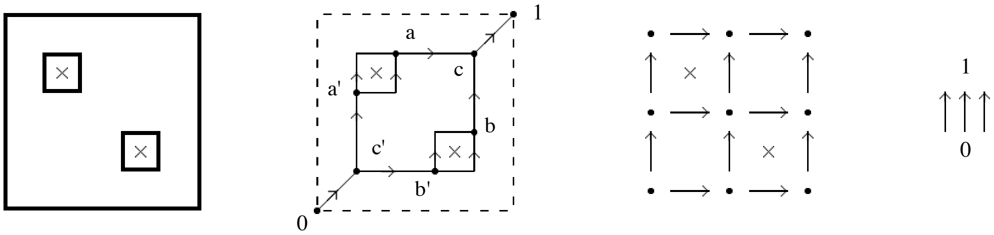
The following analysis of some planar ordered spaces is taken from [G3], Section 9.

4.1. Modelling an ordered space. In the sequel, as in 1.4, we consider *ordered* topological spaces X with minimum (0) and maximum (1) and study the pf-spectrum of the fundamental category $C = {\uparrow}\Pi_1(X)$. (A similar analysis for a preordered space can be found in [G3, 9.5].)

C inherits a privileged 'starting point' 0 (a minimal point, but possibly not the unique one, cf. 4.5) and a privileged 'ending point' 1 (which is maximal in C). Furthermore, recall that C is skeletal (since X is ordered), so that the future spectrum - if it exists - is the least full (i.e. replete) reflective subcategory of C, strictly determined as a subset of C. Objects of C (i.e., points of X) are denoted by letters x, a, b, c...; arrows of C (i.e., classes of paths of X 'up to directed homotopy', as recalled in 1.2), by Greek letters α, β, γ...

4.2. A second example. Consider, in the category p**Top**, the (compact) ordered space X: a subspace of the standard ordered square $\uparrow[0, 1]^2$ obtained by taking out two *open* squares (marked with a cross), as in the left figure below

(1)

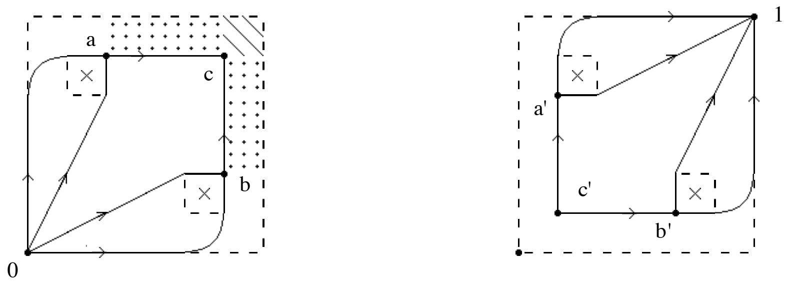


X E M Z

The fundamental category $C = \uparrow\Pi_1(X)$ is easy to determine. We prove below that its pf-spectrum is the full subcategory E, on eight vertices (where the two cells marked with a cross do not commute, while the central one does), while the spectral projective model is M and the category Z is just a coarse model (of all three). Again, as in 1.5, the projective model is isomorphic to the 'category of components' of C constructed in [FRGH].

First, we show that the category $C = \uparrow\Pi_1(X)$ has a past spectrum

(2)



P F

In fact, there are four past regularity classes of objects, each having an initial object:

(3) $[c]^- = \{x \mid x \geq c\}$ (marked with a shade),

$[a]^- = \{x \mid x \geq a\} \setminus [c]^-$, $[b]^- = \{x \mid x \geq b\} \setminus [c]^-$ (both marked with dots),

$[0]^- = X \setminus ([c]^- \cup [a]^- \cup [b]^-)$ (unmarked),

where a, b, c are effective V^--branching points and 0 is the global minimum, weakly initial in C.

These four points form the past spectrum $sp^-(C) = \{0, a, b, c\}$, as is easily verified with the characterisation 3.5a: take the full subcategory $P \subset C$ on these objects (represented in the same picture), its embedding $i^-: P \subset C$ and the projection p^- sending each point $x \in C$ to the minimum of its past regularity class. Now $i^- \dashv p^-$, with a counit-component $\varepsilon(x): i^-p^-(x) \to x$ which is uniquely determined in $\uparrow\Pi_1(X)$, since - within each of the four zones described above - there is at most one homotopy class of paths between two given points.

Symmetrically, we have the future spectrum: the full subcategory $F \subset C$ in the right figure above, on the following four objects (each of them a maximum in its future regularity class):

- 1 (the global maximum, *weakly* terminal); a', b', c' (V^+-branching points).

The projection p^+ (left adjoint to $i^+: F \subset C$) sends each point $x \in C$ to the maximum of its future regularity class (i.e. the lowest distinguished vertex $p^+(x) \geq x$); the unit-component $\eta(x): x \to i^+p^+(x)$ is, again, uniquely determined in $\uparrow\Pi_1(X)$.

Globally, we have constructed a spectral pf-presentation of C (3.6); this generates the skeletal injective model E, as the full subcategory of C on sp(C) = {0, a, b, c, a', b', c', 1}. The full subcategory $Z \subset E$ on the objects 0, 1 is isomorphic to the past spectrum of F, as well as to the future spectrum of P, hence coarse equivalent (1.3) to C and E.

Comments. The pf-spectrum E provides a category with the same past and future behaviour as C. This can be read as follows:

(a) the action begins at the 'starting point' 0, the minimum, from where we can only move to c';

(b) c' is an (effective) V^+-branching point, where we choose: either the upper/middle way or the lower/middle one;

(c) the first choice leads to a', a further V^+-branching point where we choose between the upper or the middle way; similarly, the second choice leads to the V^+-branching point b', where we choose between the lower or the middle way (the same as before);

(d) the first bifurcation considered in (c) is 'joined' at a, the second at b (V^--branching points);

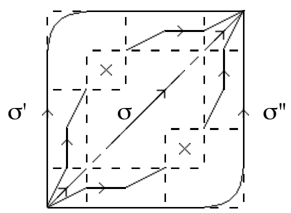
(e) the resulting 'paths' come together at c (the last V^--branching point);

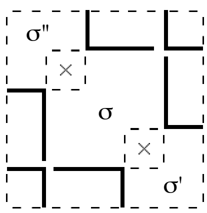
(f) from where we can only move to the 'ending point' 1, the maximum.

The 'coarse model' Z only says that in C there are *three* homotopically distinct ways of going from 0 to 1, and looses relevant information on the branching structure of C. The projective model is studied below.

4.3. The projective model. For the same category $C = \uparrow\Pi_1(X)$, the spectral projective model M, represented in the right figure below, is the full subcategory of C^2 on the 9 arrows displayed in the left figure (only three of them are labelled)

(1)





$$\begin{array}{ccccc} \sigma'' & \longrightarrow & \delta' & \longrightarrow & \beta \\ \uparrow & \times & \uparrow & & \uparrow \\ \gamma' & \longrightarrow & \sigma & \longrightarrow & \delta \\ \uparrow & & \uparrow & \times & \uparrow \\ \alpha & \longrightarrow & \gamma & \longrightarrow & \sigma' \end{array}$$

M

The projection $f(x) = (p^-x, p^+x;\ \eta x.\varepsilon x)$ (3.4), from $X = \mathrm{Ob}C$ to $\mathrm{Ob}M \subset \mathrm{Mor}C$, has thus nine equivalence classes, analytically defined in (2) and 'sketched' in the middle figure above (the thick lines are meant to suggest that a certain boundary segment belongs to a certain region, as made precise below); in each of these regions, the morphism f(x) is constant, and equal to $\alpha, \beta, \ldots$

(2) $f^{-1}(\alpha) = [0, 1/5]^2$, $\quad f^{-1}(\beta) = [4/5, 1]^2$ (closed in X),

$f^{-1}(\gamma) = \,]1/5, 3/5] \times [0, 1/5]$, $\quad f^{-1}(\gamma') = [0, 1/5] \times \,]1/5, 3/5]$,

$f^{-1}(\delta) = [4/5, 1] \times [2/5, 4/5[$, $\quad f^{-1}(\delta') = [2/5, 4/5[\times [4/5, 1]$,

$f^{-1}(\sigma) = X \cap \,]1/5, 4/5[^2$ (open in X),

$f^{-1}(\sigma') = X \cap (]3/5, 1] \times [0, 2/5[)$, $\qquad f^{-1}(\sigma'') = X \cap ([0, 2/5[\times]3/5, 1])$ $\qquad$ (open in X).

The interpretation of the projective model M is practically the same as above, in 4.2, with some differences:

(i) in M there is no distinction between the starting point and the first future branching point, as well as between the ending point and the last past branching point;

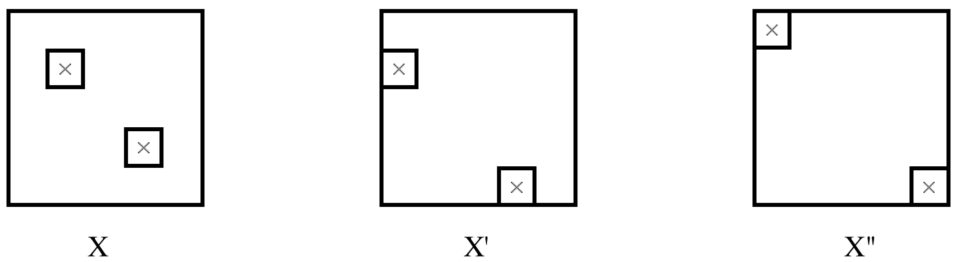
(ii) the different paths produced by the obstructions are 'distinguished' in M by three new intermediate objects: $\sigma, \sigma', \sigma''$.

Note also that - here and in many cases - one *can* also embed M in C, by choosing a suitable point of a suitable path in each homotopy class $\alpha, \beta, ...$; but there is no canonical way of doing so. This point will be studied in the next section.

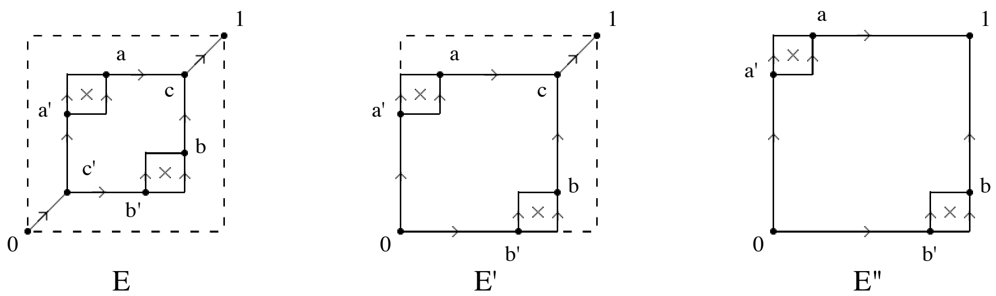
In order to compare the injective model E and the projective model M, the examples below will make clear that distinguishing 0 from c' (or c from 1) carries *some* information (like distinguishing the initial from the terminal object, in the injective model **2** of a non-pointed category having both, cf. 3.2d). According to applications, one may decide whether this information is useful or redundant.

