Modelling fundamental 2-categories for directed homotopy (*)

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Abstract. Directed Algebraic Topology is a recent field, deeply linked with ordinary and higher dimensional Category Theory. A 'directed space', e.g. an ordered topological space, has directed homotopies (generally non reversible) and fundamental *n-categories* (replacing the fundamental n-groupoids of the classical case). Finding a simple model of the latter is a non-trivial problem, whose solution gives relevant information on the given 'space'; a problem which is also of interest in general Category Theory, as it requires equivalence relations wider than categorical equivalence. Taking on a previous work on "The shape of a category up to directed homotopy", we study now the fundamental 2-category of a directed space. All the notions of 2-category theory used here are explicitly reviewed.

Introduction

Directed Algebraic Topology studies 'directed spaces' in some sense, where paths and homotopies cannot generally be reversed; for instance: ordered topological spaces, 'spaces with distinguished paths', 'inequilogical spaces', simplicial and cubical sets, etc. Its present applications deal mostly with the analysis of concurrent processes (see [FGR, FRGH, Ga, GG, Go]), but its natural range covers non reversible phenomena, in any domain.

The study of invariance under directed homotopy is far richer and more complex than in the classical case, where homotopy equivalence between 'spaces' produces a plain equivalence of their fundamental groupoids, for which one can simply take - as a minimal model - the categorical skeleton. Our directed structures have, to begin with, a *fundamental category* $\uparrow\Pi_1(X)$, which must be studied up to appropriate notions of *directed* homotopy equivalence, *wider* than ordinary categorical equivalence: the latter would often be of no use, since the fundamental category of an ordered topological space, for instance, is always skeletal (the same situation shows that the fundamental *monoids* $\uparrow\pi_1(X, x_0)$ can be trivial, without $\uparrow\Pi_1(X)$ being so; cf. 1.2). Such a study has been carried on in a previous work [G5], which will be cited as Part I; the references I.2 or I.2.3 apply, respectively, to its Section 2 or Subsection 2.3. Other references for Directed Algebraic Topology and its applications can be found there.

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In Part I, we have introduced two (dual) directed notions, which take care, respectively, of variation 'in the future' or 'from the past': *future equivalence* (a symmetric version of an adjunction, with two units) and its dual, a *past equivalence* (with two counits); and studied how to combine them. *Minimal models* of a category, up to these equivalences, have been introduced to better understand the 'shape' and properties of the category we are analysing, and of the process it represents. Part of this study is briefly recalled below, in Section 1. (The paper [FRGH] has similar goals and results, based on different categorical tools, categories of fractions.)

As already noted in Part I, this analysis captures essential facts of many *planar* ordered spaces (subspaces of the ordered plane $\uparrow\mathbf{R}^2$), but may say little about objects embedded in the ordered space $\uparrow\mathbf{R}^3$ (in the same way as π_1 cannot detect the singularity of a 2-sphere). This is why we want to develop here a similar study of the 'shape' of 2-categories, adapted to study the fundamental *2-category* $\uparrow\Pi_2(X)$ of an ordered space.

Outline. We begin with a brief review of the basic aspects of Part I (Section 1), ending with a motivation of a higher dimensional study (1.5). Lax natural transformations of 2-functors between 2-categories and the 'local adjunctions' they produce, introduced in the 70's by Bunge [Bu], Gray [Gr] and Kelly [Ke], are recalled in Section 2 - and in the Appendix (Section 7) for more technical points. Sections 3 and 4 introduce and study *future 2-equivalences* between 2-categories, a symmetric version of a local adjunction. Theorem 3.4 shows that a future 2-equivalence has a canonical factorisation in two split future 2-equivalences (an analogous 1-dimensional property was proved in Part I), so that our 2-categories can be embedded as *future 2-retracts* (a sort of locally full, locally reflective subcategory) of a common one; on the other hand, a future 2-retract and a past 2-retract of the same 2-category generate a global 2-dimensional model (4.2, 4.3). The definition of the fundamental 2-category of an ordered space is given in Section 5, and extended to more complex directed structures in 5.7-5.8. The previous notions are used in Section 6 to give a model of the fundamental 2-category of a 3-dimensional ordered space, the 'hollow cube', for which $\uparrow\Pi_1$ was already seen to give insufficient information (in 1.5).

1. One-dimensional analysis of directed spaces

We begin with a review of the basic ideas and results of Part I. A *preorder* relation is assumed to be reflexive and transitive; it is called a (partial) *order* if it is also anti-symmetric; using a *pre*order as the main notion has strong advantages, as recalled at the end of 1.1.

1.1. Homotopy for preordered spaces. The simplest topological setting where one can study directed paths and directed homotopies is likely the category p**Top** of *preordered topological spaces* and *preorder-preserving continuous mappings*; the latter will be simply called *morphisms* or *maps*, when it is understood we are in this category. (Richer settings will be recalled in Section 5).

In this setting, a (directed) *path* in the preordered space X is a map $a\colon \uparrow[0, 1] \to X$, defined on the standard directed interval $\uparrow\mathbf{I} = \uparrow[0, 1]$ (with euclidean topology and natural order). A (directed) *homotopy* $\varphi\colon f \to g\colon X \to Y$, *from* f *to* g, is a map $\varphi\colon X\times\uparrow\mathbf{I} \to Y$ coinciding with f on the lower basis of the *cylinder* $X\times\uparrow\mathbf{I}$, with g on the upper one. Of course, this (directed) cylinder is a product in p**Top**: it is equipped with the product topology *and* with the product preorder, where $(x, t) \prec (x', t')$

if $x \prec x'$ in X and $t \leq t'$ in $\uparrow\mathbf{I}$.

The fundamental category $C = \uparrow\Pi_1(X)$ has, for arrows, the classes of directed paths up to the equivalence relation *generated* by directed homotopy with fixed endpoints; composition is given by the concatenation of consecutive paths.

Note that, generally, the fundamental category of a preordered space X is *not* a preorder, i.e. can have different arrows $x \to x'$ between two given points (cf. 1.2); but any loop in X lives in a zone of equivalent points and is reversible, so that all endomorphisms of $\uparrow\Pi_1(X)$ are invertible. Moreover, if X is *ordered*, the fundamental category has no endomorphisms and no isomorphisms, except the identities, and is *skeletal*; therefore, *ordinary equivalence of categories cannot yield any simpler model*. Note also that, in this case, all the fundamental monoids $\uparrow\pi_1(X, x_0) = \uparrow\Pi_1(X)(x_0, x_0)$ are trivial. All these are crucial differences with the classical fundamental groupoid $\Pi_1(X)$ of a space, for which a model up to homotopy invariance is given by the skeleton: a family of fundamental groups $\pi_1(X, x_i)$, obtained by choosing one point in each path-connected component of X.

The fundamental category of a preordered space can be computed by a van Kampen-type theorem, as proved in [G2], Thm. 3.6, in a much more general setting ('d-spaces', defined by a family of distinguished paths).

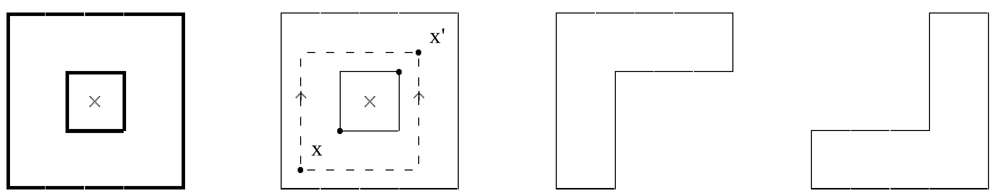
The forgetful functor U: p**Top** $\to$ **Top** to the category of topological spaces has both a left and a right adjoint, $D \dashv U \dashv C$, where DX (resp. CX) is the space X with the *discrete* order (resp. the *coarse* preorder). Therefore, U preserves limits and colimits. The standard embedding of **Top** in p**Top** will be the *coarse* one, so that all (ordinary) paths in X are directed in CX. Note that the category of *ordered* spaces does not allow for such an embedding, and has different colimits.

1.2. The fundamental category of a square annulus. An elementary example will give some idea of the analysis developed in Part I. Let us start from the standard *ordered* square $\uparrow[0, 1]^2$, with the euclidean topology and the product order

$$(x, y) \leq (x', y') \quad \text{if:} \quad x \leq x', \ y \leq y',$$

and consider the (compact) ordered subspace A obtained by taking out the *open* square $]1/3, 2/3[^2$ (marked with a cross), a sort of 'square annulus'

(1)



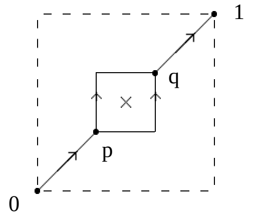
A L L'

Its directed paths are, by definition, the continuous *order-preserving* maps $\uparrow[0, 1] \to A$ defined on the standard ordered interval, and move 'rightward and upward' (in the weak sense). Directed homotopies of such paths are continuous order-preserving maps $\uparrow[0, 1]^2 \to A$. The fundamental category $C = \uparrow\Pi_1(A)$ has, for arrows, the classes of directed paths up to the equivalence relation *generated* by directed homotopy (with fixed endpoints, of course).

In our example, the fundamental category C has *some* arrow $x \to x'$ provided that $x \leq x'$ and both points are in L or L' (the closed subspaces represented above). Precisely, there are *two* arrows when $x \leq p = (1/3, 1/3)$ and $x' \geq q = (2/3, 2/3)$ (as in the last figure above), and *one* otherwise. This evident fact can be easily proved with the 'van Kampen' theorem recalled above, using the subspaces L, L' (whose fundamental category is the induced order).

Thus, the whole category C is easy to visualise and 'essentially represented' by the full subcategory E on four vertices 0, p, q, 1 (the central cell does not commute)

(1)



E

But E is far from being equivalent to C, as a category, since C *is already a skeleton*, in the ordinary sense. The situation can be analysed as follows, in E:

- the action begins at 0, from where we move to the point p,
- p is an (effective) future branching point, where we have to choose between two paths,
- which join at q, an (effective) past branching point,
- from where we can only move to 1.

(Definitions and properties of *regular* and *branching* points can be found in I.6).

In order to make precise how E can 'model' the category C, we proved in Part I (and will recall below) that E is both *future equivalent* and *past equivalent* to C, and actually is the 'join' of a minimal 'future model' with a minimal 'past model' of the latter.

1.3. Future equivalence of categories. A *future equivalence* $(f, g; \varphi, \psi)$ (I.2.1) between the categories C, D is a symmetric version of an adjunction, with two units. It consists of a pair of functors and a pair of natural transformations (i.e., directed homotopies), the *units*, satisfying a coherence condition:

(1) $f: C \rightleftarrows D :g$ $\qquad\qquad$ $\varphi: 1_C \to gf, \quad \psi: 1_D \to fg,$

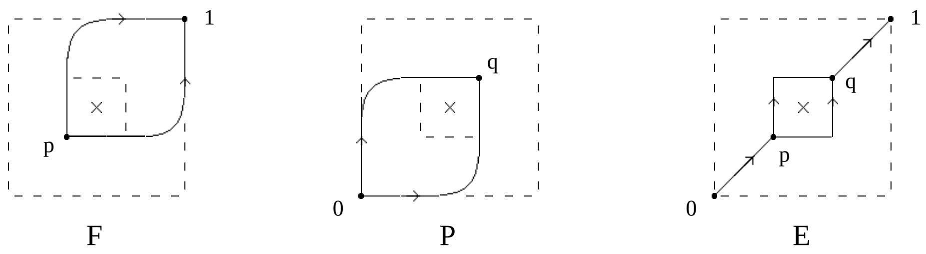
(2) $f\varphi = \psi f: f \to fgf,$ $\qquad\qquad$ $\varphi g = g\psi: g \to gfg$ $\qquad$ (*coherence*).

Note that the *directed homotopies* φ, ψ proceed *from* the identities *to* the composites gf, fg ('in the future'). Dually, *past equivalences* have *co*units, in the opposite direction.

Future equivalences compose (in the same way as adjunctions), and yield an equivalence relation of categories. A property (making sense *in* a category, or *for* a category) is said to be *future invariant* if it is preserved by future equivalences.

An adjunction $f \dashv g$ with *invertible* counit $\varepsilon: fg \cong 1$ amounts to a future equivalence with invertible $\psi = \varepsilon^{-1}$. In this case, a 'split' future equivalence, D can be identified with a full reflective subcategory of C (a *future retract*, I.2.4). But, in a general future equivalence, f need not determine g. Theorem I.2.5 shows that two categories are future equivalent if and only if they are full reflective subcategories of a third.

1.4. Minimal one-dimensional models. In our example (1.2), the category $C = {\uparrow}\Pi_1(A)$ has a least *full reflective* subcategory F, which is future equivalent to C and minimal as such; its objects are a *future branching* point p (where we must choose between different ways, out of it) and a *maximal* point 1 (where one cannot further proceed); they form the *future spectrum* $sp^+(C)$ (as defined in I.7.2)

(1) 

Dually, we have the least *full coreflective* subcategory P, on the *past spectrum* $sp^-(C) = \{0, q\}$.

Putting together the information coming from a past and a future spectrum, the *pf-spectrum* E = Sp(C) is the full subcategory of C on the set of objects $sp(C) = sp^-(C) \cup sp^+(C)$ (I.7.6); it is linked to C by a diagram formed of four commutative squares:

(2)
$$\begin{array}{ccccc} P & \underset{p^-}{\overset{i^-}{\rightleftarrows}} & C & \underset{i^+}{\overset{p^+}{\leftrightarrows}} & F \\ \| & & \uparrow f & & \| \\ P & \underset{q^-}{\overset{j^-}{\rightleftarrows}} & E & \underset{j^+}{\overset{q^+}{\leftrightarrows}} & F \end{array} \qquad\qquad \begin{array}{c} C \\ g^- \downarrow \uparrow \downarrow g^+ \\ E \end{array} \qquad (g^\alpha = j^\alpha p^\alpha).$$

Adding the two functors $g^\alpha = j^\alpha p^\alpha \colon X \to E$ ($\alpha = \pm$), E becomes a *minimal injective model* of the category C, in a precise sense, which we recall now (all this is not technically required for the sequel, but will suggest how to proceed for dimension 2, in Section 4).

First, a category E is made an *injective model* of C (I.4.1) by assigning a *pf-injection*, or *pf-embedding*, $E \leftrightarrows C$. This consists of a *full embedding* $f\colon E \to C$ (full, faithful and injective on objects) which appears at the same time in a past equivalence $(f, g^-; \varepsilon_E, \varepsilon_C)$ and in a future one $(f, g^+; \eta_E, \eta_C)$

(3) $f : E \leftrightarrows C : g^-, g^+$,

$$\varepsilon_E\colon g^-f \to 1_E, \quad \varepsilon_C\colon fg^- \to 1_C, \qquad f\varepsilon_E = \varepsilon_C f\colon fg^-f \to f, \qquad \varepsilon_E g^- = g^-\varepsilon_C\colon g^-fg^- \to g^-,$$
$$\eta_E\colon 1_E \to g^+f, \quad \eta_C\colon 1_C \to fg^+, \qquad f\eta_E = \eta_C f\colon f \to fg^+f, \qquad \eta_E g^+ = g^+\eta_C\colon g \to g^+fg^+.$$

(A coherence condition between these two structures automatically holds, I.3.3. By I.3.4, it suffices to assign the three functors f, g^-, g^+ - the first being a full embedding - together with the natural transformations ε_C and η_C, under the conditions $fg^-\varepsilon_C = \varepsilon_C fg^-$, $fg^+\eta_C = \eta_C fg^+$.)