4.4. Variations. (a) Consider the previous ordered space X (4.2) together with the spaces X' and X", obtained by taking out, from the ordered square $\uparrow[0, 1]^2$, two open squares placed in different positions, 'at' the boundary

(1)



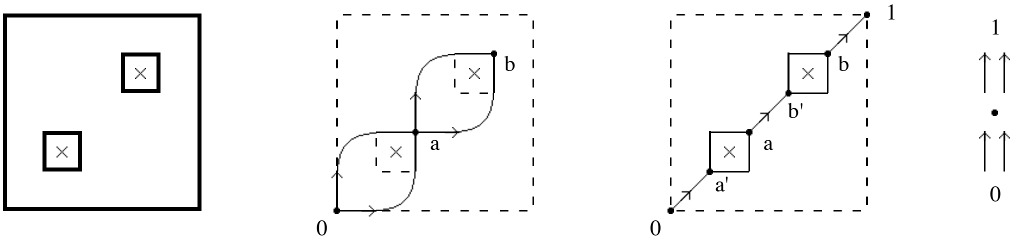
(2)



The pf-spectra E, E' and E" distinguish these situations: in the second case the starting point 0 is an *effective* future branching point, and *we must make a choice from the very beginning* (either the upper/middle way or the middle/lower one); in the last case, this remains true and moreover the ending point is an *effective* past branching point. The projective models of these three spectra coincide: we always get the category M of 4.3.

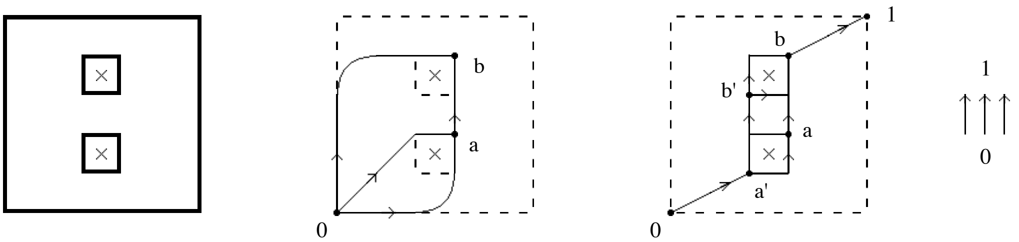
(b) The following examples show similar situations, with a different injective (and projective) model. We start again from a (compact) ordered space $X_i \subset \uparrow[0, 1]^2$, obtained by taking out two open squares

(3)



X_1 P_1 E_1 Z_1

(4)

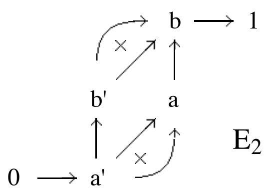


X_2 P_2 E_2 Z_2

In both cases, the past spectrum of the fundamental category $C_i = \uparrow\Pi_1(X_i)$ is the full subcategory P_i on three objects: 0 (the minimum), a, b (V^--branching points), as shown above. The future spectrum is symmetric. The pf-spectrum, generated by the previous presentation, is the full subcategory E_i on the pf-spectrum $sp(C_i) = \{0, a, b, a', b', 1\}$. Coarse models of C_i are given by the categories Z_i generated by the graphs above; in particular, Z_1 has four arrows from 0 to 1.

In the second case, E_2 is better represented "abstractly", to avoid the partial superposition of paths in the former embedding; the central cell commutes

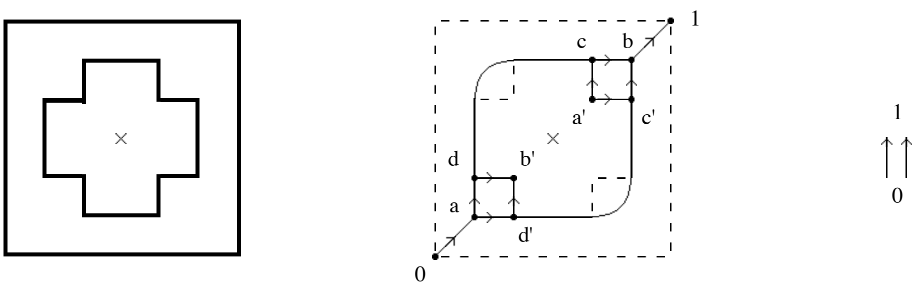
(5)



E_2

4.5. The Swiss flag. The following situation is often analysed as a basic one, in concurrency: the 'Swiss flag' $X \subset \uparrow[0, 1]^2$. See [FGR, FRGH, GG, Go] for a description of 'the conflict of resources' which it depicts, and [FRGH, p. 84] for an analysis of the fundamental category which leads to a 'category of components' similar to the projective model we get here

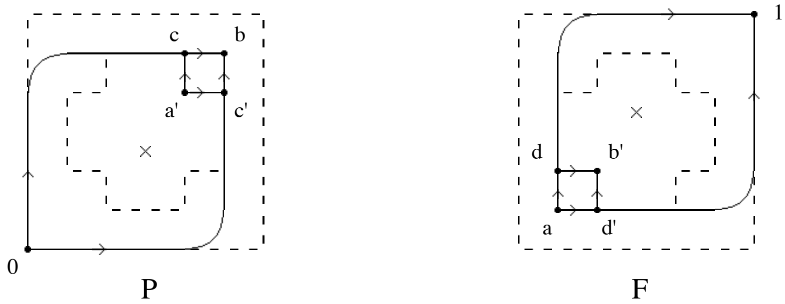
(1)



X E Z

Proceeding as above, the fundamental category $C = \uparrow\Pi_1(X)$ has an injective model E and a coarse model Z. Now, the past spectrum is the full subcategory $P \subset C$ represented below

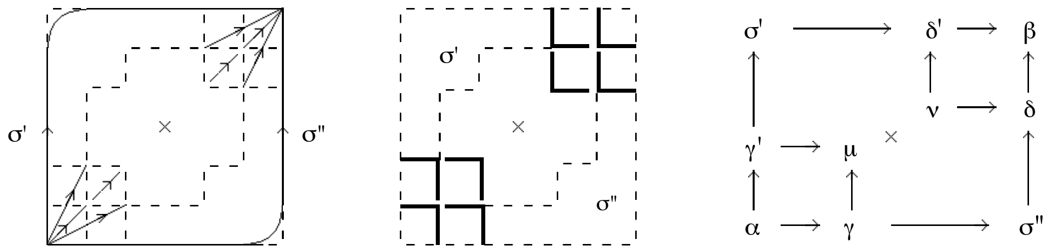
(2)



The past spectrum $sp^-(C) = \{0, a', a, b, c, c'\}$ contains two minimal points 0, a' (note that the starting point 0 is not a minimum for $\prec$) and three V^--branching points (b, c, c'). Similarly, the future spectrum is the full subcategory $F \subset C$ in the right figure above, on the future spectrum $sp^+(C) = \{a, d, d', b', 1\}$. The pf-spectrum of C is the full subcategory category E on $sp(C) = sp^-(C) \cup sp^+(C)$.

The spectral projective model M is shown below, under the same conventions as in 4.3.1

(3)



Note that it can again be embedded in C.

5. Surjective models and retracts

We begin now a new study of the fundamental category, via *surjective* models, determined by retracts. For the hollow cube, we get in this way an analysis similar to the one of [FRGH] and [Ha].

5.1. Retractile models. Given a category X, a retract $i: M \rightleftarrows X :p$ gives an idempotent endofunctor $e = ip: X \to X$; conversely, given the latter, one reconstructs M as the subcategory of objects and morphisms which e leaves fixed. The retracts of X form an ordered set, with the usual order relation of idempotents: $e' \leq e$ if $e' = ee' = e'e$; this amounts to saying that the second retract $i': M' \rightleftarrows X :p'$ factors through the first, by a (unique) retraction

$$\text{(1)} \qquad M' \underset{q}{\overset{j}{\rightleftarrows}} M \underset{p}{\overset{i}{\rightleftarrows}} X \qquad\qquad ij = i', \qquad qp = p'.$$

A *retractile model* of X will be a pf-equivalence $p: X \leftrightarrows M :g^\alpha$ (3.1)

$$\text{(2)} \quad \varepsilon: g^-p \to 1, \qquad \varepsilon_M: pg^- \to 1_M, \qquad (p\varepsilon = \varepsilon_M p, \quad \varepsilon g^- = g^-\varepsilon_M),$$

$$\eta: 1 \to g^+p, \qquad \eta_M: 1_M \to pg^+, \qquad (p\eta = \eta_M p, \quad \eta g^+ = g^+\eta_M),$$

together with a section $i: M \to X$ of p, *consistent* with the pf-equivalence in the following sense (writing $e = ip: X \to X$)

(3) $e\varepsilon e = e\varepsilon$, $\qquad$ $e\eta e = e\eta$.

Equivalently, we can assign the functors $p: X \to M$ and $i, g^-, g^+: M \to X$ and the two natural transformations at X, under the axioms

(4) $\varepsilon: g^-p \to 1_X$ (the *main counit*), $\qquad (g^-p)\varepsilon = \varepsilon(g^-p)$,

$\eta: 1_X \to g^+p$ (the *main unit*), $\qquad (g^+p)\eta = \eta(g^+p)$,

$pi = 1$, $\qquad e\varepsilon e = e\varepsilon$, $\qquad e\eta e = e\eta$.

Then, we define

(5) $\varepsilon_M = p\varepsilon i: pg^- \to 1_M$, $\qquad \eta_M = p\eta i: 1_M \to pg^+$,

and we get $\varepsilon_M p = p\varepsilon ip = p\varepsilon$, $g^-\varepsilon_M = g^-p\varepsilon i = \varepsilon g^-pi = \varepsilon g^-$, etc. The coherence condition 3.1.2 is automatically satisfied, because p is surjective on objects.

Again, such a model will often be written as $p: X \to M$ leaving the remaining structure understood. We say that a retractile model is *semi-faithful* if the functor p is faithful. Notice that p need not be full, and our pf-surjection need not be a pf-projection, as it will appear from the examples below.

We do not know whether a spectral projective model is always a retractile one. But all the ones considered in Section 4 are.

5.2. Pf-presentation and retractile models. Let us start from a pf-presentation (3.3) of the category X (not necessarily produced by spectra)

(1) $$P \underset{p^-}{\overset{i^-}{\rightleftarrows}} X \underset{i^+}{\overset{p^+}{\rightleftarrows}} F \qquad \begin{array}{ll} \varepsilon: i^-p^- \to 1_X & (p^-i^- = 1,\ p^-\varepsilon = 1,\ \varepsilon i^- = 1), \\ \eta: 1_X \to i^+p^+ & (p^+i^+ = 1,\ p^+\eta = 1,\ \eta i^+ = 1). \end{array}$$

We say that a retract $i: M \rightleftarrows X :p$ is *consistent* with this pf-presentation if the endofunctor $e = ip: X \to X$ satisfies:

(2) $e\varepsilon e = e\varepsilon$, $\qquad$ $e\eta e = e\eta$.

The next theorem shows that M is then a retractile model of X, admitting the same past and future retracts P and F. This fact will be of interest when the projection p is faithful (or - equivalently - the endofunctor e is), *while the injective and projective models associated to the given presentation* (3.4) *are not faithful.*

5.3. Theorem (Presentations and retracts). Let us be given a pf-presentation of the category X and a consistent retract $i: M \rightleftarrows X :p$ (as above, 5.2).

(a) There is an induced pf-presentation of M, so that the four squares of the following left diagram commute

$$(1)\qquad \begin{array}{ccccc} P & \underset{p^-}{\overset{i^-}{\rightleftarrows}} & X & \underset{i^+}{\overset{p^+}{\leftrightarrows}} & F \\ \| & & \downarrow p & & \| \\ P & \underset{q^-}{\overset{j^-}{\rightleftarrows}} & M & \underset{j^+}{\overset{q^+}{\leftrightarrows}} & F \end{array} \qquad\qquad \begin{array}{c} X \\ g^- \uparrow \downarrow \uparrow g^+ \\ M \end{array}$$

(b) There is an associated retractile model $p: X \rightleftarrows M :g^\alpha$, with $i: M \to X$ and

(2) $g^\alpha = i^\alpha q^\alpha = i^\alpha p^\alpha i$,

(3) $g^\alpha p = i^\alpha p^\alpha$, $\qquad g^\alpha p g^\alpha = g^\alpha \qquad (\alpha = \pm)$.

Proof. (a) Let us define the new retractions

(4) $j^\alpha = pi^\alpha$, $\qquad q^\alpha = p^\alpha i \qquad (\alpha = \pm)$,

and note that $q^\alpha p = p^\alpha ip = p^\alpha$ (by hypothesis, 5.2.2) and $q^\alpha j^\alpha = q^\alpha pi^\alpha = p^\alpha i^\alpha = 1$. Now, P becomes a past retract of M and F a future retract, with the following counit and unit:

(5) $\varepsilon' = p\varepsilon i: j^-q^- \to 1_X \qquad \eta' = p\varepsilon i: 1_X \to j^+q^+ \qquad (j^\alpha q^\alpha = pi^\alpha p^\alpha i)$,

which are easily seen to be coherent (applying the coherence of the original past and future retract, together with the hypothesis, $e\varepsilon e = e\varepsilon$ and $e\eta e = e\eta$). For instance, for P we have:

(6) $q^-\varepsilon' = q^-p\varepsilon i = (p^-\varepsilon)i = 1$, $\qquad \varepsilon' = p\varepsilon ipi^- = p(\varepsilon i^-) = 1$.

(b) Let us define the functors g^α as in (2), which implies (3). The functor p is obviously surjective on objects. The pair (p, g^-) becomes a past equivalence with the original ε and the previous ε'

(7) $\varepsilon: g^-p \to 1_X \qquad \varepsilon': pg^- \to 1_M$,

since $g^-p = i^-q^-p = i^-p^-$ and $pg^- = pi^-q^- = j^-q^-$. Again, coherence is easily verified:

(8) $p\varepsilon = p\varepsilon ip = \varepsilon'p$,

$\varepsilon g^- = (\varepsilon i^-)q^- = 1$, $\qquad g^-\varepsilon' = i^-q^-p\varepsilon i = i^-(p^-\varepsilon)i = 1$.

Symmetrically, one shows that (p, g^+, η, η') is a future equivalence between X and M, noting that $\eta: 1_X \to i^+p^+ = g^+p$ and $\eta': 1_X \to j^+q^+ = pg^+$. The conditions 5.1.3 are satisfied, by hypothesis.

Finally, the associated functor $g: M \to X^2$ (3.1.4)

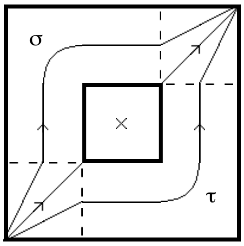
(9) $g(x) = \varepsilon g^+(x).g^-\eta'(x) = g^+\varepsilon'(x).\eta g^-(x): g^-(x) \to g^+(x)$,

can even be constant (on objects and morphisms), in cases of interest: see the hollow cube 1.6, where the past spectrum P reduces to the initial object $\bot$, the future spectrum F reduces to the terminal object $\top$, so that g(x) is always the unique morphism $\bot \to \top$. □

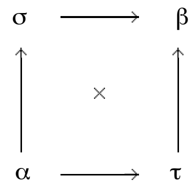
5.4. Reviewing the square annulus. Let us reconsider the fundamental category $C = \uparrow\Pi_1(X)$ of the square annulus X (1.2), and its spectral projective model $p: C \rightleftarrows M :g^\alpha$ (3.4, 3.6)

(1) $p: C \rightleftarrows M :g^\alpha$, $\qquad p(x) = \eta x.\varepsilon x: i^-p^-x \to i^+p^+x$,

(2)



X



M

The space X is decomposed into four classes

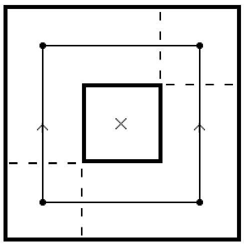
(3) $p^{-1}(\alpha) = [0, 1/3]^2$, $p^{-1}(\beta) = [2/3, 1]^2$ (closed in X),

$p^{-1}(\sigma) = X \cap ([0, 2/3[\times]1/3, 1])$, $p^{-1}(\tau) = X \cap (]1/3, 1] \times [0, 2/3[)$ (open in X).

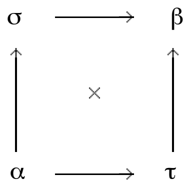
This projective model can be obtained as above, in 5.3, starting from the spectral presentation and a retract i: M $\rightleftarrows$ C :p, where i is any section of p, i.e. any choice of points in the four classes with

(4) $i(\alpha) \leq i(\sigma) \leq i(\beta)$, $i(\alpha) \leq i(\tau) \leq i(\beta)$,

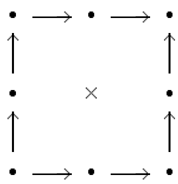
(5)



X



M



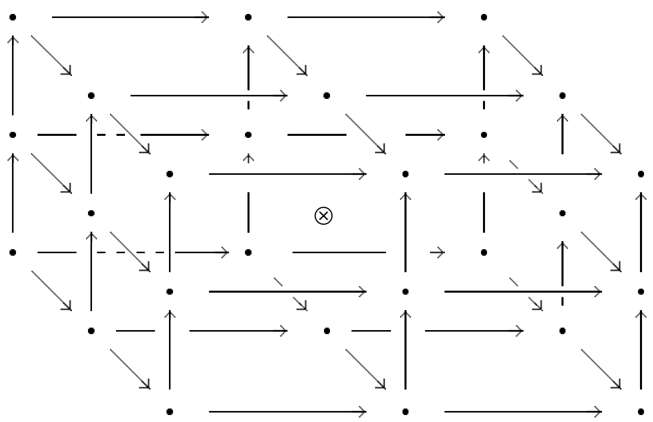
M'

The procedure 5.3 can also be applied to a larger retract i': M' $\rightleftarrows$ C :p' (see the right figure above), taking for instance the middle points of all the eight 'subsquares' of X of edge 1/3, 'around the obstruction' (with some arbitrary choice in defining the eight equivalence classes of p'). Plainly, there is no advantage in enlarging the model.

5.5. A retractile model of the hollow cube. However, similar enlargements can be of interest *when the spectral projective model is unsatisfactory* - e.g. not faithful, as it happens with the hollow cube X (1.6).

The fundamental category $C = \uparrow\Pi_1(X)$ has a reasonably simple *semi-faithful* retractile model, the full subcategory M on the 26 middle points of the 'subcubes' of edge 1/3 placed 'around the obstruction' (as suggested by the analysis of [FRGH], see figure 7). M is generated by the following graph under the condition that all squares commute except the 3 ones around the obstruction ⊗

(1)



It would be interesting to prove that this retractile model is minimal (among such).

Note also that this pf-surjection is *not* be a pf-projection: the associated functor g: M → C^2 (3.1.4) is not even injective on objects.

6. Normal quotients of categories

Generalised quotients of categories, examined here, will also be useful to construct surjective models.

6.1. Generalised congruences of categories. First, let us recall that a very general notion of *generalised congruence* in a category (also involving objects) can be found in a paper by Bednarczyk, Borzyszkowski and Pawlowski [BBP].

A *quotient* of categories will be viewed with respect to this notion, even if - as in [Ha] - we only need a particular case, *determined by the maps which we want to become identities*. More precisely, given a category X and a set A of its arrows, X/A denotes the quotient of X modulo the generalised congruence generated by declaring every arrow in A to be equivalent to the identity of its domain. (As shown in [BBP], the generalised congruences of a category form a complete lattice, hence we can always take the intersection of all the generalised congruences containing a certain relation.)

The quotient p: X → X/A is determined by the obvious universal property:

(i) for every functor f: X → Y which takes all the morphisms of A to identities, there is a unique functor f': X/A → Y such that f = f'p.

These quotients of the category X also form a complete lattice, with the same arbitrary meets as the general quotients of X (in the previous sense): the meet of a family p_i: X → X/A_i is the quotient of X modulo the set-theoretical union of the sets A_i.

6.2. Kernels and normal quotients. This particular case can be made clearer when viewed at the light of general considerations on kernels and cokernels with respect to an *assigned ideal* of 'null' arrows, studied in [G1] - independently of the existence of a zero object. (See also Ehresmann [Eh], including the Comments in the same volume, p. 845-847; and Lavendhomme [La].)

Take, in **Cat**, the ideal of *discrete* functors, i.e. those functors which send every map to an identity; or, equivalently, consider as *null objects* in **Cat** the discrete categories and say that a functor is null if it factors through such a category (we have thus a *closed* ideal, according to [G1]).