Secondly, we say that E is a *minimal injective model* of X (I.5.2) if:

(i) E is an injective model of every injective model E' of X,

(ii) every injective model E' of E is isomorphic to E.

We also say that E is a *strongly minimal injective model* if it satisfies the stronger condition (i'), together with (ii):

(i') E is an injective model of every category injectively equivalent to X,

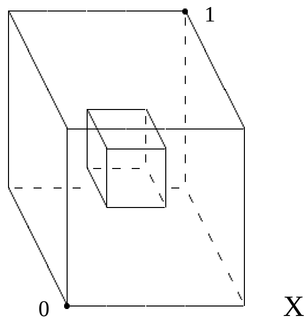
where two categories are said to be *injectively equivalent* if they can be linked by a finite chain of pf-embeddings, forward or backward (I.4.1).

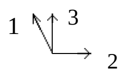
Finally, Theorems I.8.4 and I.8.6 prove that, if a category has a pf-spectrum, this is a *strongly minimal injective model* of the former, *determined up to a unique isomorphism*. (More generally, the minimal injective model of a category, when existing, is determined up to isomorphism *but the isomorphism itself need not be determined*; cf. I.5.5, I.5.6).

1.5. The hollow cube. The analysis recalled above, based on the fundamental category, gives relevant information for *planar* ordered spaces (subspaces of $\uparrow\mathbf{R}^2$), also in much more complicated examples (see I.9). It may be insufficient for higher dimensional singularities.

The simplest case (already considered in I.9.7) is a 3-dimensional analogue of our previous example, the 'hollow cube' $X \subset \uparrow[0, 1]^3$ represented below, an ordered compact space again:

(1)





$$X = \uparrow[0, 1]^3 \setminus H$$

$$H = \,]1/3, 2/3[^3.$$

The fundamental category $C = \uparrow\Pi_1(X)$ seems to say little about this space: C has an initial and a terminal object, 0 and 1, whence it is future contractible (to its object 1) and past contractible as well (to 0); its minimal injective model is the category $\mathbf{2} = \{0 \to 1\}$ (cf. I.5.4).

Now, as already remarked in Part I, this injective model is *not* faithful: the original category C is not a preorder, since C(x, y) has two arrows when x, y are suitably placed 'around' the obstruction (a phenomenon which only appears within *directed* homotopy theory). One might therefore try to extract a better information from C, using *faithful* models. However, we are not able to find any simple one (and likely, there is no finite one).

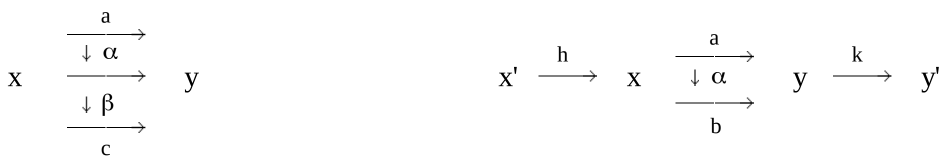
Here, we shall study the fundamental 2-category $C_2 = \uparrow\Pi_2(X)$, trying to reproduce one dimension up the previous study of $\uparrow\Pi_1(A)$, for the 'square annulus'. This will be done in Section 6, after preparing the new tools.

2. Lax natural transformations and local adjunctions

We review now the main tools of 2-dimensional category theory which will be used in this paper, (strict) 2-functors, their lax natural transformations, their modifications and local adjunctions as introduced in [Bu, Gr, Ke]. We shall mostly follow Bunge's terminology [Bu], slightly adapted (cf. 2.2).

2.1. Notation. Dealing with (strict) 2-categories, there is some advantage in beginning with the more general notion of *sesquicategory*, where we only have a vertical composition of cells and a horizontal 'whisker' composition of cells with arrows [St].

We shall use the following notation. In a sesquicategory X we have objects $x, y, \ldots$ maps $a\colon x \to y, \ldots$ and cells $\alpha\colon a \to b\colon x \to y, \ldots$ Maps have an associative composition, written ba (or $b.a$), with identities 1_x. Cells have a *main* composition $\beta\alpha\colon a \to c\colon x \to y$ (also written $\beta.\alpha$), as in the left diagram below, which is associative and has identities 1_a (the terms 'horizontal' and 'vertical' will only be used for the pastings of the associated double cells; see below)

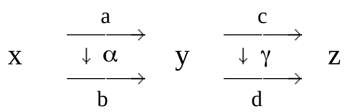
(1) 

Cells and maps have a *whisker* composition $k\alpha h\colon kah \to kbh\colon x' \to y'$ (as in the right diagram above) such that:

(2) $1_y\, \alpha\, 1_x = \alpha$, $\qquad k'(k\alpha h)h' = (k'k)\, \alpha\, (hh')$,

$k\, 1_a\, h = 1_{kah}$, $\qquad k(\beta\alpha)h = (k\beta h)(k\alpha h)$.

This sesquicategory is a 2-category if and only if the 'reduced interchange axiom' holds:

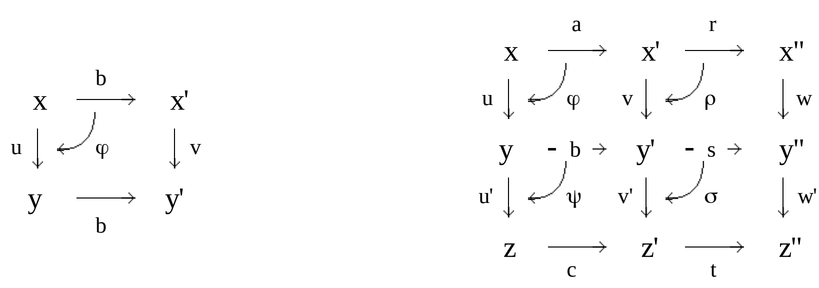
(3) 

$\qquad \gamma b.c\alpha = d\alpha.\gamma a,$

in which case, one can define the *second composition* $\gamma\circ\alpha\colon ca \to db\colon x \to z$ as the previous common result. We shall generally work with 2-categories (mostly without using the second composition).

A (strict) 2-functor $f\colon X \to Y$ takes items of X to similar items of Y, preserving identities and compositions. We shall only use such strict morphisms.

It will be useful to use pasting. This amounts to identifying a 2-category with its strict double category of quintets (due to Ehresmann), with double cells as in the left diagram below (provided by a 2-cell $\varphi\colon va \to bu$)

(4) 

and the obvious horizontal and vertical compositions (obtained from the vertical composition of 2-cells and the whisker composition of 2-cells with arrows, as in the right diagram above)

(5) $\varphi \otimes_h \rho \;=\; s\varphi.\rho a\colon wra \to sva \to sbu, \qquad\qquad \varphi \otimes_v \psi \;=\; \psi u.v'\varphi\colon v'va \to v'bu \to cu'u.$

Note that, more generally, these horizontal and vertical compositions of double cells can be defined (and are associative) for a sesquicategory X; then, X is a 2-category (satisfies the reduced interchange axiom) if and only if its double cells form a double category (i.e., the horizontal and vertical pastings satisfy the middle-four interchange axiom).

2.2. Lax natural transformations. A *lax natural transformation* $\varphi\colon f \to g\colon X \to Y$ (between strict 2-functors) assigns:

(i) to every object $x \in X$, a map $\varphi x\colon fx \to gx$ in Y,

(ii) to every map $a\colon x \to x'$ in X, a *comparison* cell $\varphi a\colon \varphi x'.fa \to ga.\varphi x$ in Y,

(1)
$$\begin{array}{ccc} fx & \xrightarrow{\;fa\;} & fx' \\ {\scriptstyle \varphi x}\downarrow & \swarrow\; \varphi a & \downarrow{\scriptstyle \varphi x'} \\ gx & \xrightarrow[\;ga\;]{} & gx' \end{array}$$

so that the following axioms hold:

(lnt.1) if $a = 1_x$, then $\varphi a = 1\colon \varphi x \to \varphi x$,

(lnt.2) if $c = ba\colon x \to x' \to x''$, then $\varphi a \otimes_h \varphi b = \varphi c$ (i.e., $\varphi c = (gb.\varphi a)(\varphi b.fa)$),

(2)
$$\begin{array}{ccccc} fx & \xrightarrow{\;fa\;} & fx' & \xrightarrow{\;fb\;} & fx'' \\ {\scriptstyle \varphi x}\downarrow & \swarrow\; \varphi a & {\scriptstyle \varphi x'}\downarrow & \swarrow\; \varphi b & \downarrow{\scriptstyle \varphi x''} \\ gx & \xrightarrow[\;ga\;]{} & gx' & \xrightarrow[\;gb\;]{} & gx'' \end{array} \quad = \quad \begin{array}{ccc} fx & \xrightarrow{\;fc\;} & fx'' \\ {\scriptstyle \varphi x}\downarrow & \swarrow\; \varphi c & \downarrow{\scriptstyle \varphi x''} \\ gx & \xrightarrow[\;gc\;]{} & gx'' \end{array}$$

(lnt.3) given a cell $\alpha\colon a \to b\colon x \to x'$ in X, then $f\alpha \otimes_v \varphi b = \varphi a \otimes_v g\alpha$ $(\varphi b(\varphi x'.f\alpha) = (g\alpha.\varphi x)\varphi a)$,

(3)
$$\begin{array}{ccc} fx & \begin{array}{c} \xrightarrow{\;fa\;} \\ \downarrow f\alpha \\ \longrightarrow \end{array} & fx' \\ {\scriptstyle \varphi x}\downarrow & \swarrow\; \varphi b & \downarrow{\scriptstyle \varphi x'} \\ gx & \xrightarrow[\;gb\;]{} & gx' \end{array} \quad = \quad \begin{array}{ccc} fx & \xrightarrow{\;fa\;} & fx' \\ {\scriptstyle \varphi x}\downarrow & \swarrow\; \varphi a & \downarrow{\scriptstyle \varphi x'} \\ gx & \begin{array}{c} \longrightarrow \\ \downarrow g\alpha \\ \xrightarrow[\;gb\;]{} \end{array} & gx' \end{array}$$

A *colax natural transformation* (the cell-dual notion) has comparison cells $ga.\varphi x \to \varphi x'.fa$. These terms agree with Bunge's paper [Bu], but differ from [Gr] and [Ke]:

	lax natural transformation:	colax natural transformation:
[Gr]:	'quasi$_d$ natural transformation',	'quasi natural transformation',
[Ke]:	'op-lax natural transformation',	'lax natural transformation'.

which take as leading notion the dual one (in [Gr], 'd' stays for down). The related notion of 'local adjunction' (2.4) seems to show that the leading form should be chosen as in [Bu], with cells $\varphi a\colon \varphi x'.fa \to ga.\varphi x$ directed *from* f *to* g (see the remark at the end of 2.5).

2.3. Modifications. A *modification* $M\colon \varphi \to \psi\colon f \to g\colon X \to Y$ (between lax natural transformations of 2-functors) assigns to every object $x \in X$ a cell $Mx\colon \varphi x \to \psi x\colon fx \to gx$ in Y, so that the following axiom holds:

(mdf) if $a\colon x \to x'$, then:

$$(1) \qquad \begin{array}{ccccc} fx & = & fx & \xrightarrow{fa} & fx' \\ \psi x \downarrow & \swarrow Mx & \varphi x \downarrow & \swarrow \varphi a & \downarrow \varphi x' \\ gx & = & gx & \xrightarrow[ga]{} & gx' \end{array} \quad = \quad \begin{array}{ccccc} fx & \xrightarrow{fa} & fx' & = & fx' \\ \psi x \downarrow & \swarrow \psi a & \psi x \downarrow & \swarrow Mx' & \downarrow \varphi x' \\ gx & \xrightarrow[ga]{} & gx' & = & fx' \end{array}$$

The calculus of lax natural transformations and modifications, under their compositions, is deferred to the Appendix, Section 7.

2.4. Local adjunctions. A *local adjunction* $(f, g; \eta, \varepsilon; L, R)$ between the 2-categories X, Y consists of a pair of 2-functors, a pair of lax natural transformations and a pair of modifications (replacing the triangular identities)

(1) $f\colon X \rightleftarrows Y \colon g$,

$\eta\colon 1_X \to gf\colon X \to X$, $\qquad \varepsilon\colon fg \to 1_Y\colon Y \to Y$ $\qquad$ (*unit*, *counit*),

$L\colon \varepsilon f.f\eta \to 1_f\colon f \to f\colon X \to Y$ $\qquad$ (*left triangular comparison*),

$R\colon 1_g \to g\varepsilon.\eta g\colon g \to g\colon Y \to X$ $\qquad$ (*right triangular comparison*).

This will be called a *coherent local adjunction* if it satisfies the coherence axioms:

$$(2) \qquad \begin{array}{ccccc} 1 & \xrightarrow{\eta} & gf & = & gf \\ \eta \downarrow & \swarrow \eta{*}\eta & \eta gf \downarrow & \swarrow Rf & \| \\ gf & \xrightarrow{gf\eta} & gfgf & \xrightarrow{g\varepsilon f} & gf \\ \| & \swarrow gL & g\varepsilon f \downarrow & \swarrow 1 & \| \\ gf & = & gf & = & gf \end{array} \; = 1_\eta, \qquad \begin{array}{ccccc} fg & = & fg & = & fg \\ \| & \swarrow 1 & f\eta g \downarrow & \swarrow fR & \| \\ fg & \xrightarrow{f\eta g} & fgfg & \xrightarrow{fg\varepsilon} & fg \\ \| & \swarrow Lg & \varepsilon fg \downarrow & \swarrow \varepsilon{*}\varepsilon & \downarrow \varepsilon \\ fg & = & fg & \xrightarrow{\varepsilon} & 1 \end{array} \; = 1_\varepsilon,$$

with obvious modifications $\eta{*}\eta$, $\varepsilon{*}\varepsilon$ (graded composition, 7.3). In a *strictly coherent local adjunction* the triangular comparisons are identities ($L = 1$, $R = 1$) and the coherence axioms reduce to $\eta{*}\eta = 1$, $\varepsilon{*}\varepsilon = 1$.

A coherent local adjunction is called a 'formal lax adjunction' in [Bu], 3.1 (where f need not be strict). A *colocal adjunction*, the cell dual notion, has colax natural transformations $\eta\colon 1_X \to gf$, $\varepsilon\colon fg \to 1_Y$ with comparisons $L\colon 1_f \to \varepsilon f.f\eta$, $R\colon g\varepsilon.\eta g \to 1_g$; it is called a 'weak quasi-adjunction' in [Gr], I.7.1, and a 'quasi-adjunction' when coherence holds. The term 'local adjunction', motivated below, appeared in [BK, BP, Ja], with similar meanings.

Local adjunctions are closed under composition ([Gr], I.7.3).

2.5. The local behaviour. The name of *local* adjunction is motivated by the fact that this structure is linked with ordinary adjunctions at the 'local' level, of hom-categories.