This ideal produces - by the usual universal properties *formulated with respect to null functors* - a notion of kernels and cokernels in **Cat**. Precisely, given a functor $f: X \to Y$, its kernel is the wide subcategory of all morphisms of X which f sends to identities of Y, while its cokernel is the quotient $Y \to Y/B$, produced by the set B of arrows reached by f.

A *normal subcategory* $X_0 \subset X$, by definition, is a kernel of some functor starting at X, or, equivalently, the kernel of the cokernel of its embedding. It is necessarily a wide subcategory, but of course there are wide subcategories which are not normal. (For instance, in the ordinal **3**, the wide subcategory consisting of the arrows $0 \to 1$ and $0 \to 2$ (with all identities) is not normal: also $1 \to 2$ is forced to become an identity in the quotient.)

Dually, a *normal quotient* $p: X \to X'$ is the cokernel of some functor with values in X (or, equivalently, the cokernel of its kernel). A normal quotient (or, more generally, any quotient modulo a generalised congruence) is always surjective on objects, as it follows easily using its factorisation through its full image, $p = jq: X \to X'' \subset X'$. (The functors p and q have clearly the same kernel, whence also q factors through p, as $q = hp$; moreover, $jh = 1$, by the universal property of p; it follows that $jhj = j$ and, cancelling the embedding, $hj = 1$.)

Now, the normal quotients of X are precisely those we are interested in. First, we already know that a normal quotient is always of the type $X \to X/A$. Conversely, given a set A of arrows of X, the quotient $X \to X/A$ is the cokernel of some functor with values in X; for instance, one can take the free category A' on the graph A and the resulting functor $A' \to X$.

The normal subcategories of a category X and its normal quotients form thus two complete lattices, anti-isomorphic via kernels and cokernels

(Similarly, the ideal in **Cat** of those functors which send all maps to isomorphisms would give, as normal quotients, the categories of fractions.)

6.3. Lemma (The 2-dimensional universal property). The normal quotient $p: X \to X/A$ satisfies (after (i) in 6.2), a 2-dimensional universal property:

(ii) for every natural transformation $\varphi: f \to g: X \to Y$ where f and g take all the morphisms of A to identities of Y, there is a unique natural transformation $\varphi': f' \to g': X/A \to Y$ such that $\varphi = \varphi' p$.

(More generally, quotient of categories in the sense recalled in 6.1 also satisfy a 2-dimensional universal property; the proof is similar.)

Proof. As in 3.1, a natural transformation $\varphi: f \to g: X \to Y$ can be viewed as a functor $\varphi: X \to Y^{\mathbf{2}}$ which, composed with Dom, Cod: $Y^{\mathbf{2}} \to Y$ gives f and g, respectively. This functor sends the object $x \in X$ to the arrow $\varphi(x): f(x) \to g(x)$ and the map $a: x \to x'$ to its naturality square

$$(1) \qquad \begin{array}{ccc} f(x) & \xrightarrow{\varphi x} & g(x) \\ {\scriptstyle fa}\downarrow & & \downarrow{\scriptstyle ga} \\ f(x') & \xrightarrow[\varphi x']{} & g(x') \end{array}$$

Therefore, if $a \in A$, $f(a)$ and $g(a)$ are identities and $\varphi(a) = \mathrm{id}\varphi(x)$. Then, by the original universal property, the functor φ factors uniquely through p, by a functor $\varphi': X/A \to Y^{\mathbf{2}}$; this is the natural transformation that we want. □

6.4. Theorem. A normal quotient $p: X \to X'$ is given.

(a) For $x, x' \in X$, $p(x) = p(x')$ if and only if there exists a finite zig-zag $(a_1,..., a_n): x \nrightarrow x'$ of morphisms a_i in Ker(p), as below (the dotted arrow recalls that this sequence is not a map of X)

$$\text{(1)} \qquad \begin{array}{ccccccccc} & & x_1 & & & & x_{2n-1} & & \\ & {}^{a_1}\nearrow & & \nwarrow^{a_2} & & {}^{a_{2n-1}}\nearrow & & \nwarrow^{a_{2n}} & \\ x = x_0 & & & & x_2 \quad \dots \quad x_{2n-2} & & & & x_{2n} = x' \end{array}$$

Since p is surjective on objects (6.2), ObX' can be identified with the quotient of ObX modulo the equivalence relation of *connection in* Ker(p). (Recall that two objects in a category are said to be *connected* if they are linked by a zig-zag of morphisms.)

(b) A morphism $z: p(x) \to p(x')$ in the quotient X' comes from a finite zig-zag as above where the *backward* arrows a_{2i} are in Ker(p), and $z = p(a_{2n-1}). ... p(a_3).p(a_1)$. Thus, every arrow of X' is a composition of arrows in the graph-image of p.

Proof. (a) Let R be the generalised congruence of X generated by Ker(p), and R_0 the equivalence relation which it induces on the objects; plainly, the equivalence relation R'_0 described above (by the existence of a diagram (1)) is contained in R_0. To show that it coincides with the latter, let R' be the relation given by R'_0 on the objects and as 'chaotic' on morphisms as possible:

(2) $a\, R'_1\, b$ if $(\mathrm{Dom}(a)\, R'_0\, \mathrm{Dom}(b)$ and $\mathrm{Cod}(a)\, R'_0\, \mathrm{Cod}(b))$.

R' is a generalised congruence of X, whose quotient is the preorder category on $\mathrm{Ob}(X)/R'_0$ having one morphism from [x] to [x'] whenever there is a chain of maps of X, composable up to R'_0, from x to x'.

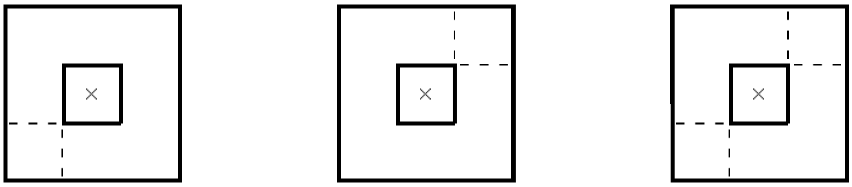
Therefore, the intersection $R \cap R'$ is again a congruence of categories. But $X/(R \cap R')$ plainly satisfies the universal property of R, whence $R = R \cap R'$, which means that $R_0 \subset R'_0$.

Finally, by the general theory of [BBP], a morphism $z: p(x) \to p(x')$ of the quotient category is the equivalence class of a finite sequence $(b_0,..., b_p)$ of maps in X which are 'composable up to the previous equivalence relation on objects'; inserting a zig-zag as above between all pairs $\mathrm{Cod}(b_{i-1})$, $\mathrm{Dom}(b_i)$ and identities where convenient, we can always form a global zig-zag (1), from x to x', where all 'backward' arrows are in A; and then $p(a_{2n-1}). ... p(a_3).p(a_1) = z$. □

6.5. Pf-equivalence of objects. We have recalled the *future regularity equivalence* relation $x \sim^+ x'$ between two objects x, x' in a category X (2.6), meaning that there is a finite zig-zag $x \nrightarrow x'$ (as in 6.4.1), made of future regular maps. Its dual is written $x \sim^- x'$, and we will write $x \sim^{\pm} x'$ to mean that both these relations hold.

But we are more interested in a stronger equivalence relation: we say that the objects x, x' are *pf-equivalent* if there is a finite zig-zag $x \nrightarrow x'$ whose maps are both *past and future regular*.

For instance, in the fundamental category of the square annulus, there are two equivalence classes for $\sim^-$ (see the left picture below) and two classes for $\sim^+$ (see the central picture)

(1) 

which produce 3 classes for $\sim^{\pm}$ (their intersections) and 4 classes of pf-equivalence, since the two L-shaped zones in the right picture above are not connected. We shall see that, if the category X has a past and a future spectrum, its pf-equivalence classes coincide with the connected components of $\sim^{\pm}$-equivalence classes (6.8c), as in the example above.

6.6. Proposition. If $p: X \to M$ is a quotient model, then $p(x) = p(x')$ implies that x, x' are pf-equivalent.

Proof. By 6.4, if $p(x) = p(x')$ there is a finite zig-zag $x \twoheadrightarrow x'$ of morphisms a_i which p takes to identities. But p, as a past and future equivalence, reflects past and future regular morphisms (Thm. 2.8), whence all a_i are past and future regular. □

6.7. Lemma (Quotients and retracts). (a) Given a retract $i: M \rightleftarrows X :p$ (with $pi = 1_M$), the following conditions are sufficient to ensure that p is a normal quotient:

(i) for every $x \in X$ there exists a zig-zag $(a_1,..., a_n)$: $ip(x) \twoheadrightarrow x$, in Ker(p),

(ii) for every map $u: x \to y$ in X there exists a zig-zag $((a_1, b_1)..., (a_n, b_n))$: $ip(u) \twoheadrightarrow u$ in the category X^2, whose arrows a_i, b_i belong to Ker(p)

(1)
$$\begin{array}{ccccccccc}
x_0 & \xrightarrow{a_1} & x_1 & \xleftarrow{a_2} & x_2 & \xrightarrow{a_3} & x_3 \; ... & \xleftarrow{a_{2n}} & x \\
{\scriptstyle ip(u)}\downarrow & & \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_2} & & \downarrow{\scriptstyle u_3} & & \downarrow{\scriptstyle u} \\
y_0 & \xrightarrow[b_1]{} & y_1 & \xleftarrow[b_2]{} & y_2 & \xrightarrow[b_3]{} & y_3 \; ... & \xleftarrow[b_{2n}]{} & y
\end{array}$$

(Point (i) is an obvious consequence of (ii); nevertheless, it makes things clearer.)

(b) In a future retract $i: F \rightleftarrows X :p$, the functor p is always a normal quotient.

Proof. (a) Let $f: X \to Y$ be any functor with $\mathrm{Ker}(p) \subset \mathrm{Ker}(f)$. Plainly, $p(x) = p(x')$ implies $f(x) = f(x')$ and $p(u) = p(u')$ implies $f(u) = f(u')$, for all objects x, x' and all maps u, u' in X. Since $p(ip) = p$, it follows that $f = f(ip) = (fi)p$. Therefore, f factors through p, obviously in a unique way.

(b) Follows easily from (a), since for every $x \in X$, the unit $\eta x: ip(x) \to x$ belongs to Ker(p) (because of the coherence condition $p\eta = 1$). For (ii), just apply the naturality of η. □

6.8. Lemma (Quotients and spectra). (a) If the category X has a future spectrum, the retraction $p^+: X \to \mathrm{Sp}^+(X)$ is a normal quotient. Its kernel is characterised by the following equivalent conditions, on a map $a: x \to x'$

(i) $p^+(a)$ is an identity,

(ii) a is future regular,

(iii) $x \sim^+ x'$.

(b) Dually, the projection $p^-: X \to Sp^-(X)$ on the past spectrum (if it exists) is a normal quotient of X, whose kernel is the set of past regular morphisms.

(c) If the category X has a past and a future spectrum, then its pf-equivalence classes coincide with the connected components of $\sim^{\pm}$-equivalence classes (connected by zig-zags of maps).

Proof. (a) After 6.7b, we have only to prove the characterisation of $\mathrm{Ker}(p^+)$. In fact, if $p^+(a)$ is an identity, then a is future regular (since p^+ reflects such morphisms), which implies (iii). Finally, if $x \sim^+ x'$, then $\eta x'.a = \eta x: x \to i^+p^+(x)$ (because $i^+p^+(x)$ is terminal in the full subcategory on $[x]^+$); but $p^+(\eta x)$ and $p^+(\eta x')$ are identities, and so is $p^+(a)$.

(c) Follows from (a) and its dual, which prove that a map $a: x \to x'$ is past and future regular if and only if $x \sim^{\pm} x'$. □

7. Surjective models and normal quotients

After retractions, normal quotients of categories also yield a way of constructing surjective models.

7.1. Quotient models. A pf-equivalence $p: X \rightleftarrows M : g^{\alpha}$ (3.1) will be said to be a *quotient model* of X if p is a normal quotient. It is thus a surjective model (p is surjective on objects, by 6.2). Again, a quotient model is said to be *semi-faithful* if the functor p is faithful.

Also here, a quotient model will often be represented as $p: X \to M$, leaving the remaining structure understood:

(1) $\varepsilon: g^-p \to 1$, $\quad \varepsilon_M: pg^- \to 1_M$, $\quad (p\varepsilon = \varepsilon_M p, \quad \varepsilon g^- = g^-\varepsilon_M)$,

$\eta: 1 \to g^+p$, $\quad \eta_M: 1_M \to pg^+$, $\quad (p\eta = \eta_M p, \quad \eta g^+ = g^+\eta_M)$.

A *minimal quotient model* is defined (up to isomorphism of categories) by the following properties (similar to the ones in 3.2):

(i) M is a quotient model of every quotient model M' of X,

(ii) every quotient model M' of M is isomorphic to M.

7.2. Pf-presentation and quotient models. Let us start from a pf-presentation (3.3) of the category X (not necessarily produced by spectra)

$$(1) \qquad P \underset{p^-}{\overset{i^-}{\rightleftarrows}} X \underset{i^+}{\overset{p^+}{\rightleftarrows}} F \qquad \begin{array}{ll} \varepsilon: i^-p^- \to 1_X & (p^-i^- = 1,\ p^-\varepsilon = 1,\ \varepsilon i^- = 1), \\ \eta: 1_X \to i^+p^+ & (p^+i^+ = 1,\ p^+\eta = 1,\ \eta i^+ = 1). \end{array}$$

We say that a normal quotient $p: M \to X$ is *consistent* with this pf-presentation if there are functors $q^-: M \to P$ and $q^+: M \to F$ satisfying the following conditions (which determine them)

(2) $q^-p = p^-$, $\qquad q^+p = p^+$,

or, equivalently, if the kernel of p (i.e. the set of morphisms which p takes to identities) is contained in the set of past and future regular maps.

The next theorem shows that M is then a quotient model of X, admitting the same past and future retracts P and F. This fact will be of interest when the projection p is faithful, *while the injective and projective models associated to the given presentation* (3.4) *are not faithful.*

7.3. Theorem (Presentations and quotients). Let us be given a pf-presentation of the category X and a *consistent* normal quotient $p: M \to X$ (as above, 7.2).

(a) There is a (uniquely determined) induced pf-presentation of M, so that the four squares of the following left diagram commute

(1)
$$\begin{array}{ccccc} P & \underset{p^-}{\overset{i^-}{\rightleftarrows}} & X & \underset{i^+}{\overset{p^+}{\leftrightarrows}} & F \\ \| & & \downarrow p & & \| \\ P & \underset{q^-}{\overset{j^-}{\rightleftarrows}} & M & \underset{j^+}{\overset{q^+}{\leftrightarrows}} & F \end{array} \qquad\qquad \begin{array}{c} X \\ g^- \uparrow \; \downarrow \; \uparrow g^+ \\ M \end{array}$$

(b) There is an associated quotient model $p: X \rightleftarrows M :g^\alpha$, with

(2) $g^\alpha \;=\; i^\alpha q^\alpha$

(3) $g^\alpha p \;=\; i^\alpha p^\alpha,$ $\qquad pg^\alpha \;=\; j^\alpha q^\alpha,$ $\qquad g^\alpha p g^\alpha \;=\; g^\alpha$ $\qquad (\alpha = \pm),$

so that, in particular, p is a past and future equivalence. (This model need not be projective).

Proof. (a) The new retractions are (and must be) defined taking $j^\alpha = pi^\alpha$ and the functors q^α such that $q^\alpha p = p^\alpha$ (7.2.2).

Now (according to 6.3(ii)), the natural transformations $p\varepsilon: (j^-q^-)p \to 1_X p$ and $p\eta: 1_X p \to (j^+q^+)p$ induce two natural transformations, characterised as below

(4) $\varepsilon': j^-q^- \to 1_M,$ $\qquad \varepsilon' p \;=\; p\varepsilon,$

$\eta': 1_M \to j^+q^+,$ $\qquad \eta' p \;=\; p\eta.$

Thus, P becomes a past retract of M (and F a future retract):

(5) $\varepsilon' j^- \;=\; \varepsilon' p i^- \;=\; p\varepsilon i^- \;=\; 1,$ $\qquad (q^-\varepsilon')p \;=\; q^- p\varepsilon \;=\; p^-\varepsilon \;=\; 1_{p^-} \;=\; (1_{q^-}).p.$

(b) Let us define the functors g^α as in (2), which implies (3). The functor p is surjective on objects. The pair (p, g^-) becomes a past equivalence with the original ε and the previous ε'

(6) $\varepsilon: g^-p \to 1_X$ $\qquad \varepsilon': pg^- \to 1_{M}.$

As to coherence, we already know that $p\varepsilon = \varepsilon' p$; moreover:

(7) $\varepsilon g^- \;=\; (\varepsilon i^-)q^- \;=\; 1,$ $\qquad g^-\varepsilon' \;=\; i^-(q^-\varepsilon') \;=\; 1.$

Symmetrically, one shows that (p, g^+, η, η') is a future equivalence between X and M, noting that $\eta: 1_X \to i^+p^+ = g^+p$ and $\eta': 1_X \to j^+q^+ = pg^+$.

For the last remark, it suffices to note that the associated functor $g: M \to X^{\mathbf{2}}$ (3.1.4)

(8) $g(x) \;=\; \varepsilon g^+(x).g^-\eta'(x) \;=\; g^+\varepsilon'(x).\eta g^-(x): g^-(x) \to g^+(x),$

can even be constant (on objects and morphisms). For instance, take the fundamental category of the hollow cube 1.6: the past spectrum P reduces to the initial object ⊥, the future spectrum F reduces to the terminal object ⊤, and one *can* take as p the functor $X \to \mathbf{1}$. (Of course, the interesting quotient model is not this one; see 7.5.)

7.4. Theorem (Spectral presentations and quotients). Assume now that 7.3.1 is the *spectral* presentation of X.

(a) If $p: X \to M$ is a normal quotient, consistent with the spectral pf-presentation of X, and *surjective on maps*, then the induced pf-presentation of M (7.3a) is also a spectral presentation. (We already know, by 7.3b, that p is a quotient model of X, in a canonical way.)

(b) Let $p: X \to M$ be a normal quotient whose kernel is *precisely* the set of morphisms which are both past and future regular in X. Then, p is a quotient model of X, consistent with the spectral presentation of X. Moreover, if p is surjective on maps, it is the *minimal* quotient model of X.

(c) The projective model $f: X \to M \subset X^2$ associated to the spectral presentation has for kernel the set of past and future regular morphisms of X. If f is a normal quotient of X surjective on maps, then it is the *minimal* quotient model of X.

Proof. (a) Note that p is a past and future equivalence (7.3). We use the characterisation of future spectra of 3.5a, to show that, since $i^+: F \rightleftarrows X :p^+$ is a future spectrum, also $j^+: F \rightleftarrows M :q^+$ is. (The dual fact holds for P.)

First, the category F has precisely one object in each future regularity class of M (since $j^+ = pi^+$ and p, as a future equivalence, gives a bijective correspondence between future regularity classes of X and M). Second, we already know that $j^+: F \rightleftarrows M :q^+$ is a future retract (7.3). Third, for $y \in M$, we have to prove that the unit-component $\eta'y: y \to j^+q^+(y)$ is the unique M-morphism with these endpoints. By hypothesis, if $b: y \to j^+q^+(y)$ is an M-morphism, there is some $a: x \to x'$ such that $p(a) = b$; note that the objects x, x' are $\sim^+$-equivalent, because p reflects this relation. Now, consider the composite

(1) $a' = \eta x'.a: x \to i^+p^+(x')$, $\qquad i^+p^+(x') = i^+p^+(x)$,

where $p(\eta x') = \eta'p(x') = \eta'j^+q^+(y)$ is an identity and $p(a') = p(a)$. This a' must be the only X-morphism from x to $i^+p^+(x)$, i.e. $a' = \eta x$ (because F is a future retract of X). Finally

(2) $b = p(a) = p(a') = p\eta x = \eta'p(x) = \eta'y$.

(b) The consistency of p is obvious, by definition (7.2), and we only have to prove that p is a *minimal* quotient model (assuming it is surjective on maps).

First, if a quotient model $p': X \to M'$ sends the morphism a to an identity, a must be past and future regular in X (since this property is reflected by all functors which are both past and future equivalences, Thm. 2.8), which means that a is sent to an identity in M; it follows that p' factors through p. Second, given a quotient model $q: M \to M'$ which sends the morphism b to an identity, then b must be past and future regular in M; but $b = p(a)$ for some a in X, which must be past and future regular, whence b is an identity; thus, the kernel of q is discrete and q is an isomorphism.