More precisely, a local adjunction $f \dashv g$ (2.4.1) induces, for every pair of objects x (in X) and y (in Y), a sort of 'preadjunction' $f' \dashv g'$ (without triangular identities):

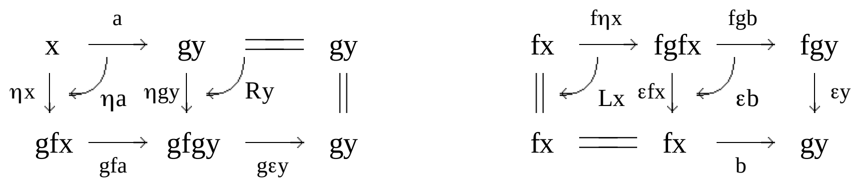
(1) $f'\colon X(x, gy) \rightleftarrows Y(fx, y) :g'$,

$f'(a\colon x \to gy) = \varepsilon y.fa\colon fx \to y$, $\qquad$ $g'(b\colon fx \to y) = gb.\eta x\colon x \to gy$,

$\eta'\colon 1 \to g'f'\colon X(x, gy) \to X(x, gy)$, $\qquad$ $\varepsilon'\colon f'g' \to 1\colon Y(fx, y) \to Y(fx, y)$,

$\eta'a\colon a \to g'f'(a) = g\varepsilon y.gfa.\eta x$, $\qquad$ $\varepsilon'b\colon f'g'(b) = \varepsilon y.fgb.f\eta x \to b$,

where the cells $\eta'a$ and $\varepsilon'b$ are defined by the following pastings

(2) 

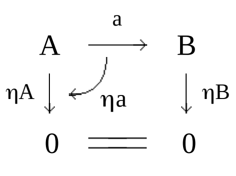
If the original local adjunction is coherent, then (1) is an adjunction (satisfies the triangular identities). On the other hand, a coherent *co*local adjunction $f \dashv g$ produces an adjunction $g' \to f'$ 'discordant' with respect to the given $f \dashv g$.

2.6. Local terminal objects. Local adjunctions produce local limits and colimits, as studied in the references cited above. In a very elementary way, let us consider the 2-category $X = \mathrm{Rel}\mathbf{Ab}$ of (additive) relations of abelian groups and the trivial 2-category $\mathbf{1}$ on one object $*$, linked by the following 2-functors forming a retraction

(1) $p\colon X \rightleftarrows \mathbf{1} :i$, $\qquad$ $i(*) = 0$ $\qquad$ $(pi = 1)$.

Adding units or counits, we can get various (co)local adjunctions (but no 2-adjunction, since X has neither a terminal nor an initial object).

(a) First, we have a strictly coherent local adjunction $p \dashv i$, with trivial counit $pi = 1$ and unit $\eta\colon 1_X \to ip$, sending an object A to the *greatest* relation $\eta A\colon A \to 0$ (with graph $A{\times}\{0\}$). The following diagram shows the comparison cells of η, on an arbitrary relation $a\colon A \to B$

(2) 

$p\eta = 1$, $\quad \eta i = 1$,

$\eta{*}\eta = 1$,

while the coherence properties ((lnt.1-3) in 2.2) are automatically satisfied, because a 2-category of relations is locally ordered: its cells are determined by their domain and codomain.

We can say that this adjunction presents the null group 0 as a *local terminal* object of X.

(b) Secondly, we have a coherent local adjunction $i \dashv p$ with trivial unit $1 = pi$ and counit $\varepsilon\colon ip \to 1_X$, sending an object A to the *least* relation $\varepsilon A\colon 0 \to A$ (with graph $\{0, 0\}$)

(3)
$$\begin{array}{ccc} 0 & = & 0 \\ {\scriptstyle \varepsilon A}\downarrow & \swarrow {\scriptstyle \varepsilon a} & \downarrow{\scriptstyle \varepsilon B} \\ A & \xrightarrow{\ a\ } & B \end{array} \qquad\qquad p\varepsilon \;=\; 1, \quad \varepsilon i \;=1, \quad \varepsilon{*}\varepsilon \;=\; 1.$$

This presents the null group 0 as a *local initial* object, in X.

(c) But we also have two coherent *co*local adjunctions $p \dashv i$ and $i \dashv p$.

The unit of the first is the *co*lax natural transformation $\eta'\colon 1_X \to ip$ where $\eta'A = (\varepsilon A)^{\#}\colon A \to 0$ is the *least* relation (see the left diagram below)

(4)
$$\begin{array}{ccc} A & \xrightarrow{\ a\ } & B \\ {\scriptstyle \eta'A}\downarrow & {\scriptstyle \eta'a}\nearrow & \downarrow{\scriptstyle \eta'B} \\ 0 & = & 0 \end{array} \qquad\qquad \begin{array}{ccc} 0 & = & 0 \\ {\scriptstyle \varepsilon'A}\downarrow & {\scriptstyle \varepsilon'a}\nearrow & \downarrow{\scriptstyle \varepsilon'B} \\ A & \xrightarrow[\ a\ ]{} & B \end{array}$$

while the counit of second is the *co*lax natural transformation $\varepsilon'\colon ip \to 1_X$, $\varepsilon'A = (\eta A)^{\#}$. These adjunctions present 0 as a *colocal terminal* and *colocal initial object*.

Also the points (a) and (b) seem to show that *lax* natural transformations, in the present sense, play a leading role with respect to the dual notion: in fact, the presentation of 0 as a local terminal object comes with *terminal* relations $A \to 0$ (terminal 'objects' in the order-category Rel**Ab**(A, 0)), while the presentation as a local initial object comes with *initial* relations $0 \to A$.

For relations of sets, a similar argument would show that the 2-functor p: Rel**Set** $\to$ **1** has two (non-isomorphic) strictly coherent local right adjoints, corresponding to the empty set and the singleton. But now, the transformations pertaining to the empty set (which is 2-terminal and 2-initial) are 2-natural.

3. Future and past 2-equivalences

As in the one-dimensional case, directed homotopy equivalence of 2-categories appears in two dual forms, detecting invariants of the *future* or the *past*.

3.1. Future 2-equivalences. We shall work with strict 2-functors, their lax natural transformations and modifications, as recalled above (or in Section 7, for their compositions).

A *future 2-equivalence* $(f, g; \varphi, \psi; F, G)$ between the 2-categories X, Y will consist of a pair of 2-functors, a pair of lax natural transformations (2.2), the *units*, and a pair of modifications (2.3), the *comparisons*

(1) $f\colon X \rightleftarrows Y :g,$

$\varphi\colon 1_X \to gf\colon X \to X,$ $\qquad$ $\psi\colon 1_Y \to fg\colon Y \to Y$ $\qquad$ (*units*),

$F\colon \psi f \to f\varphi\colon f \to fgf\colon X \to Y,$ $\qquad$ $G\colon \varphi g \to g\psi\colon g \to gfg\colon Y \to X$ $\qquad$ (*comparisons*).

We say that this future 2-equivalence is *coherent* if the following axioms hold:

(2) $\varphi*\varphi = (gF.Gf)\varphi\colon\ \varphi gf.\varphi \to gf\varphi.\varphi$, $\psi*\psi = (fG.Fg)\psi\colon\ \psi fg.\psi \to fg\psi.\psi$ (*coherence*),

$$\begin{array}{ccc} 1 & \xrightarrow{\ \varphi\ } & gf \\ {\scriptstyle\varphi}\downarrow & \swarrow \varphi*\varphi & \downarrow{\scriptstyle\varphi gf} \\ gf & \xrightarrow[\ gf\varphi\ ]{} & gfgf \end{array} \quad = \quad \begin{array}{ccccc} 1 & \xrightarrow{\ \varphi\ } & gf & = = = & gf \\ & & \| & \swarrow gF.Gf & \downarrow{\scriptstyle\varphi gf} \\ & & gf & \xrightarrow[\ gf\varphi\ ]{} & gfgf \end{array}$$

(see 7.3 for the modification $\varphi*\varphi$, a graded composite; see 7.2 for the whisker composition of the modification $gF.Gf\colon \varphi gf.\varphi \to gf\varphi.\varphi\colon gf \to gfgf$ with $\varphi\colon 1_X \to gf$).

Both notions are reflexive and symmetric; the first is also transitive (3.2). Moreover (having chosen the arrow of the comparison cells of lax natural transformations, in 2.2), the arrow of the comparison cells F, G in the previous definition cannot be inverted, if we want the result of Thm. 3.4 (see the note at the end of the proof).

A property (making sense *in* a 2-category, or *for* a 2-category) will be said to be *future 2-invariant* if it is preserved by future 2-equivalences; an elementary example will be future 2-contractibility (3.5). A future 2-equivalence between ordinary categories amounts to a future equivalence (1.3).

A coherent local adjunction $f \dashv g$ (2.4) with *invertible* counit $\varepsilon\colon fg \to 1$ and *invertible* comparisons $L\colon \varepsilon f.f\eta \to 1_f$ and $R\colon 1_g \to g\varepsilon.\eta g$ amounts to a coherent future 2-equivalence with invertible unit ψ and invertible comparisons, letting:

(3) $\varphi = \eta$, $\psi = \varepsilon^{-1}$, $F = (\psi f.L)^{-1}\colon \psi f \to f\varphi$, $G = (g\psi.R)^{-1}\colon \varphi g \to g\psi$.

This case, a 'split' future 2-equivalence, will be treated later (3.3).

Dually, a *past 2-equivalence* $(f, g; \varphi, \psi; F, G)$ has

(4) $f\colon X \rightleftarrows Y :g$,

$\varphi\colon gf \to 1_X\colon X \to X$, $\psi\colon fg \to 1_Y\colon Y \to Y$ (*counits*),

$F\colon f\varphi \to \psi f\colon fgf \to f\colon X \to Y$, $G\colon g\psi \to \varphi g\colon gfg \to g\colon Y \to X$ (*comparisons*),

and is *coherent* if:

(5) $(Gf.gF)\varphi = \varphi*\varphi\colon\ gf\varphi.\varphi \to \varphi gf.\varphi$, $(Fg.fG)\psi = \psi*\psi\colon\ fg\psi.\psi \to \psi fg.\psi$ (*coherence*).

Future 2-equivalences, being linked with (locally) reflective sub-2-categories and idempotent 2-monads (3.3), will generally be given priority with respect to the dual case (related with *co*reflective sub-2-categories and 2-*co*monads). The cell dual notion, a *cofuture 2-equivalence*, will only be considered marginally; it has *colax natural* transformations $\varphi\colon 1_X \to gf$, $\psi\colon 1_Y \to fg$ directed the same way but having opposite comparison cells (2.2) and triangular comparisons directed the other way round ($F\colon f\varphi \to \psi f$, $G\colon g\psi \to \varphi g$). Finally, notice that (in contrast with 2.5) a future 2-equivalence does *not* induce a future equivalence (nor even functors) at the level of hom-categories.

3.2. Composition. Future 2-equivalences can be composed (much in the same way as local adjunctions, in [Gr], I.7.3), which shows that *being future equivalent 2-categories* is an equivalence relation.

In fact, after (f, g; φ, ψ; F, G) (as in 3.1.1), let a second future 2-equivalence be given

(1) $h: Y \rightleftarrows Z :k$,

$\vartheta: 1_Y \to kh: Y \to Y$, $\qquad\qquad$ $\zeta: 1_Z \to hk: Z \to Z$,

$H: \zeta h \to h\vartheta: h \to hkh: Y \to Z$, $\qquad\qquad$ $K: \vartheta k \to k\zeta: k \to khk: Z \to Y$.

Their composite is defined as follows:

(2) $hf: X \rightleftarrows Z :gk$,

$g\vartheta f.\varphi: 1_X \to gk.hf$, $\qquad\qquad$ $h\psi k.\zeta: 1_Z \to hf.gk$,

$L: (h\psi k.\zeta)hf \to hf(g\vartheta f.\varphi):\ hf \to hf.gk.hf:\ X \to Z$,

$R: (g\vartheta f.\varphi)gk \to gk(h\psi k.\zeta):\ gk \to gk.hf.gk:\ Z \to X$,

where the modifications L and R are given by the following pastings, *in the 2-categories* **Lnt**(X, Z) and **Lnt**(Z, X) of 2-functors, lax natural transformations and modifications (7.1)

(3)
$$\begin{array}{ccccc} hf & = & hf & = & hf \\ \| & & \| \swarrow Hf & & \downarrow \zeta hf \\ hf & = & hf & \xrightarrow{h\vartheta f} & hkhf \\ \| \swarrow hF & & h\psi f \downarrow \swarrow h(\psi*\vartheta)f & & \downarrow h\psi khf \\ hf & \xrightarrow[hf\varphi]{} & hfgf & \xrightarrow[hfg\vartheta f]{} & hfgkhf \end{array} = L, \qquad \begin{array}{ccccc} gk & = & gk & = & gk \\ \| & & \| \swarrow Gk & & \downarrow \varphi gk \\ gk & = & gk & \xrightarrow{g\psi k} & gfgk \\ \| \swarrow gK & & g\vartheta k \downarrow \swarrow g(\vartheta*\psi)k & & \downarrow g\vartheta fgk \\ gk & \xrightarrow[gk\zeta]{} & gkhk & \xrightarrow[gkh\psi k]{} & gkhfgk \end{array} = R.$$

On the other hand, the coherent case seems not to be closed under composition.

3.3. Future 2-retracts. A particular case will be important, and also able to express the general situation (as proved below, in 3.4). A *split* future 2-equivalence *of* X *onto* X_0 (or *of* X_0 *into* X) will be a *coherent* future 2-equivalence (p, i; η, i; 1, 1) where the unit $1_{X_0} \to pi$ is an identity, as well as *both* comparisons

(1) $p: X \rightleftarrows X_0 :i$, $\qquad$ $\eta: 1_X \to ip$, $\qquad$ $pi = 1_{X_0}$,

$p\eta = 1_p$, $\qquad$ $\eta i = 1_i$,

(2) $1_\eta = \eta*\eta: \eta ip.\eta \to ip\eta.\eta$ $\qquad$ (*coherence*).

This equivalence (p, i; η, i; 1, 1) is a split epi in the category of future 2-equivalences, with section (i, p; 1, η; 1, 1) (use the composition diagram 3.2.3). We shall view i as an inclusion and X_0 as a sub-2-category of X; it is easy to see that X_0 is *locally full* in X (but not necessarily full, as shown by the examples of Section 6). Indeed, every X-cell $\alpha: a \to b: x \to x'$ *between maps of* X_0 necessarily belongs to the latter, since the lax natural transformation $\eta: 1_X \to ip$ gives the following equality (axiom (lnt.3) in 2.2)

$$
\text{(3)}\qquad
\begin{array}{ccc}
x & \overset{a}{\underset{b}{\overset{\downarrow\,\alpha}{\rightrightarrows}}} & x' \\
{\scriptstyle \eta x}\downarrow & \swarrow\, \eta b & \downarrow{\scriptstyle \eta x'} \\
ipx & \underset{ipb}{\longrightarrow} & ipx'
\end{array}
\quad = \quad
\begin{array}{ccc}
x & \overset{a}{\longrightarrow} & x' \\
{\scriptstyle \eta x}\downarrow & \swarrow\, \eta a & \downarrow{\scriptstyle \eta x'} \\
ipx & \underset{ipb}{\overset{\downarrow\, ip\alpha}{\rightrightarrows}} & ipx'
\end{array}
$$

where $\eta x, \eta x'$ are identity maps and $\eta a, \eta b$ are identity cells (by $\eta i = 1_i$).