(c) The functor f sends an object $x \in X$ to the arrow $f(x) = \eta x.\varepsilon x: i^-p^-(x) \to i^+p^+(x)$, and the arrow a to the pair $(i^-p^-(a), i^+p^+(a))$. The latter is an identity if and only if a is past and future regular in X (by 6.8). The last assertion follows from (b). □

7.5. Examples. In all the examples of Section 4, the hypothesis of 7.4c is satisfied: the projective model $f: C \to M$ associated to the spectral presentation of the fundamental category C is a normal quotient, and is thus *the* minimal quotient model of C.

This also holds for the fundamental category C of the hollow cube: the minimal quotient model of C is the category **1**, as already remarked; but here the functor $C \to \mathbf{1}$ is not faithful. On the other hand, the *semi-faithful* retractile model $p: C \to M$ of 5.5 (with 26 objects) is a normal quotient by 6.7a, and a *semi-faithful* quotient model, by 7.3b. It would be interesting to prove that this model is minimal *within the semi-faithful quotient models*.

8. The fundamental biased d-lax 2-category of a directed space

Higher singularities can also be studied with higher fundamental categories, introduced in [G4, G5, G6]. On the other hand, directed homotopy gives a geometric intuition for this subject, yielding - in the lax case - a precise criterion for the direction of comparison cells. We end with recalling one of such definitions, from [G5].

8.1. The guideline of directed homotopy. We shall work in the setting d**Top** of d-spaces, reviewed above (1.7).

In a d-space, a (directed) homotopy between two iterated concatenations of (the same) paths can only move towards a 'route' which, at each moment, has made a longer way than the initial one, as in the following cases (the homotopy will be made explicit below, in 8.2.5-7)

(1) $a \otimes (b \otimes c) \to (a \otimes b) \otimes c, \qquad 1_x \otimes a \to a \to a \otimes 1_{x'}.$

For instance, in the first case, at the instant $t = 1/2$ the second path has already reached the point $x'' = b(1)$, while the first is still in $x' = a(1)$; and the latter can certainly be moved to x'' along b.

(It is interesting to note that Mac Lane's proof of his coherence theorem for monoidal categories follows, *in the associativity part*, a directed approach which agrees with the direction of the associativity homotopy, above: a *directed path*, in the sense of [M1, Thm. 3.1], links iterated tensors with decreasing *rank*, and an iterated tensor has rank zero if and only if 'all parentheses start in front'.)

In [G5, Section 3], we also developed a richer *unbiased* approach, with n-ary concatenations and new comparisons, like the following ones

(2) $a \otimes (b \otimes c) \to a \otimes b \otimes c \to (a \otimes b) \otimes c.$

In both approaches, a relevant role is played by *reparametrisation functions*, i.e. maps $r: \uparrow\mathbf{I} \to \uparrow\mathbf{I}$ which preserve the endpoints; or, equivalently, order-preserving surjective endomappings of the standard interval, necessarily continuous. They have an n-ary concatenation

(3) $(r_1 \otimes ... \otimes r_n)(t) = (i - 1)/n + r_i(nt - i + 1)/n, \qquad \text{when } (i - 1)/n \leq t \leq i/n.$

The pointwise order $r \leq r'$ produces an *interpolating* directed 2-homotopy, by affine interpolation

(4) $\varphi_0(r, r'): r \to r': \uparrow\mathbf{I} \to \uparrow\mathbf{I}, \qquad \varphi_0(r, r')(s, t) = (1 - t).r(s) + t.r'(s).$

(Note: the notation r' has nothing to do with derivatives.) As a crucial point, its class $[\varphi_0(r, r')]$ up to 3-homotopy (with fixed boundary) is *uniquely determined* by r, r'. In fact, if $\alpha, \beta: r \to r'$ are 2-homotopies, also $(\alpha \vee \beta)(s, t) = \max(\alpha(s, t), \beta(s, t))$ is so, and - plainly - there are 3-homotopies $\alpha \to \alpha \vee \beta \leftarrow \beta$.

8.2. The construction. Let X be a d-space. The *fundamental biased d-lax 2-category* $\uparrow b\Pi_2(X)$ has the following objects, arrows, cells, elementary compositions (nullary and binary) and comparisons.

(a) An *object* is a point of X.

(b) An *arrow* $a: x \to y$ is a (directed) path $a: \uparrow\mathbf{I} \to X$ with $a(0) = x$, $a(1) = y$; the *unit-arrow* $1_x: x \to x$ is the constant path at x.

(c) A *cell* $[\alpha]: a \to a': x \to y$ is a homotopy class of homotopies of paths; more precisely, α is a (directed) 2-*homotopy* $a \to a'$ (with fixed endpoints), which means that the map $\alpha: \uparrow\mathbf{I}^2 \to X$ has the boundary represented below (the thick lines represent constant paths)

$$(1) \qquad \begin{array}{ccc} x & \xrightarrow{a} & y \\ \| & \alpha & \| \\ x & \xrightarrow[a']{} & y \end{array} \qquad\qquad \begin{array}{l} \bullet \longrightarrow s \\ \downarrow t \end{array}$$

and its homotopy class $[\alpha]$ is up to the equivalence relation generated by 3-*homotopies* $\alpha' \to \alpha''$ (with fixed boundary); the *unit-cell* $1_a: a \to a$ is the class of the trivial 2-homotopy $c_a(s, t) = a(s)$.

(d) The *main composition*, or *upper-level composition*, of $[\alpha]$ with $[\alpha']: a' \to a'': x \to y$ is defined by the pasting $\alpha \otimes_2 \alpha'$ of any two representatives, with respect to the second variable

(2) $[\alpha] \otimes_2 [\alpha']: a \to a'': x \to y$, $\qquad [\alpha] \otimes_2 [\alpha'] = [\alpha \otimes_2 \alpha']$;

$(\alpha \otimes_2 \alpha')(s, t) = \alpha(s, 2t)$ if $0 \leq t \leq 1/2$, $\qquad (\alpha \otimes_2 \alpha')(s, t) = \alpha'(s, 2t - 1)$ if $1/2 \leq t \leq 1$.

(e) The (*lower-level*) *composition* of $a: x \to y$ with $b: y \to z$ is the standard concatenation $a \otimes b: x \to z$ of the paths

(3) $(a \otimes b)(t) = a(2t)$ if $0 \leq t \leq 1/2$, $\qquad (a \otimes b)(t) = b(2t - 1)$ if $1/2 \leq t \leq 1$.

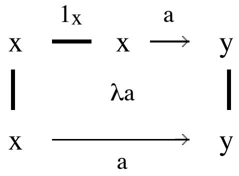
(f) The *lower-level composition* of $[\alpha]: a \to a': x \to y$ with $[\beta]: b \to b': y \to z$ is defined by the pasting $\alpha \otimes \beta$ of any two representatives, with respect to the first variable

(4) $[\alpha] \otimes [\beta]: a \otimes b \to a' \otimes b': x \to z$, $\qquad [\alpha] \otimes [\beta] = [\alpha \otimes \beta]$;

$(\alpha \otimes \beta)(s, t) = \alpha(2s, t)$ if $0 \leq s \leq 1/2$, $\qquad (\alpha \otimes \beta)(s, t) = \beta(2s - 1, t)$ if $1/2 \leq s \leq 1$.

We use abbreviations as: $x \otimes a = 1_x \otimes a = 1 \otimes a$, $a \otimes [\alpha] = 1_a \otimes [\alpha] = 1 \otimes [\alpha]$, $x \otimes [\alpha] = 1_{1_x} \otimes [\alpha]$ (when the domain-arrow of α is degenerate) and so on.

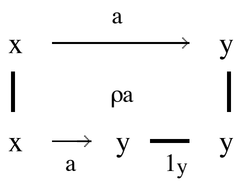
(g) For an arrow $a: x \to y$, the *left-unit* and the *right-unit comparisons* are given by the following 2-homotopies (determined by 2-homotopies λ_0, ρ_0, affine in the second variable)

(5) 

$[\lambda a]: x\otimes a \to a, \qquad \lambda a = a\circ\lambda_0: \uparrow\mathbf{I}\times\uparrow\mathbf{I} \to X,$

$\lambda_0(s, t) = (1 - t).r(s) + t.s,$

$r(s) = \max(0, 2s - 1),$

(6) 

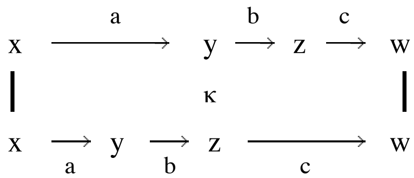
$[\rho a]: a \to a\otimes y, \qquad \rho a = a\circ\rho_0: \uparrow\mathbf{I}\times\uparrow\mathbf{I} \to X,$

$\rho_0(s, t) = (1 - t).s + t.r'(s),$

$r'(s) = \min(2s, 1).$

(h) For three consecutive arrows $a: x \to y$, $b: y \to z$, $c: z \to w$, the *associativity comparison* is expressed as follows (here, the ternary concatenation $a\otimes b\otimes c$ is only used as a shortcut in describing the 2-homotopy):

(7) $[\kappa] = [\kappa(a, b, c)]: a\otimes(b\otimes c) \to (a\otimes b)\otimes c, \qquad \kappa = (a\otimes b\otimes c)\circ\kappa_0: \uparrow\mathbf{I}\times\uparrow\mathbf{I} \to X,$



$\kappa_0: \uparrow\mathbf{I}\times\uparrow\mathbf{I} \to \uparrow\mathbf{I},$

$\kappa_0(s, t) = (1 - t).r(s) + t.r'(s),$

$$r(s) = \begin{cases} 2s/3, & \text{if } 0 \leq s \leq 1/2, \\ (4s - 1)/3, & \text{if } 1/2 \leq s \leq 1, \end{cases} \qquad r'(s) = \begin{cases} 4s/3, & \text{if } 0 \leq s \leq 1/2, \\ (2s + 1)/3, & \text{if } 1/2 \leq s \leq 1. \end{cases}$$

8.3. Definition. Abstracting the previous situation, a *biased d-lax 2-category* **A** consists of the following data and properties. (Greek letters denote now 2-cells.)

(bdl.0) A set of objects, Ob**A**.

(bdl.1) For any two objects x, y, a category $\mathbf{A}(x, y)$ of *maps* $a: x \to y$ and *cells* $\alpha: a \to b$, with *main*, or *upper-level*, composition $\alpha\otimes_2\beta: a \to b \to c$ and units $1_a: a \to a$.