Equivalently, we have a strictly coherent local adjunction $p \dashv i$ with unit $\eta: 1_X \to ip$, where the counit is an identity (as well as both comparisons, cf. 2.4). Thus, p will be called the *local reflector* of the embedding i.

Equivalently again, one can assign a *strictly idempotent coherent local monad* (e, η) on X, i.e. a 2-endofunctor e and a lax natural transformation η such that

$$
\text{(4)}\quad e: X \to X, \qquad \eta: 1_X \to e,
$$
$$
ee = e, \qquad e\eta = 1_e = \eta e, \qquad \eta * \eta = 1_\eta.
$$

Indeed, given $(i, p; \eta)$, we take $e = ip$; given (e, η), we factor $e = ip$ splitting e through the sub-2-category X_0 of X formed of the objects, arrows and cells which e leaves fixed.

Dually, a *split* past equivalence, *of* X_0 *into* X (or *of* X *onto* X_0) is a coherent past equivalence $(p, i; \varepsilon, 1; 1, 1)$ where the counit $pi \to 1_P$ and both comparisons are identities

$$
\text{(5)}\quad p: X \rightleftarrows X_0 :i, \qquad \varepsilon: ip \to 1_X, \qquad pi = 1_{X_0},
$$
$$
p\varepsilon = 1_p, \qquad \varepsilon i = 1_i, \qquad 1_\varepsilon = \varepsilon * \varepsilon: \varepsilon.ip\varepsilon \to \varepsilon.\varepsilon ip.
$$

Then, X_0 will be said to be a *past retract* of X, with *local coreflector* p (locally right adjoint to the inclusion, with trivial unit and comparisons).

3.4. Theorem [Future 2-equivalence and future 2-retracts]. (a) A future 2-equivalence $(f, g; \varphi, \psi; F, G)$ between X and Y (3.1) has a canonical factorisation in two split future 2-equivalences

$$
\text{(1)}\qquad X \underset{p}{\overset{i}{\rightleftarrows}} W \underset{j}{\overset{q}{\rightleftarrows}} Y \qquad\qquad \eta: 1_W \to ip, \quad \eta': 1_W \to jq,
$$

where X and Y are future 2-retracts of W (the *graph* of the given future 2-equivalence).

(b) Two 2-categories are future 2-equivalent if and only if they are future 2-retracts of a third.

(c) A property is future 2-invariant if and only if it is preserved by all *embeddings* of future 2-retracts, as well as by their *local reflectors*.

Proof. (a). First, we construct the 2-category W, enriching the construction of I.2.5 for 1-dimensional categories.

(i) An object is a six-tuple $(x, y; u, v; U, V)$ such that:

$$
\text{(2)}\quad u: x \to gy \text{ (in X)}, \quad v: y \to fx \text{ (in Y)}, \quad U: \varphi x \to gv.u \text{ (in X)}, \quad V: \psi y \to fu.v \text{ (in Y)}
$$

$$\begin{array}{ccc} x & = & x \\ u\downarrow & \swarrow U & \downarrow \varphi x \\ gy & \xrightarrow{gv} & gfx \end{array} \qquad\qquad \begin{array}{ccc} y & = & y \\ v\downarrow & \swarrow V & \downarrow \psi y \\ fx & \xrightarrow{fu} & fgy \end{array}$$

(ii) A morphism is a four-tuple (a, b; A, B): (x, y; u, v; U, V) → (x', y'; u', v'; U', V') such that:

(3) a: x → x' (in X), b: y → y' (in Y), A: u'a → gb.u, B: v'b → fa.v,

$$\begin{array}{ccc} x & \xrightarrow{a} & x' \\ u\downarrow & \swarrow A & \downarrow u' \\ gy & \xrightarrow[gb]{} & gy' \end{array} \qquad\qquad \begin{array}{ccc} y & \xrightarrow{b} & y \\ v\downarrow & \swarrow B & \downarrow v' \\ fx & \xrightarrow[fa]{} & fx' \end{array}$$

(a coherence condition can be added; but this is not necessary). A cell between parallel maps (α, β): (a, b; A, B) → (a', b'; A', B') is a pair such that:

(4) α: a → a' (in X), β: b → b' (in Y), (gβ.u)A = A'.u'α, (fα.v)B = B'.v'β,

$$\begin{array}{ccc} x & \xrightarrow{a} & x' \\ u\downarrow & \swarrow A & \downarrow u' \\ gy & \overset{\rightarrow}{\underset{gb'}{\downarrow g\beta}} & gy' \end{array} = \begin{array}{ccc} x & \overset{a}{\underset{a'}{\downarrow \alpha}} & x' \\ u\downarrow & \swarrow A' & \downarrow u' \\ gy & \xrightarrow[gb']{} & gy' \end{array} \qquad \begin{array}{ccc} y & \xrightarrow{b} & y' \\ v\downarrow & \swarrow B & \downarrow v' \\ fx & \overset{\rightarrow}{\underset{fa'}{\downarrow f\alpha}} & fx' \end{array} = \begin{array}{ccc} y & \overset{b}{\underset{b'}{\downarrow \beta}} & y' \\ v\downarrow & \swarrow B' & \downarrow v' \\ fx & \xrightarrow[fa']{} & fx' \end{array}$$

(iii) The composition of arrows is as follows (it is easy to see that it is 'categorical')

(5) (a', b'; A', B').(a, b; A, B) = (a'a, b'b; $A \otimes_h A'$, $B \otimes_h B'$),

$$\begin{array}{ccccc} x & \xrightarrow{a} & x' & \xrightarrow{a'} & x'' \\ u\downarrow & \swarrow A & u'\downarrow & \swarrow A' & \downarrow u'' \\ gy & \xrightarrow[gb]{} & gy' & \xrightarrow[gb']{} & gy'' \end{array} \qquad\qquad \begin{array}{ccccc} y & \xrightarrow{b} & y' & \xrightarrow{b'} & y'' \\ v\downarrow & \swarrow B & v'\downarrow & \swarrow B' & \downarrow v'' \\ fx & \xrightarrow[fa]{} & fx' & \xrightarrow[fa']{} & fx'' \end{array}$$

(iv) The main and secondary composition of cells are defined component-wise (and satisfy the axioms of 2-categories, with the obvious identities)

(6) (α', β').(α, β) = (α'α, β'β), (γ, δ)∘(α, β) = (γ∘α, δ∘β).

The construction of the 2-category W is completed. We have a split future 2-equivalence of X into W

(7) i: X ⇄ W :p $\qquad$ η: 1_W → ip,

(8) i(x) = (x, fx; φx, 1_{fx}; $1_{\varphi x}$, Fx: ψfx → fφx),

i(a: x → x') = (a, fa; φa: φx'.a → gfa.φx, 1_{fa}),

i(α: a → a') = (α, fα),

(9) $p(x, y; u, v; U, V) = x,$

$p(a, b; A, B) = a, \qquad\qquad p(\alpha, \beta) = \alpha,$

(10) $\eta(x, y; u, v; U, V) = (1_x, v;\ U: \varphi x \to gv.u,\ 1_v)$:

$(x, y; u, v; U, V) \to i(x) = (x, fx; \varphi x, 1_{fx}; 1_{\varphi x}, Fx: \psi fx \to f\varphi x),$

$\eta(a, b; A, B) = (1_a, B)\colon (a, v'b; U'a.gv'A, B) \to (a, fa.v; \varphi a.gfaU, 1)$

(11)
$$\begin{array}{ccc}
(x, y; u, v; U, V) & \xrightarrow{(a,\, b;\, A,\, B)} & (x', y'; u', v'; U', V') \\
{\scriptstyle (1_x,\, v;\, U,\, 1_v)}\downarrow & \swarrow \eta(a, b; A, B) & \downarrow{\scriptstyle (1_{x'},\, v';\, U',\, 1_{v'})} \\
(x, fx; \varphi x, 1_{fx}; 1_{\varphi x}, Fx) & \xrightarrow[(a,\, fa;\, \varphi a,\, 1_{fa})]{} & (x', fx'; \varphi x', 1_{fx'}; 1_{\varphi x}, Fx')
\end{array}$$

The correctness of the definitions of i and η is easily verified; for instance, the coherence of the lax natural transformation η with a W-cell (α, β) (property (lnt.3) of 2.2) follows from the definition of a cell, in (4). The relations $pi = 1_W$, $\eta i = 1_i$, $p\eta = 1_p$ are plain. We also have $\eta * \eta = 1_\eta$ (independently of the coherence of the original future 2-equivalence)

(12)
$$\begin{array}{ccc}
(x, y; u, v; U, V) & \xrightarrow{(1_x,\, v;\, U,\, 1_v)} & (x, fx; \varphi x, 1_{fx}; 1_{\varphi x}, Fx) \\
{\scriptstyle (1_x,\, v;\, U,\, 1_v)}\downarrow & \swarrow (1_{1_x}, 1_v) & \downarrow{\scriptstyle (1_x,\, 1_{fx};\, 1_{\varphi x},\, 1_{1_{fx}})} \\
(x, fx; \varphi x, 1_{fx}; 1_{\varphi x}, Fx) & \xrightarrow[(1_x,\, 1_{fx};\, 1_{\varphi x},\, 1_{1_{fx}})]{} & (x, fx; \varphi x, 1_{fx}; 1_{\varphi x}, Fx)
\end{array}$$

Symmetrically, there is a split future 2-equivalence of Y into W

(13) $j: Y \rightleftarrows W :q \qquad\qquad \eta': 1_W \to jq,$

$j(y) = (gy, y; 1_{gy}, \psi y;\ Gy: \varphi gy \to g\psi y,\ 1_{\psi y}),$

$j(b: y \to y') = (gb, b;\ 1_{gb},\ \psi b: \psi y'.b \to fgb.\psi y),$

$j(B) = (gB, B),$

$q(x, y; u, v; U, V) = y,$

$q(a, b; A, B) = b, \qquad\qquad q(A, B) = B,$

$\eta'(x, y; u, v; U, V) = (u, 1_y; 1_u, V: \psi y \to fu.v)$:

$(x, y; u, v; U, V) \to j(y) = (gy, y;\ 1_{gy}, \psi y;\ Gy: \varphi gy \to g\psi y, 1_{\psi y}),$

$\eta'(a, b; A, B) = (A, 1_b)\colon (u'a, b; A, V'b.fu'B) \to (gb.u, b; 1, \psi b.fgbV).$

Now, composing these two equivalences as in (1) (cf. 3.2.2-3), gives back the original future 2-equivalence $(f, g; \varphi, \psi; F, G)$

(14) $(q, j; \eta', 1; 1, 1)\,(i, p; 1, \eta; 1, 1) = (qi, pj; p\eta'i, q\eta j; q(\eta*\eta')i, p(\eta'*\eta)j),$

(15) $qi(x) = f(x), \qquad qi(a) = f(a), \qquad qi(A) = f(A),$

(16) $p\eta'i: 1_X \to pj.qi,$

$p\eta'i(x) \;=\; p\eta'(x, fx; \varphi x, 1_{fx}; 1_{\varphi x}, Fx) \;=\; p(\varphi x, 1_{fx}; 1_{\varphi x}, Fx) \;=\; \varphi x,$

$p\eta'i(a) \;=\; p\eta'(a, fa;\; \varphi a,\; 1_{fa}) \;=\; p(\varphi a, 1_{fa}) \;=\; \varphi a,$

(17) $q(\eta*\eta')i\colon f\varphi \to \psi f,$

$q(\eta*\eta')i(x) \;=\; q(\eta(\eta'i(x))) \;=\; q(\eta(\varphi x, 1_{fx}; 1_{\varphi x}, Fx)) \;=\; q(1_{\varphi x}, Fx) \;=\; Fx.$

Finally, (b) and (c) follow immediately from (a), by composing future 2-equivalences (3.2).

We also note that the proof shows the 'necessity' of the previous choices for the direction of cells (once we fix it in lax natural transformations). Indeed, the direction of the cell A in (3) must agree with the direction of $\varphi a\colon \varphi x'.a \to gfa.\varphi x$ in (8); but then, because of (10), also the arrow of U is fixed; finally, (13) determines the arrow of Gy. □

3.5. Future 2-contractible 2-categories. We say that a 2-category X is *future 2-contractible* if the 2-functor $p\colon X \to \mathbf{1}$ with values in the singleton 2-category (one object $*$ and its identities) is a future 2-equivalence.

This means that we have a 2-functor $i\colon \mathbf{1} \to X$ (amounting to an object $x_0 = i(*)$ of X), with a lax natural transformation η and a modification F

(1) $p\colon X \rightleftarrows \mathbf{1} :i,$

$\eta\colon 1_X \to ip\colon X \to X, \qquad\qquad F\colon \eta i \to 1_i\colon i \to i\colon \mathbf{1} \to X;$

note that F merely amounts to a cell $F_0 = F(*)\colon \eta x_0 \to \mathrm{id}(x_0)$. (The axiom (mdf), in 2.3, is trivially satisfied, since $\mathbf{1}$ has precisely one arrow, an identity.)

In this situation, we also say that X is *future 2-contractible to the object* x_0. Notice that the latter is not determined up to isomorphism (as shown at the end of this subsection).

We say that X is *split future 2-contractible* if $p\colon X \to \mathbf{1}$ is a split future 2-equivalence onto $\mathbf{1}$, i.e. a coherent future 2-equivalence with comparison $F = 1$. This amounts to a strictly coherent local adjunction $p \dashv i$ with unit η (and counit $pi = 1$; cf. 2.4)

(2) $i\colon \mathbf{1} \rightleftarrows X :p, \qquad\qquad \eta\colon 1_X \to ip\colon X \to X,$

$\eta i \;=\; 1_i, \qquad\qquad \eta(\eta x) \;=\; 1_{\eta x}.$

We have already seen, in 2.6, that Rel**Ab** is split future 2-contractible (to the object 0), and split past 2-contractible (to the same object); moreover, the same is true in the cell dual sense, with respect to colocal adjunctions. There are no other *split* solutions. Indeed, if (i, η) is one, the following cell shows that every component $\eta x\colon x \to i(*)$ must be the greatest relation $x \to i(*)$

(3) $$\begin{array}{ccc} x & \xrightarrow{\;a\;} & i(*) \\ {\scriptstyle \eta x}\downarrow & \geq & \downarrow{\scriptstyle \eta i\,=\,1} \\ i(*) & = & i(*) \end{array}$$

and then $\eta i = 1$ shows that $i(*) = 0$.

Similarly, the 2-category Rel**Set** is split future 2-contractible to precisely two objects (up to isomorphism): the empty set and the singleton.

3.6. Proposition. In order that the category X be split future 2-contractible to an object x_0, it suffices that the latter be equipped, for every object x, with an arrow $\eta x: x \to x_0$ which is terminal in the category $X(x, x_0)$ and such that $\eta x_0 = \mathrm{id}(x_0)$.