(bdl.2) For any object x a *lower identity* 1_x; for any triple of objects x, y, z a functor of *lower composition*

(1) $- \otimes -: \mathbf{A}(x, y) \times \mathbf{A}(y, z) \to \mathbf{A}(x, z).$

Explicitly, the functorial properties give:

(2) $1_{a\otimes b} = 1_a\otimes 1_b$ (*nullary interchange*),

(3) $(\alpha\otimes\beta)\otimes_2(\alpha'\otimes\beta') = (\alpha\otimes_2\alpha')\otimes(\beta\otimes_2\beta')$ (*binary* or *middle-four interchange*),

$$\begin{array}{ccccc}
x & \xrightarrow{\;a\;} & y & \xrightarrow{\;b\;} & z \\
\| & \alpha & \| & \beta & \| \\
x & \overset{a'}{\dashrightarrow} & y & \overset{b'}{\dashrightarrow} & z \\
\| & \alpha' & \| & \beta' & \| \\
x & \xrightarrow[a'']{} & y & \xrightarrow[b'']{} & z
\end{array}$$

(bdl.3) For any map a and for any triple (a, b, c) of consecutive maps, three cells

(4) $\lambda a\colon x{\otimes}a \to a$, $\rho a\colon a \to a{\otimes}y$ (*left* and *right-unit comparison*),

$\kappa(a, b, c)\colon\; a{\otimes}(b{\otimes}c) \to (a{\otimes}b){\otimes}c$ (*associativity comparison*),

forming three natural transformations between the obvious (ordinary) functors $\mathbf{A}(x, y) \to \mathbf{A}(x, y)$ (in the first two cases) and $\mathbf{A}(x, y) \times \mathbf{A}(y, z) \times \mathbf{A}(z, w) \to \mathbf{A}(x, w)$ (in the last).

Explicitly, the naturality properties give

(5) $(x{\otimes}\alpha) \otimes_2 (\lambda a') \;=\; (\lambda a) \otimes_2 \alpha$ (*naturality of* λ),

(6) $(\rho a) \otimes_2 (\alpha{\otimes}y) \;=\; \alpha \otimes_2 (\rho a')$ (*naturality of* ρ),

(7) $(\alpha{\otimes}(\beta{\otimes}\gamma)) \otimes_2 \kappa(a',b',c') \;=\; \kappa(a,b,c) \otimes_2 ((\alpha{\otimes}\beta){\otimes}\gamma)$ (*naturality of* κ),

$$\begin{array}{ccccc}
x & \xrightarrow{\;1_x\;} & x & \xrightarrow{\;a\;} & y \\
\| & 1 & \| & \alpha & \| \\
x & \overset{1_x}{\dashrightarrow} & x & \overset{a'}{\dashrightarrow} & y \\
\| & & \lambda a' & & \| \\
x & & \xrightarrow[a']{\qquad\qquad} & & y
\end{array}
\quad = \quad
\begin{array}{ccccc}
x & \xrightarrow{\;1_x\;} & x & \xrightarrow{\;a\;} & y \\
\| & & \lambda a & & \| \\
x & & \xrightarrow{\qquad a \qquad} & & y \\
\| & & \alpha & & \| \\
x & & \xrightarrow[a']{\qquad\qquad} & & y
\end{array}$$

$$\begin{array}{ccccc}
x & & \xrightarrow{\qquad a \qquad} & & y \\
\| & & \rho a & & \| \\
x & \overset{a}{\dashrightarrow} & y & \overset{1_y}{\dashrightarrow} & y \\
\| & \alpha & \| & 1 & \| \\
x & \xrightarrow[a']{} & y & \xrightarrow[1_y]{} & y
\end{array}
\quad = \quad
\begin{array}{ccccc}
x & & \xrightarrow{\qquad a \qquad} & & y \\
\| & & \alpha & & \| \\
x & & \xrightarrow{\qquad a' \qquad} & & y \\
\| & & \rho a' & & \| \\
x & \xrightarrow[a']{} & y & \xrightarrow[1_y]{} & y
\end{array}$$

$$\begin{array}{ccccccc}
x & \xrightarrow{\;a\;} & y & \xrightarrow{\;b\;} & z & \xrightarrow{\;c\;} & w \\
\| & \alpha & \| & \beta & \| & \gamma & \| \\
x & \longrightarrow & y & \longrightarrow & z & \longrightarrow & w \\
\| & & & \kappa & & & \| \\
x & \xrightarrow[a']{} & y & \xrightarrow[b']{} & z & \xrightarrow[c']{} & w
\end{array}
\quad = \quad
\begin{array}{ccccccc}
x & \xrightarrow{\;a\;} & y & \xrightarrow{\;b\;} & z & \xrightarrow{\;c\;} & w \\
\| & & & \kappa & & & \| \\
x & \longrightarrow & y & \longrightarrow & z & \longrightarrow & w \\
\| & \alpha & \| & \beta & \| & \gamma & \| \\
x & \xrightarrow[a']{} & y & \xrightarrow[b']{} & z & \xrightarrow[c']{} & w
\end{array}$$

(bdl.4) (*coherence*) Every diagram (universally) constructed with comparison cells, via $\otimes$- and $\otimes_2$-

compositions, commutes.

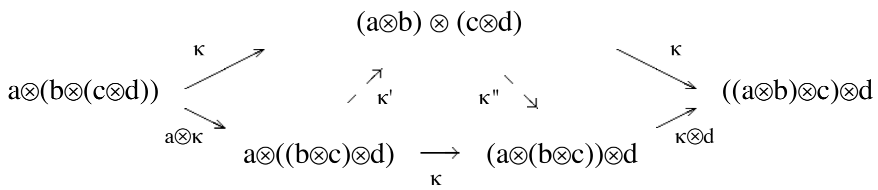
This last axiom can be made more precise using techniques developed in [G5]. One defines the functor of iterated composition along a dichotomic tree τ

(8) $< -; \tau>: \mathbf{A}(x_0, x_1) \times ... \times \mathbf{A}(x_{n-1}, x_n) \to \mathbf{A}(x_0, x_n), \qquad (a_1,..., a_n) \mapsto <a_1,..., a_n; \tau>,$

and requires that, for any two such trees, there be *at most one* natural transformation $< -; \tau> \to < -; \tau'>$ constructed with λ, ρ, α. (Moreover, the 'unbiased' approach in [G5] allows for a simpler formulation.)

8.4. Theorem [G5, 2.4] For a d-space X, the structure $\uparrow b\Pi_2(X)$ constructed above (8.2) is indeed a biased d-lax 2-category, as defined above (8.3).

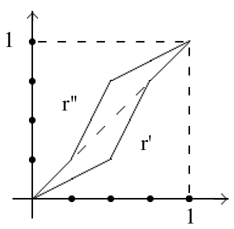
8.5. Higher comparisons. One can *enrich* the fundamental biased d-lax 2-category $\uparrow b\Pi_2(X)$ of a d-space by adding two new *higher associativity comparisons* $\kappa'(a, b, c, d)$ and $\kappa''(a, b, c, d)$, depending on *four* consecutive arrows, so to break Mac Lane's pentagon into 3 triangles

(1) 

We obtain κ', κ'' as *generalised comparison cells*, in the sense of 8.4.1 (which proves the commutativity of the diagram above). In fact, with respect to the quaternary composite $a{\otimes}b{\otimes}c{\otimes}d$:

- $(a{\otimes}b) \otimes (c{\otimes}d)$ coincides with it (and its reparametrisation function is the identity),
- $a{\otimes}((b{\otimes}c){\otimes}d)$ and $(a{\otimes}(b{\otimes}c)){\otimes}d$ have reparametrisation functions r', r'' such that $r' \leq id \leq r''$

(2) $r'(s) = \begin{cases} s/2 & \text{if } 0 \leq s \leq 1/2, \\ 2s - 3/4 & \text{if } 1/2 \leq s \leq 3/4, \\ s & \text{if } 3/4 \leq s \leq 1, \end{cases} \qquad r''(s) = \begin{cases} s & \text{if } 0 \leq s \leq 1/4, \\ 2s - 1/4 & \text{if } 1/4 \leq s \leq 1/2, \\ (s+1)/2 & \text{if } 1/2 \leq s \leq 1, \end{cases}$

(3) 

Following 8.4.2, κ' and κ'' are constructed as follows

(4) $\kappa'(a, b, c, d) = \varphi(a{\otimes}b{\otimes}c{\otimes}d; r', r) = [(a{\otimes}b{\otimes}c{\otimes}d){\circ}\kappa'_0]: \uparrow\mathbf{I}{\times}\uparrow\mathbf{I} \to X,$

$\kappa''(a, b, c, d) = \varphi(a{\otimes}b{\otimes}c{\otimes}d; r, r'') = [(a{\otimes}b{\otimes}c{\otimes}d){\circ}\kappa''_0]: \uparrow\mathbf{I}{\times}\uparrow\mathbf{I} \to X,$

$\kappa'_0(s, t) = (1 - t).r'(s) + t.s, \qquad \kappa'_0(s, t) = (1 - t).s + t.r''(s).$

One can now define an *extended biased d-lax 2-category*, including these higher associativity comparisons in the structure *and* in the axioms (bdl.3-4). Theorem 8.4 still holds in this extended sense, as we have already observed that the new comparisons lie within the generalised ones, dealt with in the proof.

8.6. Basic coherence properties. Taking into account the higher comparisons $\kappa'(a, b, c, d)$, $\kappa''(a, b, c, d)$ (8.4), we can formulate seven 'basic' coherence properties, for *extended* biased d-lax 2-categories. It would be interesting to prove that they are sufficient to ensure that 'all diagrams of comparison cells commute', so that the theory could be formulated as a first-order one.

(a) Given an object x, the cells $\lambda(1_x)\colon 1_x{\otimes}1_x \to 1_x$ and $\rho(1_x)\colon 1_x \to 1_x{\otimes}1_x$ are inverse.