Proof. Let $i: \mathbf{1} \to X$, $i(*) = x_0$. For every arrow $a: x \to x'$ in X, let $\eta a: \eta x'.a \to \eta x$ be the unique cell to the terminal arrow $\eta x: x \to x_0$. Plainly, this defines a lax natural transformation $\eta: 1 \to ip$ (2.2). Again, let $F(*): 1_{x_0} \to \eta x_0$ be the unique cell to the terminal arrow $\eta x_0: x_0 \to x_0$; this defines a modification $F: \eta i \to 1_i: i \to i: \mathbf{1} \to X$. The condition $\eta i = 1_i$ is already assumed, and $\eta(\eta x): \eta x \to \eta x$ necessarily coincides with $1_{\eta x}$. □

4. Two dimensional models

A past 2-retract and a future 2-retract of a 2-category generate a global 2-dimensional model (4.3). We also study properties of objects, invariant up to future 2-equivalence, which will be of use to construct minimal models of 2-categories (4.4, 4.5).

4.1. Injective 2-models. A *future 2-embedding of* E *into* X will be a future 2-equivalence $(f, g; \eta_E, \eta)$ where the comparison cells are identities, f is a locally full 2-embedding and additional properties hold

(1) $f: E \rightleftarrows X :g$, $\quad \eta_E: 1_E \to gf: E \to E$, $\quad \eta: 1_X \to fg: X \to X$,

$\eta f = f\eta_E: f \to fgf: E \to X$, $\quad \eta_E g = 1 = g\eta: g \to gfg: X \to E$, $\quad \eta * \eta = 1_\eta: \eta \to \eta$,

$$\begin{array}{ccc} 1 & \xrightarrow{\ \eta\ } & fg \\ {\scriptstyle\eta}\downarrow & \swarrow\ \eta*\eta & \downarrow{\scriptstyle 1} \\ fg & \xrightarrow[\ 1\ ]{} & fg \end{array}$$

In particular, $gfg = g$, so that gf and fg are idempotent endofunctors. Moreover, η_E is determined by η (the *main unit*) and $\eta_E * \eta_E = 1_{\eta_E}$ holds as well, so that our future 2-equivalence is strictly coherent (3.1).

Dually, we have the notion of a *past 2-embedding*. Combining both aspects, the 2-category E is made an *injective 2-model* of X by assigning a *pf-2-embedding of* E *into* X, i.e. a pair formed of a past 2-embedding $(f, g^-; \varepsilon_E, \varepsilon)$ and a future 2-embedding $(f, g^+; \eta_E, \eta)$ sharing the same *locally full* 2-embedding $f: E \to X$:

(2) $f : E \rightleftarrows X : g^-, g^+$,

$\varepsilon_E: g^- f \to 1_E$, $\quad \varepsilon: fg^- \to 1_X$,

$f\varepsilon_E = \varepsilon f: fg^- f \to f$, $\quad g^-\varepsilon = 1 = \varepsilon_E g^-: g^- fg^- \to g^-$, $\quad \varepsilon * \varepsilon = 1_\varepsilon: \varepsilon \to \varepsilon$,

$\eta_E: 1_E \to g^+ f$, $\quad \eta: 1_X \to fg^+$,

$\eta f = f\eta_E: f \to fg^+ f$, $\quad g^+\eta = 1 = \eta_E g^+: g^+ \to g^+ fg^+$, $\quad \eta * \eta = 1_\eta: \eta \to \eta$,

$$
\begin{array}{ccc}
fg^- & \xrightarrow{\;1\;} & fg^- \\
{\scriptstyle 1}\downarrow & \swarrow\; \varepsilon{*}\varepsilon & \downarrow{\scriptstyle \varepsilon} \\
fg^- & \xrightarrow[\varepsilon]{} & 1
\end{array}
\qquad\qquad
\begin{array}{ccc}
1 & \xrightarrow{\;\eta\;} & fg^+ \\
{\scriptstyle \eta}\downarrow & \swarrow\; \eta{*}\eta & \downarrow{\scriptstyle 1} \\
fg^+ & \xrightarrow[1]{} & fg^+
\end{array}
$$

4.2. Pf-presentations. We introduce now a second structure which combines past and future, and will produce an injective 2-model.

A *pf-2-presentation* of the category X will be a diagram consisting of a past 2-retract P and a future 2-retract F of X (3.3; both are locally full sub-2-categories)

$$
(1) \qquad P \;\underset{p^-}{\overset{i^-}{\rightleftarrows}}\; X \;\underset{i^+}{\overset{p^+}{\rightleftarrows}}\; F
$$

$$
\begin{array}{ll}
\varepsilon\colon i^-p^- \to 1_X & \qquad (p^-i^- = 1,\;\; p^-\varepsilon = 1,\;\; \varepsilon i^- = 1,\;\; \varepsilon{*}\varepsilon = 1_\varepsilon), \\
\eta\colon 1_X \to i^+p^+ & \qquad (p^+i^+ = 1,\;\; p^+\eta = 1,\;\; \eta i^+ = 1,\;\; \eta{*}\eta = 1_\eta).
\end{array}
$$

We have thus two strictly coherent local adjunctions $i^- \dashv p^-$, $p^+ \dashv i^+$.

Recall that P and F are locally full sub-2-categories of X (3.3). We form now a locally full sub-2-category E, which will be called the *injective 2-model* of X *generated* by the pf-2-presentation (1) - and will be proved to be such a model (4.3). Its objects belong to $P_0 \cup F_0$ (i.e., ObP∪ObF) while its arrows are generated by:

(a) the arrows of P and F,

(b) the components $\varepsilon x\colon i^-p^-x \to x$ for $x \in F_0$ and $\eta x\colon x \to i^+p^+x$ for $x \in P_0$.

Note that *all* the components of ε, η on items of E live in E. In fact, on an object $x \in P_0 \cup F_0$, it suffices to consider the condition (b), together with the properties $\varepsilon x = 1_x$ for $x \in P_0$ and $\eta x = 1_x$ for $x \in F_0$ (and condition (a)). Secondly, on a map $a\colon x \to x'$ in E, the thesis follows from the fact that the faces of εa and ηa belong to E, which is locally full in X

$$
(2) \qquad
\begin{array}{ccc}
i^-p^-x & \xrightarrow{\;i^-p^-a\;} & i^-p^-x' \\
{\scriptstyle \varepsilon x}\downarrow & \swarrow\; \varepsilon a & \downarrow{\scriptstyle \varepsilon x'} \\
x & \xrightarrow[a]{} & x'
\end{array}
\qquad\qquad
\begin{array}{ccc}
x & \xrightarrow{\;a\;} & x' \\
{\scriptstyle \eta x}\downarrow & \swarrow\; \eta a & \downarrow{\scriptstyle \eta x'} \\
i^+p^+x & \xrightarrow[i^+p^+a]{} & i^+p^+x'
\end{array}
$$

4.3. Theorem [Pf-2-presentations and injective 2-models]. Given a pf-2-presentation of the 2-category X (written as in 4.2.1), let E be the locally full sub-2-category of X described above (4.2) and f its embedding in X.

(a) These data can be uniquely completed to the left diagram below, with (four) commutative squares

$$(1)\qquad \begin{array}{ccccc} P & \underset{p^-}{\overset{i^-}{\rightleftarrows}} & X & \underset{i^+}{\overset{p^+}{\leftrightarrows}} & F \\ \| & & \uparrow f & & \| \\ P & \underset{q^-}{\overset{j^-}{\rightleftarrows}} & E & \underset{j^+}{\overset{q^+}{\leftrightarrows}} & F \end{array} \qquad\qquad \begin{array}{c} X \\ g^- \downarrow \uparrow \downarrow g^+ \\ E \end{array}$$

Moreover:

(b) there is a unique lax natural transformation $\varepsilon_E: j^-q^- \to 1_E$ such that $f\varepsilon_E = \varepsilon f$;

(c) there is a unique lax natural transformation $\eta_E: 1_E \to j^+q^+$ such that $f\eta_E = \eta f$;

(d) these transformations make the lower row into a pf-2-presentation of E;

(e) letting $g^\alpha = j^\alpha p^\alpha: X \to E$ $(\alpha = \pm)$, we get a pf-2-embedding $(f, g^-, g^+; \varepsilon_E, \varepsilon, \eta_E, \eta): E \to X$ making E an injective 2-model of X (*generated* by the presentation).

Proof. (a) First, we (must) take $j^+: F \subset E$ (so that $fj^+ = i^+$) and $q^+ = p^+f: E \to F$; and dually.

Now, we prove (b) to (d), completing the lower row of diagram (1) to a pf-2-presentation of E, as stated. On the right side, we already know that $q^+j^+ = p^+i^+ = 1_F$. We have already seen, at the end of 4.2, that all the components of $\eta f: f \to i^+p^+f: E \to X$ belong to E; there is thus a unique lax natural transformation $\eta_E: 1_E \to j^+q^+$ such that $f\eta_E = \eta f$; plainly, $\eta_E j^+ = 1$ and $q^+\eta_E = 1$. Similarly for $\varepsilon_E: j^-q^- \to 1_E$.

(e) Then, we define $g^\alpha = j^\alpha p^\alpha: X \to E$ and observe that:

$$(2)\quad fg^+ = fj^+p^+ = i^+p^+, \qquad\qquad g^+f = j^+p^+f = j^+q^+.$$

Therefore, we can take the lax natural transformations

$$(3)\quad \eta: 1_X \to i^+p^+ = fr^+, \qquad\qquad \eta_E: 1_E \to j^+q^+ = g^+f,$$

as the units of the pf-2-embedding $f: E \rightleftarrows X :r^\alpha$; in fact, the relations:

$$(4)\quad \eta f = f\eta_E, \qquad g^+\eta = 1 = \eta_E g^+, \qquad \eta * \eta = 1_\eta,$$

are already known, or come from $g^+\eta = j^+p^+\eta = 1$, $\eta_E g^+ = \eta_E j^+p^+ = 1$. Similarly for the counits. □

4.4. Future 2-regularity. A point x in the 2-category X will be said to be V_2^+-*regular* if it satisfies (i), O_2^+-*regular* if it satisfies (ii) and future 2-regular if it satisfies both:

(i) for any pair of 2-cells $\alpha_i: a \to a_i: x \to x'$ $(i = 1, 2;$ see the left diagram below), there exists a pair of double cells ξ_i such that $\alpha_1 \otimes_v \xi_1 = \alpha_2 \otimes_v \xi_2$,

(ii) given three 2-cells $\alpha: a \to a': x \to x'$ and $\alpha_i: a' \to a'': x \to x'$ $(i = 1, 2;$ see the right diagram below) such that $\alpha \otimes_v \alpha_1 = \alpha \otimes_v \alpha_2$, there exists a 2-cell $\xi: (u \, \frac{a''}{\bar{a}} \, u')$ such that $\alpha_1 \otimes_v \xi = \alpha_2 \otimes_v \xi$,

(1)

On the other hand, we shall say that x is V_2^+-*branching* if it is not V_2^+-regular; that it is O_2^+-*branching* if it is not O_2^+-regular; that is a *future branching* if it falls in (at least) one of the previous cases, i.e. if it is not future regular. Dually, we have V_2^--*regular*, O_2^--*regular*, *past regular* points and the corresponding *branching* points.

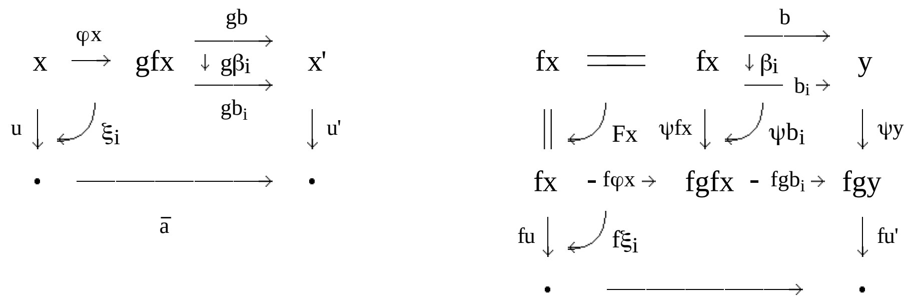
4.5. Theorem [Future 2-equivalences and regular points]. Given a future 2-equivalence $f: X \rightleftarrows Y :g$, with lax natural transformations $\varphi: 1 \to gf$, $\psi: 1 \to fg$ and comparisons $F: \psi f \to f\varphi$, $G: \varphi g \to g\psi$, we have:

(a) the functors f and g *preserve* V_2^+-regular, O_2^+-regular and future regular points,

(b) the functors f and g *preserve* V_2^+-branching, O_2^+-branching and future branching points (i.e. *reflect* V_2^+-regular, O_2^+-regular and future regular points),

Proof. The index i always takes values 1, 2.

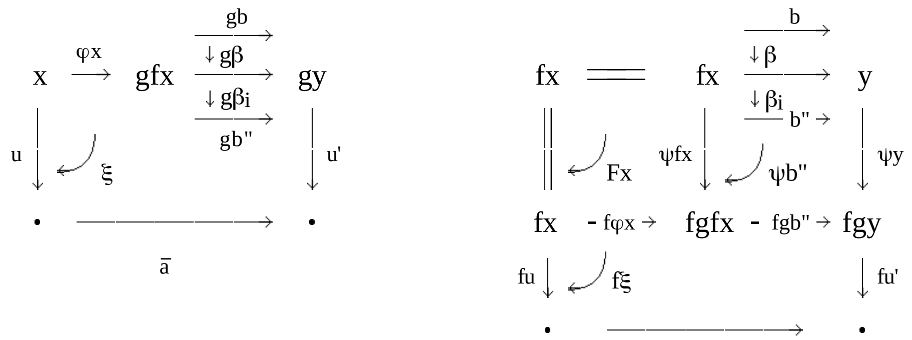
(a). Suppose that x is V_2^+-regular in X; we must prove that also fx is, in Y. Given a pair of 2-cells $\beta_i: b \to b_i: fx \to y$ in Y, as in the right diagram below, there exists in X a pair of double cells ξ_i as in the left diagram, such that $(g\beta_i.\varphi x) \otimes_v \xi_i = \xi$ (independently of i)

(1) 

Then, in the right diagram above, the double cells $\eta_i = (Fx \otimes_h \psi b_i) \otimes_v f\xi_i$ have the same vertical composition with β_i

(2) $\beta_i \otimes_v \eta_i = \beta_i \otimes_v (Fx \otimes_h \psi b_i) \otimes_v f\xi_i = (Fx \otimes_h (\beta_i \otimes_v \psi b_i)) \otimes_v f\xi_i = (Fx \otimes_h (\psi b \otimes_v fg\beta_i)) \otimes_v f\xi_i$
$\quad = (Fx \otimes_h \psi b) \otimes_v (fg\beta_i.f\varphi x) \otimes_v f\xi_i = (Fx \otimes_h \psi b) \otimes_v f\xi.$

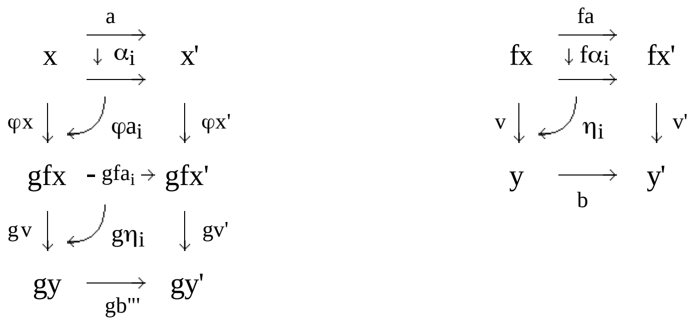
Second, suppose that x is O_2^+-regular in X, and let us prove the same of fx in Y. Take, in the right diagram below, three 2-cells $\beta: b \to b': fx \to y$ and $\beta_i: b' \to b''$, so that $\beta \otimes_v \beta_1 = \beta \otimes_v \beta_2$

(3) 

Then, in X, the composite $g\beta.\varphi x \otimes_v g\beta_i.\varphi x$ (which 'starts' at x) does not depend on i, and there exists a 2-cell ξ such that $(g\beta_i.\varphi x) \otimes_v \xi = \overline{\xi}$ (independently of i). One shows as previously that the double cell $\eta = (Fx \otimes_h \psi b'') \otimes_v f\xi$, in the right diagram above, has the same vertical composites with β_1, β_2.

(b) Assume that fx is V_2^+-regular in Y. Given a pair of 2-cells $\alpha_i: a \to a_i: x \to x'$ in the left diagram below, there exists in Y a pair of double cells η_i such that $f\alpha_i \otimes_v \eta_i = \eta$ (independently of i, see the right diagram)

(4)

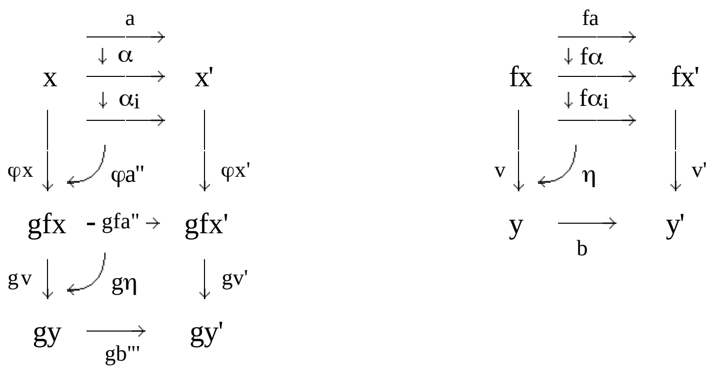


then, in the left diagram above, the double cells $\xi_i = \varphi a_i \otimes_v g\eta_i$ solve our condition for the pair α_i:

(5) $\alpha_i \otimes_v \varphi a_i \otimes_v g\eta_i \;=\; \varphi a \otimes_v gf\alpha_i \otimes_v g\eta_i \;=\; \varphi a \otimes_v g\eta.$

Finally, let fx be O_2^+-regular in Y and take, in X (as in the left diagram below), three 2-cells $\alpha: a \to a': x \to x'$ and $\alpha_i: a' \to a'': x \to x'$, so that $\alpha \otimes_v \alpha_1 = \alpha \otimes_v \alpha_2$; then there exists in Y a 2-cell η such that $f\alpha_1 \otimes_v \eta = f\alpha_2 \otimes_v \eta$. It follows, as previously, that in X the double cell $\xi = \varphi a'' \otimes_v g\eta$ has the same vertical composites with α_1, α_2

(6)



□

5. The fundamental 2-category of a preordered space

We define here the fundamental 2-category of a preordered space, and extend the construction to other settings. The index α takes values 0, 1 (written –, + in superscripts).

5.1. The structure of the directed interval. The directed interval $\uparrow$**I** is a lattice *in* p**Top**, with the following structural mappings: *faces* (δ^α), *degeneracy* (ε) *and connections* (γ^α, the binary operations of join and meet)

$$(1)\qquad \{*\} \underset{\varepsilon}{\overset{\delta^\alpha}{\rightleftarrows}} \uparrow\mathbf{I} \overset{\gamma^\alpha}{\longleftarrow} \uparrow\mathbf{I}^2$$

$$\delta^-(*) = 0, \qquad \delta^+(*) = 1, \qquad \gamma^-(t, t') = \max(t, t'), \qquad \gamma^+(t, t') = \min(t, t').$$

Actually, we are not interested in the complete axioms of lattices (e.g., the idempotence of the binary operations $\gamma^\pm$, or their full absorption laws), but only in a part of them, corresponding to a *cubical monoid* in the sense of [G1]: a set equipped with two structures of commutative monoid, so that the unit element of each is absorbent for the other. Formally, this means the following axioms (defining a cubical monoid in a monoidal category, with tensor product $\times$)

$$(2)\quad \varepsilon.\delta^\alpha = 1, \qquad \varepsilon.\gamma^\alpha = \varepsilon.(\varepsilon\times\uparrow\mathbf{I}) = \varepsilon.(\uparrow\mathbf{I}\times\varepsilon) \qquad (\textit{degeneracy}),$$

$$\gamma^\alpha.(\gamma^\alpha\times\uparrow\mathbf{I}) = \gamma^\alpha.(\uparrow\mathbf{I}\times\gamma^\alpha), \qquad \gamma^\alpha.(\delta^\alpha\times\uparrow\mathbf{I}) = 1 = \gamma^\alpha.(\uparrow\mathbf{I}\times\delta^\alpha) \qquad (\textit{associativity, unit}),$$

$$\gamma^\beta.(\delta^\alpha\times\uparrow\mathbf{I}) = \delta^\alpha.\varepsilon = \gamma^\beta.(\uparrow\mathbf{I}\times\delta^\alpha) \qquad (\textit{absorbency};\ \alpha \neq \beta).$$

Higher faces, degeneracies and connections are constructed from the structural maps, via the monoidal structure, for $1 \leq i \leq n$ and $\alpha = \pm$

$$(3)\quad \delta^\alpha_i = \uparrow\mathbf{I}^{i-1}\times\delta^\alpha\times\uparrow\mathbf{I}^{n-i}\colon \uparrow\mathbf{I}^{n-1} \to \uparrow\mathbf{I}^n, \qquad \varepsilon_i = \uparrow\mathbf{I}^{i-1}\times\varepsilon\times\uparrow\mathbf{I}^{n-i}\colon \uparrow\mathbf{I}^n \to \uparrow\mathbf{I}^{n-1},$$

$$\gamma^\alpha_i = \uparrow\mathbf{I}^{i-1}\times\gamma^\alpha\times\uparrow\mathbf{I}^{n-i}\colon \uparrow\mathbf{I}^{n+1} \to \uparrow\mathbf{I}^n \qquad (1 \leq i \leq n;\ \alpha = \pm),$$

and the *cocubical relations* (for faces, degeneracy and connections) follow from this construction and the axioms above, in (2):

$$(4)\quad \delta^\beta_j.\delta^\alpha_i = \delta^\alpha_{i+1}.\delta^\beta_j \quad (j \leq i), \qquad \varepsilon_i.\varepsilon_j = \varepsilon_j.\varepsilon_{i+1} \quad (j \leq i),$$

$$\varepsilon_j.\delta^\alpha_i = \delta^\alpha_{i-1}.\varepsilon_j \quad (j < i) \qquad \textit{or} \quad \mathrm{id} \quad (j = i) \qquad \textit{or} \quad \delta^\alpha_i.\varepsilon_{j-1} \quad (j > i),$$

$$\gamma^\beta_j.\gamma^\alpha_i = \gamma^\alpha_i.\gamma^\beta_{j+1} \quad (j > i), \qquad \gamma^\alpha_i.\gamma^\alpha_i = \gamma^\alpha_i.\gamma^\alpha_{i+1},$$

$$\varepsilon_j.\gamma^\alpha_i = \gamma^\alpha_{i-1}.\varepsilon_j \quad (j < i) \qquad \textit{or} \quad \varepsilon_i.\varepsilon_i \quad (j = i) \qquad \textit{or} \quad \gamma^\alpha_i.\varepsilon_{j+1} \quad (j > i),$$

$$\gamma^\beta_j.\delta^\alpha_i = \delta^\alpha_{i-1}.\gamma^\beta_j \quad (j < i-1) \qquad \textit{or} \quad \delta^\alpha_i.\gamma^\beta_{j-1} \quad (j > i),$$

$$\gamma^\alpha_i.\delta^\alpha_i = \mathrm{id} = \gamma^\alpha_i.\delta^\alpha_{i+1}, \qquad \gamma^\beta_i.\delta^\alpha_i = \delta^\alpha_i.\varepsilon_i = \gamma^\beta_i.\delta^\alpha_{i+1} \quad (\alpha \neq \beta).$$

5.2. The cubical set of a preordered space. Given a preordered space X, the previous structure of $\uparrow\mathbf{I}$ (and its powers, forming a cocubical object in p**Top**) produces a cubical set with connections $P_*(X)$

$$(1)\quad P_n(X) = \mathrm{p}\mathbf{Top}(\uparrow\mathbf{I}^n, X),$$

$$\partial^\alpha_i\colon P_n(X) \to P_{n-1}(X), \qquad \partial^\alpha_i(a) = a.\delta^\alpha_i\colon \uparrow\mathbf{I}^{n-1} \to X,$$

$$e_i\colon P_{n-1}(X) \to P_n(X), \qquad e_i(a) = a.\varepsilon_i\colon \uparrow\mathbf{I}^n \to X,$$

$$g^\alpha_i\colon P_n(X) \to P_{n+1}(X), \qquad g^\alpha_i(a) = a.\gamma^\alpha_i\colon \uparrow\mathbf{I}^{n+1} \to X,$$

satisfying the cubical relations (dual to the cocubical ones, listed above).

5.3. Moore paths and parallelepipeds. Let us form the *free cubical ω-category* $M_*(X)$ *on this cubical set* $P_*(X)$. A general item is a *Moore parallelepiped*, defined on a standard ordered n-parallelepiped (possibly degenerate)

(1) a: $\prod_{j=1,...n} \uparrow[0, p_j] \to X$ $(p_1,..., p_n \in \mathbf{N})$.

These maps form the component $M_n(X)$, with obvious faces and degeneracies

(2) $\partial_i^\alpha: M_n(X) \to M_{n-1}(X)$, $\partial_i^\alpha(a)(t_1,..., t_{n-1}) = a(t_1,..., \alpha p_i,..., t_{n-1})$,

$e_i: M_{n-1}(X) \to M_n(X)$, $e_i(a)(t_1,..., t_n) = a(t_1,..., \hat{t}_i,..., t_n)$,

respectively defined on the standard parallelepipeds defined by the natural numbers $(p_1,..., \hat{p}_i,..., p_n)$ and $(p_1,..., p_{i-1}, 0, ..., p_{n-1})$. The i-composition of two Moore parallelepipeds a, b (defined on $\prod \uparrow[0, p_j]$ and $\prod \uparrow[0, q_j]$), with $\partial_i^-(a) = \partial_i^+(a)$, is also obvious

(3) $$(a *_i b)(t_1,..., t_n) = \begin{cases} a(t_1,..., t_i,..., t_n) & (0 \leq t_i \leq p_i), \\ b(t_1,..., t_i - p_i,..., t_n) & (p_i \leq t_i \leq p_i + q_i), \end{cases}$$

with identities given by the degeneracies e_i.

This cubical set has 'pre-connections', whose 'degenerate' faces are constant (instead of being actual identities)

(4) $g_i^\alpha: M_n(X) \to M_{n+1}(X)$, $g_i^\alpha(a)(t_1,..., t_{n+1}) = a(t_1,..., \gamma^\alpha(t_i, t_{i+1}),..., t_{n+1})$,

where $g_i^\alpha(a)$ is defined on $\uparrow[0, p_1] \times ... \times \uparrow[0, p_i]^2 \times ... \times \uparrow[0, p_n]$.

Truncating $M_*(X)$ at dimension 2, we get a cubical 2-category (i.e., an edge-symmetric double category) with pre-connections

(5) $$M_0(X) \underset{e}{\overset{\partial^\alpha}{\leftleftarrows\!\!\rightarrow}} M_1(X) \;\leftleftarrows\!\!\rightrightarrows\; M_2(X),$$

Further, replacing $M_2(X)$ with $N_2(X) = M_2(X)/\simeq$ (modulo homotopy with fixed boundary) and leaving $N_i(X) = M_i(X)$ for $i = 0, 1$, we have again a cubical 2-category with pre-connections.

5.4. Congruences. Let C be a cubical 2-category. A *congruence* $R = (R_0, R_1, R_2)$ in C will be a triple of equivalence relations, one in each component C_0, C_1, C_2, which are:

(i) consistent with faces and degeneracies,

(ii) consistent with each i-composition law, in the following sense: if $a, b \in C_n$ and their faces ∂_i^+a and ∂_i^-b are equivalent (modulo R_{n-1}), then:

- there exist a', b' equivalent to a, b (modulo R_n) which are i-consecutive, i.e. $\partial_i^+a' = \partial_i^-b'$,

- if also a'', b'' are so, then the i-composites $a' *_i b'$ and $a'' *_i b''$ are R_n-equivalent.

Plainly, the quotient cubical set C/R (with components C_n/R_n, the induced faces and degeneracies) inherits well-defined i-composition laws, which make it into a cubical ω-category: the *quotient* of the cubical ω-category C modulo R.

5.5. The fundamental 2-category. Now, we form a double category with connections

(1) $D_*(X) = N_*(X)/R$,

identifying 'pre-identities' (cubes which are constant in some direction) with identities.

In other words, we define the congruence R (5.4) of $N_*(X)$ as follows. R_0 is the equality of $N_0(X) = |X|$; R_1 is the least equivalence relation of $N_1(X) = M_1(X)$ closed under concatenation which identifies every constant path $a: {\uparrow}[0, p] \to X$ with its faces $\{0\} \to X$; R_2 is the least equivalence relation of $N_2(X)$, closed under 1- and 2-concatenation, which identifies the class of any 'rectangle' $a: {\uparrow}[0, p] \times {\uparrow}[0, q]$, constant in direction i, with $e_i\partial_i^-a$.

The conditions 5.4 (i)-(ii) are satisfied: the only point which needs some comment is the first part of (ii), for n = 2. Assume that the 2-dimensional items $a, b: {\uparrow}[0, p] \times {\uparrow}[0, q] \to X$ have R_1-equivalent faces $\partial_2^+[a] = \partial_2^+a$, $\partial_2^-[b] = \partial_2^-b$; modifying the path $c = \partial_2^+a: {\uparrow}[0, p] \to X$ with the insertion of a constant portion at p' ($0 \leq p' \leq p$), of length $m \in \mathbf{N}$

$$(2) \quad c'(t) = \begin{cases} c(t) & (0 \leq t \leq p'), \\ c(p') & (p' \leq t \leq p'+m), \\ c(t-m) & (p'+m \leq t \leq p+m), \end{cases}$$

can be accompanied with a similar modification on a (in the first variable), obtaining a map

$$(3) \quad a': {\uparrow}[0, p+m] \times {\uparrow}[0, q] \to X, \qquad \partial_2^+a' = c'.$$

Continuing this way, we end with replacing a, b with equivalent items $\bar{a}, \bar{b}$ having $\partial_2^+\bar{a} = \partial_2^-\bar{b}$.