(b)-(d) Given two consecutive arrows a, b, we have:

(1) $\rho a \otimes \lambda b \ = \ \kappa(a, 1_y, b)\colon \ a \otimes (1_y \otimes b) \to (a \otimes 1_y) \otimes b,$

$\lambda(a \otimes b) \ = \ (\lambda a \otimes b) \circ \kappa(1_x, a, b), \qquad\qquad \rho(a \otimes b) \ = \ \kappa(a, b, 1_z) \circ (a \otimes \rho b),$

$$\begin{array}{ccc} 1_x \otimes (a \otimes b) & \xrightarrow{\ \kappa\ } & (1_x \otimes a) \otimes b \\ & {\scriptstyle\lambda(a\otimes b)}\searrow & \big\downarrow {\scriptstyle \lambda a \otimes b} \\ & & a{\otimes}b \end{array} \qquad\qquad \begin{array}{ccc} a \otimes (b \otimes 1_z) & \xrightarrow{\ \kappa\ } & (a \otimes b) \otimes 1_z \\ {\scriptstyle a\otimes\rho b}\big\uparrow & \nearrow{\scriptstyle\rho(a\otimes b)} & \\ a{\otimes}b & & \end{array}$$

(e)-(g) Given four consecutive arrows a, b, c, d, there are three commutative triangles

$$(2) \qquad \begin{array}{ccccccc} & & & (a{\otimes}b) \otimes (c{\otimes}d) & & & \\ & \overset{\kappa}{\nearrow} & \nearrow{\scriptstyle\kappa'} & & {\scriptstyle\kappa''}\searrow & \overset{\kappa}{\searrow} & \\ a{\otimes}(b{\otimes}(c{\otimes}d)) & & & & & & ((a{\otimes}b){\otimes}c){\otimes}d \\ & \underset{a\otimes\kappa}{\searrow} & & & & \underset{\kappa\otimes d}{\nearrow} & \\ & & a{\otimes}((b{\otimes}c){\otimes}d) & \xrightarrow[\kappa]{} & (a{\otimes}(b{\otimes}c)){\otimes}d & & \end{array}$$

These properties 'somehow' correspond to Mac Lane's five original coherence axioms for monoidal categories [M1]. Here, the comparisons are not invertible and the direction comes from Directed Algebraic Topology; moreover, the original pentagon has been split into three triangles, which might be of help for attacking the coherence problem. Kelly's well-known reduction result [Ke], showing that - in the classical case - the properties (a), (c), (d) follow from the others should not subsist here, since that proof strongly depends on cancellation of invertible cells (more than Mac Lane's, apparently).

Finally, let us note that, for $\mathbf{A} = {\uparrow}\mathrm{b}\Pi_2(X)$, the condition (a) holds in a strict form

(3) $1_x{\otimes}1_x \ = \ 1_x, \qquad\qquad\qquad\qquad \lambda 1_x \ = \ 1_{1_x} \ = \ \rho 1_x.$

8.7. Functoriality. A directed map $f\colon X \to Y$ induces a *strict* 2-functor

(1) $f_*\colon {\uparrow}\mathrm{b}\Pi_2(X) \to {\uparrow}\mathrm{b}\Pi_2(Y),$

$f_*(x) \ = \ f(x), \qquad\qquad f_*(a\colon x \to x') \ = \ (f{\circ}a\colon fx \to fx'), \qquad\qquad f_*[\alpha] \ = \ [f{\circ}\alpha].$

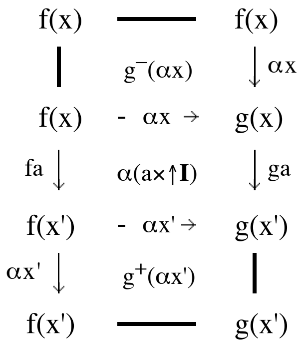
This takes objects, arrows and cells of ${\uparrow}\mathrm{b}\Pi_2(X)$ to similar items of ${\uparrow}\mathrm{b}\Pi_2(Y)$, preserving the whole structure: domains, codomains, units, compositions and comparisons (in the original or in the extended sense of 8.5): $f_*(\lambda^X a) = \lambda^Y f_*(a)$, etc.

A directed homotopy $\alpha\colon f \to g\colon X \to Y$, represented by a directed map $\alpha\colon X{\times}{\uparrow}\mathbf{I} \to Y$, induces a *lax natural transformation of 2-functors* (a notion recalled below, in 8.8)

(2) $\alpha_*: f_* \to g_*: \uparrow b\Pi_2(X) \to \uparrow b\Pi_2(Y)$, $\alpha_*(x) = \alpha(x, -): f(x) \to g(x)$,

$\alpha_*(a: x \to x') = [A''_*(a)]: f_*(a)\otimes\alpha(x') \to \alpha(x)\otimes g_*(a): f(x) \to g(x')$.

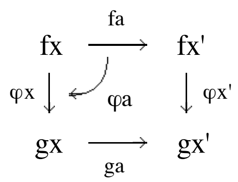
Here $A''_*(a)$ is the 2-cell associated to the *double homotopy* $\alpha\circ(a\times\uparrow\mathbf{I}): \uparrow\mathbf{I}\times\uparrow\mathbf{I} \to Y$, in the usual way, that is pasting it with the double cells $g^-(\alpha x)$ and $g^+(\alpha x')$ defined below ('lower and upper connections', a standard tool of cubical homotopical algebra), and then interchanging coordinates

(3) 
$g^-(\alpha x) = (\alpha x)(\max(s, t))$,
$g^+(\alpha x') = (\alpha x')(\min(s, t))$.

8.8. Lax natural transformations. Finally, let us recall the definition of a *lax natural transformation* $\varphi: f \to g: X \to Y$, between strict 2-functors and lax 2-categories (cf. [Bu]). It assigns

(i) to every object $x\in X$, a map $\varphi x: fx \to gx$ in Y,

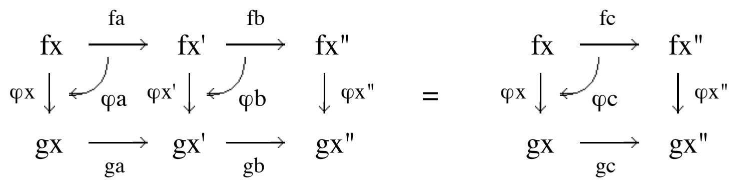
(ii) to every map $a: x \to x'$ in X, a *comparison* cell $\varphi a: fa\otimes\varphi x' \to \varphi x\otimes ga$ in Y,

(1) 

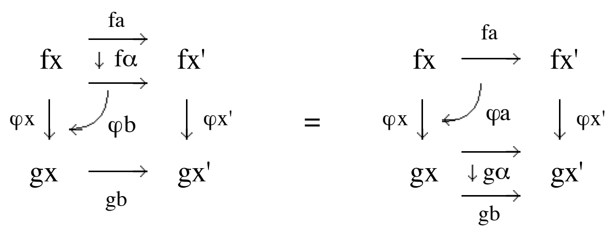
so that the following axioms hold:

(lnt.1) given $x \in X$, $\varphi(1_x) = \lambda(\varphi x)\otimes_2 \rho(\varphi x): 1_{fx}\otimes\varphi x \to \varphi x\otimes 1_{gx}$,

(lnt.2) if $c = a\otimes b: x \to x' \to x''$, then $(fa\otimes\varphi b)\otimes_2 (\varphi a\otimes gb) = \varphi c$,

(2) 

(lnt.3) given a cell $\alpha: a \to b: x \to x'$ in X, then $(f\alpha\otimes\varphi x')\otimes_2 \varphi b = \varphi a\otimes_2 (\varphi x\otimes g\alpha)$,

(3) 

References

[BBS] A. Bauer - L. Birkedal - D.S. Scott, *Equilogical spaces*, Theoretical Computer Science **315** (2004), 35-59.

[BBP] M.A. Bednarczyk - A.M. Borzyszkowski - W. Pawlowski, *Generalized congruences-epimorphisms in Cat*, Theory Appl. Categ. **5** (1999), No. 11, 266-280.

[Bu] M.C. Bunge, *Coherent extensions and relational algebras*, Trans. Amer. Math. Soc. **197** (1974), 355-390.

[Eh] C. Ehresmann, *Cohomologie à valeurs dans une catégorie dominée*, Extraits du Colloque de Topologie, Bruxelles 1964, in: C. Ehresmann, *Oeuvres complètes et commentées*, Partie III-2, 531-590, Amiens 1980.

[FGR] L. Fajstrup - E. Goubault - M. Raussen, *Algebraic topology and concurrency*, Preprint 1999.

[FRGH] L. Fajstrup, M. Raussen, E. Goubault, E. Haucourt, *Components of the fundamental category*, Appl. Categ. Structures **12** (2004), 81-108.

[Ga] P. Gaucher, *A model category for the homotopy theory of concurrency*, Homology, Homotopy Appl. **5** (2003), 549-599.

[GG] P. Gaucher - E. Goubault, *Topological deformation of higher dimensional automata*, Preprint 2001. http://arXiv.org/abs/math.AT/0107060

[Go] E. Goubault, *Geometry and concurrency: a user's guide*, in: Geometry and concurrency, Math. Structures Comput. Sci. **10** (2000), no. 4, pp. 411-425.

[G1] M. Grandis, *On the categorical foundations of homological and homotopical algebra*, Cah. Topol. Géom. Diff. Catég. **33** (1992), 135-175.

[G2] M. Grandis, *Directed homotopy theory, I. The fundamental category*, Cah. Topol. Géom. Diff. Catég. 44 (2003), 281-316. http://www.dima.unige.it/~grandis/Dht1.pdf

[G3] M. Grandis, *The shape of a category up to directed homotopy*, Theory Appl. Categ. **15** (2005) (CT2004), No. 4, 95-146. http://tac.mta.ca/tac/

[G4] M. Grandis, *Modelling fundamental 2-categories for directed homotopy*, Homology Homotopy Appl. **8** (2006), 31-70. http://intlpress.com/HHA/v8/n1/a2/

Preprint: Dip. Mat. Univ. Genova, Preprint **527** (2005)
http://www.dima.unige.it/~grandis/Shp2.pdf

[G5] M. Grandis, *Lax 2-categories and directed homotopy*, Cah. Topol. Géom. Diff. Catég., to appear. Preprint: Dip. Mat. Univ. Genova, Preprint **530** (2005).
http://www.dima.unige.it/~grandis/LCat.pdf

[G6] M. Grandis, *Absolute lax 2-categories*, Dip. Mat. Univ. Genova, Preprint **533** (2005), 22 p.
http://www.dima.unige.it/~grandis/LCat2.pdf

[Ha] E. Haucourt, *Topologie algébrique dirigée et concurrence*, Thèse de Doctorat, Université Paris 7, UFR Informatique, 2005.

[Ke] G.M. Kelly, *On Mac Lane's conditions for coherence of natural associativities, commutativities, etc.*, J. Algebra **1** (1964), 397-402.

[Kr] S. Krishnan, *A convenient category of locally ordered spaces*, presented at GETCO 2005. Unpublished Manuscript. http://www.math.uchicago.edu/~sanjeevi/

[La] R. Lavendhomme, *Un plongement pleinement fidèle de la catégorie des groupes*, Bull. Soc. Math. Belgique, **17** (1965), 153-185.

[M1] S. Mac Lane, *Natural associativity and commutativity*, Rice Univ. Studies **49** (1963), 28-46.

[M2] S. Mac Lane, *Categories for the working mathematician*, Springer, Berlin 1971.

[Sc] D. Scott, *A new category? Domains, spaces and equivalence relations*, Unpublished manuscript (1996). http://www.cs.cmu.edu/Groups/LTC/