Now, the *fundamental cubical 2-category* of X is defined as the quotient $N_*(X)/R$. The fundamental 2-category is obtained in the usual way, restricting double cells to those whose faces in direction 2 (for instance) are trivial. We have thus a functor

$$(4) \quad {\uparrow}\Pi_2: p\mathbf{Top} \to 2\text{-}\mathbf{Cat}.$$

5.6. Other directed structures. In a preordered space, every loop is reversible (as already remarked in 1.1); therefore, this setting has no 'directed circle' or 'directed torus'.

We briefly recall more complex directed structures, which allow for non-reversible loops. All of them have a directed interval $\uparrow\mathbf{I}$ with the structure considered above, so that all the previous constructions can be easily extended. Also, all of them have a *reflection* $X \mapsto X^{op}$ extending the preorder-reversion of p**Top**.

First, one could extend p**Top** by some *local* notion of ordering. The simplest way is perhaps to consider spaces equipped with a relation $\prec$ which is reflexive and locally transitive: every point has some neighbourhood on which the relation is transitive [G2, 1.4] (similar, stronger properties have been frequently used in the theory of concurrency). But a relevant internal drawback appears, which makes this setting inadequate for directed homotopy and homology: *mapping cones and suspension are lacking*. Indeed, a locally preordered space cannot have a 'point-like vortex' (where all neighbourhoods of a point contain some non-reversible loop), whence it cannot realise the cone of the directed circle (as proved in detail in [G2, 4.6]).

5.7. Inequilogical spaces. Another setting for Directed Algebraic Topology comes from a directed version of Dana Scott's equilogical spaces (see [Sc, BBS, BCRS, Ro, Rs]), which was introduced in [G4].

An *inequilogical space*, or *preordered equilogical space* $X = (X^\#, \sim_X)$ is a *preordered* topological space $X^\#$ endowed with an equivalence relation $\sim_X$ (or $\sim$); the preorder relation is generally written as $\prec_X$. The quotient $|X| = X^\#/\sim$ is viewed as a preordered topological space (with the induced preorder and topology), or a topological space, or a set, as convenient.

A *map* f: X → Y 'is' a mapping f: |X| → |Y| which admits some *continuous preorder-preserving* lifting $f'\colon X^{\#} \to Y^{\#}$. Equivalently, a map is an equivalence class f = [f'] of maps f': X → Y in p**Top** which respect the equivalence relations

(1) $\forall\, x, x' \in X\colon\ x \sim_X x' \ .\!\Rightarrow.\ f'(x) \sim_Y f'(x')$,

under the associated *pointwise* equivalence relation

(2) $f' \sim f''$ if $(\forall\, x \in X\colon\ f'(x) \sim_Y f''(x))$.

Note that there are *no mutual conditions* among topology, preorder and equivalence relation.

This category will be denoted as p**Eql**. The previous setting p**Top** embeds as a full subcategory in p**Eql**, identifying a preordered space X with the pair $(X, =_X)$. The forgetful functor

(3) |−|: p**Eql** → p**Top**, $|X| = X^{\#}/\!\sim_X$,

with values in preordered topological spaces (or spaces, or sets, *when convenient*) has already been defined, implicitly; it sends the map f: X → Y to the underlying mapping f: |X| → |Y| (also written |f|). A *point* x: {∗} → X is an element of the *underlying space* |X|.

Reversing the preorder relation gives the *reflected*, or *opposite*, inequilogical space X^{op}. This category has all limits and colimits, and is Cartesian closed (like the one of equilogical spaces). Directed homotopy is defined by the standard directed interval ↑**I**. Various models for the directed circle are considered in [G4]; the simplest is perhaps $(\uparrow\mathbf{R}, \equiv_{\mathbf{Z}})$, i.e. the quotient *in* p**Eql** of the *directed* real line modulo the action of the group of integers.

5.8. Spaces with distinguished paths. An even more general setting has been studied in [G2].

A *d-space* X = (X, dX) is a topological space equipped with a set dX of (continuous) maps a: **I** → X; these maps, called *directed paths* or *d-paths*, must contain all constant paths and be closed under concatenation and (weakly) increasing reparametrisation.

A *d-map* f: X → Y (or *map* of d-spaces) is a continuous mapping between d-spaces which preserves the directed paths: if a ∈ dX, then fa ∈ dY.

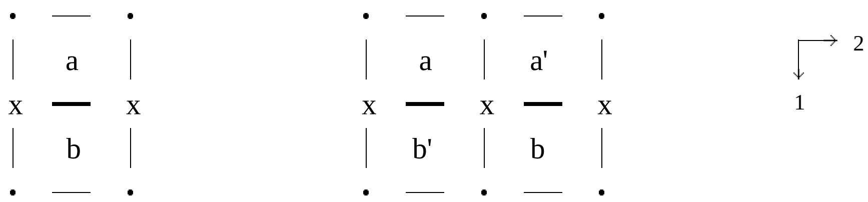
The category of d-spaces is written as d**Top**. It has all limits and colimits, constructed as in **Top** and equipped with the initial or final d-structure for the structural maps; for instance a path $\mathbf{I} \to \Pi X_i$ is directed if and only if all its components $\mathbf{I} \to X_i$ are so. The forgetful functor U: d**Top** → **Top** preserves thus all limits and colimits; a topological space is generally viewed as a d-space by its *natural* structure, where all (continuous) paths are directed (via the right adjoint to U).

Reversing d-paths, by the involution r(t) = 1 – t, yields the *reflected*, or *opposite*, d-space RX = X^{op}, where $a \in d(X^{op})$ if and only if $a^{op} = ar \in dX$.

Also here, d**Top** has all limits and colimits (constructed as in **Top** and equipped with the initial or final d-structure for the structural maps). The *standard d-interval* ↑**I** = ↑[0, 1] has directed paths given by the (weakly) increasing maps **I** → **I**. The *standard directed circle* $\uparrow\mathbf{S}^1 = \uparrow\mathbf{I}/\partial\mathbf{I}$ has the obvious d-structure, where path have to follow a precise orientation (but note that the directed structure $\uparrow\mathbf{S}^1\times\uparrow\mathbf{S}^1$ on the torus has nothing to do with orientation).

5.9. Geometrical aspects of the congruence. Defining higher fundamental categories $\uparrow\Pi_n(X)$ with $n > 2$ is even more complicated. In [G3], we considered that the problem might be solved by dividing Moore parallelepipeds modulo delays in each variable. However, this is not consistent with concatenations.

For instance, consider two cubes $a, b \in C_2$ with a common degenerate face $\partial_1^+ a = \partial_1^- b = e_1(x)$ (represented below as a thick segment). Then, their concatenation $c = a *_1 b$ is R_2-equivalent to the pasting $c' = (a *_2 a') *_1 (b' *_2 b)$, also represented below, where a' and b' are constant in direction 2

(1) 

Now, $a \equiv_2 (a *_2 a')$ and $b \equiv_2 (b' *_2 b)$, but c and c' are not equivalent modulo delays, in general. Note also that such a modification of c into c', by a sort of 'generalised delay', requires a constraint on the common face $\partial_1^+ a = \partial_1^- b$ (being degenerate). Thus, a global description of R_n (as 'attempted' in [G3]) should be very complicated.

6. Modelling a fundamental 2-category

We study the fundamental 2-category of the 'hollow cube' $X \subset \uparrow[0, 1]^3$ (1.5.1), starting from a future 2-equivalent model. The canonical basis of the vector space $\mathbf{R}^3$ is written $e_1 = (1, 0, 0)$, etc.

6.1. Bi-affine maps. We shall need to consider *biaffine* maps $\alpha: \mathbf{I}^2 \to \mathbf{R}^n$, i.e. mappings which are affine in both variables (on the standard square). Such a map gives a four-tuple of points, the images of the four vertices of the standard square, which will be called the *vertices* of the map

(1) $p_{ij} = \alpha(i, j)$ $\qquad\qquad (i, j) \in \{0, 1\}^2$.

The correspondence is bijective: given an arbitrary four-tuple (p_{ij}) of points in $\mathbf{R}^n$, the biaffine map is reconstructed by the following formula

(2) $\alpha(t_1, t_2) = (1 - t_1)(1 - t_2)\, p_{00} + t_1(1 - t_2)\, p_{10} + (1 - t_1)t_2\, p_{01} + t_1 t_2\, p_{11}$.

Moreover, we get a map $\alpha: \uparrow\mathbf{I}^2 \to \uparrow\mathbf{R}^n$ (*preserving* the canonical orders) if and only if, in $\mathbf{R}^n$:

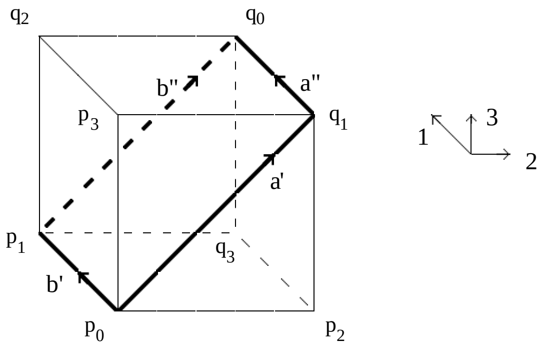
(3) $p_{00} \leq p_{10} \leq p_{11}$, $\qquad\qquad p_{00} \leq p_{01} \leq p_{11}$,

(if and only it the mapping $p: \{0, 1\}^2 \to \mathbf{R}^n$ is order-preserving).

6.2. Studying the singularity. Now, the singularity in the 'hollow cube' 1.5.1 is made evident by the existence of two different cells $\alpha, \beta: a \to b: p_0 \to q_0$ (on the 'internal boundary'), for suitable paths a, b, from $p_0 = (1/3, 1/3, 1/3)$ to $q_0 = (2/3, 2/3, 2/3)$.

Let us construct an example. The images of the cells α, β, in the picture below, cover the faces of the cubic hole: its upper-left and its lower-right half, respectively; these parts of the boundary are separated by the paths a, b (and the vectors e_i form the canonical basis of $\mathbf{R}^3$)

(1)



$p_i = p_0 + e_i/3,$

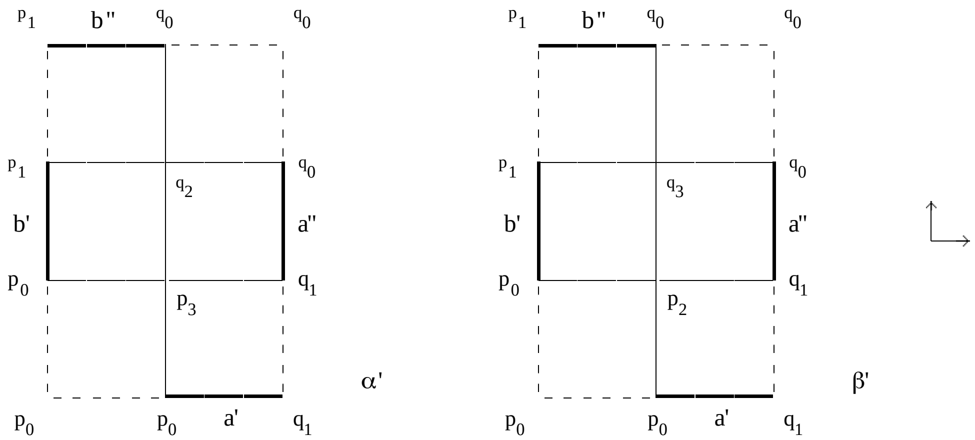
$q_i = q_0 - e_i/3,$

$a = a' * a'',$

$b = b' * b''.$

More precisely, the 2-cells α, β: a $\to$ b: $p_0 \to q_0$ can be obtained from the following *double* cells α', β': $(b' \,{}^{a'}_{b''}\, a'')$ (using connections, cf. 5.1)

(2)



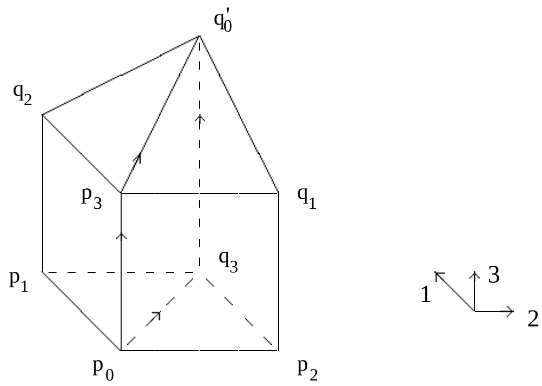
The cell α' is piecewise biaffine, made up of the pasting of six biaffine maps $\uparrow\mathbf{I}^2 \to X$, each of them determined by its four vertices in X (as specified in the left diagram); similarly for β'. Moreover, the mappings α', β' are constant on the dotted lines, while the thick lines correspond to the paths a, b. (One could add connections to explicitly describe α and β, but this would complicate the picture rather than making it clearer.)

Let us rename the paths a, b as a_1, b_1 (they go through q_1 and p_1) and the cells as α_{12} (through q_2) and α_{13} (through q_3). Permuting coordinates, we get two other similar pairs of cells (each linking a pair of paths)

(3) $\alpha_{ij}: a_i \to b_i: p_0 \to q_0$ (i, j = 1, 2, 3; $i \neq j$).

Actually, we shall need a slightly more general construct, where we replace q_0 with any $q'_0 \geq q_0$. Thus $\alpha_{31}, \alpha_{32}: a_3 \to b_3$ go from p_0 to q'_0

(4)



$a_3 = a'_3 * a''_3$

$b_3 = b'_3 * b''_3$

6.3. The future model. We define now a sub-2-category C^+ of $C_2 = \uparrow\Pi_2(X)$ which will be proved to be future 2-equivalent to C_2 (and corresponds to the future model F of the 'hollow square', in 1.4.1).

C^+ has 8 objects, forming a set V^+: the seven vertices of the cubic hole different from $q_0 = (2/3, 2/3, 2/3)$, plus q'_0, the maximum of X

(1) $p_0 = (1/3, 1/3, 1/3)$, $\quad p_i = p_0 + e_i/3$,

$q_i = q_0 - e_i/3$, $\quad q'_0 = (1, 1, 1)$ $\quad (i = 1, 2, 3)$.

After identities, it has the following twelve affine paths (determined by their vertices)

(2) $a'_i \colon p_0 \to q_i$, $\quad a''_i \colon q_i \to q'_0$,

$b'_i \colon p_0 \to p_i$, $\quad b''_i \colon p_i \to q'_0$ $\quad (i = 1, 2, 3)$,

and their six composites, piecewise affine (more precisely, their equivalence classes modulo delays, cf. 5.5)

(3) $a_i = a'_i * a''_i \colon p_0 \to q'_0$, $\quad b_i = b'_i * b''_i \colon p_0 \to q'_0$.

Finally, C^+ is locally full in C_2, which means that it has twelve non-trivial cells, precisely the six α_{ij} analytically defined in 6.2 and six other cells γ_{ij} determined by their boundary

(4) $\alpha_{ij} \colon a_i \to b_i \colon p_0 \to q'_0$,

$\gamma_{ij} \colon a_i \to b_j \colon p_0 \to q'_0$ $\quad (i, j = 1, 2, 3;\ i \neq j)$.

6.4. The retraction. Let us consider a partition of the space X into 8 zones: the points which are below precisely *one* vertex in V^+

(1) $P_0 = \downarrow p_0$, $\quad P_i = \downarrow p_i \setminus \downarrow p_0$,

$Q_1 = \downarrow q_1 \setminus (\downarrow p_2 \cup \downarrow p_3)$, $\quad Q_2 = ...,$ $\quad Q_3 = ...,$

$Q = X \setminus (\downarrow q_1 \cup \downarrow q_2 \cup \downarrow q_3)$.

Now, the 2-functor $p \colon C_2 \to C^+$ is defined as follows. It sends each point $x \in X$ to

(2) $p(x) = \min \{v \in V^+ \mid x \leq v\}$,

Let $a \colon x' \to x''$. Then there is a unique path $p(a) \colon p(x') \to p(x'')$ in C^+, excepting the case when $p(x') = p_0$ and $p(x'') = q'_0$; in this case $p(a)$ is defined to be:

(3) $a_i = a'_i * a''_i$, if a, when leaving P_0, enters Q_i,

$b_i = b'_i * b''_i$, if a, when leaving P_0, enters P_i.

Let $\alpha: a \to b: x' \to x''$. Then, there is a unique cell $p(\alpha): p(a) \to p(b)$ in C^+, unless $p(a) = a_i$ and $p(b) = b_i$ (with the same index i); in this case, there are two such cells, and we define $p(\alpha)$ as

(4) $\alpha_{ij}: a_i \to b_i$, if α meets Q_j $(j \neq i)$.

6.5. The future equivalence. The inclusion $f: C^+ \to C_2$ forms, with p, a future 2-retract

(1) $f: C^+ \rightleftarrows C_2 :p$ $\qquad pf = 1_{C^+}$, $\qquad \eta: 1_{C_2} \to fp$,

We define the lax natural transformation η. For $x \in X$, let $\eta x: x \to p(x)$ be the affine path with these endpoints (contained in the down-set $\downarrow p(x)$); for $a: x' \to x''$, let $\eta a: \eta x''.a \to pa.\eta x'$ be the 2-cell associated to the biaffine double cell $\hat{\eta}a$ (as in the left diagram below)

(2)
$$\begin{array}{ccc} x' & \xrightarrow{a} & x'' \\ {\scriptstyle \eta x'}\downarrow & \hat{\eta}a & \downarrow{\scriptstyle \eta x''} \\ px' & \xrightarrow[pa]{} & px'' \end{array} \qquad\qquad \begin{array}{ccc} x & \xrightarrow{\eta x} & px \\ {\scriptstyle \eta x}\downarrow & \hat{\eta}\eta x & \downarrow{\scriptstyle 1} \\ px & \xrightarrow[1]{} & px \end{array}$$

Plainly, $p\eta = 1_p$ and $\eta f = 1_f$; finally, the coherence condition 3.3.2 $(\eta * \eta = 1_\eta)$ is satisfied, since $\hat{\eta}\eta x$ (see the right diagram above) is the lower connection on the path ηx.

6.6. The past model. Symmetrically, we have a past 2-retract C^- of C_2, whose set of objects V^- consists of the seven vertices of the cubic hole different from $p_0 = (1/3, 1/3, 1/3)$, together with the minimum p'_0

(1) $p'_0 = (0, 0, 0)$, $\qquad p_i = p_0 + e_i/3$,

$q_i = q_0 - e_i/3$, $\qquad q_0 = (2/3, 2/3, 2/3)$ $\qquad (i = 1, 2, 3)$.

6.7. A global 2-model. Finally, the pf-2-presentation of C_2 by the future 2-retract C^+ and the past 2-retract C^- generates an injective 2-model E (4.3), on ten objects

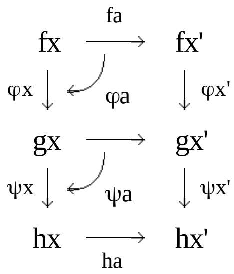
(1) $V = V^- \cup V^+ = \{p'_0, p_i, q_i, q'_0\}$ $\qquad (i = 0,..., 4)$.

7. Appendix: the calculus of lax natural transformations and modifications

We end with reviewing the various compositions of the notions recalled in Section 2. Again, we always consider (strict) 2-functors between 2-categories.

7.1. Composing transformations. (a) Given two lax natural transformations $\varphi: f \to g$, $\psi: g \to h$, the (*main*) *composition* $\psi\varphi: f \to h$ has components produced by the vertical composition of double cells in Y, and is therefore strictly associative, with strict identities

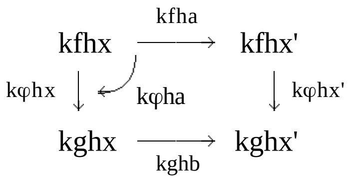
(1) $(\psi\varphi)x = \psi x.\varphi x$, $\qquad (\psi\varphi)a = \varphi a \otimes_v \psi a$,



It is easy to see that φ: f $\to$ g is invertible (for this composition) if and only if all its components φx: fx $\to$ gx and φa: φx'.fa $\to$ ga.φx are invertible in Y. Then, one defines the inverse ψ: g $\to$ f letting $\psi x = (\varphi x)^{-1}$ and $\psi a = \psi x'.(\varphi a)^{-1}.\psi x$.

(b) The *whisker composition* kφh: kfh $\to$ kgh (for h: X' $\to$ X, k: Y $\to$ Y') has the obvious components, produced by evaluation

(2) (kφh)x = kφhx, (kφh)a = kφha (a: x $\to$ x' in X'),



We have thus a sesquicategory **Lnt** of 2-categories, 2-functors and lax natural transformations, which is not a 2-category: the reduced interchange axiom (2.1) does not hold (see also 7.3).

7.2. Composing modifications. (a) First, we have an obvious *whisker composition* of modifications and 2-functors (for M: $\varphi \to \psi$: f $\to$ g: X $\to$ Y, h: X' $\to$ X and k: Y $\to$ Y')

(1) kMh: kφh $\to$ kψh: kfh $\to$ kgh: X' $\to$ Y'.

But, given the 2-categories X, Y, the important construct is the 2-category **Lnt**(X, Y) of 2-functors X $\to$ Y, their lax natural transformations and modifications. Indeed, modifications have a *main composition*

(2) M: $\varphi \to \psi$: f $\to$ g: X $\to$ Y, N: $\psi \to \xi$: f $\to$ g: X $\to$ Y,

NM: $\varphi \to \xi$, (NM)x = Nx.Mx: φx $\to$ ψx $\to$ ξx,

which is strictly associative, with obvious identities 1_φ. (For this law, the modification M is *invertible* if and only if all its component cells Mx: φx $\to$ ψx are invertible in Y, and then $M^{-1}(x) = (Mx)^{-1}$.)

(b) Moreover, we have an obvious *whisker composition* of modifications and lax natural transformations, where λ: f' $\to$ f and μ: g $\to$ g'

(3) $\mu M\lambda$: $\mu\varphi\lambda \to \mu\psi\lambda$: f' $\to$ g': X $\to$ Y,

($\mu M\lambda$)x = μx.Mx.λx: μx.φx.λx $\to$ μx.ψx.λx: f'x $\to$ g'x,

$$f'x \xrightarrow{\lambda x} fx \underset{\psi x}{\overset{\varphi x}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow Mx}}}} gx \xrightarrow{\mu x} g'x$$

The reduced interchange axiom holds, as soon as it holds in Y

(4) $$f \underset{\psi}{\overset{\varphi}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow M}}}} g \underset{\sigma}{\overset{\rho}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow R}}}} h \qquad\qquad R\psi.\rho M \;=\; \sigma M.R\varphi,$$

since its verification on a general component depends on the same property in the latter 2-category:

(5) $$fx \underset{\psi x}{\overset{\varphi x}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow Mx}}}} gx \underset{\sigma x}{\overset{\rho x}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow Rx}}}} hx.$$

7.3. Higher compositions. Coming back to **Lnt**, we already observed that this sesquicategory is *not* a 2-category. But one can define a *graded composition of lax natural transformations* $\rho*\varphi$, *as a modification*. In fact, every object x is taken to an arrow $\varphi x\colon fx \to gx$ of Y, and then to a cell $\rho(\varphi x)$ of Z

(1) $$X \underset{g}{\overset{f}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow \varphi}}}} Y \underset{s}{\overset{r}{\underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow \rho}}}} Z \qquad\qquad \begin{array}{ccc} rfx & \xrightarrow{r\varphi x} & rgx \\ {\scriptstyle \rho fx}\downarrow & \swarrow\, \rho\varphi x & \downarrow{\scriptstyle \rho gx} \\ sfx & \xrightarrow[s\varphi x]{} & sgx \end{array}$$

$$\rho*\varphi\colon \rho g.r\varphi \to s\varphi.\rho f, \qquad\qquad (\rho*\varphi)x \;=\; \rho(\varphi x)\colon\; \rho gx.r\varphi x \;\to\; s\varphi x.\rho fx\colon\; rfx \to sgx.$$

To verify the axiom (mdf), take a map $a\colon x \to x'$ in X. Then:

(2) $$\begin{array}{ccccc} rfx & = & rfx & \xrightarrow{rfa} & rfx' \\ {\scriptstyle \rho fx}\downarrow & & {\scriptstyle r\varphi x}\downarrow & \swarrow\, r\varphi a & \downarrow{\scriptstyle r\varphi x'} \\ sfx & \overset{\rho\varphi x}{\longleftarrow} & rgx & \longrightarrow & rgx' \\ {\scriptstyle s\varphi x}\downarrow & & {\scriptstyle \rho gx}\downarrow & \swarrow\, \rho ga & \downarrow{\scriptstyle \rho gx'} \\ sgx & = & sgx & \xrightarrow[sga]{} & sgx' \end{array} \quad = \quad \begin{array}{ccccc} rfx & \xrightarrow{rfa} & rfx' & = & rfx' \\ {\scriptstyle \rho fx}\downarrow & \swarrow\, \rho fa & \downarrow{\scriptstyle \rho fx'} & & \downarrow{\scriptstyle r\varphi x'} \\ sfx & \longrightarrow & sfx' & \overset{\rho\varphi x'}{\longleftarrow} & rgx' \\ {\scriptstyle s\varphi x}\downarrow & \swarrow\, s\varphi a & \downarrow{\scriptstyle s\varphi x'} & & \downarrow{\scriptstyle \rho gx'} \\ sgx & \xrightarrow[sga]{} & sgx' & = & sgx' \end{array}$$

In fact, applying (lnt.2) to the lax natural transformation ρ, the left and the right pasting give, respectively

(3) $r\varphi a \otimes_V \rho(ga.\varphi x)$, $\qquad\qquad$ $\rho(\varphi x'.fa) \otimes_V s\varphi a$,

and these result coincide, by (lnt.3), on ρ and the cell $\varphi a\colon \varphi x'.fa \to ga.\varphi x$.

Then, in the following situation (with $\psi\colon g \to h$ and $\sigma\colon s \to t$), we have

$$(4)\qquad \begin{array}{ccccc} & \overset{f}{\longrightarrow} & & \overset{r}{\longrightarrow} & \\ & \downarrow\varphi & & \downarrow\rho & \\ X & \longrightarrow & Y & \longrightarrow & Z \\ & \downarrow\psi & & \downarrow\sigma & \\ & \underset{h}{\longrightarrow} & & \underset{t}{\longrightarrow} & \end{array}$$

$$(5)\qquad \begin{array}{ccccc} rf & \xrightarrow{r\varphi} & rg & \xrightarrow{r\psi} & rh \\ \rho f\downarrow & \swarrow \rho*\varphi & \rho g\downarrow & \swarrow \rho*\psi & \downarrow\rho h \\ sf & \xrightarrow{s\varphi} & sg & \xrightarrow{s\psi} & sh \\ \sigma f\downarrow & \swarrow \sigma*\varphi & \sigma g\downarrow & \swarrow \sigma*\psi & \downarrow\sigma h \\ tf & \underset{t\varphi}{\longrightarrow} & tg & \underset{t\psi}{\longrightarrow} & th \end{array} \qquad = \qquad (\psi\varphi)*(\sigma\rho).$$

7.4. Higher compositions, II. One can also define a *higher whisker composition* of lax natural transformations and modifications.

(a) First, $\vartheta\circ M$ will be the following modification (whose general cell, shown in (2), is well defined because of (lnt.3), applied to ϑ on the cell Mx)

$$(1)\qquad \begin{array}{ccccc} & \overset{f}{\longrightarrow} & & \overset{h}{\longrightarrow} & \\ X & \varphi\downarrow\ M\ \downarrow\psi & Y & \downarrow\vartheta & Z \\ & \underset{g}{\longrightarrow} & & \underset{k}{\longrightarrow} & \end{array} \qquad\qquad \vartheta\circ M\colon \vartheta g.h\psi \to k\varphi.\vartheta f,$$

$$(2)\qquad \begin{array}{ccc} hfx & \xrightarrow{h\varphi x} & hgx \\ \vartheta fx\downarrow & \swarrow \vartheta\varphi x & \downarrow\vartheta gx \\ kfx & \longrightarrow & kgx \\ \| & \swarrow kMx & \| \\ kfx & \underset{k\psi x}{\longrightarrow} & kgx \end{array} \qquad = \qquad \begin{array}{ccc} hfx & \xrightarrow{h\varphi x} & hgx \\ \| & \swarrow hMx & \| \\ hfx & \longrightarrow & hgx \\ \vartheta fx\downarrow & \swarrow \vartheta\psi x & \downarrow\vartheta gx \\ kfx & \underset{k\psi x}{\longrightarrow} & kgx \end{array}$$

(b) Second, in the situation below, $M\circ\vartheta$ is defined as follows (whose general cell, shown in (4), is well defined because of (mdf), applied to M on the arrow $\vartheta z\colon hz \to kz$)

$$(3)\qquad \begin{array}{ccccc} & \overset{h}{\longrightarrow} & & \overset{f}{\longrightarrow} & \\ Z & \downarrow\vartheta & X & \varphi\downarrow\ M\ \downarrow\psi & Y \\ & \underset{k}{\longrightarrow} & & \underset{g}{\longrightarrow} & \end{array} \qquad\qquad M\circ\vartheta\colon \varphi k.f\vartheta \to g\vartheta.\psi h,$$

$$(4)\qquad \begin{array}{ccccc} fhz & = & fhz & \xrightarrow{f\vartheta z} & fkz \\ \psi hz\downarrow & \swarrow Mhz & \varphi hz\downarrow & \swarrow \varphi\vartheta z & \downarrow\varphi kz \\ ghz & = & ghz & \underset{g\vartheta z}{\longrightarrow} & gkz \end{array} \qquad = \qquad \begin{array}{ccccc} fhz & \xrightarrow{f\vartheta z} & fkz & = & fkz \\ \psi hz\downarrow & \swarrow \psi\vartheta z & \psi kz\downarrow & \swarrow Mkz & \downarrow\varphi kz \\ ghz & \underset{g\vartheta z}{\longrightarrow} & gkz & = & fkz \end{array}$$

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